**Definition 1.** A *functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a map $\mathrm{ob}\mathcal{C} \to \mathrm{ob}\mathcal{D}$ and a map of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$. Such that $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ and $F(g \circ f) = F(g) \circ F(f)$.

**Definition 2.** $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if for all $x, y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective. It is *full* if every such map is surjective. It is *essentially surjective* if for all $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $F(x) \cong d$.

**Definition 3.** For two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta\colon F \Rightarrow D$ is a collection of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ such that for every $x \xrightarrow{f} y$, $\eta_y \circ F(f) = G(f) \circ \eta_x$. They're natural isos if $\eta_x$ isos.

**Definition 4.** Equivalence of categories: $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. An adjoint equivalence is an equivalence where $F \dashv G$.

**Proposition 5.** The following are equivalent: $\mathcal{C}$ and $\mathcal{D}$ are equivalent, $\mathcal{C}$ and $\mathcal{D}$ are adjoint equivalent and there is $F\colon \mathcal{C} \to \mathcal{D}$ that is fully faithful and essentially surjective.

**Definition 6.** Two functors $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ are adjoint if there exist natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ with $F \xRightarrow{\mathrm{id}_F \circ \eta} FGF \xRightarrow{\epsilon \circ \mathrm{id}_F} F$ and $G \xRightarrow{\eta \circ \mathrm{id}_G} GFG \xRightarrow{\mathrm{id}_G \circ \epsilon} G$.

The forgetful functor *Forget*: $* \to$ Set has a left adjoint *Free* $\dashv$ *Forget* for $*$ being Grp, Ab, Vect. *Forget*: Ab $\to$ Grp has a left adjoint, the abelinisation of $G$. For topologies *Forget*: Top $\longrightarrow$ Set, we have $D \dashv$ *Forget* $\dashv I$, where $D$ is the discrete topology and $I$ the indiscrete topology. The forgetful functor from fields has no left adjoint. Such a left adjoint should map $\varnothing$ to an initial object in Field, but fields have no initial object.

**Definition 7** (Comma categories)**.** For $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$, and $x \in \mathcal{C}$ the category $(x \Rightarrow G)$ has objects $(y, x \xrightarrow{f} G(y))$ and morphisms $(y_1, f_1) \to (y_2, f_2)$ such that $x \xrightarrow{f_1} G(y_1) \to G(y_2)$ commutes with $f_2$. Similarly $(F \Rightarrow y)$.

**Proposition 8.** $F \dashv G$ iff $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y))$ naturally in $x, y$ iff for all $x$, $(F(x), e_x)$ is initialin $(x \Rightarrow G)$.

**Definition 9.** For a locally small category $\mathcal{C}$, the Yoneda Embedding is given by a functor $Y\colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, where $Y(x) = \mathrm{Hom}_{\mathcal{C}}(-, x)$ and $Y(x \xrightarrow{f} y) = (g \mapsto f \circ g)$.

**Definition 10.** A functor is representable if it is in the essential image of the Yoneda functor.

**Lemma 11.** The Yoneda lemma states that for any presheaf $F\colon \mathcal{C}^{op} \to \mathrm{Set}$, the map $\mathrm{Fun}(Y(x), F) \to F(x)$ given by $\eta \mapsto \eta_x(\mathrm{id}_x)$, is an isomorphism.

*Proof.* To construct the inverse, let $f \in F(x)$, then define natural transformation $\epsilon\colon Y(x) \Rightarrow F$ given by $\epsilon_y\colon Y(x)(y) \to F(y)$ and $g \mapsto F(g)(f)$. Show this is natural by $F$ preserving composition. One inverse is easy, for the other take $\eta$ arbitrary, make diagram with $x \xrightarrow{g} y$ and claim $\eta_y(g) = F(g)(\eta_x(\mathrm{id}_x))$. $\square$

**Corollary 12.** The Yoneda functor is full and faithful.

*Proof.* For $x_1, x_2$, we have that $\mathrm{Hom}(Y(x_1), Y(x_2)) \longrightarrow Y(x_1)(x_2)$ is an isomorphism by Yoneda. $\square$

**Definition 13** (Representable)**.** A presheaf is representable if it is in the essential image of the Yoneda functor.

