We work in the language $L_E = \{\bar{0}, {}^+, v, f, ', (,), -, \to, \forall, =, \leqslant, \#\}$

Definition 1. A subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula $\varphi(v_1, \ldots, v_k)$ such that $(n_1, \ldots, n_k) \in A \iff \varphi(\overline{n_1}, \ldots, \overline{n_k})$

Definition 2. A subset $A \subseteq \mathbb{N}^k$ is *provably definable* if there is $\varphi(\mathbf{x})$ such that $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$ and $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

Definition 3. A function $f \colon \mathbb{N}^k \longrightarrow \mathbb{N}$ is *definable* if $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$ is definable.
It is *weakly provably definable* from S if A is provably definable from S.
It is *provably definable* if for all $\mathbf{n} \in \mathbb{N}^k$, $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

Definition 4.

1. ${}^+ \colon \mathbb{N} \longrightarrow \mathbb{N} \setminus \{0\}$ is injective.
2. Adding and multiplying by 0 on the right: $\forall v(v + \bar{0} = \bar{0})$ and $\forall v(v \times \bar{0} = \bar{0})$
3. Addition, multiplication: $\forall v_1 \forall v_2 (v_1 + v_2^+ = (v_1 + v_2)^+)$ and $\forall v_1 \forall v_2 (v_1 \times v_2^+ = v_1 \times v_2 + v_2)$.
4. Relation $\leqslant$ is a total order, $\bar{0}$ is the least element, n^+ is the successor of n.
5. For any formula $\varphi(x)$ in one variable: $(\varphi(\bar{0}) \wedge \forall v_0(\varphi(v_0) \to \varphi(v_0^+))) \to \forall v_0(\varphi(v_0))$

1. Syntax: $\lfloor \sqrt[13]{n} \rfloor$, $k{+}{+}l$, k is prefix/suffix/substring of n and *formula sequence* last of which is σ.
2. Define `isNumeral` and `isVariable` by $\exists$. Define `isTerm` by valid sequence of term construction.
3. Identify formulas: `isAtomic`, and `isAxiomFirstOrder`.
4. So for S a definable set of formulas in Δ_i, $\text{proof}_S(\bar{n}, \bar{m})$, is Δ_i. $\Pr_S(\overline{\ulcorner\varphi\urcorner}) = (\exists x)\text{proof}_S(\overline{\ulcorner\varphi\urcorner}, x)$.
5. Define PA in Δ_1, we need the exists for the induction scheme.

Definition 5. $\Sigma_0 = \Pi_0 = \Delta_0$ formulas without unbounded quantifiers. Σ_{n+1}: formulas of the form $\exists x\varphi(x)$, with $\varphi \in \Pi_n$. Similarly, Π_{n+1} is the formulas of the form $\forall x\varphi(x)$ with $\varphi \in \Sigma_n$.
A formula ψ is provably Σ_n from S if there is a $\varphi \in \Sigma_n$, such that $S \vdash \psi \leftrightarrow \varphi$.

Lemma 6 (Diagonal Lemma)**.** For any formula $F(v_1)$ there is a formula C such that: $\text{PA} \vdash F(\overline{\ulcorner C \urcorner}) \leftrightarrow C$

Let E_n the expression with Gödel number n. Let $d(n)$ be $E_n[\bar{n}]$ and $D(m, n)$ be the formula $n = \ulcorner d(m) \urcorner$. Consider, $F(\overline{\ulcorner y \urcorner})$, then $F(\overline{\ulcorner d(y) \urcorner}) \vdash\dashv \psi(y) = \forall z(D(y, z) \to F(z))$. Let $k = \ulcorner\psi\urcorner$, $C = \psi[\bar{k}]$. Then, $C \vdash\dashv \psi(\bar{k}) \vdash\dashv F(\overline{\ulcorner d(k) \urcorner})$. But $k = \ulcorner\psi\urcorner$, so $C = E_k[\bar{k}]$ which is defined to be $d(k)$. So, $C \vdash\dashv F(\overline{\ulcorner C \urcorner})$.

Theorem 7 (Tarski)**.** Truth is *undefinable*, let $\mathbb{N} \models \text{True}(\overline{\ulcorner\varphi\urcorner})$ if and only if $\mathbb{N} \models \varphi$. Then, $F(v_1) = \neg\text{True}(v_1)$ so there is C such that $C \models\!\dashv \neg\text{True}(\overline{\ulcorner C \urcorner}) \models\!\dashv \neg C$.

