

Metric Spaces and Complex Analysis

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1 Metric Spaces

Definition 1.1 (Metric Space). A metric space $M = (X, d)$ is a set equipped with a function $d: X \times X \rightarrow \mathbb{R}$ such that:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Definition 1.2 (Continuity). A function $f: X \rightarrow Y$ is continuous at $x_0 \in X$ when $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x \in B(x_0, \delta), f(x) \in B(f(x_0), \varepsilon)$

Definition 1.3 (Uniform Continuity). A function $f: X \rightarrow Y$ is uniformly continuous if $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x \in B(x, \delta) f(z) \in B(f(x), \varepsilon)$

Definition 1.4 (Convergence). A series (x_n) converges in a metric space X if there is an $x_0 \in X$ such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N$ $d(x_n, x_0) < \varepsilon$

Lemma 1.5 (Sequential Continuity). A function $f: X \rightarrow Y$ is continuous at $a \in X$ if and only if for every sequence $(x_n) \rightarrow a, (f(x_n)) \rightarrow f(a)$

Definition 1.6 (Norm). Let V a vector space. Then $\|\cdot\|: V \rightarrow \mathbb{R}$ is a norm if:

1. $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0_V$
2. $\|\lambda v\| = |\lambda| \|v\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

Definition 1.7 (Open Set). A set $U \subseteq X$ is open if $\forall x \in U$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.

Theorem 1.8 (Topological Continuity). A function $f: X \rightarrow Y$ is continuous if and only if the pre-image of every open set is open.

Definition 1.9 (Interior). The interior of S is the largest open subset of S , defined as:

$$\text{int}(S) = \bigcup_{U \subseteq S, U \text{ open}} U$$

Definition 1.10 (Closure). The closure of a set S is the smallest closed subset containing S :

$$\overline{S} = \bigcap_{S \subseteq C, C \text{ closed}} C$$

Lemma 1.11. A function is continuous if and only if $f(\overline{S}) \subseteq \overline{f(S)}$

2 Complex Exponential

The following power series define the complex exponential and trigonometric functions:

$$\begin{aligned}\exp z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, & \sin z &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!}, & \cos z &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \\ \sinh z &= \sum_{k=0}^{\infty} \frac{z^{(2k+1)}}{(2k+1)!}, & \cosh z &= \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}\end{aligned}$$

Note:

$$\begin{aligned}\sin z &= \frac{e^{iz} - e^{-iz}}{2i}, & \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sinh z &= \frac{e^z - e^{-z}}{2}, & \cosh z &= \frac{e^z + e^{-z}}{2}\end{aligned}$$

And

$$\exp i\theta = \cos \theta + i \sin \theta$$

3 Holomorphic Functions

Definition 3.1 (Domain). A domain usually denoted U is an open, connected subset of the complex numbers.

Theorem 3.2 (Cauchy's Theorem). *Let $f: U \rightarrow \mathbb{C}$ holomorphic on a domain U . Then for all closed paths γ :*

$$\int_{\gamma} f(z) dz = 0$$

Theorem 3.3 (Deformation Theorem). *Let $f: U \rightarrow \mathbb{C}$ be holomorphic on domain U . Let two closed paths γ_1, γ_2 be homotopic. Then:*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Theorem 3.4 (Cauchy's Integral Formula). *Let $f: U \rightarrow \mathbb{C}$ holomorphic on and inside a simple, closed, positively oriented curve γ . Then for all points a on the interior of γ :*

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw$$

Theorem 3.5 (Taylor's Theorem). *All holomorphic functions on a domain can be expressed as a power series. For $f: U \rightarrow \mathbb{C}$ holomorphic on domain U and for $a \in U$, $D(a, r) \subseteq U$*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

where:

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!}$$

Theorem 3.6 (Liouville's Theorem). *Let f holomorphic on \mathbb{C} and f bounded. Then f is constant.*

Corollary 3.7. For f entire, $f(\mathbb{C})$ is dense in \mathbb{C} (i.e. $\overline{f(\mathbb{C})} = \mathbb{C}$)

Theorem 3.8 (Picard's Little Theorem). *For f non-constant entire, $f(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C} \setminus \{z\}$*

Theorem 3.9 (Fundamental Theorem of Algebra). *Let p be a non-constant polynomial with complex coefficients. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

Theorem 3.10 (Morera's Theorem). *Let f continuous on a domain U and for all closed paths γ in U*

$$\int_{\gamma} f(z) dz = 0$$

Then f is holomorphic.

Theorem 3.11 (Identity Theorem). *Let f holomorphic on domain U let $S = f^{-1}(0)$. If S contains one of its limit points then f is identically zero.*

Theorem 3.12 (Counting Zeroes). *Let f holomorphic inside and on a positively oriented closed path γ . Then the sum of zeroes counting their multiplicity is:*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(w)}{f(w)} dw$$

Theorem 3.13 (Laurent's Theorem). *Let f be a function holomorphic on $z \in \mathbb{C} | R < |z-a| < S$. Then,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

For:

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} dw$$