

Matrix monotone and convex functions

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Chapter 1

Introduction

1.1 Foreword

This master's thesis is about matrix monotone and convex functions. Matrix monotonicity and convexity are generalizations of standard monotonicity and convexity of real functions: now we are just having functions mapping matrices to matrices. Formally, f is *matrix monotone* if for any two matrices A and B such that

$$(1.1) \quad A \leq B$$

we should also have

$$(1.2) \quad f(A) \leq f(B).$$

This kind of function might be more properly called *matrix increasing* but we will mostly stick to the monotonicity for couple of reasons:

- For some reason, that is what people have been doing in the field.
- It doesn't make much difference whether we talk about increasing or decreasing functions, so we might just ignore the latter but try to symmetrize our thinking by choice of words.
- Somehow I can't satisfactorily fill the following table:

monotonic	monotonicity
increasing	?

How very inconvenient.

Matrix convexity, as you might have guessed by now, is defined as follows. A function f is *matrix convex* if for any two matrices A and B and $0 \leq t \leq 1$ we have

$$(1.3) \quad f(tA + (1-t)B) \leq tf(A) + (1-t)f(B).$$

Of course, it's not really obvious how one should make any sense of these "definitions". One quickly realizes that there two things to understand.

- How should matrices be ordered?
- How should functions act on matrices?

Both of these questions can be (of course) answered in many ways, but for both of them, there's in a way very natural answer. In both cases we can get something more general: instead of comparing matrices we can compare linear maps, and we can apply function to linear mapping.

Just to give a short glimpse of how these things might be defined, we should first fix our ground field (for matrices): let's say it's \mathbb{R} , at least for now.

For matrix ordering we should first understand which matrices are *positive*, which here, a bit confusingly maybe, means "at least zero". We say that matrix is positive if all it's eigenvalues are non-negative. Having done this, we immediately restrict ourselves to (symmetric) diagonalizable matrices with real eigenvalues, but we will later see that we can't do much "better". Also, since sum of positive matrices should be positive, we should further restrict ourselves to even stricter class of matrices, called Hermitian matrices, which correspond self-adjoint linear maps. Now everything works nicely but we still preserve non-trivial non-commutative structure.

Matrix functions, i.e. "how to apply function to matrix" is bit simpler to explain. Instead of doing something arbitrary the idea is to take real function (a function $f : \mathbb{R} \rightarrow \mathbb{R}$, say) and intepret it as function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$, *matrix function*. Polynomials extend rather naturally, and similarly analytic functions, or at least entire. Now, a perverse definition for matrix function for continuous functions would be some kind of a limit when function is uniformly approximated by polynomials (using Weierstrass approximation theorem). This works for Hermitian matrices, but one can do better: apply the function to the eigenvalues of the mapping to get another linear map.

As it turns out, much of the study of matrix monotone and convex functions is all about understanding these definitions of positive maps and matrix functions.

Lastly, one might wonder why should one be interested in the whole business of monotone and convex functions? It's all about point of view. Let's consider a very simple inequality:

For any real numbers $0 < x \leq y$ we have

$$y^{-1} \leq x^{-1}.$$

Of course, this is quite close to the axioms of the real numbers, but there's a rather fruitful interpretation. The function $(x \mapsto \frac{1}{x})$ is decreasing.

Now there's this matrix version of the previous inequality:

For any two matrices $0 < A \leq B$ we have

$$B^{-1} \leq A^{-1}.$$

This is already not trivial, and with previous interpretation in mind, could this be interpreted as the functions $(x \mapsto \frac{1}{x})$ could be *matrix decreasing*? And is this just a special case of something bigger? Yes, and that's exactly what this thesis is about.

1.2 Plan of attack

This master's thesis is a comprehensive review of the rich theory of matrix monotone and convex functions.

Master's thesis is to be structured roughly as follows.

