CSC349A Numerical Analysis

Lecture 14

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Introduction

Our next topic is the study of how a given function can be approximated by another function from a specified class of functions. The given function may be discrete or continuous. Typically the approximating function exhibits some desired properties such as:

1. Continuity
2. Easily differentiated
3. Easily integrated
4. Easily evaluated

Introduction II

Common classes of approximating functions:

1. Polynomials
2. Piecewise polynomials (splines)
3. Trigonometric sums (fourier series)

We will also study criteria for what constitutes a “good” approximating function.

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Polynomial interpolation

Recall that the general formula for an nth-order polynomial is

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

For $n + 1$ distinct data points there is one and only one order n (or less) polynomial that passes through them all. That is,

- only one line that passes through two points
- only one parabola that passes through three points, etc

Polynomial interpolation II

Polynomial Interpolation consists of determining the unique nth-order polynomial that fits the $n+1$ data points in question.

- Although the polynomial is unique there are different methods for finding it and different formats for expressing it.

Polynomial interpolation III

Formally: Let $y = f(x)$ be any given function. For any value of $n \geq 0$ and any given values $x_0, x_1, \ldots, x_n$, let $y_i = f(x_i)$. The **polynomial interpolation problem** is to determine a polynomial $P(x)$ of degree less than or equal to n for which:

$$P(x_i) = y_i \quad \text{for } i = 0, 1, \ldots, n$$

- The set of $n + 1$ data point (x_i, y_i) may be the only functional values known (that is, $f(x)$ is a **discrete function**, which could occur for example with experimental data), or
- $f(x)$ maybe be a known **continuous function**, and the $n + 1$ data points (x_i, y_i) are a finite set of values with $y_i = f(x_i)$ (samples).

Terminology

- If z is some value between 2 of the given values x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation is said to be determined by polynomial **interpolation**.

- On the other hand, if z lies outside of the interval containing all of the values x_i and if $P(z)$ is computed as an approximation to $f(z)$, then this approximation is said to be determined by polynomial **extrapolation**.

Polynomial interpolation vs. Taylor approximation

- An **interpolating polynomial** and the **Taylor polynomial** both determine polynomial approximations to $f(x)$. However, in general they are very different approximations to $f(x)$.
- An interpolating polynomial uses the information:

$$y_0 = f(x_0), y_1 = f(x_1), \ldots, y_n = f(x_n)$$

- A Taylor polynomial uses the information:

$$f(x_0), f'(x_0), \ldots, f^{(n)}(x_0)$$

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Lagrange Interpolating Polynomial

Given $(x_i, f(x_i)), 0 \leq i \leq n$, with all x_i distinct, consider the function:

$$\begin{aligned} P(x) &= \sum_{i=0}^{n} L_i(x) f(x_i) \\ &= L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots + L_n(x)f(x_n) \end{aligned}$$

where

$$\begin{aligned} L_i(x) &= \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} \\ &= \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, 2, \ldots, n \end{aligned}$$

Example 1

Derive the general, order $n = 1$, Lagrange interpolating polynomial.

Example 2

Derive the general, order $n = 2$, Lagrange interpolating polynomial.

Example 2 continued

Lagrange Interpolating Polynomial II

Since each function $L_i(x)$ is a polynomial of order n and $f(x_i)$ is a constant, $P(x)$ is a polynomial of order $\leq n$.
Also, since

$$L_i(x_i) = 1 \text{ and } L_i(x_j) = 0 \text{ if } j \neq i,$$

it follows that:

$$P(x_i) = f(x_i), \quad \text{for } i = 0, 1, 2, \ldots, n$$

that is, $P(x)$ is an interpolating polynomial for the given data.
It is called the **Lagrange interpolating polynomial**.

Example 3

Evaluate ln(2) using Lagrange polynomial interpolation, given that

$$\ln 1 = 0$$
$$\ln 4 = 1.386294$$
$$\ln 6 = 1.791760$$

Example 3 continued

Example 4

A complete elliptic integral function of the first kind is defined by

$$K(k) = \int_0^{\pi/2} \frac{dz}{\sqrt{1 - k^2 \sin^2 z}}$$

Interpolate $K(\sin 65.5^\circ)$ where

$\sin^{-1} k$	$K(k)$
65°	2.3088
66°	2.3439
67°	2.3809

Example 4 continued