These are the lecture notes for CSC349A Numerical Analysis taught by Rich Little. They roughly correspond to the material covered in each lecture in the classroom but the actual classroom presentation might deviate significantly from them depending on the flow of the course delivery. They are provide as a reference to the instructor as well as supporting material for students who miss the lectures. They are simply notes to support the lecture so the text is not detailed and they are not thoroughly checked. Use at your own risk. They are complimentary to the handouts.

1 Secant method

The advantage of the Newton method is that it provides quadratic convergence. One disadvantage is that it requires knowledge of of the derivative $f'(x)$. In many applications the derivative might not be known or impossible to derive analytically through calculus. In this case it is possible to use a discrete approximation to the derivative. One such approximation is used in the *Secant* method. We can derive the *Secant* method starting from the update equation of the Newton/Raphson method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We can approximate $f'(x_i)$ by a finite divided difference. By definition:

$$f'(x_i) = \lim_{x \to x_i} \frac{f(x) - f(x_i)}{x - x_i}$$

using

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

gives:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Notice that the *Secant* method requires two initial approximations to the root x_0 and x_1.

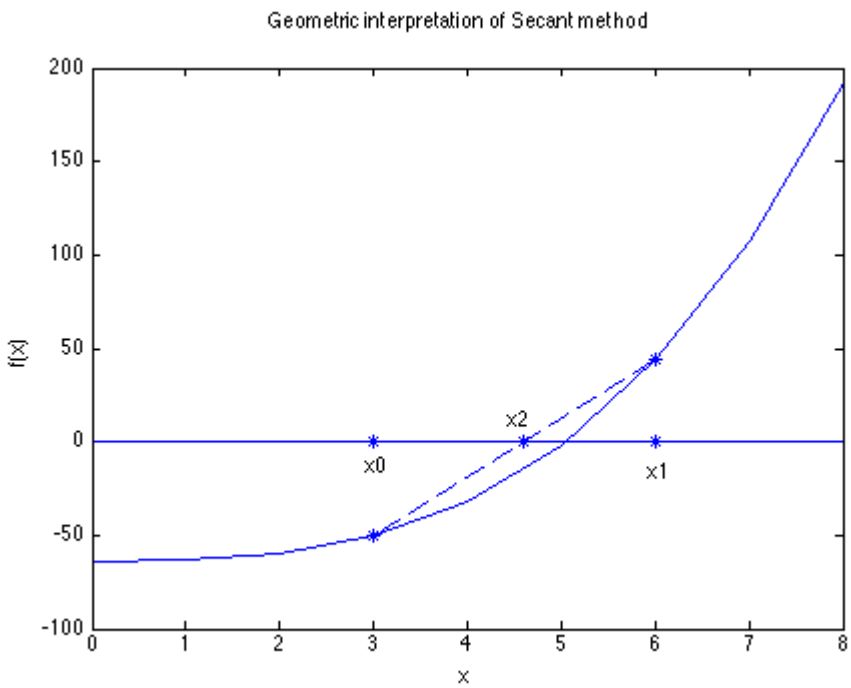

Figure 1: Geometric interpetation of the Secant method for root finding.

MATLAB Code for figure

```
x = 0:8;
fx = 0.5 * x.^3 -64;
% Plot the function
plot(x,fx);
hold on;

x0 = 3;
fx0 = 0.5 * x0.^3-64;
% Plot zero axis
plot(x, zeros(size(x)));
% plot x0 and f(x2) points
plot(x0,0,'*');
plot(x0,fx0, '*');
```

```
% plot x1 and f(x1) points
x1 = 6;
fx1 = 0.5 * x1.^3-64;
plot(x1,0,'*');
plot(x1,fx1, '*');

% plot secant line
plot([x0,x1],[fx0,fx1],'--');
x2 = x1 - fx1 *(x0-x1)/(fx0-fx1);
plot(x2,0, '*');
hold off;
```

1.1 Example of Secant Method

Estimate the root of $f(x) = e^{-x} - x$ employing initial guesses of $x_{-1} = 0$ and $x_0 = 1$. The iterative equation can be applied to compute:

i	x_i	$\varepsilon_t(\%)$
-1	0	100
0	1	76
1	0.61270	8.03
2	0.56384	0.58
3	0.56717	0.0048

Notice that the approach converges on the true root faster than *Bisection* but slower than *Newton*.

