Rings

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1 Introduction

In the "beautiful" mental position that is Hilary of the second year I decided to study rings. What is a ring? Why is it important? We will hopefully learn. But quite possibly not. Let's end this section with a quote.

If you liked it then you shoulda put a ring on it - Beyonce

2 Recap on rings

Let's start this section with the obvious thing we should define. The ring!

Definition 2.1 (Ring)**.** A ring $(R, +, \times)$ is made of a set R (also called the carrier set) and two binary operations $+ : R \times R \longrightarrow R$, $\times : R \times R \longrightarrow R$. Such that:

1. $(R, +)$ is an abelian group.
2. $\times$ is associative
3. $\times$ distributes over $+$ i.e. $\forall a, b, c \in R$

$$a \times (b + c) = a \times b + a \times c \text{ and} (a + b) \times c = a \times c + b \times c$$

If R has a multiplicative identity we call R *unital*

Theorem 2.2 (Basic properties)**.** *1. Zero is annihilating:* $\forall r \in R \; 0_R r = 0_R = r 0_R$

2. $(-x)y = -xy = x(-y)$

Proof. 1. $0_R r = (0_R + 0_R)r = 0_R r + 0_R r \implies 0_R r = 0_R$

2. $xy + (-x)y = (x - x)y = 0_R y = 0_R$ Hence by the uniqueness of the inverse of $-xy$, $(-x)y = -xy$. Similarly for $x(-y)$

$\square$

Definition 2.3 (Unit group)**.** For a *unital* ring R denote $U(R)$ the set of $r \in R$ with a multiplicative inverse.

Theorem 2.4. *$(U(R), \times)$ is a group.*

Proof.

1. Associativity is inherited
2. Identity: $1_R \in U(R)$ trivially.
3. Inverse: By definition of $U(R)$
4. Closure: Let $x, y \in U(R)$ then $\exists x^{-1}, y^{-1} \in U(R)$.
   Then $(xy)(y^{-1}x^{-1}) = 1_R \implies xy \in U(R)$

□

Definition 2.5 (Subrings)**.** Let R be a ring and $S \subseteq R$ such that S is a ring. Then S is a subring. If R is a unital ring and $1_R \in S$ then S is a unital subring.

Lemma 2.6 (Closure of intersection)**.** Let $\mathcal{Q}$ a set of subrings of a ring R. Then $\bigcap\limits_{S \in \mathcal{Q}} S$ is a subring of R.

Definition 2.7 (Generated Subring)**.**

$$S[\lambda_1, ..., \lambda_n] = \bigcap \{T : T \text{ is a subring of } R \text{ and } \lambda_1, \ldots, \lambda_n \in T \text{ and } S \subseteq T\}$$

Definition 2.8 (Homomorphisms)**.** Let $\phi : R \longrightarrow S$ where R, S rings. And $\forall x, y \in R$:

$$\phi(x + y) = \phi(x) + \phi(y) \qquad \text{and} \qquad \phi(xy) = \phi(x)\phi(y)$$

Then ϕ is a homomorphism.

If R, S are unital and $\phi(1_R) = 1_S$ then ϕ is a unital homomorphism.

Lemma 2.9 (Inverses)**.** Let $\phi : R \longrightarrow S$ a homomorphism. Then $\phi(0_R) = 0_S$ and $\forall r \in R$, $\phi(-r) = -\phi(r)$. If ϕ is unital then $\forall x \in U(R)$, $\phi(x) \in U(S)$ and $\phi(x)^{-1} = \phi(x^{-1})$.

Proof.

$$\phi(0_R) = \phi(0_R + 0_R) = \phi(0_R) + \phi(0_R) \implies \phi(0_R) = 0_S$$
$$0_S = \phi(0_R) = \phi(x) + \phi(-x) \implies \phi(-x) = -\phi(x)$$

Similarly for $\phi(x)^{-1} = \phi(x^{-1})$ □

3 Integral Domains and Polynomials

Definition 3.1 (Zero dividers)**.** $r \in R$ is a (left) *zero divider* if $\exists s \in R^*$ such that $rs = 0_R$

Definition 3.2 (Integral Domains)**.** R is an integral domain if it is a non-trivial, commutative, unital ring with no zero-divisors.

