We work in the language $L_E = \{\bar{0},\ {}^+,\ v,\ f,\ ',\ (,\ ),\ -,\ \rightarrow,\ \forall,\ =,\ \leqslant,\ \#\}$

Definition 1. A subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula $\varphi(v_1, \dots, v_k)$ such that

$$(n_1, \dots, n_k) \in A \iff \varphi(\overline{n_1}, \dots, \overline{n_k})$$

Definition 2. A subset $A \subseteq \mathbb{N}^k$ is *provably definable* if there is $\varphi(\mathbf{x})$ such that $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$ and $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

Definition 3. A function $f \colon \mathbb{N}^k \longrightarrow \mathbb{N}$ is *definable* if $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$ is definable.
It is *weakly provably definable* from S if A is provably definable from S.
It is *provably definable* if for all $\mathbf{n} \in \mathbb{N}^k$, $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

Definition 4.

1. ${}^+ \colon \mathbb{N} \longrightarrow \mathbb{N} \setminus \{0\}$ is injective.
2. Adding and multiplying by 0 on the right: $\forall v(v + \bar{0} = \bar{0})$ and $\forall v(v \times \bar{0} = \bar{0})$
3. Addition, multiplication: $\forall v_1 \forall v_2 (v_1 + v_2^+ = (v_1 + v_2)^+)$ and $\forall v_1 \forall v_2 (v_1 \times v_2^+ = v_1 \times v_2 + v_2)$.
4. Relation $\leqslant$ is a total order, $\bar{0}$ is the least element, n^+ is the successor of n.
5. For any formula $\varphi(x)$ in one variable:

$$\Big(\varphi(\bar{0}) \wedge \forall v_0\big(\varphi(v_0) \rightarrow \varphi(v_0^+)\big)\Big) \rightarrow \forall v_0\big(\varphi(v_0)\big)$$

Definition 5. For $\varphi = \sigma_0 \dots \sigma_n$ a formula of L, $\ulcorner\varphi\urcorner = \displaystyle\sum_{i=0}^{n} \ulcorner\sigma_i\urcorner 13^i$

Building $\mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner})$

1. Syntax: $\lfloor \sqrt[13]{n} \rfloor$, $k{+}{+}l$, k is prefix/suffix/substring of n and *formula sequence* last of which is σ.
2. Define `isNumeral` and `isVariable` by $\exists$. Define `isTerm` by valid sequence of term construction.
3. Identify formulas: `isAtomic`, and `isAxiomFirstOrder`.
4. So for S a definable set of formulas in Δ_i, $\mathrm{proof}_S(\bar{n}, \bar{m})$, is Δ_i. $\mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner}) = (\exists x)\mathrm{proof}_S(\overline{\ulcorner\varphi\urcorner}, x)$.
5. Define PA in Δ_1, we need the exists for the induction scheme.

Definition 6 (Quasi-substitution)**.** For $\varphi(v_i)$ and term t let $\varphi[t] = \forall v_i(v_i = t \rightarrow \varphi)$. We have $\mathrm{PA} \vdash \varphi(t) \leftrightarrow \varphi[t]$. The benefit of this definition is that it is easy to tell the Gödel number of $\varphi[t]$ from φ.

Definition 7. $\Sigma_0 = \Pi_0 = \Delta_0$ formulas without unbounded quantifiers. Σ_{n+1}: formulas of the form $\exists x \varphi(x)$, with $\varphi \in \Pi_n$. Similarly, Π_{n+1} is the formulas of the form $\forall x \varphi(x)$ with $\varphi \in \Sigma_n$.
A formula ψ is provably Σ_n from S if there is a $\varphi \in \Sigma_n$, such that $S \vdash \psi \leftrightarrow \varphi$.

