

# Metric Spaces and Complex Analysis

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September 18, 2020

# 1 Metric Spaces

**Definition 1.1** (Metric Space). A metric space  $M = (X, d)$  is a set equipped with a function  $d: X \times X \rightarrow \mathbb{R}$  such that:

1.  $d(x, y) \geq 0$  and  $d(x, y) = 0 \iff x = y$
2.  $d(x, y) = d(y, x)$
3.  $d(x, z) \leq d(x, y) + d(y, z)$

**Definition 1.2** (Continuity). A function  $f: X \rightarrow Y$  is continuous at  $x_0 \in X$  when  $\forall \varepsilon > 0 \exists \delta > 0$  such that  $\forall x \in B(x_0, \delta), f(x) \in B(f(x_0), \varepsilon)$

**Definition 1.3** (Uniform Continuity). A function  $f: X \rightarrow Y$  is uniformly continuous if  $\forall \varepsilon > 0 \exists \delta > 0$  such that  $\forall x \in B(x, \delta) f(z) \in B(f(x), \varepsilon)$

**Definition 1.4** (Convergence). A series  $(x_n)$  converges in a metric space  $X$  if there is an  $x_0 \in X$  such that for all  $\varepsilon > 0$  there is an  $N \in \mathbb{N}$  such that for all  $n > N$   $d(x_n, x_0) < \varepsilon$

**Lemma 1.5** (Sequential Continuity). A function  $f: X \rightarrow Y$  is continuous at  $a \in X$  if and only if for every sequence  $(x_n) \rightarrow a, (f(x_n)) \rightarrow f(a)$

**Definition 1.6** (Norm). Let  $V$  a vector space. Then  $\|\cdot\|: V \rightarrow \mathbb{R}$  is a norm if:

1.  $\|v\| \geq 0$  and  $\|v\| = 0 \iff v = 0_V$
2.  $\|\lambda v\| = |\lambda| \|v\|$
3.  $\|x + y\| \leq \|x\| + \|y\|$

**Definition 1.7** (Open Set). A set  $U \subseteq X$  is open if  $\forall x \in U$ , there is an  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq U$ .

**Theorem 1.8** (Topological Continuity). A function  $f: X \rightarrow Y$  is continuous if and only if the pre-image of every open set is open.

**Definition 1.9** (Interior). The interior of  $S$  is the largest open subset of  $S$ , defined as:

$$\text{int}(S) = \bigcup_{U \subseteq S, U \text{ open}} U$$

**Definition 1.10** (Closure). The closure of a set  $S$  is the smallest closed subset containing  $S$ :

$$\overline{S} = \bigcap_{S \subseteq C, C \text{ closed}} C$$

**Lemma 1.11.** A function is continuous if and only if  $f(\overline{S}) \subseteq \overline{f(S)}$

**Definition 1.12** (Homeomorphism). A homeomorphism between two metric spaces is a continuous bijection with a continuous inverse.

**Definition 1.13** (Cauchy Sequence). A sequence  $(x_n)$  in a metric space  $X$  is Cauchy if for every  $\varepsilon > 0$  there is an  $N \in \mathbb{N}$  such that for all  $n, m > N$   $d(x_n, x_m) < \varepsilon$ .

**Lemma 1.14.** Convergent sequences are Cauchy. Cauchy sequences are bounded.

**Definition 1.15** (Completeness). A metric space is complete if every Cauchy sequence converges.

**Lemma 1.16.** A subset of a complete metric space is complete if and only if the subset is closed.

**Lemma 1.17.** Let  $X$  complete and  $D_1, D_2, \dots$  closed with  $D_1 \supseteq D_2 \supseteq \dots$  and  $\text{diam}(D_k) \rightarrow 0$ . Then  $\bigcap_{k \in \mathbb{N}} D_k = \{x\}$ .