1. Introduction

- Introduction to the problem, motivation
- Brief definition of the matrix monotonicity and convexity
- Past and present
 - Loewner's original work, Loewner-Heinz -inequality
 - Students: Dobsch' and Krauss'
 - Subsequent simplifications and further results: Bendat-Sherman, Wigner-Neumann, Koranyi, etc.
 - Donoghue's work
 - Later proofs: Krein-Milman, general spectral theorem, interpolation spaces, short proofs etc.
 - Development of the convex case
 - Recent simplifications, integral representations
 - Operator inequalities
 - Multivariate case, other variants
 - Further open problems?
- Scope of the thesis

2. Preliminaries (partially to be dumped to appendix?)

- Divided differences: basic definition, properties, representations and smoothness
- Positive matrices
 - Setup: finite (vs infinite) dimensional inner product spaces over \mathbb{C} (vs \mathbb{R}), basic facts
 - Linear maps, adjoint, congruence, self-adjoint maps, spectral theorem: finite and infinite dimensional
 - Good properties of spectrum
 - Positive maps: basic properties (cone structure, Sylvester's criterion etc.)
- Matrix functions
 - Several definitions
 - Derivatives of matrix functions
- Divided differences: basic definition, properties, representations and smoothness
- Pick functions
 - Basic definitions and properties
 - Pick-Nevanlinna representations theorem
 - Pick matrices/ determinants
 - Compactness
 - Pick-Nevanlinna interpolation theorem
- Regularizations
 - Basic properties
 - Lemmas needed for some of the proofs

3. Monotonic and convex matrix functions

- Basics
 - Basic definitions and properties (cone structure, pointwise limits, compositions etc.)
 - Classes P_n, K_n and their properties
 - $-1/x$
 - One directions of Loewner's theorem
 - Examples and non-examples
- Pick matrices/determinants vs matrix monotone and convex functions

- Proofs for (sufficiently) smooth functions
 - Smoothness properties
 - Ideas, simple cases
 - General case by induction and regularizations
 - Global characterizations
 - Putting everything together: we get original characterization of Loewner and determinant characterization
4. Local characterizations
- Dobsch (Hankel) matrix: basic properties, easy direction (original and new proof)
 - Integral representations
 - Introducing the general weight functions for monotonicity and convexity (and beyond?)
 - Non-negativity of the weights
 - Proof of integral representations
 - Proof of local characterizations
5. Structure of the classes P_n and K_n , interpolating properties (?)
- Strict inclusions, strict smoothness conditions
 - Strictly increasing functions
 - Extreme values
 - Interpolating properties
6. Loewner's theorem
- Preliminary discussion, relation to operator monotone functions
 - Loewner's original proof
 - Pick-Nevanlinna proof
 - Bendat-Sherman proof
 - Krein-Milman proof
 - Koranyi proof
 - Discussion of the proofs

- Convex case
7. Alternative characterizations (?)
 - Some discussion, maybe proofs
 8. Bounded variations (?)
 - Dobsch' definition, basic properties
 - Decomposition, Dobsch' theorems

1.3 Some random ideas

1. It's easy to see that [Something]. Actually, it's so so easy that we have no excuse for not doing it.
2. TODO: Maximum of two matrices (at least as big), $(a + b)/2 + \text{abs}(a - b)/2$

Chapter 2

Preliminaries

As mentioned in the introduction, in order to understand matrix monotone and convex functions we first have to understand positive matrices and matrix functions (and some other things).

2.1 Positive matrices

This section is titled “positive matrices”, although “positive maps” might be more appropriate title. We are mostly going to deal with finite-dimensional objects, but many of the ideas could be generalized infinite-dimensional settings, where matrices lose their edge. Also, one should always ask whether it really clarifies the situation to introduce concrete matrices: matrices are good at hiding the truly important properties of linear mappings. The words “matrix” and “linear map” are used somewhat synonymously, although one should always remember that the former are just special representations for the latter.

2.1.1 Motivation

How should one order matrices? What should we require from ordering anyway?

