APPM 4360 Project

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Abstract

Determining solutions for BVPs (e.g., Laplace) in an arbitrary region within the complex plane can be greatly simplified if this region is mapped to the upper half plane, or the unit disk. This technique is called Conformal Mapping and is used ubiquitously to solve problems in physical applications, such as flow over an airfoil. The contour of an airplane wing is the main method by which modern airplanes generate lift. Simulating and understanding this lift is paramount to safe and efficient aircraft operation. Due to the unusual shape of airfoils, common methods of evaluation use a conformal map that transform a circle in the z-plane into a (Joukowsky) airfoil in the mapped w-plane. In this paper, we outline the theory behind conformal mappings in order to establish a solution to the problem of airflow over a general Joukowsky airfoil. Several examples are provided to highlight important theorems and characteristics regarding conformal maps. The geometry of the Joukowsky transformation is discussed and the lift per unit length around a general Joukowsky airfoil is determined.

1 Introduction

The need for a transformation of a problem from one coordinate system to another is ubiquitous in theory and application. One such transformation is the conformal mapping. A conformal map is a transformation from one coordinate space, z, to another, w, which conserves certain geometrical properties of the original space (details provided in section 2). While it is certainly not necessary for complex numbers to be used to create such a map, the use of complex numbers provides a framework for the verification and development of such a mapping.

For example, two functions $u = u(x, y)$ and $v = v(x, y)$ can be mapped from A $->$ B using the transformation

$$f(u, v) = Au(x, y) + Bv(x, y) \tag{1}$$

where A and B are real constants. Such a transformation is guaranteed to be linearly invertible given that the determinant of the Jacobian matrix is non-zero, that is if, $u_x v_y - u_y v_x \neq 0$ [Gro11]. Similar to real functions, the invertibility of a transformation is very useful. Univalent complex transformations are especially nice to work with as a one-to-one correspondence between the original and the image space allows one to easily toggle between the spaces. However, when modelling certain physical problems, such as the lift caused by airflow around an airfoil, there are certain physical forces which need to be modeled in both spaces. It is desired that such a modeling framework would preserve the angles of line intersections between spaces. However, in general, a mapping as defined in (1) will not generally have this property. Proof 6.3 demonstrates the necessity of complex variables to this application.

It turns out that in the case of airflow around an airfoil, the property of a map being conformal is highly desirable due to importance of streamline interactions which determine the lift and drag of an aircraft [Hun21]. A streamline is the path along which a particle travels over a wing, or cylinder. To transform a circle in the z plane to an airfoil shape in the w plane, we use a Joukowsky transformation (section 4.1). It is an important detail to note that analyzing streamlines to deduce how fluid interacts with these obstructions allows one to determine the lift generated by the airfoil. This motivates the angle preservation property provided by a Conformal map.

This finally elucidates the importance of complex numbers to this application. As stated previously, a mapping from (1) is invertible but not conformal in general, however, if we redefine (1) as:

$$g(u, v) = Au(x, y) + iBv(x, y) \tag{2}$$

where g is a complex function, and g is analytic and whose derivative is non-zero, then it will be invertible and conformal wherever those assumptions hold. The exact reason why this is the case is a fundamental property of complex numbers and their derivatives. Recall from [AF21], the definition of analyticity; $h'(z) = u_x(x, y) + iv_x(x, y)$ must exist. This definition for the derivative yields a set of partial differential equations, the Cauchy-Riemann equations, that must be satisfied. This is a much more restrictive condition than the definition of derivatives provided by real variables. This more stringent set of criteria lays the groundwork for the introduction of complex variables [Gro11].

A very important result of conformal mapping theory is Riemann's Mapping Theorem. This theorem states that any simply connected domain in the complex plane, with the exception of the entire complex plane and the extended complex plane can be mapped onto the unit disk centered at the origin, or onto the upper half plane. There is no general method to construct such a transformation, however, there do exist methods to construct such transformations if the regions we wish to map meet certain criteria. For example, the Schwarz-Christoffel theorem provides a method for mapping a piecewise linear polygon in w space, and the domain interior to this shape, to the real axis in the z space, and the upper half plane (section 3.1.1).

Now that the fundamental importance of complex variables to this application has been introduced, we move into an exploration of the key properties that this transformation provides.

