

We work in the language $L_E = \{\bar{0}, +, v, f, ', (,), -, \rightarrow, \forall, =, \leq, \#\}$

Definition 1. A subset $A \subseteq \mathbb{N}^k$ is *definable* if there is a formula $\varphi(v_1, \dots, v_k)$ such that

$$(n_1, \dots, n_k) \in A \iff \varphi(\bar{n}_1, \dots, \bar{n}_k)$$

Definition 2. A subset $A \subseteq \mathbb{N}^k$ is *provably definable* if there is $\varphi(\mathbf{x})$ such that $S \vdash \varphi(\mathbf{n}) \iff \mathbf{n} \in A$ and $S \vdash \neg\varphi(\mathbf{n}) \iff \mathbf{n} \notin A$

Definition 3. A function $f: \mathbb{N}^k \rightarrow \mathbb{N}$ is *definable* if $A = \{\mathbf{x}, f(\mathbf{x}) \mid \mathbf{x} \in \mathbb{N}^k\}$ is definable.

It is *weakly provably definable* from S if A is provably definable from S .

It is *provably definable* if for all $\mathbf{n} \in \mathbb{N}^k$, $S \vdash \forall v(\varphi(\bar{\mathbf{n}}, v) \leftrightarrow f(\bar{\mathbf{n}}) = v)$

Definition 4. 1. $^+ : \mathbb{N} \rightarrow \mathbb{N} \setminus \{0\}$ is injective.

2. Adding and multiplying by 0 on the right: $\forall v(v + \bar{0} = \bar{0})$ and $\forall v(v \times \bar{0} = \bar{0})$

3. Addition, multiplication: $\forall v_1 \forall v_2(v_1 + v_2^+ = (v_1 + v_2)^+)$ and $\forall v_1 \forall v_2(v_1 \times v_2^+ = v_1 \times v_2 + v_2)$.

4. Relation \leq is a total order, $\bar{0}$ is the least element, n^+ is the successor of n .

5. For any formula $\varphi(x)$ in one variable:

$$(\varphi(\bar{0}) \wedge \forall v_0(\varphi(v_0) \rightarrow \varphi(v_0^+))) \rightarrow \forall v_0(\varphi(v_0))$$

Definition 5. For $\varphi = \sigma_0 \dots \sigma_n$ a formula of L , $\ulcorner \varphi \urcorner = \sum_{i=0}^n \ulcorner \sigma_i \urcorner 13^i$

Building $\text{Pr}_S(\ulcorner \varphi \urcorner)$

1. Syntax: $\lfloor \sqrt[n]{13} \rfloor$, $k++l$, k is prefix/suffix/substring of n and *formula sequence* last of which is σ .

2. Define `isNumeral` and `isVariable` by \exists . Define `isTerm` by valid sequence of term construction.

3. Identify formulas: `isAtomic`, and `isAxiomFirstOrder`.

4. So for S a definable set of formulas in Δ_i , $\text{proof}_S(\bar{n}, \bar{m})$, is Δ_i . $\text{Pr}_S(\ulcorner \varphi \urcorner) = (\exists x)\text{proof}_S(\ulcorner \varphi \urcorner, x)$.

5. Define PA in Δ_1 , we need the exists for the induction scheme.

Definition 6 (Quasi-substitution). For $\varphi(v_i)$ and term t let $\varphi[t] = \forall v_i(v_i = t \rightarrow \varphi)$. We have $\text{PA} \vdash \varphi(t) \leftrightarrow \varphi[t]$.

Definition 7. $\Sigma_0 = \Pi_0 = \Delta_0$ formulas without unbounded quantifiers. Σ_{n+1} : formulas of the form $\exists x\varphi(x)$, with $\varphi \in \Pi_n$. Similarly, Π_{n+1} is the formulas of the form $\forall x\varphi(x)$ with $\varphi \in \Sigma_n$.

A formula ψ is provably Σ_n from S if there is a $\varphi \in \Sigma_n$, such that $S \vdash \psi \leftrightarrow \varphi$.

Lemma 8 (Diagonal Lemma). For any formula $F(v_1)$ there is a formula C such that:

$$\text{PA} \vdash F(\ulcorner C \urcorner) \leftrightarrow C$$

Let E_n the expression with Gödel number n .

Let $d(n)$ be $E_n[\bar{n}]$ and $D(m, n)$ be the formula $n = \ulcorner d(m) \urcorner$.