I could just give you the answer, but instead I try to explain why it is standard in the first place.

We would definitely like to have natural total order on the space of matrices, but it turns out that there are no natural choices for that. Partial order is next best thing. Recall that a partial order on a set X is a binary relation \leq on such that

1. $x \leq x$ for any $x \in X$.
2. For any $x, y \in X$ for which $x \leq y$ and $y \leq x$, necessarily $x = y$.

3. If for some $x, y, z \in X$ we have both $x \leq y$ and $y \leq z$, also $x \leq z$.

The third point is the main point, the first two are just there preventing us from doing something crazy. But we can do better: this partial order on matrices should also respect addition.

4. For any $x, y, z \in X$ such that $x \leq y$, we should also have $x + z \leq y + z$.

There's another way to think about this last point. Instead of specifying order among all the pairs, we just say which matrices are positive: matrix is positive if and only if it's at least 0.

If we know all the positive matrices, we know all the "orderings". To figure out whether $x \leq y$, we just check whether $0 = x - x \leq y - x$, i.e. whether $y - x$ is positive. Also, positive matrices are just differences of the form $y - x$ where $x \leq y$. Now, conditions on the partial order are reflected to the set of positive matrices.

1'. 0 (zero matrix) is positive.

2'. If both x and $-x$ are positive, then $x = 0$.

3'. If both x and y are positive, so is their sum $x + y$.

Here 3' is kind of combination of 3 and 4.

The terminology here is rather unfortunate. Natural ordering of the reals satisfies all of the above with obvious interpretation of positive numbers, which however differs from the standard definition: 0 is itself positive in our above definition. This is undoubtedly confusing, but what can you do? For real numbers we have total order, so every number is either zero, strictly positive or strictly negative, so when we say non-negative, it literally means "not negative": we get all the positive numbers and zero. But with partial orders we might get more. So the main reasons why we are using this terminology are

1. It's short.

Also, now that we have decided to preserve the word "positive" for "at least zero" one might be tempted to preserve "strictly positive" for "at least zero, but not zero". We won't do that, we save that phrase for something more important.

Back to business. When we try to define ordering of the matrices, everything of course depends on the ground field. It hardly makes any sense to order matrices over \mathbb{F}_p : even 1×1 matrices, namely (canonically) the elements of \mathbb{F}_p defy reasonable ordering. But real numbers, for instance, have ordering, so there's a serious change that all real matrices could be ordered.

We will first try to order all real square matrices. (Actually, we won't even try to order non-square matrices.) 1×1 matrices are easy to order, but as soon one moves to larger matrices, one faces difficult decisions:

$$\text{Is } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} ?$$

That's okay, we don't necessarily have to order all pairs of matrices. But there are other problems. We would like the ordering of the matrices to be independent of the choice matrix representation. If we flip basis vectors we change the rows and columns of the matrix, so change it's sign! So if we want

- Positivity is readable from the matrix with respect to an arbitrary basis, alone.
- Positivity doesn't depend on the chosen basis.

there is no way to assign sign to the previous matrix.

Let's consider diagonalizable matrices: by change of basis these can be written as diagonal matrices, but these diagonal values, eigenvalues, need not be real anymore. One could try to ignore the complex bases, but how often really is ignoring the complex structure satisfactory. Another approach would be only to consider matrices with real eigenvalues. There's problem here though: sum of two matrices with real eigenvalues need not have real eigenvalues, even if the former are diagonalizable! Sad as it may sound, it shouldn't be too surprising since it's not very clear that eigenvalues should have any reasonable behaviour with respect to addition. Of course, one shouldn't just take my word for it: here are concrete examples:

MATRIX A and B

First two matrices have eigenvalues TODO, TODO and TODO, TODO, respectively, but their sum has characteristic polynomial TODO, that most definitely has no real zeros.

There is however very special class of matrices, corresponding to special class of linear mappings, which satisfy our requirements.