2 Conformal Transformations

Suppose C is a differentiable arc in the complex z plane. C can be defined parametrically, where its x and y components can be expressed as functions of a real parameter $s \in [a, b]$:

$$C := z(s) = x(s) + iy(s)$$

Furthermore, suppose that the function $f(z) = u(x, y) + iv(x, y) = w$ is analytic in a region D, where $C \subset D$. The image of the arc C is therefore given by:

$$\tilde{C} := w(s) = u(x(s), y(s)) + iv(x(s), y(s))$$

Since C is a differentiable arc, $x(s)$ and $y(s)$ are differentiable. Thus, by the analyticity of $f(z)$, $\tilde{C}$ is also a differentiable arc. In this case, it can be shown that for any point z_0 on C such that $f'(z_0) \neq 0$, the tangent of the arc C at z_0 is rotated by the angle $arg(f'(z_0))$ (proof 1 in Appendix). The transformation thus preserves the angles between such differentiable arcs. This is the defining characteristic of a conformal mapping[1]. We illustrate the conformal property in the case of a straight line arc C, mapped by the linear transformation $f(z) = az + b$ for constants $a, b \in \mathbb{C}$.

Tangent Rotation Linear Example: Let $C := z(s) = se^{i\varphi}$, and $f(z) = az + b$ for constants $a, b \in \mathbb{C}$.

$$\begin{aligned} \Rightarrow w(s) &= a \cdot z(s) + b \\ &= |a|e^{iarg(a)} \cdot se^{i\varphi} + b \\ &= |a|se^{i(arg(a)+\varphi)} + b \end{aligned}$$

After this transformation, the arc originates at the point $b \in \mathbb{C}$, and grows out from b as a function of s with an angle $arg(s) + \varphi$ with respect to the real axis. Thus, before the transformation, every point z_0 on C had a tangent with an argument equal to φ, and after the transformation, the quantity $arg(f'(z_0)) = arg(a)$ was added to this argument (refer to figure 1 on the following page).

[1]Refer to the Appendix for the formal definition (Theorem 5.2.1)

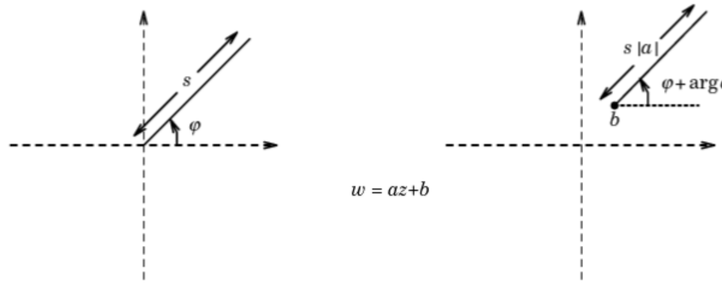


Figure 1: Tangent Rotation Linear Example [AF21]

As a consequence of this rotation property, since the tangents of any two arcs intersecting at z_0 will both be rotated by the same amount as a result of the transformation, the angle of intersection between the two arcs before and after transformation will remain constant.

Preservation of Arc Intersection Angle Example: To demonstrate this angle preserving property of conformal transformations, consider a rectangular region in the $\mathbb{C}$ plane bounded by the contour $R_z := z = x + iy$ defined by the region enclosed by the line segments AB, BC, CD, and DA, where $A = 0$, $B = 1$, $C = 1 + i2\pi$, and $D = i2\pi$ (figure 2). Note that the function $f(z) = e^z$ is entire, and at any point z_0 on or inside R_z, $f'(z_0) = e^z \neq 0$. Thus, $f(z)$ defines a conformal map. On segment AB, $z = x$, thus $f(z) = e^x$, $x \in [0, 1]$, and maps $AB \mapsto A'B'$, a line on the real axis connecting $z = 1$ and $z = e$. Similarly, on the segment CD, $z = x + i2\pi$, for $x \in [1, 0]$, so $f(z)$ maps $CD \mapsto C'D'$, a line on the real axis connecting $z = e$ and $z = 1$. On the segment BC, $z = 1 + iy$, for $y \in [0, 2pi]$. Thus $f(z) = e^{1+iy} = e \cdot e^{iy}$, which defines a circle of radius e centered at the origin. Similarly, f maps $DA \mapsto D'A'$, a circle of radius 1 centered at the origin. One can also observe how the images of any two parallel segments of the rectangle, are also parallel.

