Metric Spaces and Complex Analysis

Giannis Tyrovolas

September 18, 2020

1 Intro

Definition 1.1 (Domain)**.** A domain usually denoted U is an open, connected subset of the complex numbers.

Theorem 1.2 (Cauchy's Theorem)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ holomorphic on a domain U. Then for all closed paths γ:*

$$\int_\gamma f(z)dz = 0$$

Theorem 1.3 (Deformation Theorem)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ be holomorphic on domain U. Let two closed paths γ_1, γ_2 be homotopic. Then:*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Theorem 1.4 (Cauchy's Integral Formula)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ holomorphic on and inside a simple, closed, positively oriented curve γ. Then for all points a on the interior of γ:*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\mathrm{d}w$$

Proof. Since the interior of γ is open there is an $r > 0$ such that $D(a, r)$ is contained in the interior of γ. Then by the deformation theorem:

$$\int_\gamma \frac{f(w)}{w-a}\mathrm{d}w = \int_{\gamma(a,r)} \frac{f(w)}{w-a}\mathrm{d}w = I$$

Without loss of generality let $g(w) = f(w - a)$. Then, for $u = w - a$

$$\begin{aligned} I &= \int_{\gamma(a,r)} \frac{f(w)}{w-a}\mathrm{d}w \\ &= \int_{\gamma(0,r)} \frac{g(u)}{u}\mathrm{d}u \\ &= \int_0^{2\pi} \frac{g(re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= i\int_0^{2\pi} g(re^{i\theta})d\theta \end{aligned}$$

Hence,

$$\begin{aligned} |I - 2\pi i\, g(0)| &= |i\int_0^{2\pi} g(re^{i\theta}) - g(0)d\theta| \\ &\leqslant 2\pi \sup_{\theta\in[0,2\pi)} |g(re^{i\theta}) - g(0)| \\ &\to 0 \end{aligned}$$

as r tends to 0 by the continuity of f.

Hence $f(a) = g(0) = \frac{1}{2\pi i} I$ and $I = \int_{\gamma(a,r)} \frac{f(w)}{w-a} \mathrm{d}w$

□

Theorem 1.5 (Taylor's Theorem)**.** *All holomorphic functions on a domain can be expressed as a power series. For $f \colon U \longrightarrow \mathbb{C}$ holomorphic on domain U and for $a \in U$, $D(a,r) \subseteq U$*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where:

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}$$

Theorem 1.6 (Liouville's Theorem)**.** *Let f holomorphic on $\mathbb{C}$ and f bounded. Then f is constant.*

Corollary 1.7. For f entire, $f(\mathbb{C})$ is dense in $\mathbb{C}$ (i.e. $f(\bar{\mathbb{C}}) = \mathbb{C}$)

Theorem 1.8 (Picard's Little Theorem)**.** *For f non-constant entire, $f(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C} \setminus \{z\}$*

Theorem 1.9 (Fundamental Theorem of Algebra)**.** *Let p be a non-constant polynomial with complex coefficients. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

Theorem 1.10 (Morera's Theorem)**.** *Let f continuous on a domain U and for all closed paths γ in U*

$$\int_{\gamma} f(z) dz = 0$$

Then f is holomorphic.

Proof. □

Theorem 1.11 (Identity Theorem)**.** *Let f holomorphic on domain U let $S = f^{-1}(0)$. If S contains one of it's limit points then f is identically zero.*

Theorem 1.12 (Counting Zeroes)**.** *Let f holomorphic inside and on a positively oriented closed path γ. Then the sum of zeroes counting their multiplicity is:*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(w)}{f(w)} dw$$

Theorem 1.13 (Laurent's Theorem)**.** *Let f be a function holomorphic on $z \in \mathbb{C} | R < |z-a| < S$. Then,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

For:

$$c_n = \frac{1}{2\pi i} \int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} dw$$