Theorem 1 (Upwards Lowenheim-Skolem)**.** For an infinite L-structure $\underline{A}$ and $\kappa \geqslant |A| + |L|$ there is an L-structure $\underline{B}$ such that $\underline{A} \preccurlyeq \underline{B}$.

Theorem 2 (Downwards Lowenheim-Skolem)**.** For $\underline{B}$ an L-structure, $S \subseteq B$, there exists $\underline{A}$ such that $S \subseteq A$, $|A| \leqslant \max(|S|, |L|)$ and $\underline{A} \preccurlyeq \underline{B}$.

Definition 3. For a theory T, and variables $\mathbf{x}$, a *partial type* P is a set of formulas where $T \cup P$ is consistent.

Example 4. For $T = \mathrm{Th}(\langle \mathbb{Z}, +, -, 0, 1\rangle)$, $P(x) = \{\exists y(y + y \ldots + y = x)\} \cup \{x \neq 0\}$, is a partial type. This can be proven by compactness.

Definition 5. For a theory T, a type P is principal if for some $\theta(\mathbf{x})$, $T \cup \theta(\mathbf{x}) \models P$ and $T \cup \theta$ is consistent.

Theorem 6 (Omitting types)**.** Let T complete and L countable. Let P a countable set of non-principal types. Then, there is a countable model of T omitting every type in P.

Proof. To construct the model, expand the language with countable constants, enumerate the sentences, formulas and closed tuples. At T_{3n+1} add σ_n or $\neg\sigma_n$ depending on consistency, add $\neg\exists y \varphi_n(y)$ or $\varphi_n(c_k)$ add $\psi(\mathbf{x}) \in P$ such that $T_{3n+2} \cup \neg\psi(\mathbf{t}_n)$ is consistent. Such a ψ exists because adding constant doesn't un-principal a type and for $\theta = \sigma_n \wedge (x = c)$, $T \cup \theta(\mathbf{x}) \not\models \varphi(\mathbf{x})$, so $T \cup \{\sigma, \neg\varphi(c)\}$. $\square$

Theorem 7. If $A \leqslant B$ then for every *quantifier free* $\varphi(x_1, \ldots x_n)$,

$$\varphi^{\underline{B}} \cap A^k = \varphi^{\underline{A}}.$$

If $A \preccurlyeq B$ then this is true for all formulas φ.

Proposition 8 (Tarski-Vaught criterion)**.** If $\underline{A} \leqslant \underline{B}$ and for $\varphi(\mathbf{x}, y)$ and $\mathbf{a} \in A^n$, $\underline{B} \models \varphi(\mathbf{a}, d)$ for $d \in B$ then $\underline{B} \models \varphi(\mathbf{a}, c)$ for $c \in A$, then $\underline{A} \preccurlyeq \underline{B}$.

Definition 9. For $\underline{A}$ let $L_A = L \cup \{c_a \mid a \in A\}$. $\underline{A_A}$ is an L_A-structure. The *diagram* $\mathrm{Diag}(\underline{A})$ is all q.f. L_A sentences true in A_A.

Theorem 10. There is a 1-1 correspondence between models of $\mathrm{Diag}(\underline{A}) \cup T$ and pairs $(\underline{B}, \underline{A})$ where $\underline{B} \models T$ and $\underline{A} \leqslant \underline{B}$.

Proof. Let $\underline{C} \models \mathrm{Diag}(A) \cup T$, $\underline{B} = \underline{C}_{|L}$, so $\underline{B} \models T$, build $f\colon A \to B$, $a \mapsto c_a^{\underline{C}}$. Then f is an embedding as for q.f. φ, $\underline{A} \models \varphi(\mathbf{a})$, $\varphi(\mathbf{c}_a) \in \mathrm{Diag}(\underline{A}) \Rightarrow \underline{C} \models \varphi(\mathbf{c}_a) \Rightarrow \underline{C} \models \varphi(\mathbf{c}_a^{\underline{C}}) \Rightarrow \underline{C} \models \varphi(f(\mathbf{a})) \Rightarrow \underline{B} \models \varphi(f(\mathbf{a}))$. If $A \not\models \varphi(\mathbf{a}) \Rightarrow A \models \neg\varphi(\mathbf{a}) \Rightarrow B \models \neg\varphi(\mathbf{a}) \Rightarrow B \not\models \varphi(\mathbf{a})$. $\square$

Theorem 11. For a theory T, $\underline{A} \models T_\forall$ if and only if there exists $\underline{B} \models T$ with $\underline{A} \leqslant \underline{B}$.

