These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provide as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts. Many thanks to all the guidance and materials I received from Dale Olesky who has taught this course for many years and George Tzanetakis.

1 Barycentric Lagrange Interpolation

Lagrange can suffer from round-off error instability due to the number of computations, but there is a form of Lagrange where we can reduce the number of computations significantly. Recall, given $(x_i, f(x_i)), 0 \leq i \leq n$,

$$P(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where

$$L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, 2, \ldots, n$$

Note,

$$\begin{aligned}
L_i(x) &= \prod_{j=0,j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \\
&= \prod_{j=0,j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \cdot \frac{x - x_i}{x - x_i} \\
&= \prod_{j=0}^{n} (x - x_j) \cdot \prod_{j=0,j\neq i}^{n} \frac{1}{x_i - x_j} \cdot \frac{1}{x - x_i} \\
&= \prod_{j=0}^{n} (x - x_j) \cdot \frac{1}{\prod_{j=0,j\neq i}^{n} (x_i - x_j)} \cdot \frac{1}{x - x_i}
\end{aligned}$$

Let $L(x) = \prod_{j=0}^{n}(x - x_j)$ and $w_i = \frac{1}{\prod_{j=0, j\neq i}^{n}(x_i - x_j)}$, for each $i = 0, 1, 2, \ldots, n$, then

$$L_i(x) = L(x)\frac{w_i}{x - x_i}, \quad \text{for } i = 0, 1, 2, \ldots, n$$

and

$$P(x) = L(x)\sum_{i=0}^{n}\frac{w_i}{x - x_i}f(x_i)$$

Note, if $x = x_i$, then $P(x_i) = f(x_i)$ so there is no need to calculate. Just test for x_i in computation.

1.1 Example 1

Evaluate ln(2) using Lagrange polynomial interpolation, given that

$$\begin{aligned}
\ln 1 &= 0 \\
\ln 4 &= 1.386294 \\
\ln 6 &= 1.791760
\end{aligned}$$

Here, $x_0 = 1, x_1 = 4, x_2 = 6$ and $f(x_0) = 0, f(x_1) = 1.386294, f(x_2) = 1.791760$, (Note: this is example 3 from previous lecture again). We calculate $L(x)$ and each w_i and then construct $P(x)$.

$$\begin{aligned}
L(x) &= (x - x_0)(x - x_1)(x - x_2) = (x - 1)(x - 4)(x - 6) \\
w_0 &= \frac{1}{(x_0 - x_1)(x_0 - x_2)} = \frac{1}{(1 - 4)(1 - 6)} = \frac{1}{15} \\
w_1 &= \frac{1}{(x_1 - x_0)(x_1 - x_2)} = \frac{1}{(4 - 1)(4 - 6)} = -\frac{1}{6} \\
w_2 &= \frac{1}{(x_2 - x_0)(x_2 - x_1)} = \frac{1}{(6 - 1)(6 - 4)} = \frac{1}{10}
\end{aligned}$$

So,

$$
\begin{aligned}
P(x) &= L(x)\sum_{i=0}^{n}\frac{w_i}{x-x_i}f(x_i) \\
&= (x-1)(x-4)(x-6)\left[\frac{\frac{1}{15}(0)}{x-1}+\frac{-\frac{1}{6}(1.386294)}{x-4}+\frac{\frac{1}{10}(1.79176)}{x-6}\right] \\
&= (x-1)(x-4)(x-6)\left[\frac{-0.231049}{x-4}+\frac{0.179176}{x-6}\right]
\end{aligned}
$$

and thus,

$$
\begin{aligned}
P(2) &= (2-1)(2-4)(2-6)\left[\frac{-0.231049}{2-4}+\frac{0.179176}{2-6}\right] \\
&= (1)(-2)(-4)\left[\frac{-0.231049}{-2}+\frac{0.179176}{-4}\right] \\
&= 0.565844
\end{aligned}
$$

2 Finding the coefficients of the interpolating polynomial

Although the Langrange polynomial is well-suited to solving intermediate values it does not give you a polynomial in simple form

$$P(x) = a_0 + a_1x + \cdots + a_nx^n$$

The coefficients of such an interpolating polynomial can be determined by solving a system of linear equations.

Given a function $f(x)$ and distinct points $x_0, x_1, ..., x_n$, let $P(x)$ be the polynomial of degree $\leq n$ for which $P(x_i) = f(x_i)$ for $i = 0, 1, ..., n$.

