Definition 1. A *category* $\mathcal{C}$, consists of the following data:

1. A collection of *objects* $\mathrm{ob}\,\mathcal{C}$,
2. For every two objects $x, y \in \mathrm{ob}\,C$ a collection of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(x, y)$.
3. For every $x \in \mathcal{C}$, the identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$.
4. A composition map $\circ\colon \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(x, z)$

Such that, for all $x, y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$:

$$f \circ \mathrm{id}_x = f \;\; \mathrm{id}_y \circ f = f$$

And for all x, y, z, v with $f \in \mathrm{Hom}_{\mathcal{C}}(x, y), g \in \mathrm{Hom}_{\mathcal{C}}(y, z), h \in Homzv$:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Definition 2. A *functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a map $\mathrm{ob}\mathcal{C} \to \mathrm{ob}\mathcal{D}$ and a map of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$. Such that $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Definition 3. $F\colon \mathcal{C} \to \mathcal{D}$ is *faithful* if for all $x, y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ is injective. It is *full* if every such map is surjective.

Definition 4. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is essentially surjective if for all $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $F(x) \cong d$.

Definition 5. For two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta\colon F \Rightarrow D$ is a collection of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ such that for every $x \xrightarrow{f} y$, $\eta_y \circ F(f) = G(f) \circ \eta_x$.

It is a natural isomorphism if all morphisms η_x are isomorphisms.

Definition 6. Equivalence of categories: $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. An adjoint equivalence is an equivalence where $F \dashv G$.

Proposition 7. The following are equivalent: $\mathcal{C}$ and $\mathcal{D}$ are equivalent, $\mathcal{C}$ and $\mathcal{D}$ are adjoint equivalent and there is $F\colon \mathcal{C} \to \mathcal{D}$ that is fully faithful and essentially surjective.

Definition 8. For a locally small category $\mathcal{C}$, the Yoneda Embedding is given by a functor $Y\colon \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, where $Y(x) = \mathrm{Hom}_{\mathcal{C}}(-, x)$ and $Y(x \xrightarrow{f} y) = (g \mapsto f \circ g)$.

Definition 9. A functor is representable if it is in the essential image of the Yoneda functor.

Lemma 10. The Yoneda lemma states that for any presheaf $F\colon \mathcal{C}^{op} \to \mathrm{Set}$, the map $\mathrm{Fun}(Y(x), F) \to F(x)$ given by $\eta \mapsto \eta_x(\mathrm{id}_x)$, is an isomorphism.

Proof. To construct the inverse, let $f \in F(x)$, then define natural transformation $\epsilon\colon Y(x) \Rightarrow F$ given by $\epsilon_y\colon Y(x)(y) \to F(y)$ and $g \mapsto F(g)(f)$. Show this is natural by F preserving composition. One inverse is easy, for the other take η arbitrary, make diagram with $x \xrightarrow{g} y$ and claim $\eta_y(g) = F(g)(\eta_x(\mathrm{id}_x))$. □

Corollary 11. The Yoneda functor is full and faithful.

Proof. For x_1, x_2, we have that $\mathrm{Hom}(Y(x_1), Y(x_2)) \longrightarrow Y(x_1)(x_2)$ is an isomorphism by Yoneda. □

Proposition 12. A formal right adjoint to $F\colon \mathcal{C} \to \mathcal{D}$ is a functor $G^{formal}\colon \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ with $y \mapsto (x \mapsto \mathrm{Hom}_{\mathcal{D}}(F(x), y))$. A right adjoint G to F exists if and only if $G^{formal}(y)$ is representable for all y.

Definition 13. Let $D\colon I \to \mathcal{C}$. A limit of D is an object $\lim_I D \in \mathcal{C}$ along with maps $f_i\colon \lim_I D \to D(i)$, such that for every $g\colon i \to j$, $D(g) \circ f_i = f_j$. It is universal as for any other object W with compatible maps $W \to D(i)$, there is a unique morphism $W \to \lim_I F$.

Proposition 14. The diagonal functor $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is $\Delta(x)(i) = x$. Then $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(\Delta(W), F) \cong \mathrm{Hom}_{\mathcal{C}}(W, \lim_I F)$, so $\Delta \vdash \lim_I$.

Proposition 15. Suppose $\mathcal{C}$ has limits for diagrams of shape I and J. Then it has limits of diagrams of shape $I \times J$ and

$$\lim_{I\times J} F \cong \lim_I \lim_J F \cong \lim_J \lim_I F$$

For the proof use Δ as an adjoint.

Theorem 16. $\mathcal{C}$ has limits iff it has products and equalisers. $\mathcal{C}$ has finite limits if it has binary products, final object and equalisers.

Proof. For $F\colon I \to \mathcal{C}$, for every morphism $f\colon i \to j$, let $\prod_{k\in I} F(k) \to F(j)$ the projection map and the composite map $\prod_{k\in I} F(k) \to F(i) \xrightarrow{F(f)} F(j)$. Then, by the universal property of the product we get unique maps:

$$\prod_{k\in I} F(k) \rightrightarrows \prod_{(i\to j)\in \mathrm{Fun}([1],J)} F(j)$$

By the equalisers, we get E the limit of F. $\square$

Limits		Colimits	
Final	Set : $\{1\}$, Grp : $\{e\}$	**Initial**	Set : $\varnothing$, Grp : $\{e\}$
Products	$\times$ in Grp, Set, Vect	**Co-products**	Set: $\sqcup$, Grp: free product, Ab: $\times$
Equal	Set:x with $f(x) = g(x)$,	**Coeq**	Set: $Y/f(x) \sim g(x)$, Grp: $Y/S\ f(x)g(x)^{-1}$
Pullback	$\{(x,y) \mid f(x) = g(y)\}$	**Pushout**	

Theorem 17. Let $F \vdash G$ then F preserves colimits and G preserves limits.