- Matrix Vector Multiplication

Want $\underline{x} \longrightarrow \boxed{A} \longrightarrow \underline{y} = A\underline{x}$ for $\underline{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $\underline{y} \in \mathbb{R}^m$

- ~~Inner~~ Dot Product View e.g.

$$\underline{y} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}}_{\underline{x}} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \end{bmatrix} \underline{x}$$

$$= \begin{bmatrix} \underline{a}_1^T \underline{x} \\ \underline{a}_2^T \underline{x} \end{bmatrix} = \begin{bmatrix} \langle \underline{a}_1, \underline{x} \rangle \\ \langle \underline{a}_2, \underline{x} \rangle \end{bmatrix}$$

- Note: $\underline{y}_{m\times 1} = A_{m\times n}\, \underline{x}_{n\times 1}$

- In general:

$$\underline{y} = A\underline{x} = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_m^T \end{bmatrix} \underline{x} = \begin{bmatrix} \underline{a}_1^T \underline{x} \\ \vdots \\ \underline{a}_m^T \underline{x} \end{bmatrix}$$

$y_k = a_k^T \underline{x}$ for $k = 1, \dots, m$

Dot Product View.

- Alternatively e.g.: Lin. Combo of Cols. of A View

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1}}_{\underline{x}} = \begin{bmatrix} 1x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 1x_1 \\ 4x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 6x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$= x_1 \underline{a}_1 + x_2 \underline{a}_2 + x_3 \underline{a}_3$$

- In general

$$A = [\underline{a}_1 \cdots \underline{a}_n]_{m\times n} \rightarrow A\underline{x} = x_1 \underline{a}_1 + \cdots + x_n \underline{a}_n$$

$a_\ell \in \mathbb{R}^m$

$$= \sum_{\ell=1}^{n} x_\ell \underline{a}_\ell$$

Column-Linear-Combo of Columns of A View of $A\underline{x}$.

Matrix Multiplication

$$C_{m\times n} = A_{m\times p} B_{p\times n}$$

1) Inner Product View

$$A = [\underline{a}_1, \dots, \underline{a}_\ell, \dots, \underline{a}_p] = \begin{bmatrix} \underline{\alpha}_1^T \\ \vdots \\ \underline{\alpha}_k^T \\ \vdots \\ \underline{\alpha}_m^T \end{bmatrix}$$

$$B = [\underline{b}_1, \dots, \underline{b}_\ell, \dots, \underline{b}_n] = \begin{bmatrix} \underline{\beta}_1^T \\ \vdots \\ \underline{\beta}_k^T \\ \vdots \\ \underline{\beta}_p^T \end{bmatrix}$$

Inner Product View

$$C = AB = \begin{bmatrix} \alpha_1^T \\ \vdots \\ \alpha_k^T \\ \vdots \\ \underline{\alpha}_m^T \end{bmatrix} [\underline{b}_1 \cdots \underline{b}_\ell \cdots \underline{b}_n]$$

$$= \begin{bmatrix} \underline{\alpha}_1^T \underline{b}_1 & \cdots & \underline{\alpha}_1^T \underline{b}_\ell & \cdots & \underline{\alpha}_1^T \underline{b}_n \\ \vdots & & & & \\ \alpha_k^T \underline{b}_1 & \cdots & \alpha_k^T b_\ell & \cdots & \alpha_k^T b_n \\ \alpha_m^T b_1 & \cdots & \alpha_m^T b_\ell & \cdots & \alpha_m^T b_n \end{bmatrix}$$

$$C_{k\ell} = \alpha_k^T \underline{b}_\ell \qquad k \in \{1, \dots, m\}$$
$$\ell \in \{1, \dots, n\}$$

e.g. $C = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}_{B} = \begin{bmatrix} 1\cdot0+2\cdot1+3\cdot2 & 1\cdot3+2\cdot4+3\cdot5 \\ 4\cdot0+5\cdot1+6\cdot2 & 4\cdot3+5\cdot4+6\cdot5 \end{bmatrix}$

2) Cols. of C viewed as lin. combos. of cols of A.

$$C = AB$$

Consider: $\quad c_1 = Ab_1 \cdots \underline{c}_l = A\underline{b}_l \cdots \underline{c}_n = A\underline{b}_n$

$$[\underline{c}_1 \cdots \underline{c}_l \cdots \underline{c}_n] = [\underbrace{\boxed{Ab_1}}_{}\cdots Ab_l \cdots Ab_n]$$

lin. combo. of cols of A

$$= A \underbrace{[\underline{b}_1 \cdots \underline{b}_l \cdots \underline{b}_n]}_{B}$$

$$\underline{c}_l = A\underline{b}_l = [\underline{a}_1 \cdots \underline{a}_r \cdots \underline{a}_p] \begin{bmatrix} b_{1l} \\ \vdots \\ b_{rl} \\ \vdots \\ b_{pl} \end{bmatrix}$$

$$= b_{1l}\,\underline{a}_1 + \cdots + b_{rl}\,\underline{a}_r + \cdots + b_{pl}\,\underline{a}_p$$