Definition 8. Primitive recursive functions contain $\underline{\text{zero}}$ and $\underline{\text{succ}}$.
Composition: For $g \colon \mathbb{N}^a \to \mathbb{N}$ and for $1 \le i \le a$ $f_i \colon \mathbb{N}^k \to \mathbb{N}$, $h(\mathbf{n}) = g(f_1(\mathbf{n}), \ldots, f_a(\mathbf{n}))$ is PR.
Recursion: For $g \colon \mathbb{N}^k \to \mathbb{N}, h \colon \mathbb{N}^{k+2} \to \mathbb{N}$, $f \colon \mathbb{N}^{k+1} \to \mathbb{N}$ is primitive recursive
$f(\mathbf{n}, 0) = g(\mathbf{n})$ and $f(\mathbf{n}, m+1) = h(\mathbf{n}, m, f(\mathbf{n}, m))$.
Minimilisation: For $g \colon \mathbb{N}^{k+1} \to \mathbb{N}$ let $f \colon \mathbb{N}^k \to \mathbb{N}$, $f(\mathbf{n})$ be the minimum m such that $g(\mathbf{n}, m) = 0$ and $\perp$ otherwise.

Proposition 9. A is decidable $\iff$ A is Δ_1-definable, A is r.e. $\iff$ A is Σ_1-definable.

Proposition 10 (Provability rules)**.** For S a *provably definable* set of assumptions:
1^{st} rule: If $S \vdash \varphi$ then $\text{PA} \vdash \Pr_S(\overline{\ulcorner\varphi\urcorner})$.
2^{nd} rule: $\text{PA} \vdash \Pr_S(\overline{\ulcorner\varphi \to \psi\urcorner}) \to (\Pr_S(\overline{\ulcorner\varphi\urcorner}) \to \Pr_S(\overline{\ulcorner\psi\urcorner}))$.
3^{rd} rule: If $\text{PA} \subseteq S$ then $\text{PA} \vdash \Pr_S(\overline{\ulcorner\varphi\urcorner}) \to \Pr_S(\overline{\ulcorner \Pr_S(\overline{\ulcorner\varphi\urcorner}) \urcorner})$.
Additionally, $S \vdash \varphi$ if and only if $\mathbb{N} \models \Pr_S(\overline{\ulcorner\varphi\urcorner})$.

Definition 11. A set S of assumptions is *n-inconsistent* if for some Σ_n formula $\exists x\psi(x)$, $S \vdash \exists x\psi(x)$ but for all $m \in \mathbb{N}$, $S \vdash \neg\psi(\bar{m})$. It is *n-consistent* if it is not n-inconsistent. Formulas S are Σ_n*-complete* if every Σ_n sentence true in $\mathbb{N}$ is provable from S. Formulas S are Σ_n*-sound* if every Σ_n sentence provable from S is true in $\mathbb{N}$.

Definition 12 (Weaker arithmetics)**.** $\mathcal{Q}$ is PA without the induction schema, so it is finitely axiomatisable. $\mathcal{R}$ is the collection of all valid sentences of the form $\bar{m} + \bar{n} = \bar{k}$, $\bar{m} \times \bar{n} = \bar{k}$, $\bar{m} \neq \bar{n}$, $\forall v_1(v_1 \le \bar{n} \to (v_1 = \bar{0} \vee \ldots \vee \bar{n}))$ and $\forall v_1(v_1 \le \bar{n} \vee \bar{n} \le v_1)$. Clearly, for every $r \in \mathcal{R}$, $\mathcal{Q} \vdash r$ and $q \in \mathcal{Q}$, $\text{PA} \vdash q$.

Proposition 13. $\mathcal{R}$ is Σ_0-complete. Hence, so is $\mathcal{Q}$ and PA.

Proposition 14. If S is Σ_0-complete then it is Σ_1-complete. Hence, $\mathcal{R}$, $\mathcal{Q}$ and PA are Σ_1-complete.

Theorem 15 (1^{st} Incompleteness)**.** There exists a Π_1 sentence G such that if PA is consistent then PA $\nvdash G$, if PA is 1-consistent then PA $\nvdash \neg G$.