- They have real eigenvalues.
- They are all diagonalizable.
- They are closed under sum, and (real) scalar multiplication.

2.1.2 Hermitian matrices

Hermitian matrices/mappings will be our non-commutative playing ground in which we can define rather natural partial order. Fix any finite-dimensional inner-product space $(V, \langle \cdot, \cdot \rangle)$ over \mathbb{R} or \mathbb{C} . For any $v \in V \setminus \{0\}$ we may define the corresponding projection, denoted by P_v by setting $P_v(x) = \langle x, v \rangle / \langle v, v \rangle v$.

Definition 2.1. Let V be an finite-dimensional innerproduct space over \mathbb{R} or \mathbb{C} . Now set of *Hermitian maps* of V , denoted by $\mathcal{H}(V)$, is defined as

$$\text{span}\{P_v \mid v \in V \setminus \{0\}\},$$

where the span is \mathbb{R} -linear, i.e. Hermitian maps are the maps of the form

$$\sum_{i=1}^m \lambda_i P_{v_i},$$

for some positive integer m , $v_i \in V$ and $\lambda_i \in \mathbb{R}$.

This is not the standard definition, and in a way it's horrible, but it very directly answers our needs. Also note that since for any non-zero α and $v \in V \setminus \{0\}$ we have $P_v = P_{\alpha v}$, we could just as well only allow vectors of norm one in our definition of Hermitian maps. This is of course just to simplify notation. Remember that we want to have diagonalizable maps with real eigenvalues. Projections are very prototypical examples of such maps and since we want real eigenvalues that is simply what we require in the definition. Lastly, one of course wants now to take the span to have linear structure.

It is however very surprising that such construction works: basis vectors (projections) satisfy our requirements, but there are no reasons to expect that these would be preserved in addition. This is guaranteed by the following theorem.

Theorem 2.2 (Spectral theorem). *Let V be an n -dimensional inner-product space over \mathbb{R} or \mathbb{C} , and $A \in \mathcal{H}(V)$. Then there exists real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$ and orthonormal vectors v_1, v_2, \dots, v_n such that*

$$A = \sum_{i=1}^n \lambda_i P_{v_i}.$$

It follows that vectors v_1, v_2, \dots, v_n are eigenvectors of A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Indeed, for any v_j we have

$$Av_j = \sum_{i=1}^n \lambda_i P_{v_i} v_j = \sum_{i=1}^n \lambda_i \langle v_j, v_i \rangle v_i = \sum_{i=1}^n \lambda_i \delta_{ij} v_i = \lambda_j v_j.$$

Proof (of the Spectral theorem). We proof the statement by induction on the dimension of the space V . The case $n = 0$ is trivial: the span itself is trivial, and it's very easy to express 0-mapping as a empty sum.

Now fix a positive integer n . First note A has at least one eigenvector over \mathbb{C} , as every other linear mapping. Let v_1 be that eigenvector (of lenght one) with corresponding eigenvalue λ_1 . Write $A = \sum_{i=1}^m c_i P_{u_i}$ for $c_1, c_2, \dots, c_m \in \mathbb{R}$ and $u_1, u_2, \dots, u_m \in V \setminus \{0\}$ are of norm 1. Now the definition of the eigenvector and eigenvalue rewrites to

$$Av_1 = \sum_{i=1}^m c_i \langle v_1, u_i \rangle u_i = \lambda_1 v_1.$$

Take inner product with v_1 from both sides to get

$$\sum_{i=1}^m c_i \langle v_1, u_i \rangle \langle u_i, v_1 \rangle = \lambda_1 \langle v_1, v_1 \rangle.$$

Since $\langle v_1, u_i \rangle \langle u_i, v_1 \rangle = |\langle v_1, u_i \rangle|^2$, the left-hand side is real, so is right-hand side, and finally so is λ_1 .