$\ldots ?$

e.g. $C = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}_{B}$

$$= \Bigg[\underbrace{\left[0\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]}_{\underline{c}_1 \text{ (1st col of } C)} \underbrace{\left[3\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 6 \end{bmatrix} \right]}_{\underline{c}_2} \Bigg]$$

3) Rows of C viewed as lin. combos. of the rows of B.

$$C = AB$$

$$\underbrace{\begin{bmatrix} \underline{c}_1^T \\ \vdots \\ \underline{c}_k^T \\ \vdots \\ \underline{c}_m^T \end{bmatrix}}_{C} = \underbrace{\begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_k^T \\ \vdots \\ \underline{a}_m^T \end{bmatrix}}_{A} B = \begin{bmatrix} \underline{a}_1^T B \\ \vdots \\ \underline{a}_k^T B \\ \vdots \\ \underline{a}_m^T B \end{bmatrix}$$

$\underline{c}_k^T = k^{th}$ row of C

$$\underline{c}_k^T = \underline{a}_k^T B = \begin{bmatrix} a_{k1} \cdots a_{kr} \cdots a_{kp} \end{bmatrix} \begin{bmatrix} \underline{\beta}_1^T \\ \vdots \\ \underline{\beta}_r^T \\ \vdots \\ \underline{\beta}_{p}^T \end{bmatrix}$$

$$\underline{c}_k^T = a_{k1}\underline{\beta}_1^T + \cdots + a_{kr}\underline{\beta}_r^T + a_{kp}\underline{\beta}_p^T$$

$$= \sum_{r=1}^{p} a_{kr}\underline{\beta}_r^T$$

e.g. $C = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}_{B}$

$$= \begin{bmatrix} [\,1[0\ 3] + 2[1\ 4] + 3[2\ 5]\,] \\ [\,4[0\ 3] + 5[1\ 4] + 6[2\ 5]\,] \end{bmatrix}$$

- 4^{th} View

Detour: Outer Product (Dyadic Product)

$$\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1} \qquad \underline{b} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}_{2\times1}$$

$$\underline{a}\underline{b}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1} \begin{bmatrix} 4 & 5 \end{bmatrix}_{1\times2} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} 1\,[4 \;\; 5] \\ 2\,[4 \;\; 5] \\ 3\,[4 \;\; 5] \end{bmatrix} = \begin{bmatrix} 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & 5\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \underline{a}\,[\underline{b}_1 \cdots \underline{b}_n]$$

→ In general:

4) $$\underline{a}\underline{b}^T = \underline{a}\,[b_1 \cdots b_n] = [\,\underline{a}\,b_1 \;\cdots\; \underline{a}\,b_n\,]_{m\times n}$$

only 1 indep. col. All others are scalar multiples.

Now, $C = AB$ viewed as sum of outer products of cols. of A & rows of B.

$$C = [\underline{a}_1 \cdots \underline{a}_\ell \cdots \underline{a}_p] \begin{bmatrix} \underline{\beta}_1^T \\ \vdots \\ \underline{\beta}_\ell^T \\ \vdots \\ \underline{\beta}_p^T \end{bmatrix}$$

$\beta_1 \in \mathbb{R}^n$

$a_\ell \in \mathbb{R}^m$

$$= \underline{a}_1\underline{\beta}_1^T + \cdots + \underline{a}_\ell\underline{\beta}_\ell^T + \cdots + \underline{a}_p\underline{\beta}_p^T$$

$\underset{m\times1}{\underline{a}_\ell}\;\underset{1\times n}{\underline{\beta}_\ell^T}$

$$C = \sum_{\ell=1}^{p} \underline{a}_\ell\,\underline{\beta}_\ell^T$$

4) cont'd

e.g. $C = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}_{B}$

$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} \begin{bmatrix} 2 & 5 \end{bmatrix}$

$=$

Concept	Symbolic notation	Index (Einstein) notation
Vector–matrix product	$y = Ax$ wikipedia	$y_i = \sum_j a_{ij} x_j$ or $y_i = a_{ij} x_j$ brown
Inner product	$x^\top y$ wikipedia	$\sum_i x_i y_i = x_i y_i$ brown
Quadratic form	$x^\top A x$ wikipedia	$\sum_{i,j} a_{ij} x_i x_j = a_{ij} x_i x_j$ brown
Trace/Frobenius	$\mathrm{tr}(A^\top B)$ math.uwaterloo	$\sum_{i,j} a_{ij} b_{ij}$ brown
Jacobian of linear map	$\frac{\partial (Ax)}{\partial x} = A$ (numerator layout) wikipedia	$\frac{\partial y_i}{\partial x_k} = a_{ik}$ so the Jacobian has entries a_{ik} wikipedia

3 Matrix Multiplication

Definition 3 Let $\mathbf{A}$ be $m \times n$, and $\mathbf{B}$ be $n \times p$, and let the product $\mathbf{AB}$ be

$$\mathbf{C} = \mathbf{AB} \tag{3}$$

then $\mathbf{C}$ is a $m \times p$ matrix, with element (i, j) given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{4}$$

for all $i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, p$.