Proof. Let G such that PA $\vdash G \leftrightarrow \neg \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$. If PA $\vdash G$ by 1^{st} provability, PA $\vdash \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$ and PA $\vdash \neg\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$, contradicting the consistency of PA.
If PA $\vdash \neg G$, then PA $\vdash \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$, but $\mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner}) \equiv \exists x \mathrm{proof}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner}, x) \equiv \exists x \exists y \varphi(\overline{\ulcorner G \urcorner}, x, y)$ for $\varphi \in \Sigma_0$. Also, $\exists n(\exists x \leq n \wedge \exists y \leq n)\varphi(\overline{\ulcorner G \urcorner}, x, y) \models\dashv \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$. By 1-consistency, for some $m \in \mathbb{N}$, PA $\nvdash \neg(\exists x \leq \bar{m} \wedge \exists y \leq \bar{m})\varphi(\overline{\ulcorner G \urcorner}, x, y)$ and by Σ_0 completeness, PA $\vdash (\exists x \leq \bar{m} \wedge \exists y \leq \bar{m})\varphi(\overline{\ulcorner G \urcorner}, x, y)$. By Σ_0-soundness $\mathbb{N} \models \exists y \exists x \varphi(\overline{\ulcorner G \urcorner}, x, y)$ so $\mathbb{N} \models \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner G \urcorner})$, so PA $\vdash G$. $\square$

Theorem 16 (Rosser's)**.** Let PA $\subseteq S$ any provably definable consistent set of sentences. Then there is a sentence G such that $S \nvdash G$ and $S \nvdash \neg G$.

Proof. Let $H(x) = \exists y(\mathrm{proof}_S(\overline{\ulcorner \neg \urcorner x}, y) \wedge \forall z(z \leq y \rightarrow \neg \mathrm{proof}_S(\bar{x}, z)))$. Pick G, PA $\vdash G \leftrightarrow H(\overline{\ulcorner G \urcorner})$. If $S \vdash G$ then for some m, PA $\vdash \mathrm{proof}_S(\overline{\ulcorner G \urcorner}, \bar{m})$. But $S \vdash G$ implies $S \vdash H(\overline{\ulcorner G \urcorner})$, so there must be $r < m$ that encodes a refutation of G, so r is a *standard* natural number. So, we can prove $S \vdash \neg G$.
If $S \vdash \neg G$, let m the Godel number of the proof. But $S \vdash \neg H(\overline{\ulcorner G \urcorner})$ so there is $r < m$ such that r encodes a proof of G, so r is a *standard* natural number so $S \vdash G$. $\square$

Theorem 17 (2^{nd} Incompleteness)**.** Let PA $\subseteq S$ a provably definable set of sentences.
If $S \vdash G \leftrightarrow \neg \mathrm{Pr}_S(\overline{\ulcorner G \urcorner})$, then for any φ, $S \vdash \neg \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \neg \mathrm{Pr}_S(\overline{\ulcorner G \urcorner})$.

Proof. $S \vdash G \rightarrow (\neg G \rightarrow X)$. Now, $S \vdash \mathrm{Pr}_S(\overline{\ulcorner G \urcorner}) \rightarrow \neg G$, so $S \vdash G \rightarrow (\mathrm{Pr}_S(\overline{\ulcorner G \urcorner}) \rightarrow X)$ applying the 1st, 2nd provability and MP: $S \vdash \mathrm{Pr}_S(\overline{\ulcorner G \urcorner}) \rightarrow (\mathrm{Pr}_S(\overline{\ulcorner \mathrm{Pr}_S(\overline{\ulcorner G \urcorner}) \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner X \urcorner}))$. By 3rd provability and HS, $S \vdash \mathrm{Pr}_S(\overline{\ulcorner G \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner X \urcorner})$, now apply contrapositive. $\square$

Theorem 18 (Lob's Theorem)**.** Let PA $\subseteq S$ provably definable. Then, from $S \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \varphi$ we can deduce $S \vdash \varphi$.

Proof. Let $S \vdash \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}) \rightarrow \varphi$ and L such that $S \vdash L \leftrightarrow (\mathrm{Pr}_S(\overline{\ulcorner L \urcorner}) \rightarrow \varphi)$. Then $S \vdash \mathrm{Pr}_S(\overline{\ulcorner L \urcorner}) \rightarrow (\mathrm{Pr}_S(\overline{\ulcorner \mathrm{Pr}_S(\overline{\ulcorner L \urcorner}) \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner}))$. By 3rd provability and HS, $S \vdash \mathrm{Pr}_S(\overline{\ulcorner L \urcorner}) \rightarrow \mathrm{Pr}_S(\overline{\ulcorner \varphi \urcorner})$, so $S \vdash \mathrm{Pr}_S(\overline{\ulcorner L \urcorner}) \rightarrow \varphi$ which is defined as L, so $S \vdash L$, $S \vdash \mathrm{Pr}_S(\overline{\ulcorner L \urcorner})$ so $S \vdash \varphi$. $\square$

Proposition 19. If $\varphi \in \Sigma_1$, then PA $\vdash \varphi \rightarrow \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \varphi \urcorner})$ and PA $\vdash \forall x(\varphi(x) \rightarrow \mathrm{Pr}_{\mathrm{PA}}(\overline{\ulcorner \varphi(x) \urcorner}))$.