The idea is then to factorize A to two parts: projection corresponding to v_1 and an orthogonal part living in v_1^\perp . \square

Chapter 3

Pick-Nevanlinna functions

Pick-Nevanlinna function is an analytic function defined in upper half-plane with a non-negative real part. Such functions are sometimes also called Herglotz or \mathbb{R} functions but we will often call them just *Pick functions*. The class of Pick functions is denoted by \mathcal{P} .

Pick functions have many interesting properties related to positive matrices and that is why they are central objects to the theory of matrix monotone functions.

3.1 Basic properties and examples

Most obvious examples of Pick functions might be functions of the form $\alpha z + \beta$ where $\alpha, \beta \in \mathbb{R}$ and $\alpha \geq 0$. Of course one could also take $\beta \in \overline{\mathbb{H}}_+$. Actually real constants are the only Pick functions failing to map $\mathbb{H}_+ \rightarrow \mathbb{H}_+$: non-constant analytic functions are open mappings.

Sum of two Pick functions is a Pick function and one can multiply a Pick function by non-negative constant to get a new Pick function. Same is true for composition.

The map $z \mapsto -\frac{1}{z}$ is evidently a Pick function. Hence are also all functions of the form

$$\alpha z + \beta + \sum_{i=1}^N \frac{m_i}{\lambda_i - z},$$

where N is non-negative integer, $\alpha, m_1, m_2, \dots, m_N \geq 0$, $\beta \in \mathbb{H}_+$ and $\lambda_1, \dots, \lambda_N \in \mathbb{H}_+$. So far we have constructed our function by adding simple poles to the closure of lower half-plane. We could further add poles of higher order at lower half plane, and change residues and so on, but then we have to be a bit more careful.

There are (luckily) more interesting examples: all the functions of the form x^p where $0 < p < 1$ are Pick functions. To be precise, one should choose branch for the previous so that they are real at positive real axis. Also \log yields Pick function when branch

is chosen properly i.e. naturally again. Another classic example is \tan . Indeed, by the addition formula

$$\begin{aligned}\tan(x + iy) &= \frac{\tan(x) + \tan(iy)}{1 - \tan(x)\tan(iy)} = \frac{\tan(x) + i \tanh(y)}{1 - i \tan(x) \tanh(y)} \\ &= \frac{\tan(x)(1 + \tanh^2(y))}{1 + \tan^2(x) \tanh^2(y)} + i \frac{(1 + \tan^2(x)) \tanh(y)}{1 + \tan^2(x) \tanh^2(y)},\end{aligned}$$

and y and $\tanh(y)$ have the same sign.

We observe the following useful fact.

Proposition 3.1. *If $(\varphi_i)_{i=1}^n$ is a sequence of Pick functions converging locally uniformly, the limit function is also a Pick function.*

Proof. Locally uniform limits of analytic functions are analytic. Also the limit function has evidently non-negative imaginary part. \square

This is one of the main reasons we include real constants to Pick functions, although they are exceptional in many ways. Note that for any $z \in \mathbb{H}_+$ we have $\log(z) = \lim_{p \rightarrow 0+} (z^p - 1)/p$: \log can be understood as a limit of Pick functions. There's actually a considerable strengthening of the previous result.

Proposition 3.2. *If $(\varphi_i)_{i=1}^n$ is a sequence of Pick functions converging pointwise, the limit function is also a Pick function.*

We will not prove this quite surprising result yet.

3.2 Schur functions

As we have noticed, Pick functions need not be injections or surjections. Some are both: simple examples are functions of the form $\alpha z + \beta$ and $\frac{\alpha}{\lambda - z} + \beta$ for $\alpha > 0$ and $\beta, \lambda \in \mathbb{R}$. And that's all.

