Rings

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1 Introduction

In the "beautiful" mental position that is Hilary of the second year I decided to study rings. What is a ring? Why is it important? We will hopefully learn. But quite possibly not. Let's end this section with a quote.

$\boxed{\text{If you liked it then you shoulda put a ring on it - Beyonce}}$

2 Recap on rings

Let's start this section with the obvious thing we should define. The ring!

Definition 2.1 (Ring)**.** A ring $(R, +, \times)$ is made of a set R (also called the carrier set) and two binary operations $+ : R \times R \longrightarrow R$, $\times : R \times R \longrightarrow R$. Such that:

1. $(R, +)$ is an abelian group.
2. $\times$ is associative
3. $\times$ distributes over $+$ i.e. $\forall a, b, c \in R$

$$a \times (b + c) = a \times b + a \times c \text{ and} (a + b) \times c = a \times c + b \times c$$

If R has a multiplicative identity we call R *unital*

Theorem 2.2 (Basic properties)**.** *1. Zero is annihilating:* $\forall r \in R\ 0_R r = 0_R = r 0_R$

2. $(-x)y = -xy = x(-y)$

Proof. 1. $0_R r = (0_R + 0_R)r = 0_R r + 0_R r \implies 0_R r = 0_R$

2. $xy + (-x)y = (x - x)y = 0_R y = 0_R$ Hence by the uniqueness of the inverse of $-xy$, $(-x)y = -xy$. Similarly for $x(-y)$

$\square$

Definition 2.3 (Unit group)**.** For a *unital* ring R denote $U(R)$ the set of $r \in R$ with a multiplicative inverse.

Theorem 2.4. *$(U(R), \times)$ is a group.*

Proof.

1. Associativity is inherited
2. Identity: $1_R \in U(R)$ trivially.
3. Inverse: By definition of $U(R)$
4. Closure: Let $x, y \in U(R)$ then $\exists x^{-1}, y^{-1} \in U(R)$.
   Then $(xy)(y^{-1}x^{-1}) = 1_R \implies xy \in U(R)$

□

Definition 2.5 (Subrings)**.** Let R be a ring and $S \subseteq R$ such that S is a ring. Then S is a subring. If R is a unital ring and $1_R \in S$ then S is a unital subring.

Lemma 2.6 (Closure of intersection)**.** Let $\mathcal{Q}$ a set of subrings of a ring R. Then $\bigcap\limits_{S \in \mathcal{Q}} S$ is a subring of R.

Definition 2.7 (Generated Subring)**.**

$$S[\lambda_1, ..., \lambda_n] = \bigcap \{T : T \text{ is a subring of } R \text{ and } \lambda_1, \ldots, \lambda_n \in T \text{ and } S \subseteq T\}$$

Definition 2.8 (Homomorphisms)**.** Let $\phi : R \longrightarrow S$ where R, S rings. And $\forall x, y \in R$:

$$\phi(x + y) = \phi(x) + \phi(y) \qquad \text{and} \qquad \phi(xy) = \phi(x)\phi(y)$$

Then ϕ is a homomorphism.

If R, S are unital and $\phi(1_R) = 1_S$ then ϕ is a unital homomorphism.

Lemma 2.9 (Inverses)**.** Let $\phi : R \longrightarrow S$ a homomorphism. Then $\phi(0_R) = 0_S$ and $\forall r \in R$, $\phi(-r) = -\phi(r)$. If ϕ is unital then $\forall x \in U(R)$, $\phi(x) \in U(S)$ and $\phi(x)^{-1} = \phi(x^{-1})$.

Proof.

$$\phi(0_R) = \phi(0_R + 0_R) = \phi(0_R) + \phi(0_R) \implies \phi(0_R) = 0_S$$
$$0_S = \phi(0_R) = \phi(x) + \phi(-x) \implies \phi(-x) = -\phi(x)$$

Similarly for $\phi(x)^{-1} = \phi(x^{-1})$ □