Metric Spaces and Complex Analysis

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1 Intro

Definition 1.1 (Domain)**.** A domain usually denoted U is an open, connected subset of the complex numbers.

Theorem 1.2 (Cauchy's Theorem)**.** *Let $f \colon U \longrightarrow \mathbb{C}$ holomorphic on a domain U. Then for all closed paths γ:*

$$\int_\gamma f(z)dz = 0$$

Theorem 1.3 (Deformation Theorem)**.** *Let $f \colon U \longrightarrow \mathbb{C}$ be holomorphic on domain U. Let two closed paths γ_1, γ_2 be homotopic. Then:*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Theorem 1.4 (Cauchy's Integral Formula)**.** *Let $f \colon U \longrightarrow \mathbb{C}$ holomorphic on and inside a simple, closed, positively oriented curve γ. Then for all points a on the interior of γ:*

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-a} \mathrm{d}w$$

Proof. Since the interior of γ is open there is an $r > 0$ such that $D(a, r)$ is contained in the interior of γ. Then by the deformation theorem:

$$\int_\gamma \frac{f(w)}{w-a} \mathrm{d}w = \int_{\gamma(a,r)} \frac{f(w)}{w-a} \mathrm{d}w = I$$

Without loss of generality let $g(w) = f(w - a)$. Then, for $u = w - a$

$$\begin{aligned} I &= \int_{\gamma(a,r)} \frac{f(w)}{w-a} \mathrm{d}w \\ &= \int_{\gamma(0,r)} \frac{g(u)}{u} \mathrm{d}u \\ &= \int_0^{2\pi} \frac{g(re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= i \int_0^{2\pi} g(re^{i\theta}) d\theta \end{aligned}$$

Hence,

$$\begin{aligned} |I - 2\pi i\, g(0)| &= |i \int_0^{2\pi} g(re^{i\theta}) - g(0) d\theta| \\ &\leqslant 2\pi \sup_{\theta \in [0, 2\pi)} |g(re^{i\theta}) - g(0)| \\ &\to 0 \end{aligned}$$

as r tends to 0 by the continuity of f.

Hence $f(a) = g(0) = \frac{1}{2\pi i} I$ and $I = \int_{\gamma(a,r)} \frac{f(w)}{w-a} \mathrm{d}w$

$\square$