Lemma 3.3. If R is an integral domain then if $x \in R^*$ $xy = xz \implies y = z$

Proof.

$$0_R = xy - xz = x(y - z) \implies y - z = 0 \implies y = z \text{ since } x \text{ is not a zero-divider}$$

□

This proof is refered to the notes as "cute". Its a 7 at best.

Theorem 3.4. *A finite integral domain is a field.*

Proof. Consider $R \longrightarrow R$, $x \mapsto ax$ where $a \in R^*$. This map is injective by cancellation. Since R is finite it is also surjective and hence $\exists r \in R$ such that $ar = 1_R$. By commutativity it is a two sided inverse. So a has an inverse and since a is arbitrary R is a field. □

Basic stuff about polynomials, what you'd expect without having done any rings.

Definition 3.5 (Polynomial)**.** Let R be a non-trivial commutative, unital ring. Then we write $R[X]$ the set of R-polynomials with coefficients in R and variable X. These are of the form:

$$p(X) = \sum_{i=0}^{\infty} r_i X^i$$

where $r_i \in R$ and $r_i \in R^*$ for finitely many i.

Two polynomials are equal if all their coefficients are equal. Also let polynomials p, q with coefficients a_i, b_i. Then:

$$(p + q)(X) = \sum_{i=0}^{\infty}(a_i + b_i)X^i \quad (pq)(X) = \sum_{i=0}^{\infty}(\sum_{j=0}^{i} a_j b_{i-j})X^i$$

Theorem 3.6. *The following are equivalent:*

1. *R is an integral domain*
2. *$R[X]$ is an integral domain*
3. *$p, q \in R[X]^* \implies pq \in R[X]^*$ and $\deg pq = \deg p + \deg q$*
4. *A polynomial of degree d has at most d roots*

Proof. Most of these are trivial. (2) implies (1) by considering constant polynomials. (3) implies (2) by definition. Now, (1) implies (3) since for $\deg p = n$, $\deg q = m$ the coefficient $n+m$th coefficient of pq is $a_n b_m$. Since R is an integral domain and $a_n, b_m \neq 0_R$, $a_n b_m \neq 0_R \implies \deg pq = n + m$

For (4) implying (1): Consider the polynomial $p(X) = rX$, $r \neq 0_R$. Since $p(X)$ has only one root and 0_R is a root there are no other roots. So R is an integral domain.

Now for (1) + (2) $\implies$ (4). $\square$

4 Ideals

Reminder: an *ideal* is subring which can be a kernel of a ring homorphism. It is a subring which is closed under multiplication with the *entire* ring.

These ideals can be broken up in several ways.

Definition 4.1 (Proper)**.** A *proper* ideal is just an ideal which is not equal to the ring. Think of proper subsets.

Definition 4.2 (Principal)**.** An ideal is principal if it is generated by one element.

Example. In $\mathbb{Z}[X]$, $\langle 2\,, X\rangle$ is not principal. There's no $p \in \mathbb{Z}[X]$ such that $2p = X$ or $Xp = 2$. But for $\mathbb{F}[X]$ where $\mathbb{F}$ is a field every ideal is principal.

Definition 4.3 (Prime)**.** A proper ideal is *prime* if $ab \in I \implies a \in I$ or $b \in I$.

Example. For $R = \mathbb{Z}$, $\langle p\rangle$ is prime for any p prime. That is partly how the definition is given.

Definition 4.4 (Maximal)**.** A proper ideal is maximal if there is no strictly "bigger" proper ideal. Basically if you add an element which is not in the ring

Example. For $R = \mathbb{Z}$ $\langle 5\rangle$ is a maximal ideal, but $\langle 4\rangle$ is not because, $\langle 4\rangle \subseteq \langle 2\rangle$.

Now we can use the information for each of these ideals to determine increasingly precise things about R/I. In all the following examples R is a commutative unital ring and I is proper.

Proposition 4.5. I is prime $\iff$ R/I is an integral domain.

Let's think why this makes sense. If R/I was not an integral domain then we could pick two elements $a + I, b + I$ and $(ab) + I \in I$ hence $a, b \neq I$ but $ab \in I$. A contradiction. Similarly for the other direction.

Proposition 4.6. I is maximal $\iff$ R/I is a field.

The idea is that fields only have one ideal. That alongside with the correspondence theorem means that there are no ideals in R/I hence it is a field.

Corollary 4.7. A maximal ideal is prime.