Then,

$$
\begin{aligned}
a_0 + a_1x_0 + \cdots + a_nx_0^n &= f(x_0) \\
a_0 + a_1x_1 + \cdots + a_nx_1^n &= f(x_1) \\
&\vdots \\
a_0 + a_1x_n + \cdots + a_nx_n^n &= f(x_n)
\end{aligned}
$$

In matrix form, solve

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

So, if $n = 2$, we let $P(x) = a_0 + a_1 x + a_2 x^2$ and solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

2.1 Example 1

Let $f(x) = \sin x$, $x_0 = 0.2$, $x_1 = 0.5$, and $x_2 = 1$ and find the interpolating polynomial. Here, we want to solve the following system,

$$\begin{bmatrix} 1 & 0.2 & (0.2)^2 \\ 1 & 0.5 & (0.5)^2 \\ 1 & 1 & 1^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sin(0.2) \\ \sin(0.5) \\ \sin(1) \end{bmatrix}$$

I solved this in MATLAB using the following commands,

```
>> A=[1 .2 (.2)^2; 1 .5 (.5)^2; 1 1 1]

A =

    1.0000    0.2000    0.0400
    1.0000    0.5000    0.2500
    1.0000    1.0000    1.0000

>> b=[sin(.2);sin(.5);sin(1)]

b =

    0.1987
    0.4794
    0.8415
```

```
>> X=linsolve(A,b)

X =

   -0.0150
    1.1211
   -0.2647
```

Note, I get the same result with

```
>> X=A\b

X =

   -0.0150
    1.1211
   -0.2647
```

This implies that the interpolating polynomial is

$$P(x) = -0.0150 + 1.1211x - 0.2647x^2$$

3 Uniqueness

An interpolating polynomial can be specified in many different forms. For example the form can be $a(x - x_2)^2 + b(x - x_2) + c$ or using the Lagrange form for $n = 2$:

$$\begin{aligned} P(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}f(x_2) \end{aligned}$$

or simply as $P(x) = Ax^2 + Bx + C$.

We will show that all of these forms are identical as the interpolating polynomial is unique.

Theorem: Given any $n + 1$ distinct points $x_0, x_1, \ldots, x_n$ and any $n + 1$ values $f(x_0), f(x_1), \ldots, f(x_n)$, there exists a unique polynomial $P(x)$ of degree $\leq n$ such that

$$P(x_i) = f(x_i) \text{ for } 0 \leq i \leq n$$

Proof:
Existence: by construction of the Lagrange interpolating polynomial
Uniqueness:

Suppose there exist two polynomials $P(x)$ and $Q(x)$ of degree $\leq n$ such that:

$$P(x_i) = Q(x_i) = f(x_i), 0 \leq i \leq n$$

Consider the function

$$R(x) = P(x) - Q(x)$$

which is also a polynomial of degree $\leq n$. But $R(x_i) = 0$ for $0 \leq i \leq n$. That is $R(x)$ has $n+1$ distinct zeros. This implies that $R(x) = 0$ for all x and therefore $P(x) = Q(x)$.

4 Error term of polynomial interpolation

Theorem:
Let $x_0, x_1, \ldots x_n$ be any distinct points in $[a, b]$. Let $f(x) \in C^{n+1}[a, b]$ and let $P(x)$ interpolate $f(x)$ at x_i.

Then for each $\hat{x} \in [a, b]$, there exists a value ξ in (a, b) such that

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (\hat{x} - x_i)$$

for example for $n = 3$

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(4)}(\xi)}{24} (\hat{x} - x_0)(\hat{x} - x_1)(\hat{x} - x_2)(\hat{x} - x_3)$$

The limitation of this error bound for polynomial interpolation is the need to find an upper bound for $f^{(n+1)}(x)$ on $[a, b]$.

5 The Runge Phenomenon

The following example is the classical example to illustrate the oscillatory nature and thus the unsuitability of high order interpolating polynomials.

Example: Consider the problem of interpolating

$$f(x) = \frac{1}{1 + 25x^2}$$

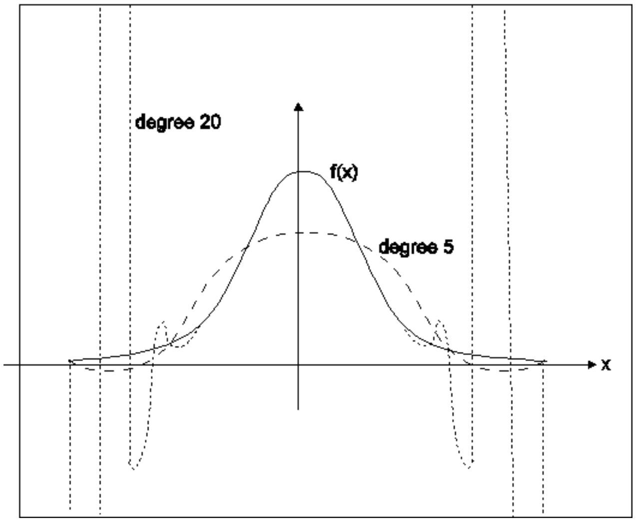

Figure 1: The Runge function

on the interval $[-1, 1]$ at $n + 1$ equally-spaced points x_i by the interpolating polynmial $P_n(x)$.

Runge's Theorem:

- Runge proved that as $n \to \infty$, $P_n(x)$ diverges from $f(x)$ for all values of x such that $0.726 \leq |x| < 1$ (except for the points of interpolation x_i).
- The interpolating polynomials do approximate $f(x)$ well for $|x| < 0.726$.
- One way to see that the difference between $f(x)$ and $P_n(x)$ becomes arbitrarily large as n becomes large is to consider the error term for polynomial interpolation

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

As $n \to \infty$, it can be shown that $f(x) - P_n(x) \to \infty$ (at some points x in $[-1, 1]$).