Proposition 1 *Let $\mathbf{A}$ be $m \times n$, and $\mathbf{x}$ be $n \times 1$, then the typical element of the product*

$$\mathbf{z} = \mathbf{Ax} \tag{5}$$

is given by

$$z_i = \sum_{k=1}^{n} a_{ik} x_k \tag{6}$$

for all $i = 1, 2, \ldots, m$. Similarly, let $\mathbf{y}$ be $m \times 1$, then the typical element of the product

$$\mathbf{z}^T = \mathbf{y}^T \mathbf{A} \tag{7}$$

is given by

$$z_i = \sum_{k=1}^{n} a_{ki} y_k = (A^T)_{ik} \, y_k \tag{8}$$

$\Rightarrow z = A^T y$

$z^T = y^T A$

for all $i = 1, 2, \ldots, n$. Finally, the scalar resulting from the product

$$\alpha = \mathbf{y}^T \mathbf{Ax} \tag{9}$$

is given by

$$\alpha = \sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk} y_j x_k \tag{10}$$

Proof: *These are merely direct applications of Definition 3.* q.e.d.

Proposition 2 *Let* $\mathbf{A}$ *be* $m \times n$, *and* $\mathbf{B}$ *be* $n \times p$, *and let the product* $\mathbf{AB}$ *be*

$$\mathbf{C} = \mathbf{AB} \tag{11}$$

then

$$\mathbf{C}^T = \mathbf{B}^T \mathbf{A}^T \tag{12}$$

Proof: *The typical element of* $\mathbf{C}$ *is given by*

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{13}$$

By definition, the typical element of $\mathbf{C}^T$, *say* d_{ij}, *is given by*

$$d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ki} a_{jk} \tag{14}$$

Hence,

$$\mathbf{C}^T = \mathbf{B}^T \mathbf{A}^T \qquad = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} \tag{15}$$

q.e.d.

5 Matrix Differentiation

In the following discussion I will differentiate matrix quantities with respect to the elements of the referenced matrices. Although no new concept is required to carry out such operations, the element-by-element calculations involve cumbersome manipulations and, thus, it is useful to derive the necessary results and have them readily available [2].

Convention 3
Let

$$\mathbf{y} = \psi(\mathbf{x}), \tag{23}$$

where $\mathbf{y}$ is an m-element vector, and $\mathbf{x}$ is an n-element vector. The symbol

$\psi(x): \mathbb{R}^n \to \mathbb{R}^m$

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix} \tag{24}$$

will denote the $m \times n$ matrix of first-order partial derivatives of the transformation from $\mathbf{x}$ to $\mathbf{y}$. Such a matrix is called the Jacobian matrix of the transformation $\psi()$.

Notice that if $\mathbf{x}$ is actually a scalar in Convention 3 then the resulting Jacobian matrix is a $m \times 1$ matrix; that is, a single column (a vector). On the other hand, if $\mathbf{y}$ is actually a scalar in Convention 3 then the resulting Jacobian matrix is a $1 \times n$ matrix; that is, a single row (the transpose of a vector).

Almost—more precisely, a Jacobian is the m×n matrix of first-order partial derivatives of a vector-valued map ψ: ℝ^n → ℝ^m, arranged with the i-th row equal to the gradient of the i-th output component and the j-th column giving sensitivities with respect to x_j across outputs. Beyond being "elementwise derivatives," the Jacobian represents the linear map (the differential) that best approximates ψ near a point: ψ(x) ≈ ψ(x0) + Jψ(x0)(x − x0) for x near x0. When m=1, the Jacobian reduces to the row form of the gradient; when n=m and one takes its determinant, that determinant is the "Jacobian" used in change of variables. wikipedia +2

Proposition 5 *Let*

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{25}$$

where $\mathbf{y}$ *is* $m \times 1$, $\mathbf{x}$ *is* $n \times 1$, $\mathbf{A}$ *is* $m \times n$, *and* $\mathbf{A}$ *does not depend on* $\mathbf{x}$, *then*

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{A} \tag{26}$$

Proof: *Since the* ith *element of* $\mathbf{y}$ *is given by*

Def 3, Prop 1

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \tag{27}$$

it follows that

$$\frac{\partial y_i}{\partial x_j} = a_{ij} \tag{28}$$

for all $i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n$. *Hence*

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{A} \tag{29}$$

q.e.d.