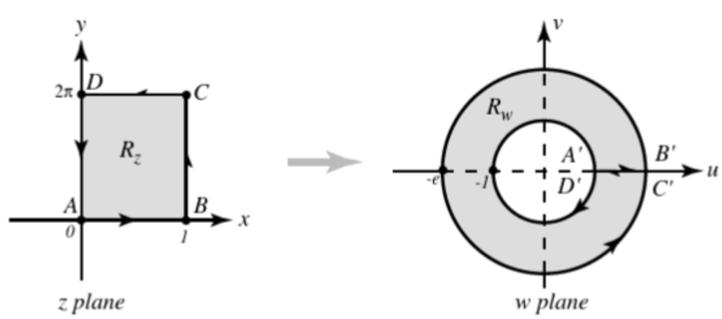


Figure 2: Preservation of Arc Intersection Angle Example [AF21]

3 Ideal Fluid Flow

We will now review the theory behind ideal fluid flow. We begin with the necessary assumptions: that the motion of the fluid in any plane parallel to the z plane is identical; that the flow is steady, i.e. the motion of the fluid is a function of its position only, and not time; that the fluid is in-compressible and in-viscid; and that the flow is irrotational, meaning along any closed contour C, the circulation of the fluid is zero.

Let u_1, u_2 be components of the gradient vector of a potential $\phi : \mathbf{u} = \nabla\phi$, and suppose they satisfy the equation

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$$

Then, ϕ must be harmonic, i.e., satisfy Laplace, which implies that there must exist a conjugate function ψ such that $\Omega(z) = \phi(x, y) + i\psi(x, y)$ is analytic. $\Omega(z)$ is referred to as the Complex Velocity Potential. Whereas the velocity is given by ϕ, streamlines of the flow are found as constant curves of ψ. The derivative of Ω, given by

$$\Omega'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

and is the conjugate of the velocity vector field:

$$\overline{\Omega'(z)} = \frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y} = u_1 + iu_2$$

3.1 Applications

We first apply this theory of conformal mapping to analyze how ideal fluid flow is obstructed by a wall. We do this by applying the **Schwarz-Christoffel Theorem** to map a slit of height s (see figure 3) and the area above it in the w plane, to the real axis and the upper half of the z plane.

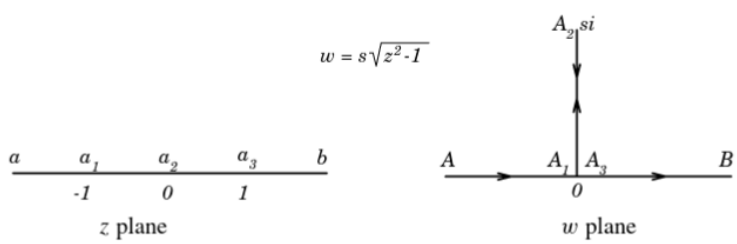


Figure 3: Slit of Height s [AF21]

3.1.1 Statement of Schwarz-Christoffel Theorem

Let Γ be a piecewise linear boundary of a polygon in the w plane, with interior angles at successive vertices of $\alpha_1\pi$,...,$\alpha_n\pi$ (see figure 4). Let the polygon's vertices in the w plane be A_1,...,A_n. Then the following transformation maps Γ to the real axis and the upper half of the z plane, where a_z,...,a_n are the locations on the real line to which A_1,...,A_n are mapped:

$$\frac{dw}{dz} = \gamma(z - a_1)^{\alpha_1 - 1}(z - a_2)^{\alpha_2 - 1} \cdots (z - a_n)^{\alpha_n - 1} = 0$$

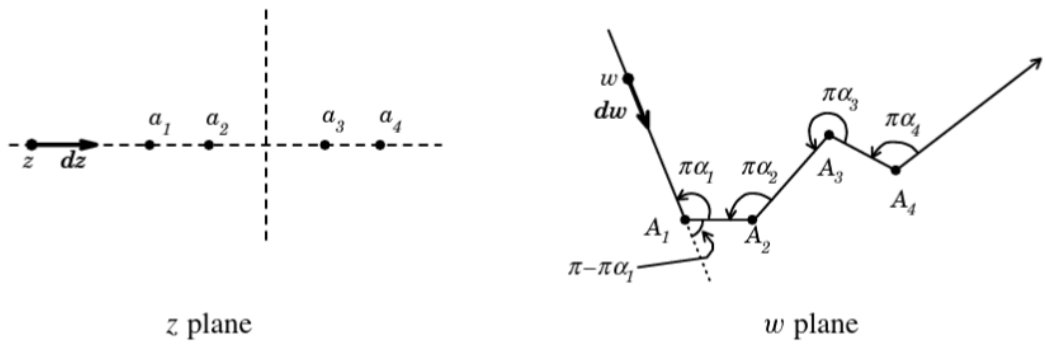


Figure 4: Piece-wise linear contour in w plane [AF21]