**Definition 1.18** (Lipschitz Continuity). A map  $f: X \rightarrow Y$  is Lipschitz continuous if for all  $x_1, x_2 \in X$ ,

$$d_Y(f(x_1), f(x_2)) \leq M d_X(x_1, x_2)$$

For  $M \in [0, 1)$ ,  $f$  is a contraction.

**Theorem 1.19** (Contraction Mapping Theorem). *Let  $f: X \rightarrow X$  a contraction and  $X$  complete and non-empty. Then  $f$  has a unique fixed point.*

*Proof.*

□

## 2 Complex Exponential

The following power series define the complex exponential and trigonometric functions:

$$\begin{aligned} \exp z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, & \sin z &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!}, & \cos z &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \\ \sinh z &= \sum_{k=0}^{\infty} \frac{z^{(2k+1)}}{(2k+1)!}, & \cosh z &= \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \end{aligned}$$

Note:

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, & \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sinh z &= \frac{e^z - e^{-z}}{2}, & \cosh z &= \frac{e^z + e^{-z}}{2} \end{aligned}$$

And

$$\exp i\theta = \cos \theta + i \sin \theta$$

### 3 Holomorphic Functions

**Definition 3.1** (Domain). A domain usually denoted  $U$  is an open, connected subset of the complex numbers.

**Theorem 3.2** (Cauchy's Theorem). Let  $f: U \rightarrow \mathbb{C}$  holomorphic on a domain  $U$ . Then for all closed paths  $\gamma$ :

$$\int_{\gamma} f(z)dz = 0$$

**Theorem 3.3** (Deformation Theorem). Let  $f: U \rightarrow \mathbb{C}$  be holomorphic on domain  $U$ . Let two closed paths  $\gamma_1, \gamma_2$  be homotopic. Then:

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

**Theorem 3.4** (Cauchy's Integral Formula). Let  $f: U \rightarrow \mathbb{C}$  holomorphic on and inside a simple, closed, positively oriented curve  $\gamma$ . Then for all points  $a$  on the interior of  $\gamma$ :

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} dw$$

**Theorem 3.5** (Taylor's Theorem). All holomorphic functions on a domain can be expressed as a power series. For  $f: U \rightarrow \mathbb{C}$  holomorphic on domain  $U$  and for  $a \in U$ ,  $D(a, r) \subseteq U$

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where:

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a, r)} \frac{f(w)}{(w-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}$$

**Theorem 3.6** (Liouville's Theorem). Let  $f$  holomorphic on  $\mathbb{C}$  and  $f$  bounded. Then  $f$  is constant.

**Corollary 3.7.** For  $f$  entire,  $f(\mathbb{C})$  is dense in  $\mathbb{C}$  (i.e.  $\overline{f(\mathbb{C})} = \mathbb{C}$ )

**Theorem 3.8** (Picard's Little Theorem). For  $f$  non-constant entire,  $f(\mathbb{C}) = \mathbb{C}$  or  $\mathbb{C} \setminus \{z\}$

**Theorem 3.9** (Fundamental Theorem of Algebra). Let  $p$  be a non-constant polynomial with complex coefficients. Then there exists  $a \in \mathbb{C}$  such that  $p(a) = 0$ .

**Theorem 3.10** (Morera's Theorem). Let  $f$  continuous on a domain  $U$  and for all closed paths  $\gamma$  in  $U$

$$\int_{\gamma} f(z)dz = 0$$

Then  $f$  is holomorphic.

**Theorem 3.11** (Identity Theorem). *Let  $f$  holomorphic on domain  $U$  let  $S = f^{-1}(0)$ . If  $S$  contains one of its limit points then  $f$  is identically zero.*

**Theorem 3.12** (Counting Zeroes). *Let  $f$  holomorphic inside and on a positively oriented closed path  $\gamma$ . Then the sum of zeroes counting their multiplicity is:*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(w)}{f(w)} dw$$

**Theorem 3.13** (Laurent's Theorem). *Let  $f$  be a function holomorphic on  $z \in \mathbb{C} | R < |z - a| < S$ . Then,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

*For:*

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w - a)^{n+1}} dw$$