Proof. ($\Rightarrow$) There is $\underline{A} \leqslant \underline{B}$ iff $\underline{B} \models \mathrm{Diag}(\underline{A}) \cup T$. iff finitely satisfiable iff $T + \varphi$ for $\varphi \in \mathrm{Diag}(\underline{A})$ is satisfiable iff $T \not\models \neg\varphi(c_1, c_2, \ldots, c_n)$ iff $T \not\models \forall \mathbf{x} \neg\varphi(\mathbf{x})$. But $A \models \exists \mathbf{x} \varphi(\mathbf{x})$ so $\forall \mathbf{x} \neg\varphi(\mathbf{x}) \notin T_\forall$. $\square$

Corollary 12. The theory of fields is not universal as, $\underline{Z} \leqslant \underline{Q}$ but $\underline{Q}$ is a field and $\underline{Z}$ is not.

Theorem 13. Sentence σ is universal if and only if for all $B \models \sigma$ and $A \leqslant B$, $A \models \sigma$.

Example 14. For F the theory of fields, $F_\forall$ is the theory of integral domains. That is because every integral domain can be embedded in a field.

Theorem 15. For a chain $\underline{A_1} \leqslant \underline{A_2} \leqslant \ldots$, let $\underline{A^*}$ be the limit of the chain. Then every AE sentence σ which holds for all $\underline{A_i}$, holds for $\underline{A^*}$.

Definition 16. Theory T admits quantifier elimination if for any formula $\theta(\mathbf{x})$, there exists a quantifier free formula $\tilde{\theta}(\mathbf{x})$ such that:

$$T \models \forall \mathbf{x}(\theta \leftrightarrow \tilde{\theta})$$

Theorem 17. If L has no constant or function symbols and T admits Q.E. then T is complete.

Example 18.

- $\mathrm{Th}(\langle \mathbb{Q}, < \rangle)$ admits QE and so is complete.
- ACF admits QE. But, the only thing ACF does not decide is the field characteristic. Hence, ACF_p for p prime or zero is complete.

- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1 \rangle)$ does not admit Q.E. Atomic sentences with one variable define only, finite and cofinite sets. But $\varphi(x) = \exists y (y^2 = x)$ defines the positive numbers.
- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1, < \rangle)$ admits Q.E. by Tarski. It is complete because the order is complete and so determines equality.

Remark. If T admits Q.E. and $\underline{A_1}, \underline{A_2} \models T$ and $\underline{A_1} \leqslant \underline{A_2}$ then $\underline{A_1} \preccurlyeq \underline{A_2}$.

Theorem 19. If it exists, there is only one way to extend a universal theory to a Q.E. theory. Prove by taking $\underline{A_1} \models S$ and $\underline{A_1} \leqslant \underline{B_1} \models T$ and build chains. The limits are equal and $\underline{A_1} \preccurlyeq A_2 \preccurlyeq C$.

Theorem 20 (Equivalence)**.**

1. T has Q.E.
2. Any partial isomorphism between models of T is elementary. It is enough to consider isomorphisms on finitely generated subsets.
3. For any $\mathcal{M} \models T$ and any $\mathbf{a} \in \mathcal{M}^n$, $T \cup \mathrm{diag}(\mathbf{a})$ is complete.

Definition 21. A theory T for a cardinal κ is κ-categorical if there exist models $\underline{A}, \underline{B} \models T$ with $|A| = |B| = \kappa$ and this implies $A \cong B$.

Proposition 22 (Los-Vaught Test)**.** If T has no finite models, and for $\kappa \geqslant |L| + \aleph_0$, T is κ-categorical, then T is complete.