A result stated in [AF21] shows that if $f(z)$ is univalent (one-to-one), and is applied to a boundary between two simply connected domains, then up to three of the polygons vertices, A_1, A_1, A_3 can be associated with arbitrary points on the real line a_1, a_1, a_3, preserving order and orientation. γ is a complex number which can be solved for using the requirements of our transformation to map $A_i \mapsto a_i$.

3.1.2 Fluid Flow over a Wall

We map the vertices of Γ according to the following (see figure 3):

$$A_1 \mapsto a_1 = -1, \alpha_1 = 1/2$$
$$A_2 \mapsto a_2 = 0, \alpha_2 = 2$$
$$A_3 \mapsto a_3 = 1, \alpha_3 = 1/2$$

Then we apply the theorem to compute a general form for the transform:

$$\frac{dw}{dz} = \gamma(z+1)^{-1/2} z(z-1)^{-1/2} = ... = \frac{i\gamma z}{\sqrt{1-z^2}}$$

$$\Rightarrow w = \int \frac{dw}{dz} = \int \frac{i\gamma z}{\sqrt{1-z^2}} = \frac{i\gamma}{2} \int \frac{du}{\sqrt{1-u}} = \gamma(z^2-1)^{1/2} + c$$

Now we use the fact that $f(z=0) \mapsto si$ and $f(z=1) \mapsto 0$ to solve for γ and c, finding the final form of the transformation. In doing this, we address the multivaluedness of the transformation by defining principal branch's from $\frac{-\pi}{2} \leq \theta_p < \frac{3\pi}{2}$, thus branch cuts from $(-1, \text{inf})$ and $(1, \text{inf})$ parallel to the negative imaginary z axis.

$$\begin{aligned}
&z = 0 \Rightarrow w = si = \gamma(-1)^{1/2} + c \\
&z = 1 \Rightarrow w = 0 = c \\
&\Rightarrow c = 0 \text{ and } \gamma = s \\
&\Rightarrow w = s(z^2-1)^{1/2} \\
&\Rightarrow z = ((\frac{w}{s})^2 + 1)^{1/2}
\end{aligned}$$

Now that we have a conformal transformation, we can easily find the complex potential associated with uniform fluid flow being obstructed by a wall. In the z plane, recall the equation for uniform flow:

$$\Omega(z) = u_0 z$$

Since our transformation is angle preserving, we can simply substitute our inverse transform in the place of z to recover the complex potential of the fluid flow in the w plane:

$$\Rightarrow \Omega(z) = \Omega(((\frac{w}{s})^2 + 1)^{1/2}) =: \Omega(w)$$

With this new complex potential, one can identify the stream function and the velocity potential by separating $\Omega(w)$ into its real and imaginary parts, where $\Omega(w) = \Phi(u,v) + i\Psi(u,v)$ and $w = u + iv$. This process yields the streamlines and fluid flow described in figure 5 below:

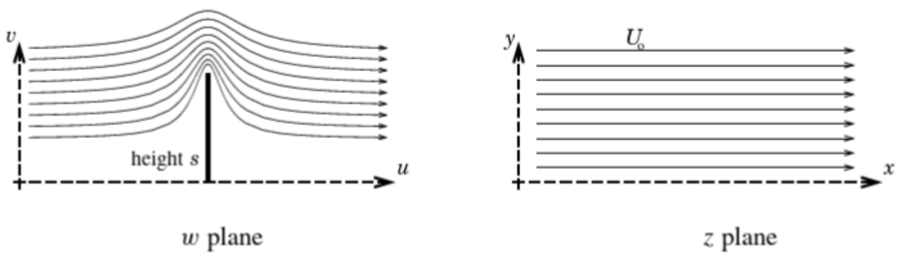


Figure 5: Fluid flow over a wall [AF21]