Proposition 6 *Let*

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{30}$$

where $\mathbf{y}$ *is* $m \times 1$, $\mathbf{x}$ *is* $n \times 1$, $\mathbf{A}$ *is* $m \times n$, *and* $\mathbf{A}$ *does not depend on* $\mathbf{x}$, *as in Proposition 5. Suppose that* $\mathbf{x}$ *is a function of the vector* $\mathbf{z}$, *while* $\mathbf{A}$ *is independent of* $\mathbf{z}$. *Then*

$$\frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \mathbf{A}\frac{\partial \mathbf{x}}{\partial \mathbf{z}} \tag{31}$$

Proof: *Since the* ith *element of* $\mathbf{y}$ *is given by*

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \tag{32}$$

for all $i = 1, 2, \ldots, m$, *it follows that*

$$\frac{\partial y_i}{\partial z_j} = \sum_{k=1}^{n} a_{ik} \frac{\partial x_k}{\partial z_j} \tag{33}$$

but the right hand side of the above is simply element (i, j) *of* $\mathbf{A}\frac{\partial \mathbf{x}}{\partial \mathbf{z}}$. *Hence*

$$\frac{\partial \mathbf{y}}{\partial \mathbf{z}} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{z}} = \mathbf{A}\frac{\partial \mathbf{x}}{\partial \mathbf{z}} \tag{34}$$

q.e.d.

chain rule holds

Proposition 7 *Let the scalar α be defined by*

$$\alpha = \mathbf{y}^T \mathbf{A} \mathbf{x} \tag{35}$$

where $\mathbf{y}$ *is* $m \times 1$, $\mathbf{x}$ *is* $n \times 1$, $\mathbf{A}$ *is* $m \times n$, *and* $\mathbf{A}$ *is independent of* $\mathbf{x}$ *and* $\mathbf{y}$, *then*

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{y}^T \mathbf{A} \tag{36}$$

and

$$\frac{\partial \alpha}{\partial \mathbf{y}} = \mathbf{x}^T \mathbf{A}^T \tag{37}$$

Proof: *Define*

$$\mathbf{w}^T = \mathbf{y}^T \mathbf{A} \tag{38}$$

and note that

$$\alpha = \mathbf{w}^T \mathbf{x} \tag{39}$$

Hence, by Proposition 5 we have that

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{w}^T = \mathbf{y}^T \mathbf{A} \tag{40}$$

which is the first result. Since α is a scalar, we can write

$$\alpha = \alpha^T = \mathbf{x}^T \mathbf{A}^T \mathbf{y} \tag{41}$$

and applying Proposition 5 as before we obtain

$$\frac{\partial \alpha}{\partial \mathbf{y}} = \mathbf{x}^T \mathbf{A}^T \tag{42}$$

q.e.d.

HW 0 Answers...

Proposition 8 *For the special case in which the scalar* α *is given by the quadratic form*

$$\alpha = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{43}$$

where $\mathbf{x}$ *is* $n \times 1$, $\mathbf{A}$ *is* $n \times n$, *and* $\mathbf{A}$ *does not depend on* $\mathbf{x}$, *then*

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{x}^T \left(\mathbf{A} + \mathbf{A}^T \right) \tag{44}$$

Proof: *By definition*

$$\alpha = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j \tag{45}$$

Def 3
Prop 1

Differentiating with respect to the kth *element of* $\mathbf{x}$ *we have*

$$\frac{\partial \alpha}{\partial x_k} = \underbrace{\sum_{j=1}^{n} a_{kj} x_j}_{i=k} + \underbrace{\sum_{i=1}^{n} a_{ik} x_i}_{j=k} \tag{46}$$

i or j must $= k$ for x_j or x_i to survive, respectively

for all $k = 1, 2, \ldots, n$, *and consequently,*

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \underbrace{\mathbf{x}^T \mathbf{A}^T}_{= Ax} + \underline{\mathbf{x}^T \mathbf{A}} = \mathbf{x}^T \left(\mathbf{A}^T + \mathbf{A} \right) \tag{47}$$

q.e.d.

Proposition 9 *For the special case where* $\mathbf{A}$ *is a symmetric matrix and*

$$\alpha = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{48}$$

where $\mathbf{x}$ *is* $n \times 1$, $\mathbf{A}$ *is* $n \times n$, *and* $\mathbf{A}$ *does not depend on* $\mathbf{x}$, *then*

$$\frac{\partial \alpha}{\partial \mathbf{x}} = 2\mathbf{x}^T \mathbf{A} \tag{49}$$

Proof: *This is an obvious application of Proposition 8.* q.e.d.