**Proposition 14.** A formal right adjoint to $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $G^{formal}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ with $y \mapsto (x \mapsto \mathrm{Hom}_{\mathcal{D}}(F(x), y))$. A right adjoint $G$ to $F$ exists if and only if $G^{formal}(y)$ is representable for all $y$.

**Definition 15.** Let $D\colon I \to \mathcal{C}$. A limit of $D$ is an object $\lim_I D \in \mathcal{C}$ along with maps $f_i\colon \lim_I D \to D(i)$, such that for every $g\colon i \to j$, $D(g) \circ f_i = f_j$. It is universal as for any other object $W$ with compatible maps $W \to D(i)$, there is a unique morphism $W \to \lim_I F$.

**Proposition 16.** The diagonal functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is $\Delta(x)(i) = x$. Then $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(\Delta(W), F) \cong \mathrm{Hom}_{\mathcal{C}}(W, \lim_I F)$, so $\mathrm{colim}_I \dashv \Delta \dashv \lim_I$.

**Proposition 17.** Suppose $\mathcal{C}$ has limits for diagrams of shape $I$ and $J$. Then it has limits of diagrams of shape $I \times J$ and

$$\lim_{I\times J} F \cong \lim_{I}\lim_{J} F \cong \lim_{J}\lim_{I} F$$

For the proof use $\Delta$ as an adjoint.

**Theorem 18.** $\mathcal{C}$ has limits iff it has products and equalisers. $\mathcal{C}$ has finite limits if it has binary products, final object and equalisers.

*Proof.* For $F\colon I \to \mathcal{C}$, for every morphism $f\colon i \to j$, let $\prod_{k\in I} F(k) \to F(j)$ the projection map and the composite map $\prod_{k\in I} F(k) \to F(i) \xrightarrow{F(f)} F(j)$. Then, by the universal property of the product we get unique maps:

$$\prod_{k\in I} F(k) \rightrightarrows \prod_{(i\to j)\in \mathrm{Fun}([1],J)} F(j)$$

By the equalisers, we get $E$ the limit of $F$. $\square$

**Definition 19.** A morphism $f\colon X \to Y$ is a *monomorphism* if for every $g, h\colon Y \to Z$, $f\circ g = f\circ h$ implies $g = h$. For an epimorphism $f$, $g \circ f = h\circ f$ implies $g = h$.

**Definition 20.** Equalisers are regular monomorphisms, coequalisers are regular epimorphisms.

| Limits | | Colimits | |
|---|---|---|---|
| **Final** | Set : $\{1\}$, Grp : $\{e\}$ | **Initial** | Set : $\varnothing$, Grp : $\{e\}$ Fields: None |
| **Products** | $\times$ in Grp, Set, Vect | **Co-products** | Set: $\sqcup$, Grp: free product, Ab: $\times$ |
| **Equal** | Set:$x$ with $f(x) = g(x)$, | **Coeq** | Set: $Y/f(x) \sim g(x)$, Grp: $Y/S\ f(x)g(x)^{-1}$ |
| **Pullback** | $\{(x, y) \mid f(x) = g(y)\}$ | **Pushout** | |

**Theorem 21.** Let $F \dashv G$ then $F$ preserves colimits and $G$ preserves limits.

**Theorem 22.** $F\colon \mathcal{C} \to \mathcal{D}$ for $\mathcal{C}, \mathcal{D}$ locally small, and $\mathcal{C}$ has small colimits. $F \dashv G$ iff $F$ preserves colimits and for all $x$, $(F \Rightarrow x)$ the solution set holds. A category $\mathcal{C}$ satisfied it if: there is $I$ small, $\{c_i\}_{i\in I}$ such that for each $x \in \mathcal{C}$ there is $c_i$ with $\mathrm{Hom}_{\mathcal{C}}(x, c_i)$ non-empty.