4 Airfoils

4.1 Joukowsky Transformation

In this section the method of the Joukowsky transformation is explored. This transformation maps circles in the z plane, to shapes resembling airfoils in the w plane. The form that will be examined in this paper is:

$$w = z + \frac{1}{z}\,, \qquad z \neq 0 \tag{3}$$

We demonstrate that this mapping is analytic and subsequently that it is conformal for $z \neq \pm 1$. Recalling that z is defined as $x + iy$, it follows that:

$$w(z(x,y)) = x + iy + \frac{1}{x+iy} = x + iy + \frac{x-iy}{x^2+y^2}$$

$$= \big(x + \frac{x}{x^2+y^2}\big) + i\big(y - \frac{y}{x^2+y^2}\big)$$

$$= u(x,y) + iv(x,y)$$

where u and v are the real and imaginary components of $w(z)$. Using the above simplification, adherence to the Cauchy-Riemann equations can be verified:

$$u_x = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y$$

$$v_x = \frac{2xy}{(x^2 + y^2)^2} = -\frac{-2xy}{(x^2 + y^2)^2} = -u_y$$

Thus, the transformation is analytic for the implied assumption $z \neq 0$. As such, its derivative can be directly calculated:

$$w'(z) = 1 - \frac{1}{z^2}$$

which is conformal for $w'(z) \neq 0$[2], $z \neq \pm 1$. The points at which this (or any) conformal mapping equals 0 are known as *critical* points. Critical points have a special property relating to the mapping of external angles which will be important in the next section in which the Joukowsky airfoils are developed. The Joukowsky can also be shown to satisfy the Laplace equation, a proof of which will be provided in the appendix (proof 3).

4.2 Formulation of Airfoils

As previously mentioned, the critical points of $z = \pm 1$ play an important role in mapping a circle to an airfoil geometry. To demonstrate their significance, first consider the mapping of a circle centered at the origin which encloses both critical points ($z = \pm 1$). Here, $z = re^{i\theta}$, $r > 1$, and so $w = (r + \frac{1}{r})cos\theta + i(r - \frac{1}{r})sin\theta = Acos\theta + iBsin\theta$, an ellipse; however, if the circle is shifted along the -Re(z) axis such that it intersections the +Re(z) axis at the critical point $z = 1$, the mapped geometry changes dramatically, as shown in Figures 6 and 7 on the next page.

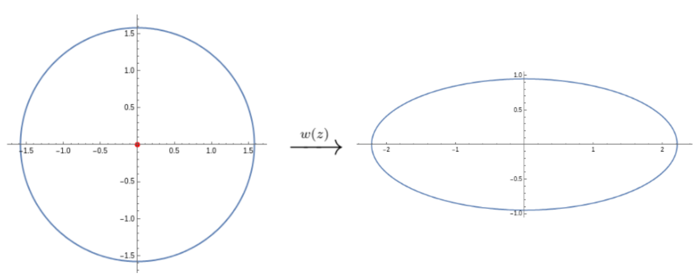


Figure 6 : transform with center at origin

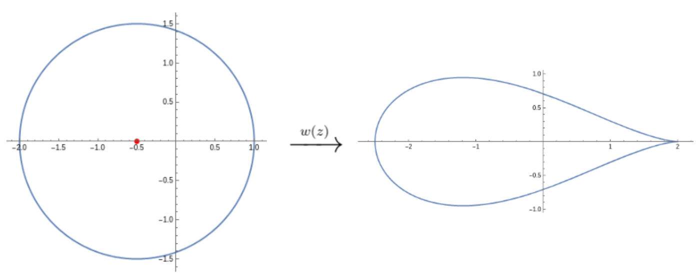


Figure 7 : transform with center on -Re(z) axis at $z = -1$

[2]see Appendix, Theorem 5.2.1

This end behavior near $w(z=1)$ results from the creation of a vertex in the w-plane due the intersection of the circle in the z-plane with its critical point. The exterior angle at the vertex is then doubled from $\theta_z = \pi$ to $\theta_w = 2\pi$[3] since only the first derivative of w vanishes at $z = 1$, resulting in a convergence point in the w-plane. [AF21]. The characteristic tail of modern airfoils can then be achieved by raising the circle's center up off the -Re(z) axis, while maintaining the intersection point at $z = 1$. This causes the airfoil shape to curl above the +Re(w) axis, a result that is produced by the conformal nature of the mapping. This phenomenon shall be discussed in regards to the following figure:

