Metric Spaces and Complex Analysis

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1 Metric Spaces

1.1 General

Definition 1.1 (Metric Space)**.** A metric space $M = (X, d)$ is a set equipped with a function $d\colon X \times X \longrightarrow \mathbb{R}$ such that:

1. $d(x, y) \geqslant 0$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leqslant d(x, y) + d(y, z)$

Definition 1.2 (Continuity)**.** A function $f\colon X \longrightarrow Y$ is continuous at $x_0 \in X$ when $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in B(x_0, \delta),\ f(x) \in B(f(x_0), \varepsilon)$

Definition 1.3 (Uniform Continuity)**.** A function $f\colon X \longrightarrow Y$ is uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall z \in B(x, \delta)\ f(z) \in B(f(x), \varepsilon)$

Definition 1.4 (Convergence)**.** A series (x_n) converges in a metric space X if there is an $x_0 \in X$ such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n > N\ d(x_n, x_0) < \varepsilon$

Lemma 1.5 (Sequential Continuity)**.** A function $f\colon X \longrightarrow Y$ is continuous at $a \in X$ if and only if for every sequence $(x_n) \to a$, $(f(x_n)) \to f(a)$

Definition 1.6 (Norm)**.** Let V a vector space. Then $\|.\|\colon V \longrightarrow \mathbb{R}$ is a norm if:

1. $\|v\| \geqslant 0$ and $\|v\| = 0 \iff v = 0_V$
2. $\|\lambda v\| = |\lambda| \|v\|$
3. $\|x + y\| \leqslant \|x\| + \|y\|$

1.2 Toplogy

Definition 1.7 (Open Set)**.** A set $U \subseteq X$ is open if $\forall x \in U$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.

Theorem 1.8 (Topological Continuity)**.** *A function $f\colon X \longrightarrow Y$ is continuous if and only if the pre-image of every open set is open.*

Definition 1.9 (Interior)**.** The interior of S is the largest open subset of S, defined as:

$$int(S) = \bigcup_{U \subseteq S,\ U \text{ open}} U$$

Definition 1.10 (Closure)**.** The closure of a set S is the smallest closed subset containing S:

$$\overline{S} = \bigcap_{S \subseteq C,\ C \text{ closed}} C$$

Lemma 1.11. A function is continuous if and only if $f(\overline{S}) \subseteq \overline{f(S)}$

Definition 1.12 (Isometry)**.** An isometry is a bijection between two metric spaces tha preserves distances. I.e. $f\colon X \longrightarrow Y$ such that $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$

Definition 1.13 (Homeomorphism)**.** A homeomorphism between two metric spaces is a continuous bijection with a continuous inverse.

1.3 Completeness

Definition 1.14 (Cauchy Sequence)**.** A sequence (x_n) in a metric space X is Cauchy if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m > N$ $d(x_n, x_m) < \varepsilon$.

Lemma 1.15. Convergent sequences are Cauchy. Cauchy sequences are bounded.

Definition 1.16 (Completeness)**.** A metric space is complete if every Cauchy sequence converges.

Lemma 1.17. A subset of a complete metric space is complete if and only if the subset is closed.

Lemma 1.18. Let X complete and $D_1, D_2, \dots$ closed with $D_1 \supseteq D_2 \supseteq \dots$ and $diam(D_k) \to 0$. Then $\bigcap_{k\in\mathbb{N}} D_k = \{x\}$.

Definition 1.19 (Lipschitz Continuity)**.** A map $f\colon X \longrightarrow Y$ is Lipschitz continuous if for all $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leqslant M d_X(x_1, x_2)$$

For $M \in [0, 1)$, f is a contraction.

Theorem 1.20 (Contraction Mapping Theorem)**.** *Let $f\colon X \longrightarrow X$ a contraction and X complete and non-empty. Then f has a unique fixed point.*

1.4 Connectedness

Definition 1.21 (Connectedness)**.** A metric space is connected if it cannot be split into two disjoint non-trivial open sets.

Lemma 1.22. The following are equivalent:

1. X is connected
2. Any continuous function $f\colon X \longrightarrow \{0, 1\}$ is constant
3. The only *clopen* sets are X and $\varnothing$

Lemma 1.23. Let A_i connected with non-empty intersection. Then $\bigcup_{i\in I} A_i$ is connected.
Let A connected with $A \subseteq B \subseteq \overline{A}$. Then B is connected.

Theorem 1.24. *Continuity preserves connectedness*

Theorem 1.25 (Connected sets in $\mathbb{R}$)**.** *A subset of* $\mathbb{R}$ *is connected if and only if it is a "general" interval.*

Corollary 1.26. Intermediate Value theorem.

Definition 1.27 (Path)**.** A continuous function $\gamma\colon [0,1] \longrightarrow M$.

Definition 1.28 (Path Connectedness)**.** A metric space X is path-connected if there exists a path between every two points of X

Proposition 1.29. Path connectedness implies connectedness

Proposition 1.30. For *open* subsets of *normed vector spaces*, connectedness implies path connectedness.

