Metric Spaces and Complex Analysis

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1 Complex Analysis and Holomorphic Functions

Definition 1.1 (Domain)**.** A domain usually denoted U is an open, connected subset of the complex numbers.

Theorem 1.2 (Cauchy's Theorem)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ holomorphic on a domain U. Then for all closed paths γ:*

$$\int_\gamma f(z)dz = 0$$

Theorem 1.3 (Deformation Theorem)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ be holomorphic on domain U. Let two closed paths γ_1, γ_2 be homotopic. Then:*

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Theorem 1.4 (Cauchy's Integral Formula)**.** *Let $f\colon U \longrightarrow \mathbb{C}$ holomorphic on and inside a simple, closed, positively oriented curve γ. Then for all points a on the interior of γ:*

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\mathrm{d}w$$

Theorem 1.5 (Taylor's Theorem)**.** *All holomorphic functions on a domain can be expressed as a power series. For $f\colon U \longrightarrow \mathbb{C}$ holomorphic on domain U and for $a \in U$, $D(a,r) \subseteq U$*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

where:

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(w)}{(w-a)^{n+1}} = \frac{f^{(n)}(a)}{n!}$$

Theorem 1.6 (Liouville's Theorem)**.** *Let f holomorphic on $\mathbb{C}$ and f bounded. Then f is constant.*

Corollary 1.7. For f entire, $f(\mathbb{C})$ is dense in $\mathbb{C}$ (i.e. $\overline{f(\mathbb{C})} = \mathbb{C}$)

Theorem 1.8 (Picard's Little Theorem)**.** *For f non-constant entire, $f(\mathbb{C}) = \mathbb{C}$ or $\mathbb{C} \setminus \{z\}$*

Theorem 1.9 (Fundamental Theorem of Algebra)**.** *Let p be a non-constant polynomial with complex coefficients. Then there exists $a \in \mathbb{C}$ such that $p(a) = 0$.*

Theorem 1.10 (Morera's Theorem)**.** *Let f continuous on a domain U and for all closed paths γ in U*

$$\int_\gamma f(z)dz = 0$$

Then f is holomorphic.

Theorem 1.11 (Identity Theorem)**.** *Let f holomorphic on domain U let $S = f^{-1}(0)$. If S contains one of it's limit points then f is identically zero.*

Theorem 1.12 (Counting Zeroes)**.** *Let f holomorphic inside and on a positively oriented closed path γ. Then the sum of zeroes counting their multiplicity is:*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(w)}{f(w)}dw$$

Theorem 1.13 (Laurent's Theorem)**.** *Let f be a function holomorphic on $z \in \mathbb{C}|R < |z-a| < S$. Then,*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

For:

$$c_n = \frac{1}{2\pi i}\int_{\gamma(a,r)}\frac{f(w)}{(w-a)^{n+1}}dw$$