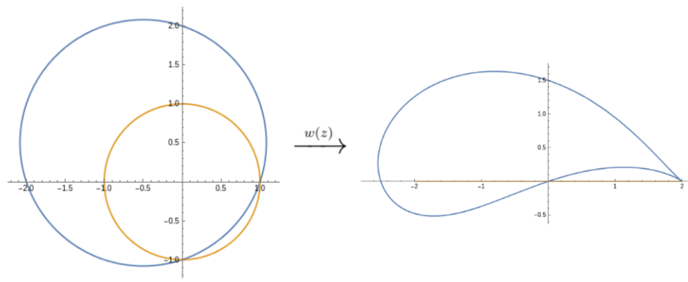


Figure 8 : transformation of circled centered above -Re(z) axis

The Real and Imaginary parts of the mapping are given parametrically by the equations:

$$\begin{aligned}
\mathrm{Re}(w) =& (r cos(\theta) - c_x)\Big(1 + \frac{1}{r^2 + 2r(c_y sin(\theta) - c_x cos(\theta)) + (c_x^2 + c_y^2)}\Big) \\
\mathrm{Im}(w) =& (r sin(\theta) + c_y)\Big(1 - \frac{1}{r^2 + 2r(c_y sin(\theta) - c_x cos(\theta)) + (c_x^2 + c_y^2)}\Big)
\end{aligned}$$

where,

r is the radius of the circle in the z-plane centered at $(-c_x, ic_y)$

and θ corresponds to $z = re^{i\theta}$

For the mapping shown in Figure 8, $r = \frac{\sqrt{10}}{2}$ and $(-c_x, ic_y) = (-\frac{1}{2}, i\frac{1}{2})$. Note the angles of intersection between the unit circle (orange) and shifted circle (blue) at $z = -1$ (really $z = -i$). As $w'(z = -i) \neq 0$, these angles are preserved between the mapped unit circle (the line segment along the Re(w) axis from $w = -2$ to $w = 2$) and the mapped airfoil.

Now, because the unit circle is mapped to a horizontal line the angles of intersection can only be preserved by one of two curves, each perpendicular to each other (i.e. opposite signed slopes). Suppose then that the curve of the mapped airfoil passed through this line segment with a positive slope. This implies that the airfoil tail did not curve above the +Re(w) axis but instead passed up through the origin from below. The mapped airfoil curve would then continue into the 2^{nd} quadrant of the w-plane. However, it was determined using *Mathematica* that both $\mathrm{Re}(re^{i\frac{5\pi}{4}})$, $\mathrm{Im}(re^{i\frac{5\pi}{4}}) < 0$. This means that if the mapped airfoil did not curl above the +Re(w) axis, then it would have to pass back through the -Re(w) axis to arrive at the point $\big(\mathrm{Re}(re^{i\frac{5\pi}{4}}), \mathrm{Im}(re^{i\frac{5\pi}{4}})\big)$. But, in doing so the mapped airfoil curve would also pass back through the transformed unit circle, implying that there was another intersection of the two circles in the z-plane which there is not. Thus, the airfoil tail must

[3]see Appendix, Theorem 5.3.1

curve above the $+\text{Re}(w)$ axis due to the conformal nature of the Joukowsky mapping along with the fact that shifted circle only intersects the unit circle once in the z-plane.

4.3 Transformed Fluid Flow

Having determined the necessary requirements for the Joukowsky transformation, we now move on to the problem of fluid flow around the transformed airfoil. The appropriate fluid flow equation for this problem is that of *uniform flow* with a *doublet source* and a non-zero *circulation*. This last property is thus a departure from ideal fluid flow, in which required an irrotational assumption. The Complex Velocity Potential then becomes [Bre06]:

$$\Omega(z) = v_0 z + \frac{v_0 r^2}{z} - \frac{i\gamma}{2\pi} ln\left(\frac{z}{r}\right) \tag{4}$$

Here $v_0 z$ provides a Uniform Flow component with flow velocity v_0, $\frac{v_0 r^2}{z}$ adds a Doublet Flow to the circle of radius r and $-\frac{i\gamma}{2\pi} ln(\frac{z}{r})$ includes Circulation with a strength of γ.