1.5 Compactness

Definition 1.31 (Sequential Compactness)**.** A metric space is sequentially compact if every sequence has a convergent subsequence.

Lemma 1.32. Let $Z \subseteq X$

1. Z sequentially compact implies that Z is closed and bounded
2. Z is closed and X is compact then Z is compact.

Theorem 1.33. *The cartesian product of compact metric spaces is compact.*

Corollary 1.34. A closed and bounded subset of $\mathbb{R}^n$ is compact.

Theorem 1.35. *A metric space is sequentially compact if and only if it is complete and totally bounded.*

Definition 1.36 (Compactness)**.** A metric space is compact if every open cover has a finite subcover.

Proposition 1.37 (Heine-Borel)**.** The interval $[a,b]$ is compact.

Lemma 1.38 (Compactness with closed sets)**.** A metric space is compact if and only if for every family of closed sets $\{C_i | i \in I\}$ for which every finite intersection is non-empty then

$$\bigcap_{i \in I} C_i \neq \varnothing$$

Theorem 1.39 (Equivalence of compactness)**.** *A metric space is compact if and only if it is sequentially compact.*

Definition 1.40 (Equicontinuity)**.** Let X a metric space and $\mathcal{F}$ is a collection of real-valued functions on X. Then $\mathcal{F}$ is equicontinuous if for any $\varepsilon > 0$ there is a δ such that for all $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$ for all $f \in \mathcal{F}$, $|f(x_1) - f(x_2)| \leqslant \varepsilon$.

Definition 1.41 (Uniformly bounded)**.** A family of continuous functions $\mathcal{F} = f \in \mathcal{F} \colon f \colon X \longrightarrow \mathbb{R}$ is uniformly bounded if there is an M such that for all x and f, $|f(x)| \leqslant M$

Theorem 1.42 (Arzela-Ascoli)**.** *Let X a compact metric space and $\mathcal{F}$ an equicontinuous and uniformly bounded collection of continuous functions. Then any sequence of functions f_n has a subsequence which converges uniformly on X.*

2 Complex Exponential

The following power series define the complex exponential and trigonometric functions:

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{(2k+1)}}{(2k+1)!}, \qquad \cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$\sinh = \sum_{k=0}^{\infty} \frac{z^{(2k+1)}}{(2k+1)!}, \qquad \cosh = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}$$

Note:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

And

$$\exp i\theta = \cos\theta + i \sin\theta$$

3 Holomorphic Functions

Definition 3.1 (Domain)**.** A domain usually denoted U is an open, connected subset of the complex numbers.

Theorem 3.2 (Cauchy's Theorem)**.** *Let $f \colon U \longrightarrow \mathbb{C}$ holomorphic on a domain U. Then for all closed paths γ:*

$$\int_\gamma f(z)dz = 0$$

Theorem 3.3 (Deformation Theorem)**.** *Let $f \colon U \longrightarrow \mathbb{C}$ be holomorphic on domain U. Let two closed paths γ_1, γ_2 be homotopic. Then:*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Theorem 3.4 (Cauchy's Integral Formula)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ holomorphic on and inside a simple, closed, positively oriented curve γ. Then for all points a on the interior of γ:*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\mathrm{d}w$$

Theorem 3.5 (Taylor's Theorem)**.** *All holomorphic functions on a domain can be expressed as a power series. For $f\colon U \longrightarrow \mathbb{C}$ holomorphic on domain U and for $a \in U$, $D(a, r) \subseteq U$*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where:

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}$$

Theorem 3.6 (Liouville's Theorem)**.** *Let f holomorphic on $\mathbb{C}$ and f bounded. Then f is constant.*

Corollary 3.7. For f entire, $f(\mathbb{C})$ is dense in $\mathbb{C}$ (i.e. $\overline{f(\mathbb{C})} = \mathbb{C}$)

Theorem 3.8 (Picard's Little Theorem)**.** *For f non-constant entire, $f(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C} \setminus \{z\}$*

Theorem 3.9 (Fundamental Theorem of Algebra)**.** *Let p be a non-constant polynomial with complex coefficients. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

Theorem 3.10 (Morera's Theorem)**.** *Let f continuous on a domain U and for all closed paths γ in U*

$$\int_\gamma f(z)dz = 0$$

Then f is holomorphic.

Theorem 3.11 (Identity Theorem)**.** *Let f holomorphic on domain U let $S = f^{-1}(0)$. If S contains one of it's limit points then f is identically zero.*

Theorem 3.12 (Counting Zeroes)**.** *Let f holomorphic inside and on a positively oriented closed path γ. Then the sum of zeroes counting their multiplicity is:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(w)}{f(w)}dw$$

Theorem 3.13 (Laurent's Theorem)**.** *Let f be a function holomorphic on $z \in \mathbb{C} | R < |z-a| < S$. Then,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

For:

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}}dw$$