Before trying to understand why is that, we have to change the point of view. All the previous functions are rational functions, but even more is true: they are all Möbius transformations. Möbius transformations are analytic bijections of extended complex plane i.e. Riemann sphere, to itself. Our examples all exactly those Möbius transformation which map the extended real axis to itself, and don't change the orientation, so the upper half-plane is mapped to itself and not to the lower half-plane. When viewed as a part of the Riemann sphere, upper half-plane is just a hemisphere. Of course it shouldn't matter too much which hemisphere we are looking at, so we could also consider mappings from unit disc to itself (or closed unit disc, to be precise). These mappings are called *Schur*

functions and class of Schur functions is denoted by \mathcal{S} . It's then natural to conjecture that bijective Schur functions are exactly the Möbius transformations which map unit circle to unit circle, and don't change the orientation so that the inside is mapped to the inside.

These claims are easily derivable from each other as follows. Consider the pair of Möbius transformations

$$\begin{aligned}\xi : \mathbb{D} &\rightarrow \mathbb{H}_+ & \xi(z) &= i \frac{1-z}{1+z} \\ \eta : \mathbb{H}_+ &\rightarrow \mathbb{D} & \eta(z) &= \frac{i-z}{i+z}.\end{aligned}$$

They are inverses of each other and map the (open) unit disc to upper half-plane and back, respectively. Now take any bijective Schur function $\psi : \mathbb{D} \rightarrow \mathbb{D}$. Then $\varphi = \xi \circ \psi \circ \eta$ is bijective Pick function. Similarly one could invert $\psi = \eta \circ \varphi \circ \xi$. This means that bijections can be paired: if all bijective Pick functions are Möbius transformations, so are all bijective Schur functions, since non-Möbiusness on one side would give rise to non-Möbiusness on the other side.

Still before proving anything we should think about this relation a bit further. We noticed that every bijective Pick function has a corresponding Schur function pair. This correspondence is however by no means unique, it was merely our choice to choose such ξ and η . Still, there is need to restrict ourselves to bijections anymore. If one takes **any** Schur function $\psi : \mathbb{D} \rightarrow \mathbb{D}$ we can form the corresponding Pick function by taking $\varphi = \xi \circ \psi \circ \eta$. This gives rise to bijection $\mathcal{S} \rightarrow \mathcal{P}$, and the inverse should be rather obvious by now. Of course, it's not a big surprise that there would be such bijection, that is to say that the sets are equal in size, but our bijection preserves composition of functions. All this is to say that in some sense these classes are almost the same.

One should be a bit more careful here though: we have included also real constant functions to our class \mathcal{P} and we should also add unimodular constants to \mathcal{S} . For these the bijection doesn't quite work; we can mostly do a natural extension, but then one would be forced to map the constant function -1 to the constant ∞ . This means that we should add the constant infinity function to our Pick functions. We will not do this, as it would change the whole business to Riemann sphere, since it will bring other technical problems, but we will try to indicate when you should think about this extension.

If one only thinks about composition one can of course do lot more. Take any simply connected domain in $U \subset \mathbb{C}$. By Riemann mapping theorem there's a analytic bijection $\xi_U : \mathbb{D} \rightarrow U$. For the domain U we could define similar class of functions, and via ξ_U and it's inverse we could connect the classes. Again, one should be a bit careful with the boundary.

In many ways Pick and Schur functions are most natural of these classes: they are

closed under addition and multiplication, respectively. Also, they both contain the identity of the respective operations, so these properties are barely true.

3.3 Schwarz lemma

Chapter 4

Monotone and Convex matrix functions

We already introduced monotone and convex matrix functions in the introduction, but now that we have properly defined and discussed underlying structures we should take a deeper look. As mentioned, monotone and convex matrix functions are sort of generalizations for the standard properties of reals, and this is why we should understand which of the phenomena for the real functions carry to matrix functions and which do not.

We will start with the matrix monotone functions; much of the discussion carries quite directly to the convex case.

4.1 Basic properties of the matrix monotone functions

We first state the definition.

Definition 4.1. Let $(a, b) \subset \mathbb{R}$ be an open, possibly unbounded interval and n positive integer. We say that $f : (a, b) \rightarrow \mathbb{R}$ is n -monotone or matrix monotone of order n , if for any $A, B \in \mathcal{H}_{(a,b)}$, such that $A \leq B$ we have $f(A) \leq f(B)$.

