CSC349A Numerical Analysis

Lecture 9

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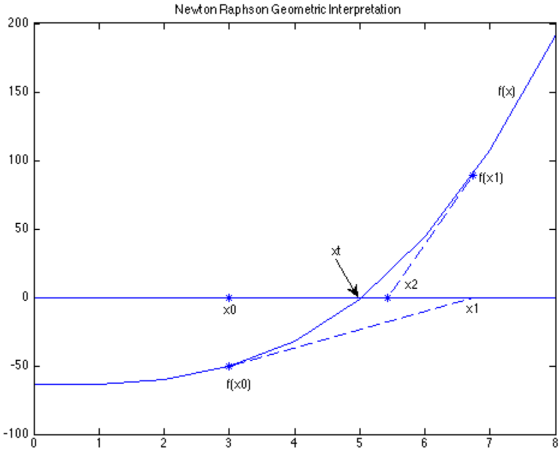
Geometric derivation

- The real roots of a function $f(x)$ occur when the graph of the function intersects with the x-axis.
- The main idea behind the Newton/Raphson method for root finding is given an initial approximation x_0 to a zero of $f(x)$ to approximate the graph of f(x) at x_0 by the tangent line - essentially linearizing the function in that area.

An illustrative example



Newton-Raphson Formula

Each iteration we approximate the root x_t with x_{i+1} based on previous approximation x_i using,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{1}$$

with the hope that

$$\lim_{i \to \infty} x_i = x_t \tag{2}$$

Example 1

Estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$. Note that the true root is $x_t = 0.56714329$.

Example 1 continued

Estimate the root of $f(x) = e^{-x} - x$ employing an initial guess of $x_0 = 0$. The iterative equation can be applied to compute:

i	x_i	$\varepsilon_t(\%)$
0	0	100
1	0.5	11.8
2	0.566311003	0.147
3	0.567143165	0.0000220
4	0.567143290	$< 10^{-8}$

Notice that the approach rapidly converges on the true root much faster than it would using *Bisection*.

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Derivation using Taylor's theorem

Recall the Taylor theorem for $f(x)$ with $n = 1$ expanded about $a = x_i$:

$$f(x) = f(x_i) + f'(x_i)(x - x_i) + \frac{f''(\xi)}{2}(x - x_i)^2 \qquad (3)$$

for some value ξ between x and x_i.

Convergence

The derivation of the Newton/Raphson method gives insight into how fast Newton's method converges:
First we evaluate the Taylor Theorem at $x = x_t$, an exact zero:

$$0 = f(x_t) = f(x_i) + (x_t - x_i)f'(x_i) + \frac{(x_t - x_i)^2}{2} f''(\xi) \quad (4)$$

Newton's method $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ can be rewritten as:

$$0 = f(x_i) + (x_{i+1} - x_i)f'(x_i) \quad (5)$$

Convergence II

If we subtract the last two equations (4,5) then we get:

$$0 = (x_t - x_{i+1})f'(x_i) + \frac{(x_t - x_i)^2}{2} f''(\xi) \tag{6}$$

and if we let $E_{i+1} = x_t - x_{i+1}$ and $E_i = x_t - x_i$ denote the error in x_{i+1}, x_i then we have:

$$0 = E_{i+1} f'(x_i) + \frac{E_i^2}{2} f''(\xi) \quad \text{thus} \quad \frac{E_{i+1}}{E_i^2} = \frac{-f''(\xi)}{2f'(x_i)}$$

Order of convergence

Definition

(not in textbook)

If a sequence $x_0, x_1, x_2, x_3, \ldots$ converges to x_t that is $\lim_{i\to\infty} x_i = x_t$ and $E_i = x_t - x_i$, then the order of converge of the sequence is α if there are constants $\lambda > 0$ and $\alpha \geq 1$ such that:

$$\lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^{\alpha}} = \lambda \tag{7}$$

In general, λ and α depend on the algorithm used to compute x_i, on $f(x)$, and on the multiplicity of the zero x_t.

Most common case:

$\alpha = 1$ linear convergence

For large i, $|E_{i+1}| \approx \lambda |E_i|$

In this case, successive errors decrease approximately by a constant amount:

$$\begin{aligned} |E_{i+1}| &\approx \lambda |E_i| \\ |E_{i+2}| &\approx \lambda |E_{i+1}| \approx \lambda^2 |E_i| \\ |E_{i+3}| &\approx \lambda |E_{i+2}| \approx \lambda^3 |E_i| \\ \text{etc} & \end{aligned}$$

Errors $|E_{i+1}| \to 0$, that is $\lim_{i\to\infty} x_i = x_t$ only if $0 < \lambda < 1$.