Consider, $F(\ulcorner \bar{y} \urcorner)$, then $F(\ulcorner d(y) \urcorner) \vdash \psi(y) = \forall z(D(y, z) \rightarrow F(z))$. Let $k = \ulcorner \psi \urcorner$, $C = \psi[\bar{k}]$. Then, $C \vdash \psi(\bar{k}) \vdash F(\ulcorner d(k) \urcorner)$. But $k = \ulcorner \psi \urcorner$, so $C = E_k[\bar{k}]$ which is defined to be $d(k)$. So, $C \vdash F(\ulcorner C \urcorner)$.

So, *truth is undefinable*, let $\mathbb{N} \models \text{True}(\ulcorner \varphi \urcorner)$ if and only if $\mathbb{N} \models \varphi$. Then, $F(v_1) = \neg \text{True}(v_1)$ so there is C such that $C \models \neg \text{True}(\ulcorner C \urcorner) \models \neg C$.

Recursive Functions

Definition 9. Primitive recursive functions contain zero and succ and are closed under, projection, composition and primitive recursion. Recursive functions are closed under minimilisation as well.

Proposition 10. Equivalences: A is a decidable set $\iff A$ is Δ_1 -definable.

A is a recursively enumerable set $\iff A$ is Σ_1 -definable.

Provability Rules

For S a provably definable set of assumptions.

1. If $S \vdash \varphi$ then $\text{PA} \vdash \text{Pr}_S(\overline{\varphi})$.
2. $\text{PA} \vdash \text{Pr}_S(\overline{\varphi \rightarrow \psi}) \rightarrow (\text{Pr}_S(\overline{\varphi}) \rightarrow \text{Pr}_S(\overline{\psi}))$.
3. If $\text{PA} \subseteq S$ then $\text{PA} \vdash \text{Pr}_S(\overline{\varphi}) \rightarrow \text{Pr}_S(\overline{\text{Pr}_S(\overline{\varphi})})$

Additionally, $S \vdash \varphi$ if and only if $\mathbb{N} \models \text{Pr}_S(\overline{\varphi})$.

Let $\text{PA} \subseteq S$ a provably definable set of sentences. Then, there is a formula G , $\mathbb{N} \models G$ but $S \not\vdash G$.

Definition 11. A set S of assumptions is n -inconsistent if for some Σ_n formula $\exists x\psi(x)$, $S \vdash \exists x\psi(x)$ but for all $m \in \mathbb{N}$, $S \vdash \neg\psi(\bar{m})$. It is n -consistent if it is not n -inconsistent.

Definition 12. Formulas S are Σ_n -complete if every Σ_n sentence true in \mathbb{N} is provable from S .

Definition 13 (Weaker arithmetics). \mathcal{Q} is PA without the induction schema, so it is finitely axiomatisable. \mathcal{R} is the collection of all valid sentences of the form $\bar{m} + \bar{n} = \bar{k}$, $\bar{m} \times \bar{n} = \bar{k}$, $\bar{m} \neq \bar{n}$, $\forall v_1(v_1 \leq \bar{n} \rightarrow (v_1 = \bar{0} \vee \dots \vee \bar{n}))$ and $\forall v_1(v_1 \leq \bar{n} \vee \bar{n} \leq v_1)$.

Proposition 14. For every $r \in \mathcal{R}$, $\mathcal{Q} \vdash r$.

Proposition 15. \mathcal{R} is Σ_0 -complete. Hence, so is \mathcal{Q} and PA.

Proposition 16. If S is Σ_0 -complete then it is Σ_1 -complete. Hence, \mathcal{R} , \mathcal{Q} and PA are Σ_1 -complete.

Theorem 17 (1st Incompleteness). There exists a Π_1 sentence G such that if PA is consistent then $\text{PA} \not\vdash G$, if PA is 1-consistent then $\text{PA} \not\vdash \neg G$.

Theorem 18 (Rosser's). Let $\text{PA} \subseteq S$ any provably definable consistent set of sentences. Then there is a sentence G such that $S \not\vdash G$ and $S \not\vdash \neg G$.

Theorem 19 (2nd Incompleteness). Let $\text{PA} \subseteq S$ a provably definable set of sentences.

If $S \vdash G \leftrightarrow \neg \text{Pr}_S(\overline{G})$, then for any φ , $S \vdash \neg \text{Pr}_S(\overline{\varphi}) \rightarrow \neg \text{Pr}_S(\overline{G})$.

So, $S \vdash \neg \text{Pr}_S(\overline{\varphi})$ implies $S \vdash G$. But, if S is consistent $S \not\vdash G$.

In particular, $S \not\vdash \neg \text{Pr}_S(\overline{\bar{0} = \bar{1}})$ which is Con_S .

Theorem 20 (Lob's Theorem). Let $\text{PA} \subseteq S$ provably definable. Then, from $S \vdash \text{Pr}_S(\overline{\varphi}) \rightarrow \varphi$ we can deduce $S \vdash \varphi$.