2 Order of convergence of Secant and Bisection

The order of convergence of the Secant method derives from the following limit,

$$\lim_{i\to\infty} \left| \frac{E_{i+1}}{E_i E_{i-1}} \right| = \left| \frac{f''(x_t)}{2f'(x_t)} \right| \tag{1}$$

This gives a relationship **between 3 successive errors**. However, this does indicate the **order** α of the Secant method, which requires that the errors of

2 successive approximations be related by

$$\lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^\alpha} = \lambda, \quad \text{for some constant } \lambda \tag{2}$$

It can be shown in fact that,

$$\lim_{i\to\infty} \left| \frac{E_{i+1}}{E_i E_{i-1}} \right| = \lim_{i\to\infty} \frac{|E_{i+1}|}{|E_i|^\alpha} = \left| \frac{f''(x_t)}{2f'(x_t)} \right| \tag{3}$$

where

$$\alpha = 1 + \frac{1}{\alpha} \implies \alpha^2 - \alpha - 1 = 0 \implies \alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$$

which is the **order of the Secant method**.
Note: this value α is known as the "golden ratio", and occurs in many places in nature as well as many diverse applications.

An alternate definition of **linear convergence** (similar to the definition given in class except that it does not involve a limit) is:

$$|E_i| \le c|E_{i-1}| \text{ or } |x_t - x_i| \le c|x_t - x_{i-1}|$$

for some constant c such that $0 < c < 1$.

Applying this inequality recursively gives

$$|x_t - x_i| \le c^i |x_t - x_0|$$

From page 2 of Handout Number 8, for the Bisection method we had:

$$|x_t - x_i| \le \left(\frac{1}{2}\right)^i \Delta x^0, \text{where } \Delta x^0 = x_u - x_l$$

and $[x_l, x_u]$ is the initial interval. This implies linear convergence with the above definition, and $c = \frac{1}{2}$.

3 Multiple Roots and the Multiplicity of a Zero

If Newton's method converges to a zero x_t of $f(x)$, a necessary condition for quadratic convergence is that $f'(x_t) \neq 0$. We now relate this condition on the derivative of $f(x)$ to the multiplicity of the zero x_t.

Theorem 1. *(not in textbook)*

If x_t is a zero of any analytic function $f(x)$, then there exists a positive integer m and a function $q(x)$ such that :

$$f(x) = (x - x_t)^m q(x), \quad \textit{where} \quad \lim_{x \to x_t} q(x) \neq 0$$

(In particular, if $q(x_t)$ is defined, note that $q(x_t) \neq 0$.) The value m is called the **multiplicity** of the zero x_t. If $m = 1$, then x_t is called a **simple zero** of $f(x)$.

Example 1 Consider

$$f(x) = x^4 = 9.5x^3 + 18x^2 - 56x - 160 = (x + 4)^3(x - 2.5)$$

The zero at $x_t = -4$ has $m = 3$ (here $q(x) = x - 2.5$ and $q(-4) \neq 0$).
The zero at $x = 2.5$ has $m = 1$ (here $q(x) = (x + 4)^3$ and $q(2.5) \neq 0$).

Example 2 Consider

$$f(x) = e^x - x - 1$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$. This zero has multiplicity m=2 since $f(x) = (x - 0)^2 q(x)$ with $q(x) = \frac{e^x - x - 1}{x^2}$.

Using l'Hospital's rule we have:

$$\lim_{x \to 0} q(x) = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = 0.5$$

Theorem 2. *(not in textbook)*

Suppose that $f(x)$ and $f'(x)$ are continuous on some interval $[a, b]$, and that $x_t \in (a, b)$ and $f(x_t) = 0$. Then x_t is a simple zero of $f(x)$ if and only if $f'(x_t) \neq 0$.

Proof. $\Rightarrow$ Suppose first that x_t is a simple zero of $f(x)$. Then

$$f(x) = (x - x_t)q(x), \text{where } \lim_{x \to x_t} q(x) \neq 0$$

Therefore,

$$f'(x) = q(x) + (x - x_t)q'(x)$$

and thus

$$f'(x_t) = q(x_t) \neq 0$$

$\Leftarrow$ For the converse, suppose that $f'(x_t) \neq 0$. Then by Taylor's Theorem expansion about $a = x_t$,

$$f(x) = f(x_t) + (x - x_t)f'(\xi) = (x - x_t)f'(\xi) \text{ since } f(x_t) = 0$$

for some value ξ between x and x_t. Thus $q(x) = f'(\xi)$ and $\lim_{x \to x_t} q(x) = \lim_{x \to x_t} f'(\xi) = f'(x_t) \neq 0$. Hence x_t is a simple zero (that is, the multiplicity is $m = 1$).