**Definition 23** (Monads)**.** A monad $T\colon \mathcal{C} \to \mathcal{C}$ is a functor with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and multiplication $\mu\colon T^2 \Rightarrow T$, satisfying: $T^3 \xRightarrow{id_T\circ\mu} T^2 \overset{\mu}{\Rightarrow} T$ is equal to $T^3 \xRightarrow{\mu\circ\mathrm{id}_T} T^2 \overset{\mu}{\Rightarrow} T$ and $T \xRightarrow{\mathrm{id}_T\circ\eta} T^2 \overset{\mu}{\Rightarrow} T$ and $T \xRightarrow{\eta\circ\mathrm{id}_T} T^2 \overset{\mu}{\Rightarrow} T$ are both equal to the identity.

**Proposition 24.** Let $F \dashv G$ with $F\colon \mathcal{C} \to \mathcal{D}$, then $GF$ is a monad in $\mathcal{C}$ and $FG$ is a comonad in $\mathcal{D}$. For the proof, let $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\mu\colon GFGF \xRightarrow{\mathrm{id}_G\circ\epsilon\circ\mathrm{id}_F} GF$. Diagrams for $T^2$ and then append $G$ and $F$.

**Definition 25.** For a monad $T$ an algebra $\mathrm{Alg}_T(\mathcal{C})$ is the category with objects $(x, T(x) \xrightarrow{\alpha_x} x)$ for $x \in \mathcal{C}$ with $\alpha_x$ such that $T^2(x) \xrightarrow{\mu_x} T(x) \xrightarrow{\alpha_x} x$ is equal to $T^2(x) \xrightarrow{T(\alpha_x)} T(x) \xrightarrow{\alpha_x} x$ and $x \xrightarrow{\eta_x} T(x) \xrightarrow{\alpha_x} x$ is the identity. The morphisms $(x, \alpha_x) \to (y, \alpha_y)$ are given by $f\colon x \to y$ such that $T(x) \xrightarrow{\alpha_x} x \xrightarrow{f} y$ and $T(x) \xrightarrow{T(f)} T(y) \xrightarrow{\alpha_y} y$ commutes.

**Proposition 26.** The forgetful functor $F\colon \mathrm{Alg}_T(\mathcal{C}) \to \mathcal{C}$ has a left adjoint $L$ and $FL = T$. We have $L(x) = (T(x), \mu_x)$. Write down the commutative square for a morphism $T(x) \xrightarrow{f} A$ and then $T(x) \xrightarrow{T(\eta_x)} T^2(x) \xrightarrow{\mu_x} T(x)$ commutes. So $f$ is uniquely determined by $x \xrightarrow{\eta_x} Tx \xrightarrow{f} A$ giving an isomorphism of Homs.

**Definition 27.** For $F \dashv G$ let $G_{enh}\colon \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ with $G_{enh}(x) = (G(x), GFG(x) \xrightarrow{G(\epsilon_x)} G(x))$. $G$ is *monadic* if there is $F \dashv G$ and for $T = GF$, $G_{enh}\colon \mathcal{D} \to \mathrm{Alg}_T(\mathcal{C})$ is an equivalence.

**Definition 28.** A functor $G\colon \mathcal{C} \to \mathcal{D}$ is *conservative* if when $G(f)$ is an isomorphism so is $f$. The forgetful functors from Ab, Grp, Vect are conservative, the one from Top is not.

**Definition 29.** A fork is a cocone $x \rightrightarrows y \xrightarrow{e} z$ so that $e\circ g = e \circ f$. It is split if there are $s\colon z \to y$, $t\colon y \to x$ such that $es = \mathrm{id}_z$, $ft = \mathrm{id}_y$ and $gt = se$. Split forks are coequalisers.

**Definition 30.** Morphisms $f, g\colon x \to y$ are a split pair if their coequaliser exists and is split. They are $G$ split if $G(f), G(g)$ is split.

**Theorem 31.** A functor $G\colon \mathcal{D} \to \mathcal{C}$ is *monadic* if and only if: it has a left adjoint, it is conservative and every $G$-split pair admits a coequaliser in $\mathcal{D}$ and is preserved by $G$.