

Types

Definition 1. For a theory T , and variables \mathbf{x} , a *partial type* P is a set of formulas where $T \cup P$ is consistent.

Example 2. For $T = \text{Th}(\langle \mathbb{Z}, +, -, 0, 1 \rangle)$, $P(x) = \{\exists y(y + y \dots + y = x)\} \cup \{x \neq 0\}$, is a partial type. This can be proven by compactness.

Definition 3. For a theory T , a type P is principal if for some $\theta(\mathbf{x})$, $T \cup \theta(\mathbf{x}) \models P$ and $T \cup \theta$ is consistent.

Theorem 4. If P is not principal it is omitted in some model of T . If P is principal and T is complete then every model of T realises P .

Embeddings

Theorem 5. If $A \leq B$ then for every *quantifier free* $\varphi(x_1, \dots, x_n)$,

$$\varphi^B \cap A^k = \varphi^A.$$

If $A \preceq B$ then this is true for all formulas φ .

Preservation Theorems

Theorem 6. For a theory T , $\underline{A} \models T_\forall$ if and only if there exists $\underline{B} \models T$ with $\underline{A} \leq \underline{B}$.

Corollary 7. The theory of fields is not universal as, $\underline{\mathbb{Z}} \leq \underline{\mathbb{Q}}$ but $\underline{\mathbb{Q}}$ is a field and $\underline{\mathbb{Z}}$ is not.

Theorem 8. Sentence σ is universal if and only if for all $B \models \sigma$ and $A \leq B$, $A \models \sigma$.

Example 9. For F the theory of fields, F_\forall is the theory of integral domains. That is because every integral domain can be embedded in a field.

Theorem 10. For a chain $\underline{A}_1 \leq \underline{A}_2 \leq \dots$, let \underline{A}^* be the limit of the chain. Then every AE sentence σ which holds for all \underline{A}_i , holds for \underline{A}^* .

Quantifier elimination

Definition 11. Theory T admits quantifier elimination if for any formula $\theta(\mathbf{x})$, there exists a quantifier free formula $\tilde{\theta}(\mathbf{x})$ such that:

$$T \models \forall \mathbf{x}(\theta \leftrightarrow \tilde{\theta})$$

Theorem 12. If L has no constant or function symbols and T admits Q.E. then T is complete.

Example 13. • $\text{Th}(\langle \mathbb{Q}, < \rangle)$ admits QE and so is complete.

• ACF admits QE. But, the only thing ACF does not decide is the field characteristic. Hence, ACF_p for p prime or zero is complete.

• $\text{Th}(\langle \mathbb{R}, +, -, \times, 0, 1 \rangle)$ does not admit Q.E. Atomic sentences with one variable define only, finite and cofinite sets. But $\varphi(x) = \exists y(y^2 = x)$ defines the positive numbers.

• $\text{Th}(\langle \mathbb{R}, +, -, \times, 0, 1, < \rangle)$ admits Q.E. by Tarski. It is complete because the order is complete and so determines equality.

Remark. If T admits Q.E. and $\underline{A}_1, \underline{A}_2 \models T$ and $\underline{A}_1 \leq \underline{A}_2$ then $\underline{A}_1 \preceq \underline{A}_2$.

Theorem 14. If it exists, there is only one way to extend a universal theory to a Q.E. theory. Prove by taking $S \models \underline{A}_1$ and $\underline{A}_1 \leq \underline{B}_1 \models T$ and build chains. The limits are equal and $\underline{A}_1 \preceq \underline{A}_2 \preceq C$.

Theorem 15 (Equivalence). 1. T has Q.E.

2. Any partial isomorphism between models of T is elementary. It is enough to consider isomorphisms on finitely generated subsets.

3. For any $\mathcal{M} \models T$ and any $\mathbf{a} \in \mathcal{M}^n$, $T \cup \text{diag}(\mathbf{a})$ is complete.

1 Categoricity

Definition 16. A theory T for a cardinal κ is κ -categorical if there exist models $\underline{A}, \underline{B} \models T$ and for $|A| = |B| = \kappa$, $A \cong B$.

Proposition 17. If T has no finite models, and for $\kappa \geq |L| + \aleph_0$ T is κ -categorical, then T is complete.

Example 18. 1. Theory of equality $T_=_$ is categorical for every cardinal. So T_∞ is complete.

2. Vect_K is categorical for every $\kappa \geq |K|$, so $\text{Vect}_K \cup T_\infty$ is complete.

3. DLO is \aleph_0 -categorical and has no finite models. Proof by back and forth lemma.

Definition 19 (Atomic Model). \underline{A} is an *atomic* model of a *complete* theory T if for any $\mathbf{a} \in A^n$ there is $\varphi(\mathbf{x})$ such that $\underline{A} \models \varphi(\mathbf{a})$ and for any $\psi(\mathbf{x})$:

$$T \models \forall x(\varphi \rightarrow \psi) \text{ or } T \models \forall x(\varphi \rightarrow \neg\psi)$$

Definition 20. A model $\underline{A} \models T$ is *homogeneous* if for any $\mathbf{a}, \mathbf{b} \in A^n$ that satisfy the same formulas, there is an automorphism $\alpha: A \rightarrow A$ such that $\alpha(a_i) = b_i$.

Definition 21. A model $A \models T$ is *prime* if for any model $B \models T$, A embeds elementarily to B .

Proposition 22. *Countable* atomic models are isomorphic. In fact, every finite partial isomorphism can be extended to an isomorphism. They are also prime and homogeneous.

Definition 23 (Type).