Quadratic convergence

For $\alpha = 2$ we have quadratic convergence. For large i,
$|E_{i+1}| \approx \lambda |E_i|^2$
After some error $|E_i| < 1$, convergence is rapid as the number of correct significant digits approximately doubles with each iteration e.g if $|E_i| = 10^{-t}$, then $|E_{i+1}| \approx \lambda 10^{-2t}$.

Convergence of Newton's method

For Newton's method above:

$$\frac{E_{i+1}}{E_i^2} = \frac{-f''(\xi)}{2f'(x_i)}$$

for some ξ between x_i and x_{i+1}.

$$\lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^2} = \lim_{i\to\infty} \frac{|f''(\xi)|}{2|f'(x_i)|} = \frac{|f''(x_t)|}{2|f'(x_t)|}$$

which is a constant λ provided that $f'(x_t) \neq 0$.
Result: Newton's method converges quadratically to a zero x_t provided that $f'(x_t) \neq 0$

Implementation

function root = Newton(x_0, ε, `imax`, $f(x)$, $f'(x)$)
$i \leftarrow 1$
output heading
while $i \leq$ `imax`
    $root \leftarrow x_0 - f(x_0)/f'(x_0)$
    output i, root
    if $|1 - x_0/root| < \varepsilon$
        exit
    end if
    $i \leftarrow i + 1$
    $x_0 \leftarrow root$
end while
output "failed to converge"

Implementation Note

Note that in general, using $|f(root)| < \varepsilon$ is not a suitable test for convergence (instead of testing approximation error). The reason is that $|f(root)| < \varepsilon$ does not imply that the value *root* is within distance of ε of an exact root x_t.

Newton Convergence

Theorem: If Newton's method is applied to $f(x) = 0$ producing a sequence x_i that converges to a root x_t, and if $f'(x_t) \neq 0$, then then order of convergence is 2.

- If $f'(x_t) = 0$ and Newton's method convergences to a root x_t, then we will see later that the order of convergence is NOT quadratic.

Example 1

An illustration of the quadratic convergence of Newton's method. Here $f(x) = cos(x) - x$. This was computed in MATLAB, so at most 16 correct digits are possible. The bold digits are all correct.

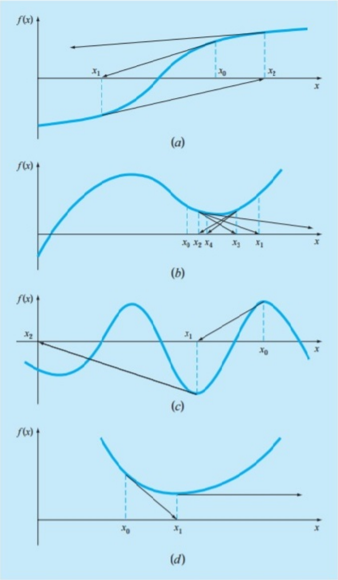
i	x_i	no. of correct digits
0	$\frac{\pi}{4} = 0.\mathbf{7}85398$	1
1	$0.\mathbf{739}5361$	3
2	$0.\mathbf{7390851}78$	7
3	$0.\mathbf{73908513321516}10$	14
4	$0.\mathbf{7390851332151606}$	16

Example 2 (Newton)

Root of $x^3 + 4x^2 - 10 = 0$ with $p_0 = -100$.

$p_0 = -100$	
$p_1 = -67.12$	
$p_2 = -45.21$	
$\dots$	$\dots$
$p_{14} = -2.54$	
$p_{15} = -3.14$	
$p_{16} = -2.80$	
$\dots$	$\dots$
$p_{21} = 1.9405$	no. of correct digits
$p_{22} = \mathbf{1}.4793$	1
$p_{23} = \mathbf{1.3}711$	2
$p_{24} = \mathbf{1.365}25$	4
$p_{25} = \mathbf{1.36523000}11$	9

Four Cases of Poor Convergence



Newton Convergence

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Theorem

Suppose that $f(x), f'(x)$ and $f''(x)$ all exist and are continuous on some interval $[a, b]$, that $x_t \in [a, b]$ is a root of $f(x) = 0$, and that $f'(x_t) \neq 0$. Then there exists a value $\delta > 0$, such that Newton's method converges for all initial approximations $x_0 \in [x_t - \delta, x_t + \delta]$.

Note

In general there is no way to determine such a value δ. This theorem only says that for all such functions $f(x)$, such a value δ exists. Even if the value of δ is extremely small, there is an interval of values around the root x_t such that if x_0 (the initial approximation) lies in this interval, then Newton's method will converge.
Thus the interpretation of the above theorem is that Newton's method always converges if the initial approximation x_0 is sufficiently close to the root x_t.