# CSC349A Numerical Analysis

## Lecture 10

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# Introduction

- The advantage of the Newton method is that it provides quadratic convergence.
- One disadvantage is that it requires knowledge of the derivative $f'(x)$.
- In many applications the derivative might not be known or impossible to derive analytically through calculus.
- In this case it is possible to use a discrete approximation to the derivative. One such approximation is used in the *Secant* method.

# Secant derivation

We can derive the *Secant* method starting from the update equation of the Newton/Raphson method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

We can approximate $f'(x_i)$ by a finite divided difference:

$$f'(x_i) = \lim_{x \to x_i} \frac{f(x) - f(x_i)}{x - x_i}$$

using

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

gives:

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

# Secant Geometry

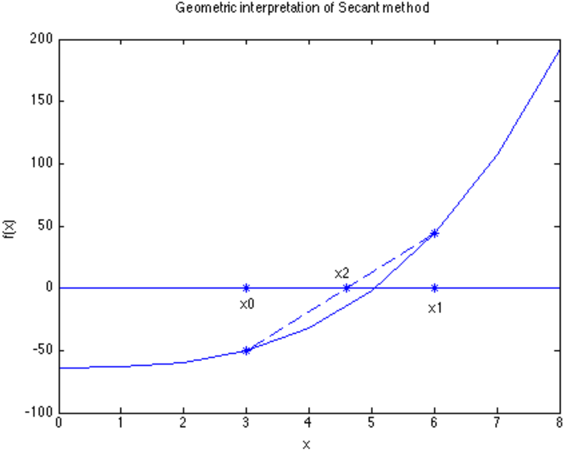


Figure: Geometric interpetation of the Secant method for root finding.

# Example of Secant Method

Estimate the root of $f(x) = e^{-x} - x$ employing initial guesses of $x_{-1} = 0$ and $x_0 = 1$. Recall $x_t = 0.56714329...$

# Example of Secant Method continued

Estimate the root of $f(x) = e^{-x} - x$ employing initial guesses of $x_{-1} = 0$ and $x_0 = 1$. The iterative equation can be applied to compute:

| $i$ | $x_i$ | $\varepsilon_t(\%)$ |
|---|---|---|
| $-1$ | 0 | 100 |
| 0 | 1 | 76 |
| 1 | 0.61270 | 8.03 |
| 2 | 0.56384 | 0.58 |
| 3 | 0.56717 | 0.0048 |

Notice that the approach converges on the true root faster than *Bisection* but slower than *Newton*.

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# Order of convergence of the Secant method

The order of convergence of the Secant method derives from the following limit,

$$\lim_{i\to\infty}\left|\frac{E_{i+1}}{E_i E_{i-1}}\right| = \left|\frac{f''(x_t)}{2f'(x_t)}\right| \tag{1}$$

This gives a relationship **between 3 successive errors**. However, this does indicate the **order** $\alpha$ of the Secant method, which requires that the errors of 2 successive approximations be related by

$$\lim_{i\to\infty}\frac{|E_{i+1}|}{|E_i|^{\alpha}} = \lambda, \quad \text{for some constant } \lambda \tag{2}$$

# Secant and Golden Ratio

It can be shown in fact that,

$$\lim_{i\to\infty}\left|\frac{E_{i+1}}{E_iE_{i-1}}\right| = \lim_{i\to\infty}\frac{|E_{i+1}|}{|E_i|^{\alpha}} = \left|\frac{f''(x_t)}{2f'(x_t)}\right| \tag{3}$$

where

$$\alpha = 1 + \frac{1}{\alpha} \implies \alpha^2 - \alpha - 1 = 0 \implies \alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$$

which is the **order of the Secant method**.
**Note:** this value $\alpha$ is known as the "golden ratio", and occurs in many places in nature as well as many diverse applications.

