We work in the language $L_E = \{\bar{0},\ ^+,\ v,\ f,\ ',\ (,\ ),\ -,\ \rightarrow,\ \forall,\ =,\ \leqslant,\ \#\}$

Definition 1. A subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula $\varphi(v_1, \ldots, v_k)$ such that

$$(n_1, \ldots, n_k) \in A \iff \varphi(\overline{n_1}, \ldots, \overline{n_k})$$

Definition 2. A subset $A \subseteq \mathbb{N}^k$ is *provably definable* if there is $\varphi(\mathbf{x})$ such that $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$ and $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

Definition 3. A function $f \colon \mathbb{N}^k \longrightarrow \mathbb{N}$ is *definable* if $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$ is definable.
It is *weakly provably definable* from S if A is provably definable from S.
It is *provably definable* if for all $\mathbf{n} \in \mathbb{N}^k$, $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

Definition 4.

1. $^+ \colon \mathbb{N} \longrightarrow \mathbb{N} \setminus \{0\}$ is injective.
2. Adding and multiplying by 0 on the right: $\forall v(v + \bar{0} = \bar{0})$ and $\forall v(v \times \bar{0} = \bar{0})$
3. Addition, multiplication: $\forall v_1 \forall v_2 (v_1 + v_2^+ = (v_1 + v_2)^+)$ and $\forall v_1 \forall v_2 (v_1 \times v_2^+ = v_1 \times v_2 + v_2)$.
4. Relation $\leqslant$ is a total order, $\bar{0}$ is the least element, n^+ is the successor of n.
5. For any formula $\varphi(x)$ in one variable:

$$\Big(\varphi(\bar{0}) \wedge \forall v_0 \big(\varphi(v_0) \rightarrow \varphi(v_0^+)\big)\Big) \rightarrow \forall v_0 \big(\varphi(v_0)\big)$$

Definition 5. For $\varphi = \sigma_0 \ldots \sigma_n$ a formula of L, $\ulcorner \varphi \urcorner = \displaystyle\sum_{i=0}^{n} \ulcorner \sigma_i \urcorner 13^i$

Definition 6. $\Sigma_0 = \Pi_0 = \Delta_0$ formulas without unbounded quantifiers. Σ_{n+1}: formulas of the form $\exists x \varphi(x)$, with $\varphi \in \Pi_n$. Similarly, Π_{n+1} is the formulas of the form $\forall x \varphi(x)$ with $\varphi \in \Sigma_n$.
A formula ψ is provably Σ_n from S if there is a $\varphi \in \Sigma_n$, such that $S \vdash \psi \leftrightarrow \varphi$.

Lemma 7 (Diagonal Lemma)**.** For any formula $F(v_1)$ there is a formula C such that:

$$\mathrm{PA} \vdash F(\ulcorner C \urcorner) \leftrightarrow C$$

Provability Rules

1. If $S \vdash \varphi$ then $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner})$.
2. $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \rightarrow \psi \urcorner}) \rightarrow (\mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner \psi \urcorner}))$.
3. $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \urcorner})$

Definition 8. A set S of assumptions is *n-inconsistent* if for some Σ_n formula $\exists x \psi(x)$, $S \vdash \exists x \psi(x)$ but for all $m \in \mathbb{N}$, $S \vdash \neg\psi(\overline{m})$. It is *$n$-consistent* if it is not n-inconsistent.