

We work in the language  $L_E = \{\bar{0}, +, v, f, ', (, ), -, \rightarrow, \forall, =, \leq, \#\}$

**Definition 1.** A subset  $A \subseteq \mathbb{N}^k$  is *definable* if there is a formula  $\varphi(v_1, \dots, v_k)$  such that

$$(n_1, \dots, n_k) \in A \iff \varphi(\bar{n}_1, \dots, \bar{n}_k)$$

**Definition 2.** A subset  $A \subseteq \mathbb{N}^k$  is *provably definable* if there is  $\varphi(\mathbf{x})$  such that  $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$  and  $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

**Definition 3.** A function  $f: \mathbb{N}^k \rightarrow \mathbb{N}$  is *definable* if  $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$  is definable.

It is *weakly provably definable* from  $S$  if  $A$  is provably definable from  $S$ .

It is *provably definable* if for all  $\mathbf{n} \in \mathbb{N}^k$ ,  $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

**Definition 4.** 1.  $^+ : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$  is injective.

2. Adding and multiplying by 0 on the right:  $\forall v(v + \bar{0} = \bar{0})$  and  $\forall v(v \times \bar{0} = \bar{0})$

3. Addition, multiplication:  $\forall v_1 \forall v_2(v_1 + v_2^+ = (v_1 + v_2)^+)$  and  $\forall v_1 \forall v_2(v_1 \times v_2^+ = v_1 \times v_2 + v_2)$ .

4. Relation  $\leq$  is a total order,  $\bar{0}$  is the least element,  $n^+$  is the successor of  $n$ .

5. For any formula  $\varphi(x)$  in one variable:

$$(\varphi(\bar{0}) \wedge \forall v_0(\varphi(v_0) \rightarrow \varphi(v_0^+))) \rightarrow \forall v_0(\varphi(v_0))$$

**Definition 5.** For  $\varphi = \sigma_0 \dots \sigma_n$  a formula of  $L$ ,  $\ulcorner \varphi \urcorner = \sum_{i=0}^n \ulcorner \sigma_i \urcorner 13^i$

**Building  $\text{Pr}_S(\ulcorner \varphi \urcorner)$**

1. Syntax:  $\lfloor \sqrt[n]{13} \rfloor$ ,  $k++l$ ,  $k$  is prefix/suffix/substring of  $n$  and *formula sequence* last of which is  $\sigma$ .

2. Define `isNumeral` and `isVariable` by  $\exists$ . Define `isTerm` by valid sequence of term construction.

3. Identify formulas: `isAtomic`, and `isAxiomFirstOrder`.

4. So for  $S$  a definable set of formulas in  $\Delta_i$ ,  $\text{proof}_S(\bar{n}, \bar{m})$ , is  $\Delta_i$ .  $\text{Pr}_S(\ulcorner \varphi \urcorner) = (\exists x)\text{proof}_S(\ulcorner \varphi \urcorner, x)$ .

5. Define PA in  $\Delta_1$ , we need the exists for the induction scheme.

**Definition 6** (Quasi-substitution). For  $\varphi(v_i)$  and term  $t$  let  $\varphi[t] = \forall v_i(v_i = t \rightarrow \varphi)$ . We have  $\text{PA} \vdash \varphi(t) \leftrightarrow \varphi[t]$ .

**Definition 7.**  $\Sigma_0 = \Pi_0 = \Delta_0$  formulas without unbounded quantifiers.  $\Sigma_{n+1}$ : formulas of the form  $\exists x\varphi(x)$ , with  $\varphi \in \Pi_n$ . Similarly,  $\Pi_{n+1}$  is the formulas of the form  $\forall x\varphi(x)$  with  $\varphi \in \Sigma_n$ .

A formula  $\psi$  is provably  $\Sigma_n$  from  $S$  if there is a  $\varphi \in \Sigma_n$ , such that  $S \vdash \psi \leftrightarrow \varphi$ .

**Lemma 8** (Diagonal Lemma). For any formula  $F(v_1)$  there is a formula  $C$  such that:

$$\text{PA} \vdash F(\ulcorner C \urcorner) \leftrightarrow C$$

Let  $E_n$  the expression with Gödel number  $n$ .

Let  $d(n)$  be  $E_n[\bar{n}]$  and  $D(m, n)$  be the formula  $n = \ulcorner d(m) \urcorner$ .

