Types

Definition 1. For a theory T, and variables $\mathbf{x}$, a *partial type* P is a set of formulas where $T \cup P$ is consistent.

Example 2. For $T = \mathrm{Th}(\langle \mathbb{Z}, +, -, 0, 1\rangle)$, $P(x) = \{\exists y(y + y \ldots + y = x)\} \cup \{x \neq 0\}$, is a partial type. This can be proven by compactness.

Definition 3. For a theory T, a type P is principal if for some $\theta(\mathbf{x})$, $T \cup \theta(\mathbf{x}) \models P$ and $T \cup \theta$ is consistent.

Theorem 4. If P is not principal it is omitted in some model of T. If P is principal and T is complete then every model of T realises P.

Embeddings

Theorem 5. If $A \leqslant B$ then for every *quantifier free* $\varphi(x_1, \ldots x_n)$,

$$\varphi^{\underline{B}} \cap A^k = \varphi^{\underline{A}}.$$

If $A \preceq B$ then this is true for all formulas φ.

Preservation Theorems

Theorem 6. For a theory T, $\underline{A} \models T_\forall$ if and only if there exists $\underline{B} \models T$ with $\underline{A} \leqslant \underline{B}$.

Corollary 7. The theory of fields is not universal as, $\underline{Z} \leqslant \underline{Q}$ but $\underline{Q}$ is a field and $\underline{Z}$ is not.

Theorem 8. Sentence σ is universal if and only if for all $B \models \sigma$ and $A \leqslant B$, $A \models \sigma$.

Example 9. For F the theory of fields, $F_\forall$ is the theory of integral domains. That is because every integral domain can be embedded in a field.

Theorem 10. For a chain $\underline{A_1} \leqslant \underline{A_2} \leqslant \ldots$, let $\underline{A^*}$ be the limit of the chain. Then every AE sentence σ which holds for all $\underline{A_i}$, holds for $\underline{A^*}$.

Quantifier elimination

Definition 11. Theory T admits quantifier elimination if for any formula $\theta(\mathbf{x})$, there exists a quantifier free formula $\tilde{\theta}(\mathbf{x})$ such that:

$$T \models \forall \mathbf{x}(\theta \leftrightarrow \tilde{\theta})$$

Theorem 12. If L has no constant or function symbols and T admits Q.E. then T is complete.

Example 13.

- $\mathrm{Th}(\langle \mathbb{Q}, < \rangle)$ admits QE and so is complete.
- ACF admits QE. But, the only thing ACF does not decide is the field characteristic. Hence, ACF_p for p prime or zero is complete.
- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1\rangle)$ does not admit Q.E. Atomic sentences with one variable define only, finite and cofinite sets. But $\varphi(x) = \exists y(y^2 = x)$ defines the positive numbers.
- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1, < \rangle)$ admits Q.E. by Tarski. It is complete because the order is complete and so determines equality.

Remark. If T admits Q.E. and $\underline{A_1} \models T$, $\underline{A_2} \models T$ then $\underline{A_1} \preceq \underline{A_2}$.

Theorem 14. If it exists, there is only one way to extend a universal theory to a Q.E. theory. Prove by taking $S \models \underline{A_1}$ and $\underline{A_1} \leqslant \underline{B_1} \models T$ and build chains. The limits are equal and $\underline{A_1} \preceq A_2 \preceq C$.