We will denote the space of n -monotone functions on open interval (a, b) by $P_n(a, b)$. One immediately sees that that all the matrix monotone functions are monotone as real functions.

Proposition 4.2. *If $f \in P_n(a, b)$, f is increasing.*

Proof. Take any $a < x \leq y < b$. Now for $xI, yI \in \mathcal{H}_{(a,b)}^n$ we have $xI \leq yI$ so by definition

$$f(x)I = f(xI) \leq f(yI) = f(y)I,$$

from which it follows that $f(x) \leq f(y)$. This is what we wanted. \square

Actually, increasing functions have simple and expected role in n -monotone matrices.

Proposition 4.3. *Let (a, b) be an open interval and $f : (a, b) \rightarrow \mathbb{R}$. Then the following are equivalent:*

- (i) f is increasing.
- (ii) $f \in P_1(a, b)$.
- (iii) For any positive integer n and commuting $A, B \in \mathcal{H}_{(a,b)}^n$ such that $A \leq B$ we have $f(A) \leq f(B)$.

Proof. TODO \square

The equivalence of the first two is almost obvious and from this point on we shall identify 1-monotone and increasing functions. But the third point is very important: it is exactly the non-commutative nature which makes the classes of higher order interesting.

Let us then have some examples.

Proposition 4.4. *For any positive integer n , open interval (a, b) and $\alpha, \beta \in \mathbb{R}$ such that $\alpha \geq 0$ we have that $(x \mapsto \alpha x + \beta) \in P_n(a, b)$.*

Proof. Assume that for $A, B \in \mathcal{H}_{(a,b)}$ we have $A \leq B$. Now

$$f(B) - f(A) = (\alpha B + \beta I) - (\alpha A + \beta I) = \alpha(B - A).$$

Since by assumption $B - A \geq 0$ and $\alpha \geq 0$, also $\alpha(B - A) \geq 0$, so by definition $f(B) \geq f(A)$. This is exactly what we wanted. \square

That was easy. It's not very easy to come up with other examples, though. Most of the common monotone functions fail to be matrix monotone. Let's try some non-examples.

Proposition 4.5. *Function $(x \mapsto x^2)$ is not n -monotone for any $n \geq 2$ and any open interval $(a, b) \subset \mathbb{R}$.*

Proof. Let us first think what goes wrong with the standard proof for the case $n = 1$.

Note that if $A \leq B$,

$$B^2 - A^2 = (B - A)(B + A)$$

is positive as a product of two positive matrices (real numbers).

There are two fatal flaws here when $n > 1$.

- $(B - A)(B + A) = B^2 - A^2 + (BA - AB)$, not $B^2 - A^2$.

- Product of two positive matrices need not be positive.

Note that both of these objections result from the non-commutativity and indeed, both would be fixed should A and B commute.

Let's write $B = A + H$ ($H \geq 0$). Now we are to investigate

$$(A + H)^2 - A^2 = AH + HA + H^2.$$

Note that $H^2 \geq 0$, but as we have seen in TODO, $AH + HA$ need not be positive! Also, if H is small enough, H^2 is negligible compared to $AH + HA$. We are ready to formulate our proof strategy: find $A \in \mathcal{H}_{a,b}^n$ and \mathbb{H}_+^n such that $AH + HA \not\geq 0$. Then choose parameter $t > 0$ so small that $A + tH \in \mathcal{H}^n(a, b)$ and

$$(A + tH)^2 - A^2 = t(AH + HA + tH^2) \not\geq 0$$

and set the pair $(A, A + tH)$ as the counterexample.

TODO □

In a similar manner one could show the similar statement for the functions $(x \mapsto x^k)$.

At this point several other important properties of the matrix monotone functions should be clear.