Consider,  $F(\ulcorner \bar{y} \urcorner)$ , then  $F(\ulcorner d(y) \urcorner) \vdash \psi(y) = \forall z(D(y, z) \rightarrow F(z))$ . Let  $k = \ulcorner \psi \urcorner$ ,  $C = \psi[\bar{k}]$ . Then,  $C \vdash \psi(\bar{k}) \vdash F(\ulcorner d(k) \urcorner)$ . But  $k = \ulcorner \psi \urcorner$ , so  $C = E_k[\bar{k}]$  which is defined to be  $d(k)$ . So,  $C \vdash F(\ulcorner C \urcorner)$ .

So, *truth is undefinable*, let  $\mathbb{N} \models \text{True}(\ulcorner \varphi \urcorner)$  if and only if  $\mathbb{N} \models \varphi$ . Then,  $F(v_1) = \neg \text{True}(v_1)$  so there is  $C$  such that  $C \models \neg \text{True}(\ulcorner C \urcorner) \models \neg C$ .

## Recursive Functions

**Definition 9.** Primitive recursive functions contain zero and succ and are closed under, projection, composition and primitive recursion. Recursive functions are closed under minimilisation as well.

**Proposition 10.** Equivalences:  $A$  is a decidable set  $\iff A$  is  $\Delta_1$ -definable.

$A$  is a recursively enumerable set  $\iff A$  is  $\Sigma_1$ -definable.

## Provability Rules

For  $S$  a provably definable set of assumptions.

1. If  $S \vdash \varphi$  then  $\text{PA} \vdash \text{Pr}_S(\overline{\ulcorner \varphi \urcorner})$ .
2.  $\text{PA} \vdash \text{Pr}_S(\overline{\ulcorner \varphi \rightarrow \psi \urcorner}) \rightarrow (\text{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \text{Pr}_S(\overline{\ulcorner \psi \urcorner}))$ .
3. If  $\text{PA} \subseteq S$  then  $\text{PA} \vdash \text{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \text{Pr}_S(\overline{\ulcorner \text{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \urcorner})$

Additionally,  $S \vdash \varphi$  if and only if  $\mathbb{N} \models \text{Pr}_S(\overline{\ulcorner \varphi \urcorner})$ .

Let  $\text{PA} \subseteq S$  a provably definable set of sentences. Then, there is a formula  $G$ ,  $\mathbb{N} \models G$  but  $S \not\vdash G$ .

**Definition 11.** A set  $S$  of assumptions is *n-inconsistent* if for some  $\Sigma_n$  formula  $\exists x \psi(x)$ ,  $S \vdash \exists x \psi(x)$  but for all  $m \in \mathbb{N}$ ,  $S \vdash \neg \psi(\bar{m})$ . It is *n-consistent* if it is not *n-inconsistent*.

**Definition 12.** Formulas  $S$  are  $\Sigma_n$ -complete if every  $\Sigma_n$  sentence true in  $\mathbb{N}$  is provable from  $S$ .

**Definition 13** (Weaker arithmetics).  $\mathcal{Q}$  is PA without the induction schema, so it is finitely axiomatisable.  $\mathcal{R}$  is the collection of all valid sentences of the form  $\bar{m} + \bar{n} = \bar{k}$ ,  $\bar{m} \times \bar{n} = \bar{k}$ ,  $\bar{m} \neq \bar{n}$ ,  $\forall v_1 (v_1 \leq \bar{n} \rightarrow (v_1 = \bar{0} \vee \dots \vee \bar{n}))$  and  $\forall v_1 (v_1 \leq \bar{n} \vee \bar{n} \leq v_1)$ .

**Proposition 14.** For every  $r \in \mathcal{R}$ ,  $\mathcal{Q} \vdash r$ .

**Proposition 15.**  $\mathcal{R}$  is  $\Sigma_0$ -complete. Hence, so is  $\mathcal{Q}$  and PA.

**Proposition 16.** If  $S$  is  $\Sigma_0$ -complete then it is  $\Sigma_1$ -complete. Hence,  $\mathcal{R}$ ,  $\mathcal{Q}$  and PA are  $\Sigma_1$ -complete.

**Theorem 17.** There exists a  $\Pi_1$  sentence  $G$  such that if PA is consistent then  $\text{PA} \not\vdash G$ , if PA is 1-consistent then  $\text{PA} \not\vdash \neg G$ .