

Lambda

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Why study Lambda Calculus? Well it is the most mathematical and symbolic way to talk about computation. Last year I loved it as a concept. Now we'll see how it actually goes.

1 Introduction

Definition 1.1 (Lambdas). A function $f(x) = y$ can be represented by $\lambda x.y$. For example the identity function is described by $\lambda x.x$. Note, we would usually define a function with a domain and a co-domain. This is an ambiguity which will be explored later.

To apply a function we simply adjoin it to a variable. So for instance,

$$(\lambda x.x^2 + x) 3 \rightarrow 3^2 + 3 \rightarrow 12$$

For functions $f(x, y) = z$ we can use currying. Take $f(x, y) = 2x + y$. Then:

$$\begin{aligned} f &= \lambda x.\lambda y.2x + y \\ f\ 3 &= \lambda y.6 + y \\ f\ 3\ 2 &= 8 \end{aligned}$$

But what is a function? And when can two functions be equal? Well the extensional definition of a function is the following.

Definition 1.2 (Extensional equality). Two functions f, g are equal if and only if $\forall x f(x) = g(x)$. Note that again we have not defined the domain of f and g .

So although $\lambda x.x + x$ and $\lambda x.2x$ are extensionally equal, one might say that intensionally they are not as the instructions given are different.

Now the pure untyped lambda calculus has no types, does not distinguish between data and so on. It can be described by the context-free grammar:

$$s ::= x \mid \lambda x.s \mid s\ s$$

Suprisingly this can encode any possible function. Weird.

The way we “compute” in λ -calculus is through reductions. But reduction as a term can be misleading. Take for instance:

$$\Omega := (\lambda x.xx)(\lambda y.yy)$$

Then:

$$\Omega \rightarrow (xx)[(\lambda y.yy)/x] \rightarrow (\lambda y.yy)(\lambda y.yy) = \Omega$$

Which looks like a non-terminating loop. Also we will prove later that every lambda term has a fixed point. I.e.

$$\forall \lambda\text{-terms } g \ \exists x_0 \text{ s.t. } g \ x_0 = x_0$$