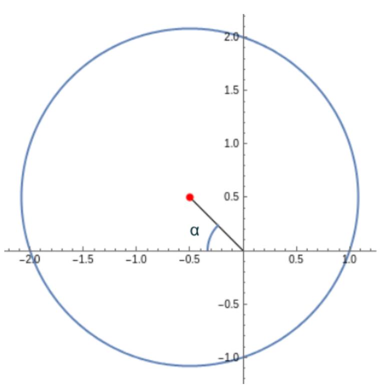
In its current state, 4 models fluid flow around a circle that is centered at the origin in the z-plane. To find the desired solution, we must therefore shift the center of the circle and apply the Joukowsky transformation. The shifted and transformed complex velocity potentials are given in 5 and 6 below:

$$\Omega(z - z_0) = v_0(z - z_0) + \frac{v_0 r^2}{z - z_0} - \frac{i\gamma}{2\pi} ln\left(\frac{z - z_0}{r}\right) \tag{5}$$

$$\Omega(w(z - z_0)) = v_0(z - z_0 + \frac{1}{z - z_0}) + \frac{v_0 r^2}{z - z_0 + \frac{1}{z - z_0}} - \frac{i\gamma}{2\pi} ln\left(\frac{z - z_0 + \frac{1}{z - z_0}}{r}\right) \tag{6}$$

$$= \phi(u(x, y), v(x, y)) + i\psi(u(x, y), v(x, y))$$

Here, ϕ and ψ again represent the velocity potential and stream function, respectively, as discussed in 3. Through methods outside the scope of knowledge of the authors of this paper, $\overline{\Omega'(z)}$ is found to be [Bre06]:



$$v_1 + iv_2 = \left(v_0\big(1 - \frac{1}{e^{-i2\alpha}}\big) + \frac{i\gamma}{2\pi R e^{-i\alpha}}\right)\frac{z^2}{z^2 - 1} \tag{7}$$

where,

α is an angle associated with the shifted circle in the z-plane, shown in the adjacent figure

Now according to 7, the stream velocity will grow without bound as z approaches its critical point at 1. Note here that $z = -1$ is not considered because fluid flow only has physical significance at and outside the boundary of the original and transformed geometries. As previously discussed, $z = 1$ maps to the trailing vertex of the airfoil in the w-plane and thus an infinite velocity at this point is inconsistent with observed reality. As such, one must restrict the velocity at this point. A sufficient condition is to require that the velocity vanish, a restriction known as the *Kutta condition* which determines the following value of the circulation strength, γ [Bre06]:

$$\gamma = -4v_0\pi r sin(\alpha)$$

This is very convenient given that there exists an equation for lift as a function of γ. Thus the lift per unit length of the transformed airfoil is given by [Bre06]:

$$L = -\rho\gamma v_0 = 4\pi\rho v_0^2 r sin(\alpha)$$

5 Conclusion and Future Work

In this work, we begin by motivating the use of complex numbers for modelling real valued problems in application. One such application of complex numbers is conformal maps, which are mappings from one space to another where the angles between streamlines are preserved. We demonstrate that analyticity in the complex number system provides a strong theoretical framework for the verification of such maps. We give a proof for the angle preserving properties provided by conformal maps and demonstrate how these properties are manifested in simple cases. We then show how a conformal map can be constructed using the Schwarz-Christoffel method and illustrate its usefulness in solving Ideal Fluid Flow. This culminates in an exploration of airflow around an wing. We choose to introduce the Joukowsky transformation to model this situation. The analyticity, and thus the validity of the conformal mapping of the Joukowsky transformation is verified. Then we we provide an application of the Joukowsky map to airflow over an airfoil. The properties and parameters involved in this model are elaborated on and demonstrate the usefulness of complex numbers in real-valued problems in aerospace. This field of study is ripe for further development, one such area being the large family of non-Joukowskian shapes or non-ideal fluid flow based scenarios which can be left for further investigation. A thorough analysis of the proof behind the Schwarz-Christoffel theorem is desired. Conformal maps, while not strictly requiring complex variables, would be untenable without the their contribution. This beautiful example of imaginary numbers improving our understanding of the real world is a valuable lesson in the practical usefulness of mathematics and its applications.