# Bisection convergence

An alternate definition of **linear convergence**:

$$|E_i| \leq c|E_{i-1}| \text{ or } |x_t - x_i| \leq c|x_t - x_{i-1}|$$

for some constant $c$ such that $0 < c < 1$.
Applying this inequality recursively gives

$$|x_t - x_i| \leq c^i|x_t - x_0|$$

For the Bisection method we had (Handout 8, pg. 2):

$$|x_t - x_i| \leq \left(\frac{1}{2}\right)^i \Delta x^0, \text{where } \Delta x^0 = x_u - x_l$$

and $[x_l, x_u]$ is the initial interval. This implies linear convergence with the above definition, and $c = \frac{1}{2}$.

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# Introduction

If Newton's method converges to a zero $x_t$ of $f(x)$, a necessary condition for quadratic convergence is that $f'(x_t) \neq 0$ . We now relate this condition on the derivative of $f(x)$ to the multiplicity of the zero $x_t$.

# Multiplicity

**Theorem**

*If $x_t$ is a zero of any analytic function $f(x)$, then there exists a positive integer $m$ and a function $q(x)$ such that :*

$$f(x) = (x - x_t)^m q(x), \quad \textit{where} \quad \lim_{x \to x_t} q(x) \neq 0$$

(In particular, if $q(x_t)$ is defined, note that $q(x_t) \neq 0$.) The value $m$ is called the **multiplicity** of the zero $x_t$. If $m = 1$, then $x_t$ is called a **simple zero** of $f(x)$.

# Example 1

Function
$f(x) = x^4 + 9.5x^3 + 18x^2 - 56x - 160 = (x+4)^3(x-2.5)$
has two zeroes, determine the multiplicity of each.

# Example 2

Let $f(x) = e^x - x - 1$. Since $f(0) = 0$, $x_t = 0$ is a zero of $f(x)$. What is it's multiplicity?

# Simple Zero Theorem

**Theorem**

*Suppose that $f(x)$ and $f'(x)$ are continuous on some interval $[a, b]$, and that $x_t \in (a, b)$ and $f(x_t) = 0$. Then $x_t$ is a simple zero of $f(x)$ if and only if $f'(x_t) \neq 0$.*

# Simple zero - Proof $\Rightarrow$

# Simple zero - Proof $\Leftarrow$

# Corrolary

The following result follows directly from the above Theorem and our previous result about the quadratic convergence of Newton's method.

**Corrolary**

If Newton's method converges to a simple zero $x_t$ of $f(x)$, then the order of convergence is 2.

In order to determine whether or not Newton's method converges quadratically to a zero $x_t$ of $f(x)$, you only need to know whether the multiplicity of $x_t$ is 1 or is $\geq 2$. The following result is more general than the above Theorem, and enables you to determine the exact multiplicity of a zero.

# Multiplicity and derivatives

**Theorem**

*Suppose that $f(x)$ and its first $m$ derivatives are continuous on some interval $[a, b]$ that contains a zero $x_t$ of $f(x)$. Then the multiplicity of $x_t$ is $m$ if and only if*
$f(x_t) = f'(x_t) = f''(x_t) = \cdots = f^{(m-1)}(x_t) = 0$ *but*
$f^{(m)}(x_t) \neq 0$.

# Example 3

Use the above theorem to show that $f(x) = e^x - x - 1$ has root $x_t = 0$ of multiplicity 2.

# Significance of multiplicity

- Bracketing methods, such as the Bisection method, cannot be used to compute zeros of **even** multiplicity.
- Newton's method and the Secant method both converge only linearly (order of convergence is $\alpha = 1$) if the multiplicity $m$ is $\geq 2$.

# Variant

A quadratically convergent algorithm for computing a zero $x_t$ of any (unknown) multiplicity of a function $f(x)$ is obtained by applying Newton's method to the new function.

$$u(x) = \frac{f(x)}{f'(x)}$$

rather than to $f(x)$. This is true since if $f(x) = (x - x_t)^m q(x)$ and $m \geq 2$, then

$$u(x) = \frac{f(x)}{f'(x)} = \frac{(x - x_t)q(x)}{mq(x) + (x - x_t)q'(x)}$$

has a simple zero $(m = 1)$ at $x_t$. By evaluating $u'(x)$, this new algorithm can be written as:

$$x_{i+1} = x_i - \frac{u(x_i)}{u'(x_i)} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f''(x_i)}$$