CSC349A Numerical Analysis

Lecture 15

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Barycentric Lagrange Interpolation

Lagrange can suffer from round-off error instability due to the number of computations, but there is a form of Lagrange where we can reduce the number of computations significantly. Recall, given $(x_i, f(x_i)), 0 \leq i \leq n$,

$$P(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where

$$L_i(x) = \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } i = 0, 1, 2, \ldots, n$$

Barycentric Lagrange Interpolation II

Note,

$$
\begin{aligned}
L_i(x) &= \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \\
&= \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} \cdot \frac{x - x_i}{x - x_i} \\
&= \prod_{j=0}^{n} (x - x_j) \cdot \prod_{j=0, j\neq i}^{n} \frac{1}{x_i - x_j} \cdot \frac{1}{x - x_i} \\
&= \prod_{j=0}^{n} (x - x_j) \cdot \frac{1}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)} \cdot \frac{1}{x - x_i}
\end{aligned}
$$

Barycentric Lagrange Interpolation III

Let $L(x) = \prod_{j=0}^{n}(x - x_j)$ and $w_i = \frac{1}{\prod_{j=0,j\neq i}^{n}(x_i - x_j)}$, for each $i = 0, 1, 2, \ldots, n$, then

$$L_i(x) = L(x)\frac{w_i}{x - x_i}, \quad \text{for } i = 0, 1, 2, \ldots, n$$

and

$$P(x) = L(x)\sum_{i=0}^{n}\frac{w_i}{x - x_i}f(x_i)$$

Note, if $x = x_i$, then $P(x_i) = f(x_i)$ so there is no need to calculate. Just test for x_i in computation.

Example 1

Evaluate ln(2) using Lagrange polynomial interpolation, given that

$$\ln 1 = 0$$
$$\ln 4 = 1.386294$$
$$\ln 6 = 1.791760$$

Example 1 continued

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Finding the coefficients of the interpolating polynomial

Although the Langrange polynomial is well-suited to solving intermediate values it does not give you a polynomial in simple form

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n$$

The coefficients of such an interpolating polynomial can be determined by solving a system of linear equations.

Finding the coefficients of the interpolating polynomial

Given a function $f(x)$ and distinct points $x_0, x_1, \ldots, x_n$, let $P(x)$ be the polynomial of degree $\leq n$ for which $P(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$.
Then,

$$
\begin{aligned}
a_0 + a_1 x_0 + \cdots + a_n x_0^n &= f(x_0) \\
a_0 + a_1 x_1 + \cdots + a_n x_1^n &= f(x_1) \\
&\vdots \\
a_0 + a_1 x_n + \cdots + a_n x_n^n &= f(x_n)
\end{aligned}
$$

Finding the coefficients of the interpolating polynomial

In matrix form, solve

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \dots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix}$$

So, if $n = 2$, we let $P(x) = a_0 + a_1 x + a_2 x^2$ and solve

$$\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \end{bmatrix}$$

Example 2

Let $f(x) = \sin x$, $x_0 = 0.2$, $x_1 = 0.5$, and $x_2 = 1$ and find the interpolating polynomial.

Note

An interpolating polynomial can be specified in many different forms.

1. For example the form can be $a(x-x_2)^2 + b(x-x_2) + c$ or
2. using the Lagrange form for $n=2$:

$$\begin{aligned} P(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) \\ &+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2) \end{aligned}$$

   or

3. simply as $P(x) = Ax^2 + Bx + C$.

We will show that all of these forms are identical as the interpolating polynomial is unique.

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Uniqueness

Theorem: Given any $n+1$ distinct points $x_0, x_1, \ldots, x_n$ and any $n+1$ values $f(x_0), f(x_1), \ldots, f(x_n)$, there exists a unique polynomial $P(x)$ of degree $\leq n$ such that

$$P(x_i) = f(x_i) \text{ for } i = 0, 1, ..., n$$

Proof:
Existence: by construction of the Lagrange interpolating polynomial

Uniqueness proof

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Error term of polynomial interpolation

Theorem:
Let $x_0, x_1, \ldots x_n$ be any distinct points in $[a, b]$. Let $f(x) \in C^{n+1}[a, b]$ and let $P(x)$ interpolate $f(x)$ at x_i.
Then for each $\hat{x} \in [a, b]$, there exists a value ξ in (a, b) such that

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (\hat{x} - x_i)$$

for example for $n = 3$

$$f(\hat{x}) = P(\hat{x}) + \frac{f^{(4)}(\xi)}{24} (\hat{x} - x_0)(\hat{x} - x_1)(\hat{x} - x_2)(\hat{x} - x_3)$$

The limitation of this error bound for polynomial interpolation is the need to find an upper bound for $f^{(n+1)}(x)$ on $[a, b]$.

The Runge Phenomenon

The following example is the classical example to illustrate the oscillatory nature and thus the unsuitability of high order interpolating polynomials.
Example: Consider the problem of interpolating

$$f(x) = \frac{1}{1 + 25x^2}$$

on the interval $[-1, 1]$ at $n + 1$ equally-spaced points x_i by the interpolating polynomial $P_n(x)$.

Graphs of $f(x)$, $P_5(x)$, and $P_{20}(x)$

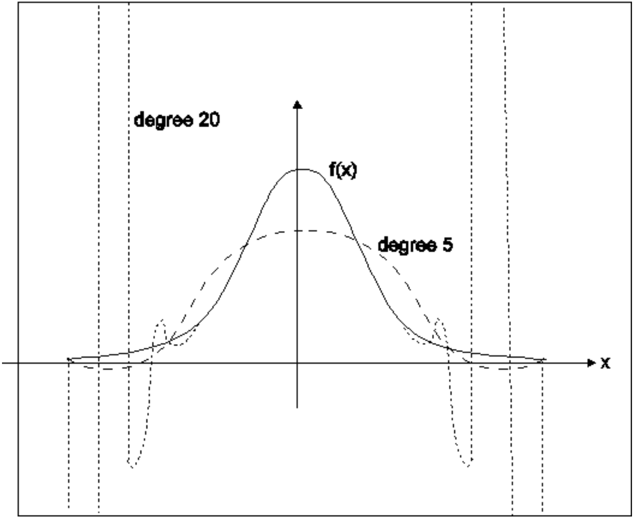

Figure: The Runge function

Runge's Theorem

- Runge proved that as $n \to \infty$, $P_n(x)$ diverges from $f(x)$ for all values of x such that $0.726 \leq |x| < 1$ (except for the points of interpolation x_i).
- The interpolating polynomials do approximate $f(x)$ well for $|x| < 0.726$.
- One way to see that the difference between $f(x)$ and $P_n(x)$ becomes arbitrarily large as n becomes large is to consider the error term for polynomial interpolation

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

As $n \to \infty$, it can be shown that $f(x) - P_n(x) \to \infty$ (at some points x in $[-1, 1]$).