Lemma 8 (Diagonal Lemma)**.** For any formula $F(v_1)$ there is a formula C such that:

$$\mathrm{PA} \vdash F(\overline{\ulcorner C \urcorner}) \leftrightarrow C$$

Let E_n the expression with Gödel number n.
Let $d(n)$ be $E_n[\bar{n}]$ and $D(m, n)$ be the formula $n = \ulcorner d(m) \urcorner$.
Consider, $F(\overline{\ulcorner y \urcorner})$, then $F(\overline{\ulcorner d(y) \urcorner}) \vdash\dashv \psi(y) = \forall z(D(y, z) \rightarrow F(z))$. Let $k = \ulcorner\psi\urcorner$, $C = \psi[\bar{k}]$. Then, $C \vdash\dashv \psi(\bar{k}) \vdash\dashv F(\overline{\ulcorner d(k) \urcorner})$. But $k = \ulcorner\psi\urcorner$, so $C = E_k[\bar{k}]$ which is defined to be $d(k)$. So, $C \vdash\dashv F(\overline{\ulcorner C \urcorner})$.

So, *truth is undefinable*, let $\mathbb{N} \models \mathrm{True}(\overline{\ulcorner\varphi\urcorner})$ if and only if $\mathbb{N} \models \varphi$. Then, $F(v_1) = \neg\mathrm{True}(v_1)$ so there is C such that $C \models\!\dashv \neg\mathrm{True}(\overline{\ulcorner C \urcorner}) \models\!\dashv \neg C$.

Recursive Functions

Definition 9. Primitive recursive functions contain $\underline{\mathrm{zero}}$ and $\underline{\mathrm{succ}}$ and are closed under, projection, composition and primitive recursion. Recursive functions are closed under minimilisation as well.

Proposition 10. Equivalences: A is a decidable set $\iff$ A is Δ_1-definable.
A is a recursively enumerable set $\iff$ A is Σ_1-definable.

Provability Rules

For S a *provably definable* set of assumptions.

1. If $S \vdash \varphi$ then PA $\vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner})$.
2. PA $\vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \to \psi \urcorner}) \to (\mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pr}_S(\overline{\ulcorner \psi \urcorner}))$.
3. If PA $\subseteq S$ then PA $\vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \to \mathrm{Pr}_S(\overline{\ulcorner \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \urcorner})$

Additionally, $S \vdash \varphi$ if and only if $\mathbb{N} \models \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner})$.
Let PA $\subseteq S$ a provably definable set of sentences. Then, there is a formula G, $\mathbb{N} \models G$ but $S \nvdash G$.

Definition 11. A set S of assumptions is *n-inconsistent* if for some Σ_n formula $\exists x \psi(x)$, $S \vdash \exists x \psi(x)$ but for all $m \in \mathbb{N}$, $S \vdash \neg\psi(\overline{m})$. It is *$n$-consistent* if it is not n-inconsistent.

Definition 12. Formulas S are *Σ_n-complete* if every Σ_n sentence true in $\mathbb{N}$ is provable from S.

Definition 13 (Weaker arithmetics)**.** $\mathcal{Q}$ is PA without the induction schema, so it is finitely axiomatisable. $\mathcal{R}$ is the collection of all valid sentences of the form $\bar{m} + \bar{n} = \bar{k}$, $\bar{m} \times \bar{n} = \bar{k}$, $\bar{m} \neq \bar{n}$, $\forall v_1(v_1 \leq \bar{n} \to (v_1 = \bar{0} \vee \ldots \vee \bar{n}))$ and $\forall v_1(v_1 \leq \bar{n} \vee \bar{n} \leq v_1)$.

Proposition 14. For every $r \in \mathcal{R}$, $\mathcal{Q} \vdash r$.

Proposition 15. $\mathcal{R}$ is Σ_0-complete. Hence, so is $\mathcal{Q}$ and PA.

Proposition 16. If S is Σ_0-complete then it is Σ_1-complete. Hence, $\mathcal{R}$, $\mathcal{Q}$ and PA are Σ_1-complete.

