We work in the language $L_E = \{\bar{0},\ ^+,\ v,\ f,\ ',\ (,\ ),\ -,\ \rightarrow,\ \forall,\ =,\ \leqslant,\ \#\}$

Definition 1. A subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula $\varphi(v_1, \dots, v_k)$ such that

$$(n_1, \dots, n_k) \in A \iff \varphi(\overline{n_1}, \dots, \overline{n_k})$$

Definition 2. A subset $A \subseteq \mathbb{N}^k$ is *provably definable* if there is $\varphi(\mathbf{x})$ such that $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$ and $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

Definition 3. A function $f\colon \mathbb{N}^k \longrightarrow \mathbb{N}$ is *definable* if $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$ is definable.
It is *weakly provably definable* from S if A is provably definable from S.
It is *provably definable* if for all $\mathbf{n} \in \mathbb{N}^k$, $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

Definition 4.

1. $^+\colon \mathbb{N} \longrightarrow \mathbb{N} \setminus \{0\}$ is injective.
2. Adding and multiplying by 0 on the right: $\forall v(v + \bar{0} = \bar{0})$ and $\forall v(v \times \bar{0} = \bar{0})$
3. Addition, multiplication: $\forall v_1 \forall v_2(v_1 + v_2^+ = (v_1 + v_2)^+)$ and $\forall v_1 \forall v_2(v_1 \times v_2^+ = v_1 \times v_2 + v_2)$.
4. Relation $\leqslant$ is a total order, $\bar{0}$ is the least element, n^+ is the successor of n.
5. For any formula $\varphi(x)$ in one variable:

$$\Big(\varphi(\bar{0}) \wedge \forall v_0\big(\varphi(v_0) \rightarrow \varphi(v_0^+)\big)\Big) \rightarrow \forall v_0\big(\varphi(v_0)\big)$$

Definition 5. For $\varphi = \sigma_0 \dots \sigma_n$ a formula of L, $\ulcorner\varphi\urcorner = \displaystyle\sum_{i=0}^{n} \ulcorner\sigma_i\urcorner 13^i$

Building $\mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner})$

1. Syntax: $\lfloor \sqrt[13]{n} \rfloor$, $k{+}{+}l$, k is prefix/suffix/substring of n and *formula sequence* last of which is σ.
2. Define `isNumeral` and `isVariable` by $\exists$. Define `isTerm` by valid sequence of term construction.
3. Identify formulas: `isAtomic`, and `isAxiomFirstOrder`.
4. So for S a definable set of formulas in Δ_i, $\mathrm{proof}_S(\bar{n}, \bar{m})$, is Δ_i. $\mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner}) = (\exists x)\mathrm{proof}_S(\overline{\ulcorner\varphi\urcorner}, x)$.
5. Define PA in Δ_1, we need the exists for the induction scheme.

Definition 6 (Quasi-substitution)**.** For $\varphi(v_i)$ and term t let $\varphi[t] = \forall v_i(v_i = t \rightarrow \varphi)$. We have $\mathrm{PA} \vdash \varphi(t) \leftrightarrow \varphi[t]$.

Definition 7. $\Sigma_0 = \Pi_0 = \Delta_0$ formulas without unbounded quantifiers. Σ_{n+1}: formulas of the form $\exists x\varphi(x)$, with $\varphi \in \Pi_n$. Similarly, Π_{n+1} is the formulas of the form $\forall x\varphi(x)$ with $\varphi \in \Sigma_n$.
A formula ψ is provably Σ_n from S if there is a $\varphi \in \Sigma_n$, such that $S \vdash \psi \leftrightarrow \varphi$.

Lemma 8 (Diagonal Lemma)**.** For any formula $F(v_1)$ there is a formula C such that:

$$\mathrm{PA} \vdash F(\overline{\ulcorner C\urcorner}) \leftrightarrow C$$

Let E_n the expression with Gödel number n.
Let $d(n)$ be $E_n[\bar{n}]$ and $D(m,n)$ be the formula $n = \ulcorner d(m)\urcorner$.
Consider, $F(\overline{\ulcorner y\urcorner})$, then $F(\overline{\ulcorner d(y)\urcorner}) \vdash\dashv \psi(y) = \forall z(D(y,z) \rightarrow F(z))$. Let $k = \ulcorner\psi\urcorner$, $C = \psi[\bar{k}]$. Then, $C \vdash\dashv \psi(\bar{k}) \vdash\dashv F(\overline{\ulcorner d(k)\urcorner})$. But $k = \ulcorner\psi\urcorner$, so $C = E_k[\bar{k}]$ which is defined to be $d(k)$. So, $C \vdash\dashv F(\overline{\ulcorner C\urcorner})$.

So, *truth is undefinable*, let $\mathbb{N} \models \mathrm{True}(\overline{\ulcorner\varphi\urcorner})$ if and only if $\mathbb{N} \models \varphi$. Then, $F(v_1) = \neg\mathrm{True}(v_1)$ so there is C such that $C \models\!\dashv \neg\mathrm{True}(\overline{\ulcorner C\urcorner}) \models\!\dashv \neg C$.

Recursive Functions

Definition 9. Primitive recursive functions contain zero and succ and are closed under, projection, composition and primitive recursion. Recursive functions are closed under minimilisation as well.

Proposition 10. Equivalences: A is a decidable set $\iff$ A is Δ_1-definable.
A is a recursively enumerable set $\iff$ A is Σ_1-definable.

Provability Rules

For S a *provably definable* set of assumptions.

1. If $S \vdash \varphi$ then $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner})$.
2. $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner\varphi \to \psi\urcorner}) \to (\mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner}) \to \mathrm{Pr}_S(\overline{\ulcorner\psi\urcorner}))$.
3. $\mathrm{PA} \vdash \mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner}) \to \mathrm{Pr}_S(\overline{\ulcorner \mathrm{Pr}_S(\overline{\ulcorner\varphi\urcorner}) \urcorner})$

Definition 11. A set S of assumptions is *n-inconsistent* if for some Σ_n formula $\exists x \psi(x)$, $S \vdash \exists x \psi(x)$ but for all $m \in \mathbb{N}$, $S \vdash \neg\psi(\bar{m})$. It is *$n$-consistent* if it is not n-inconsistent.