Topology and Groups Notes

Giannis Tyrovolas

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1 Simplicial Approximation Theorem

The simplicial approximation theorem has lots of steps. Which is why I'm going to lay it all out with my comments here.

First, we'll define the star of a point in a simplicial complex.

Definition 1.1 (Star)**.** Let $x \in |K|$ then the star of x in $|K|$ is the union of all open simplices that contain x, i.e:

$$st_K(x) = \bigcup \{\text{inside of } \sigma | \sigma \text{ is a simplex of } |K|, x \in \sigma\}$$

Lemma 1.2. The star of a point is open in the topological realisation of the simplex.

Note: We can't prove this as a union of open sets as the inside of a simplex need not be open.

Now we will use simplicial maps to induce maps which are homotopic to continuous maps.

Theorem 1.3. *Let $f\colon |K| \longrightarrow |L|$ continuous. Suppose that for all $v \in V(K)$ there is a vertext $g(v) \in L$ such that:*

$$f(st_K(v)) \subseteq st_L(g(v))$$

Then $g\colon K \longrightarrow L$ is a simplicial map and for a suitable $|g|\colon |K| \longrightarrow |L|$, $|g| \simeq f$

Proof. Let $x \in K$ be inside $\sigma = (v_0, \dots, v_n)$ with

$$\begin{aligned} & x = \sum \lambda_i v_i \\ \implies \quad & x \in st_K(v_i) \text{ for each } i = 0, \dots, n \\ \implies \quad & f(x) \in st_L(g(v_i)) \text{ by hypothesis} \\ \implies \quad & f(x) \in \text{ inside of } (g(v_0), \dots, g(v_n), \dots) \end{aligned}$$

but $(g(v_0), \dots, g(v_n))$ is itself a simplex of L. Now define $|g|(x) = \sum \lambda_i g(v_i)$

We claim that $|g|$ is continuous and homotopic to f by the straight line homotopy. □

This is the bulk of the theorem, but we need some more ingredients.

Definition 1.4 (Standard metric)**.** The standard metric on a simplicial complex $|K|$ is defined as:

$$d(\sum \lambda_i v_i, \sum \mu_i v_i) = \sum |\lambda_i - \mu_i|$$

Definition 1.5 (Coarsness)**.** The coarsness of a subdivision K' of K is:

$$\sup\{d_K(x, y)\colon x, y \in st_{K'} | v \in V(K')\}$$

The coarsness is a measure of the biggest simplex in the subdivision with respect to the distance in the *original* simplex.

We claim that every finite simplicial complex can be subvided in arbitrarily small subvisions.

Lemma 1.6 (Legesgue Covering Lemma)**.** For a compact topological metric space X and any open cover $\mathcal{U}$ there exists a $\delta > 0$ such that any subset with diameter less than δ is entirely contained in an element of $\mathcal{U}$.