Types

Definition 1. For a theory T, and variables $\mathbf{x}$, a *partial type* P is a set of formulas where $T \cup P$ is consistent.

Example 2. For $T = \mathrm{Th}(\langle \mathbb{Z}, +, -, 0, 1 \rangle)$, $P(x) = \{\exists y(y + y \ldots + y = x)\} \cup \{x \neq 0\}$, is a partial type. This can be proven by compactness.

Definition 3. For a theory T, a type P is principal if for some $\theta(\mathbf{x})$, $T \cup \theta(\mathbf{x}) \models P$ and $T \cup \theta$ is consistent.

Theorem 4. If P is not principal it is omitted in some model of T. If P is principal and T is complete then every model of T realises P.

Embeddings

Theorem 5. If $A \leqslant B$ then for every *quantifier free* $\varphi(x_1, \ldots x_n)$,

$$\varphi^{\underline{B}} \cap A^k = \varphi^{\underline{A}}.$$

If $A \preceq B$ then this is true for all formulas φ.

Preservation Theorems

Theorem 6. For a theory T, $\underline{A} \models T_\forall$ if and only if there exists $\underline{B} \models T$ with $\underline{A} \leqslant \underline{B}$.

Corollary 7. The theory of fields is not universal as, $\underline{Z} \leqslant \underline{Q}$ but $\underline{Q}$ is a field and $\underline{Z}$ is not.

Theorem 8. Sentence σ is universal if and only if for all $B \models \sigma$ and $A \leqslant B$, $A \models \sigma$.

Example 9. For F the theory of fields, $F_\forall$ is the theory of integral domains. That is because every integral domain can be embedded in a field.

Theorem 10. For a chain $\underline{A_1} \leqslant \underline{A_2} \leqslant \ldots$, let $\underline{A^*}$ be the limit of the chain. Then every AE sentence σ which holds for all $\underline{A_i}$, holds for $\underline{A^*}$.

Quantifier elimination

Definition 11. Theory T admits quantifier elimination if for any formula $\theta(\mathbf{x})$, there exists a quantifier free formula $\tilde{\theta}(\mathbf{x})$ such that:

$$T \models \forall \mathbf{x}(\theta \leftrightarrow \tilde{\theta})$$

Theorem 12. If L has no constant or function symbols and T admits Q.E. then T is complete.

Example 13.

- $\mathrm{Th}(\langle \mathbb{Q}, < \rangle)$ admits QE and so is complete.
- ACF admits QE. But, the only thing ACF does not decide is the field characteristic. Hence, ACF_p for p prime or zero is complete.
- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1 \rangle)$ does not admit Q.E. Atomic sentences with one variable define only, finite and cofinite sets. But $\varphi(x) = \exists y(y^2 = x)$ defines the positive numbers.
- $\mathrm{Th}(\langle \mathbb{R}, +, -, \times, 0, 1, < \rangle)$ admits Q.E. by Tarski. It is complete because the order is complete and so determines equality.

Remark. If T admits Q.E. and $\underline{A_1}, \underline{A_2} \models T$ and $\underline{A_1} \leqslant \underline{A_2}$ then $\underline{A_1} \preceq \underline{A_2}$.

Theorem 14. If it exists, there is only one way to extend a universal theory to a Q.E. theory. Prove by taking $S \models \underline{A_1}$ and $\underline{A_1} \leqslant \underline{B_1} \models T$ and build chains. The limits are equal and $\underline{A_1} \preceq A_2 \preceq C$.

Theorem 15 (Equivalence)**.**

1. T has Q.E.
2. Any partial isomorphism between models of T is elementary. It is enough to consider isomorphisms on finitely generated subsets.
3. For any $\mathcal{M} \models T$ and any $\mathbf{a} \in \mathcal{M}^n$, $T \cup \mathrm{diag}(\mathbf{a})$ is complete.

Categoricity

Definition 16. A theory T for a cardinal κ is κ-categorical if there exist models $\underline{A}, \underline{B} \models T$ and for $|A| = |B| = \kappa$, $A \cong B$.

Proposition 17. If T has no finite models, and for $\kappa \geqslant |L| + \aleph_0$ T is κ-categorical, then T is complete.

Example 18.
1. Theory of equality $T_=$ is categorical for every cardinal. So T_∞ is complete.
2. Vect_K is categorical for every $\kappa \geqslant |K|$, so $\mathrm{Vect}_K \cup T_\infty$ is complete.
3. DLO is $\aleph_0$-categorical and has no finite models. Proof by back and forth lemma.

Definition 19 (Atomic Model)**.** $\underline{A}$ is an *atomic* model of a *complete* theory T if for any $\mathbf{a} \in A^n$ there is $\varphi(\mathbf{x})$ such that $\underline{A} \models \varphi(\mathbf{a})$ and for any $\psi(\mathbf{x})$:

$$T \models \forall x(\varphi \to \psi) \text{ or } T \models \forall x(\varphi \to \neg\psi)$$

Definition 20. A model $\underline{A} \models T$ is *homogeneous* if for any $\mathbf{a}, \mathbf{b} \in A^n$ that satisfy the same formulas, there is an automorphism $\alpha\colon A \longrightarrow A$ such that $\alpha(a_i) = b_i$.

Definition 21. A model $A \models T$ is *prime* if for any model $B \models T$, A embeds elementarily to B.

Proposition 22. *Countable* atomic models are isomorphic. In fact, every finite partial isomorphism can be extended to an isomorphism. They are also prime and homogeneous.

Definition 23 (Type)**.** The *n-type* of an n-tuple $\mathbf{a} \in A^n$ is the set of formulas satisfied by $\mathbf{a}$, denoted by $\mathrm{tp}_A(\mathbf{a})$. In particular, it is a partial type for the $\mathrm{Th}(\underline{A})$. It is complete as $\varphi(\mathbf{x}) \in \mathrm{tp}_A(\mathbf{a})$ or $\neg\varphi(\mathbf{x}) \in \mathrm{tp}_A(\mathbf{a})$.

Proposition 24. For a complete theory T the atomic models realise the fewest types.

Proposition 25. For a countable language L, Prime $\iff$ Countable and Atomic.

Corollary 26. The prime models of T are isomorphic, by uniqueness of countable & atomic.

Proposition 27. If for each n the set of n-types is countable, then T has a prime model.

Definition 28. A countable model $\mathcal{M} \models T$ is *universal*, if every countable model embeds elementarily into $\mathcal{M}$.

Theorem 29 (Ryll-Nardzewski)**.** Let T complete and L-countable. Then, T is $\aleph_0$-categorical $\iff$ every countable model is prime $\iff$ every countable model is atomic $\iff$ every type is principal $\iff$ there are only finitely many n-types $\iff$ n-formulas $\varphi(\mathbf{x})$ up to T equivalence is finite $\iff$ every countable model is universal $\iff$ a countable model is prime and universal $\iff$ every countable model is universal and homogeneous.

Definition 30. A *saturated* model is a model that realises all n-types and is homogeneous. Equivalently: If $\mathcal{M}$ is saturated, for all $B \subseteq \mathcal{M}$ and $|B| < |\mathcal{M}|$, $\mathcal{M}_B$ realises all 1-types of $\mathrm{Th}(\mathcal{M}_B)$.

Definition 31. A group G applied to a G-set is oligomorphic if there are finitely many orbits of G.