CSC349A Numerical Analysis

Lecture 6

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Truncation Errors

Truncation errors occur when some exact mathematical procedure is replaced by a finite approximation. Examples:

- Approximation of a function by a finite number of terms:

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

- Approximation of a derivative by a finite difference:

$$\frac{dv}{dt}\Big|_{t=t_i} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

- Approximation of a definite integral by a finite sum:

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Taylor's theorem

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Taylor's Theorem is the fundamental tool for deriving and analyzing numerical approximation formulas in this course.

- It states that any "smooth" function (one with a sufficient number of derivatives) can be approximated by a polynomial, and it includes an error (remainder) term that indicates how accurate the polynomial approximation is.
- Taylor's theorem also provides a means to estimate the value of a function $f(x)$ at some point x_{i+1} using the values of $f(x)$ and its derivatives at some nearby point x_i.

Taylor's theorem

Theorem

Let $n \geq 0$ and let a be any constant. If $f(x)$ and its first $n+1$ derivatives are continuous on interval containing x and a, then:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_n$$

where R_n is the remainder term.

Taylor polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

is a polynomial of degree n in x, and is called the **Taylor polynomial approximation of degree** n for f(x) expanded about a.

Truncation error

The remainder term, R_n, is the **truncation error** of the Taylor polynomial approximation to $f(x)$.

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

where ξ is some value between x and a.

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Example 1

(a) Determine the Taylor polynomial approximation of order $n = 3$ for $f(x) = \ln(x + 1)$ expanded about $a = 0$.

Example 1 continued

(b) Using (a) approximate $\ln(1.25)$.

Example 1 continued

(c) Determine a good upper bound on the error of the approximation in (b).

Truncation error on interval

Rather than just using the Taylor polynomial approximation to estimate the value of a function at one specified point, it is more common to use the polynomial approximation **for an entire interval of values x**. In such a case, it is also desirable to be able to determine the accuracy (that is an upper bound for the error).

Example 1 continued

(d) Determine a good upper bound on the error of approximating $\ln(x+1)$ for any x in the interval $[0, 0.5]$.

The remainder term

Consider the remainder term when $n = 0$, expanded about some a. Then, $f(x) \approx f(a)$ with error term $R_0 = f'(\xi)(x - a)$ for some ξ between x and a.

The Mean-Value Theorem

Definition

Let f be continuous on interval $[a, b]$ and differentiable on (a, b). Then, there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Take another look at Taylor

What are we doing here?

- We know $f(a)$, $f'(a)$,$f''(a)$, etc.
- We approxiamte some neighbouring $f(x)$.
- That is, if we know the value of a function and its derivatives at some point, we can predict its value at some other (nearby) point.
- What does this describe?

An iterative process for approximating a function

Let $x_{i+1} = x_i + h$ so that $h = x_{i+1} - x_i$. Then Taylor's theorem for $f(x)$ expanded about x_i, and evaluated at $x = x_{i+1}$ is:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f'''(x_i)}{3!}h^3 + \cdots + \frac{f^n(x_i)}{n!}h^n + R_n$$

Example 2

Use the alternative form of Taylor's theorem, with $n = 1$, to derive an approximation of $f'(x_i)$ with an associated error term.

Approximating first derivative

This gives the first derivative approximation

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

that was used in Chapter 1; it also gives the trunaction error of this finite difference approximation to the derivative, namely $-\frac{R_1}{h} = -\frac{f''(\xi)}{2}h$. As this is some constant times h, we say that this truncation error is $O(h)$.

Example 3

The Taylor polynomial approximation for $f(x) = e^x$ expanded about $a = 0$ is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

With remainder term,

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Example 3

The truncation error of any Taylor polynomial approximation is small when x is close to a (note that $R_n = 0$ when $x = a$) and will increase as x gets further away from a. Also, as n increases, the Taylor polynomial approximations become better and better approximations to $f(x)$, provided of course that $f^{(n+1)}(x)$ is bounded on some interval containing x and a.

Alternative form of remainder

Alternative form for the remainder R_n which can also be defined as:

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt \qquad (1)$$

Derivation

The derivation is based on the first theorem of mean for integrals which states that if a function f is continuous and integrable on an interval containing α and x, then there exists a point ξ between α and x such that:

$$\int_{\alpha}^{x} g(t)dt = g(\xi)(x - \alpha) \tag{2}$$