$\square$

The following result follows directly from the above Theorem and our previous result about the quadratic convergence of Newton's method.
Corrolary If Newton's method converges to a simple zero x_t of $f(x)$, then the order of convergence is 2.

In order to determine whether or not Newton's method converges quadratically to a zero x_t of $f(x)$, you only need to know whether the multiplicity of x_t is 1 or is ≥ 2. The following result is more general than the above Theorem, and enables you determine the exact multiplicity of a zero.

Theorem 3. *Suppose that $f(x)$ and its first m derivatives are continuous on some interval $[a, b]$ that contains a zero x_t of $f(x)$. Then the multiplicity of x_t is m if and only if $f(x_t) = f'(x_t) = f''(x_t) = \cdots = f^{(m-1)}(x_t) = 0$ but $f^{(m)}(x_t) \neq 0$.*

Example
Consider

$$f(x) = e^x - x - 1$$

Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 1$.
Since $f'(x) = e^x - 1$ and $f'(0) = 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m \geq 2$.
Since $f''(x) = e^x$ and $f''(0) \neq 0$, $x_t = 0$ is a zero of $f(x)$ with multiplicity $m = 2$.

Significance of multiplicity concerning root-finding algorithms

- Bracketing methods, such as the Bisection method, cannot be used to compute zeros of **even** multiplicity.
- Newton's method and the Secant method both converge only linearly (order of convergence is $\alpha = 1$) if the multiplicity m is ≥ 2.
- A quadratically convergent algorithm for computing a zero x_t of any (unknown) multiplicity of a function $f(x)$ is obtained by applying Newton's method to the new function.

$$u(x) = \frac{f(x)}{f'(x)}$$

rather than to $f(x)$. This is true since if $f(x) = (x - x_t)^m q(x)$ and $m \geq 2$, then

$$u(x) = \frac{f(x)}{f'(x)} = \frac{(x - x_t)q(x)}{mq(x) + (x - x_t)q'(x)}$$

has a simple zero ($m = 1$) at x_t. By evaluating $u'(x)$, this new algorithm can be written as:

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$

Example 5: Let $f(x) = (x - 3)(x - 1)^2 = x^3 - 5x^2 + 7x - 3$. Let $x_0 = 0$ and solve with Newton's method and the modified Newton's method. Compare. Do it again for $x_0 = 4$.

In MATLAB,

```
>> Newton(0,1e-8,30,@Ex5,@Ex5Prime)
 iteration approximation
      0 0.0000000000000000
      1 0.4285714285714286
      2 0.6857142857142857
      3 0.8328654004954585
      4 0.9133298932566344
      5 0.9557832929657391
```

```
      6 0.9776551012729385
      7 0.9887661675236611
      8 0.9943674406865725
      9 0.9971797713100611
     10 0.9985888935423050
     11 0.9992941981316302
     12 0.9996470368288750
     13 0.9998235028456799
     14 0.9999117475293338
     15 0.9999558727910671
     16 0.9999779361539637
     17 0.9999889680091290
     18 0.9999944840012569
     19 0.9999972419768860
     20 0.9999986210059997
     21 0.9999993105703031
     22 0.9999996551849977
     23 0.9999998274423416
     24 0.9999999123701399
     25 0.9999999554463016
     26 0.9999999753813109
     27 0.9999999844006620

ans =

    1.0000

>> NewtonMod(0,1e-8,30,@Ex5,@Ex5Prime,@Ex5PrimePrime)
 iteration approximation
      0 0.0000000000000000
      1 1.1052631578947369
      2 1.0030816640986033
      3 1.0000023814938155
      4 1.0000000000373122
      5 1.0000000000746248

ans =
```

```
    1.0000

>> Newton(4,1e-8,30,@Ex5,@Ex5Prime)
 iteration approximation
      0 4.0000000000000000
      1 3.3999999999999999
      2 3.0999999999999996
      3 3.0086956521739134
      4 3.0000746407911918
      5 3.0000000055706231
      6 3.0000000000000000

ans =

     3

>> NewtonMod(4,1e-8,30,'Ex5','Ex5Prime','Ex5PrimePrime')
 iteration approximation
      0 4.0000000000000000
      1 2.6363636363636367
      2 2.8202247191011240
      3 2.9617282104948424
      4 2.9984787191881508
      5 2.9999976821826633
      6 2.9999999999946287
      7 2.9999999999999996

ans =

    3.0000
```