Proof. Take $\mathcal{M} \models T$, $|\mathcal{M}| = \kappa$. Then, for any sentence σ, $\mathcal{M} \models \sigma$ or $\mathcal{M} \models \neg\sigma$, wlog let it be σ. Then, $T \cup \{\neg\sigma\}$ has no model of cardinality κ, by the Lowenheim-Skolems $T \cup \{\neg\sigma\}$ has no infinite models. $\square$

Example 23.

1. Theory of equality $T_=$ is categorical for every cardinal. So T_∞ is complete.
2. Vect_K is categorical for every $\kappa > |K|$, so $\mathrm{Vect}_K \cup T_\infty$ is complete. But, $\mathrm{Vect}_\mathbb{Q}$ is not $\aleph_0$-categorical.
3. DLO is $\aleph_0$-categorical and has no finite models. Proof by back and forth lemma. It is not $\aleph_1$-categorical, take $\mathbb{R} \sqcup \mathbb{Q} \ncong \mathbb{R}$.

Definition 24 (Atomic Model)**.** $\underline{A}$ is an *atomic* model of a *complete* theory T if for any $\mathbf{a} \in A^n$ there is $\varphi(\mathbf{x})$ such that $\underline{A} \models \varphi(\mathbf{a})$ and for any $\psi(\mathbf{x})$: $T \vDash \forall x(\varphi \to \psi)$ or $T \vDash \forall x(\varphi \to \neg\psi)$

Definition 25. A model $\underline{A} \models T$ is *homogeneous* if for any $\mathbf{a}, \mathbf{b} \in A^n$ that satisfy the same formulas, there is an automorphism $\alpha \colon A \longrightarrow A$ such that $\alpha(a_i) = b_i$.

Definition 26. A model $A \models T$ is *prime* if for any model $B \models T$, A embeds elementarily to B.

Proposition 27. *Countable* atomic models are isomorphic. In fact, every finite partial isomorphism can be extended to an isomorphism. They are also prime and homogeneous.

Definition 28 (Type)**.** The *n-type* of an n-tuple $\mathbf{a} \in A^n$ is the set of formulas satisfied by $\mathbf{a}$, denoted by $\mathrm{tp}_A(\mathbf{a})$. $\mathrm{tp}_A(\mathbf{a})$ is a partial type for the $\mathrm{Th}(\underline{A})$. It is complete as $\varphi(\mathbf{x}) \in \mathrm{tp}_A(\mathbf{a})$ or $\neg\varphi(\mathbf{x}) \in \mathrm{tp}_A(\mathbf{a})$.

Proposition 29. For a complete theory T the atomic models realise the fewest types.

Proposition 30. For a countable language L, Prime $\iff$ Countable and Atomic.

Corollary 31. The prime models of T are isomorphic, by uniqueness of countable & atomic.

Proposition 32. If for each n the set of n-types is countable, then T has a prime model.

Definition 33. A countable model $\mathcal{M} \models T$ is *universal*, if every countable model embeds elementarily into $\mathcal{M}$.

Theorem 34 (Ryll-Nardzewski)**.** Let T complete and L-countable. Then, T is $\aleph_0$-categorical $\iff$ every countable model is prime $\iff$ every countable model is atomic $\iff$ every type is principal $\iff$ there are only finitely many n-types $\iff$ n-formulas $\varphi(\mathbf{x})$ up to T equivalence is finite $\iff$ every countable model is universal $\iff$ a countable model is prime and universal $\iff$ every countable model is universal and homogeneous.

Definition 35. A *saturated* model is a model that realises all n-types and is homogeneous. Equivalently: If $\mathcal{M}$ is saturated, for all $B \subseteq \mathcal{M}$ and $|B| < |\mathcal{M}|$, $\mathcal{M}_B$ realises all 1-types of $\mathrm{Th}(\mathcal{M}_B)$.

Proposition 36. If $\mathcal{M}$ is saturated and countable, it is universal and unique up to isomorphism.

Definition 37. A group G applied to a G-set is oligomorphic if there are finitely many orbits of G.

Proposition 38. T is $\aleph_0$-categorical if and only if for a countable $\mathcal{M}$, $\mathrm{Aut}(\mathcal{M})$ is oligomorphic.