Theorem 17 (1^{st} Incompleteness)**.** There exists a Π_1 sentence G such that if PA is consistent then PA $\nvdash G$, if PA is 1-consistent then PA $\nvdash \neg G$.

Theorem 18 (Rosser's)**.** Let PA $\subseteq S$ any provably definable consistent set of sentences. Then there is a sentence G such that $S \nvdash G$ and $S \nvdash \neg G$.

Theorem 19 (2^{nd} Incompleteness)**.** Let PA $\subseteq S$ a provably definable set of sentences.
If $S \vdash G \leftrightarrow \neg\mathrm{Pr}_S(\overline{\ulcorner G \urcorner})$, then for any φ, $S \vdash \neg\mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \to \neg\mathrm{Pr}_S(\overline{\ulcorner G \urcorner})$.

So, $S \vdash \neg\mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner})$ implies $S \vdash G$. But, if S is consistent $S \nvdash G$.
In particular, $S \nvdash \neg\mathrm{Pr}_S(\overline{\ulcorner \bar{0} = \bar{1} \urcorner})$ which is Con_S.

Theorem 20 (Lob's Theorem)**.** Let PA $\subseteq S$ provably definable. Then, from $S \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \to \varphi$ we can deduce $S \vdash \varphi$.

Definition 21 (Godel-Lob Logic)**.** Symbols: countably many propositional variables, $\bot$, $\to$, $\Box$.
Formulae: propositional variables, $\bot$. For φ, ψ formulae, $\varphi \to \psi$ and $\Box\varphi$ are formulae. Logical axioms: Propositional tautologies, where $\bot$ is contradiction, $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$, and $\Box(\Box\varphi \to \varphi) \to \varphi$.
Rules of inference: Modus ponens and necessitation $\vdash \varphi$ implies $\vdash \Box\varphi$.

Proposition 22 (Substitution)**.** Let $\varphi, \psi, \chi, \theta$ formulae. Let θ' formula θ where some instances of χ are replaced with ψ. Then: $\vdash (\varphi \to (\psi \leftrightarrow \chi)) \to (\varphi \to (\theta \leftrightarrow \theta'))$.

Proposition 23 (Modalised substitution)**.** Let $X = X(p)$ with instances of p bound by $\Box$.
Then $\vdash \Box(p \leftrightarrow q) \to (X(p) \leftrightarrow X(q))$.

Theorem 24 (Fixed-point theorem)**.** Let $A(p)$ with p bound by $\Box$. Then there is X with letters only from $A(\cdot)$ such that $\vdash X \leftrightarrow A(X)$. X is "unique": $\vdash (\Box(p \leftrightarrow A(p)) \wedge \Box(q \leftrightarrow A(q))) \to \Box(p \leftrightarrow q)$.

Proposition 25 (GL Incompleteness)**.** 1^{st} Incompleteness: There is a formula G such that: $\vdash G \leftrightarrow \neg\Box G$.
2^{nd} Incompleteness: For any A, B we have $\vdash \Box\neg\Box A \to \Box B$.

Proof. Consider $A(p) = \neg\Box p$, then G is a fixed point such that $\vdash G \leftrightarrow \neg\Box G$.
For the 2^{nd} we have $\vdash \neg\Box A \to (\Box A \to A)$ by propositional calculus. So, $\vdash \Box(\neg\Box A \to (\Box A \to A))$ by necessitation. By second provability rule and axiom 2: $\vdash \Box\neg\Box A \to \Box A$. By the correspondence $\Box, \mathrm{Pr}$: $\vdash \Box A \to \Box\Box A$. So, $\vdash \Box\neg\Box A \to \Box\Box A$. Now for any B we have $\vdash \Box\neg\Box A \to \Box\Box A \to \Box B$. So by hypothetical syllogism, $\vdash \Box\neg\Box A \to \Box B$. $\Box$