Definition 20 (Strengthenings)**.** $\omega-$rule: If for all $n \in \mathbb{N}$, $S \vdash \varphi(\bar{n})$ then $S \vdash \forall x \varphi(x)$. $\mathcal{R}^\omega$ is complete.
URP: for $F(v_1)$ a formula add axiom $\forall n \mathrm{Pr}_{PA}(\ulcorner \forall v_1(v_1 = 0 \urcorner \ulcorner \bar{n} \urcorner \ulcorner \rightarrow F(v_1) \urcorner) \rightarrow \forall n F(n)$. URP $\vdash G$.

Definition 21 (Godel-Lob Logic)**.** Symbols: countably many propositional variables, $\perp$, $\rightarrow$, $\Box$.
Formulae: propositional variables, $\perp$. For φ, ψ formulae, $\varphi \rightarrow \psi$ and $\Box\varphi$ are formulae. Logical axioms: Propositional tautologies, where $\perp$ is contradiction, $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$, and $\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$.
Rules of inference: Modus ponens and necessitation $\vdash \varphi$ implies $\vdash \Box\varphi$.

Proposition 22 (Substitution)**.** Let $\varphi, \psi, \chi, \theta$ formulae. Let θ' formula θ where some instances of χ are replaced with ψ. Then: $\vdash (\varphi \rightarrow (\psi \leftrightarrow \chi)) \rightarrow (\varphi \rightarrow (\theta \leftrightarrow \theta'))$. Let $X = X(p)$ with instances of p bound by $\Box$. Then $\vdash \Box(p \leftrightarrow q) \rightarrow (X(p) \leftrightarrow X(q))$.

Theorem 23 (Fixed-point theorem)**.** Let $A(p)$ with p bound by $\Box$. Then there is X with letters only from $A(\cdot)$ such that $\vdash X \leftrightarrow A(X)$. X is "unique": $\vdash (\Box(p \leftrightarrow A(p)) \wedge \Box(q \leftrightarrow A(q))) \rightarrow \Box(p \leftrightarrow q)$.
For $A(p) = \Box B(p)$, $\vdash \Box B(\top) \leftrightarrow A(\Box B(\top)) \equiv \Box B(\Box B(\top))$.
For $A(p) = D(C_1, \ldots, C_n)$, find $F_i \leftrightarrow \Box C_i(D(F_1, \ldots, F_n))$. Let $G_i(q) \leftrightarrow \Box C_i(D(G_1(q), \ldots, G_n(q), q)$.
Then, $G_{n+1} \leftrightarrow \Box C_{n+1}(D(G_1(F_{n+1}), \ldots, G_n(F_{n+1}), F_{n+1}))$ and $F_i = G_i(F_{n+1})$.

Proposition 24 (GL Incompleteness)**.** 1^{st} Incompleteness: There is a formula G such that: $\vdash G \leftrightarrow \neg\Box G$.
2^{nd} Incompleteness: For any A, B we have $\vdash \Box\neg\Box A \rightarrow \Box B$.

Proof. Consider $A(p) = \neg\Box p$, then G is a fixed point such that $\vdash G \leftrightarrow \neg\Box G$.
For the 2^{nd} we have $\vdash \neg\Box A \rightarrow (\Box A \rightarrow A)$ by propositional calculus. So, $\vdash \Box(\neg\Box A \rightarrow (\Box A \rightarrow A))$ by necessitation. By second provability rule and axiom 2: $\vdash \Box\neg\Box A \rightarrow \Box A$. By the correspondence $\Box$, Pr: $\vdash \Box A \rightarrow \Box\Box A$. So, $\vdash \Box\neg\Box A \rightarrow \Box\Box A$. Now for any B we have $\vdash \Box\neg\Box A \rightarrow \Box\Box A \rightarrow \Box B$. So by hypothetical syllogism, $\vdash \Box\neg\Box A \rightarrow \Box B$. $\square$