Proposition 4.6. *For any positive integer n and open interval (a, b) the set $P_n(a, b)$ is a convex cone, i.e. it is closed under taking summation and multiplication by non-negative scalars.*

Proof. This is easy: closedness under summation and scalar multiplication with non-negative scalars correspond exactly to the same property of positive matrices. □

We should be a bit careful though. As we saw with the square function example, product of two n -monotone functions need not be n -monotone in general, even if they are both positive functions; similar statement holds for increasing functions. Similarly, taking maximums doesn't preserve monotonicity.

Proposition 4.7. *Maximum of two n -monotone functions need not be n -monotone for $n \geq 2$.*

Proof. Again, let's think what goes wrong with the standard proof for $n = 1$.

Fix open interval (a, b) , positive integer $n \geq 2$ and two functions $f, g \in P^n(a, b)$. Take any two $A, B \in \mathcal{H}_{(a,b)}^n$ with $A \leq B$. Now $f(A) \leq f(B) \leq \max(f, g)(B)$ and Now $f(A) \leq f(B) \leq \max(f, g)(B)$. It follows that

$$\max(f, g)(A) = \max(f(A), g(A)) \leq \max(f, g)(B),$$

as we wanted.

Here the flaw is in the expression $\max(f(A), g(A))$: what is maximum of two matrices? This is an interesting question and we will come back to it a bit later, but it turns out that however you try to define it, you can't satisfy the above inequality.

We still need proper counterexamples though. Let's try $f \equiv 0$ and $g = \text{id}$. So far the only n -monotone functions we know are affine functions so that's essentially our only hope for counterexamples.

TODO □

Similarly we have composition and pointwise limits.

Proposition 4.8. *If $f : (a, b) \rightarrow (c, d)$ and $g : (c, d) \rightarrow \mathbb{R}$ are n -monotone, so is $g \circ f : (a, b) \rightarrow \mathbb{R}$.*

Proof. Fix any $A, B \in \mathcal{H}_{(a,b)}^n$ with $A \leq B$. By assumption $f(A) \leq f(B)$ and $f(A), f(B) \in \mathcal{H}_{(c,d)}^n$ so again by assumption, $g(f(A)) \leq g(f(B))$, our claim. □

Proposition 4.9. *If n -monotone functions $f_i : (a, b) \rightarrow \mathbb{R}$ converge pointwise to $f : (a, b) \rightarrow \mathbb{R}$ as $i \rightarrow \infty$, also f is n -monotone.*

Proof. As always, fix $A, B \in \mathcal{H}_{(a,b)}^n$ with $A \leq B$. Now by assumption

$$f(B) - f(A) = \lim_{i \rightarrow \infty} f_i(B) - \lim_{i \rightarrow \infty} f_i(A) = \lim_{i \rightarrow \infty} (f_i(B) - f_i(A)) \geq 0,$$

so also $f \in P_n(a, b)$. □

We shall be using especially the previous result a lot.

One of the main properties of the classes of matrix monotone functions has still avoided our discussion, namely the relationship between classes of different orders. We already noticed that matrix monotone functions of all orders all monotonic, or $P_n(a, b) \subset P_1(a, b)$ for any $n \geq 1$. It should not be very surprising that we can make much more precise inclusions.

Proposition 4.10. *For any open interval (a, b) and positive integer n we have $P_{n+1}(a, b) \subset P_n(a, b)$.*

Proof. TODO □

One might ask whether these inclusions are strict. It turns out they are, as long as our interval is not the whole \mathbb{R} . We will come back to this.

There are also more trivial inclusions: $P_n(a, b) \subset P_n(c, d)$ for any $(a, b) \supset (c, d)$. More interval, more matrices, more restrictions, less functions. To be precise, we only allowed functions with domain (a, b) to the class $P_n(a, b)$, so maybe one should say instead something like: if $(a, b) \supset (c, d)$ and $f \in P_n(a, b)$, then also $f|_{(c,d)} \in P_n(c, d)$. We will try not to worry too much about these technicalities.