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6 Appendix

6.1 Relevant Theorems

- **Theorem 2.1.1**: The function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ of a region in the complex plane if and only if the partial derivatives $u_x, u_y, v_x,$ and v_y are continuous and satisfy the Cauchy-Riemann conditions. [AF21]
- **Theorem 5.2.1**: Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane. For any point $z \in D$ for which $f'(z) \neq 0$, this mapping is *conformal*, that is, it preserves the angle between two differentiable arcs. [AF21]
- **Theorem 5.3.1**: Assume that $f(z)$ is analytic and not constant in a domain D of the complex z-plane. Suppose that $f'(z_0) = f''(z_0) = ... = f^{n-1}(z_0) = 0$, while $f^n(z_0) \neq 0$, $z_0 \in D$. Then the mapping $z \to f(z)$ magnifies n times the angle between two intersecting differentiable arcs that meet at z_0. [AF21]

6.2 Proof 1

Proof. We want to show that under the analytic transformation $f(z)$, the tangent through a point z_0 residing on an arbitrary arc within this domain of analyticity is rotated by an angle $arg(f'(z_0))$ as a result of the transformation. However, this is the case only when $f'(z_0) \neq 0$.

We first the define the derivative of the arc with respect to the real parameter s: $\frac{dz(s)}{ds}$

$$\frac{dz(s)}{ds} = \frac{dx(s)}{ds} + i\frac{dy(s)}{ds}$$

Suppose $f(z)$ is analytic in an open neighborhood around $z_0 \equiv z(s_0)$. The derivative of the image of C, $w(s) = f(z(s))$, can be found using the chain rule:

$$\begin{aligned}\frac{dw(s_0)}{ds} &= f'(z_0)\frac{dz(s_0)}{ds} \\ &= f'(z_0) \cdot z'(s_0) \\ &= |f'(z_0)|e^{iarg(f'(z_0))} \cdot |z'(s_0)|e^{iarg(z'(s_0))} \\ &= |f'(z_0)| \cdot |z'(s_0)| \cdot e^{i(arg(f'(z_0))+arg(z'(s_0)))}\end{aligned}$$

Thus $arg(w'(s_0)) = arg(f'(z_0)) + arg(z'(s_0))$, i.e., the argument of the derivative of the transformation at point z_0 is added to the argument of the tangent of the image. □

6.3 Proof 2

Proof. Here we motivate the usefulness of the CR Equations. This proof summarizes proof given in a lecture by Professor Herbert Gross, MIT [Gro11]. Namely it shows that without the CR-equations, Laplace's equation does not hold for some general T(u,v).

$$u = u(x, y) \tag{8}$$
$$v = v(x, y) \tag{9}$$
$$T_x = T_{uu}u_x + T_{vv}v_x \tag{10}$$
$$T_y = T_{uu}u_y + T_{vv}v_y \tag{11}$$
$$T_{xx} + T_{yy} = T_{uu}(u_x^2 + u_y^2) + T_{vv}(v_x^2 + v_y^2) \tag{12}$$
$$+ 2T_{uv}(u_xv_x + u_yv_y) + T_u(u_{xx} + u_{yy}) + T_v(v_{xx} + v_{yy}) \tag{13}$$

But notice for $f = u + iv$, where u and v are analytic, we get:

$$f = u + iv \tag{14}$$
$$|f'|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2 = v_y^2 + v_x^2 \tag{15}$$
$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0 \tag{16}$$

which gives:

$$f = u + iv \tag{17}$$
$$|f'|^2 = u_x^2 + v_x^2 = u_x^2 + u_y^2 = v_y^2 + v_x^2 \tag{18}$$
$$u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0 \tag{19}$$

and finally yields:

$$T_{xx} + T_{yy} = [T_{uu} + T_{vv}]|f'(z)^2| \tag{20}$$
$$\tag{21}$$

and for f'(z) $\neq$ 0:

$$T_{xx} + T_{yy} = 0 \iff T_{uu} + T_{vv} = 0 \tag{22}$$
$$\tag{23}$$

□

6.4 Proof 3

Proof. Here we demonstrate that the Joukowsky also satisfies the Laplace equation. Recall that:

$$u_x = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2} = v_y \tag{24}$$
$$v_x = \frac{2xy}{(x^2 + y^2)^2} = -\frac{-2xy}{(x^2 + y^2)^2} = -u_y \tag{25}$$
$$\tag{26}$$

Computation of u_{xx} and u_{yy} yields:

$$u_{xx} + u_{yy} = \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3} - \frac{2x(x^2 - 3y^2)}{(x^2 + y^2)^3} = 0 \tag{27}$$

$$\tag{28}$$

$\square$