A Geometric Interpretation of the Instantaneous Frequency

Federico Milano

University College Dublin

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What is the Frequency?

We know thus that the frequency of a power system varies from point to point after a large perturbation.

At this point, we need to ask ourselves two fundamental questions:

- What is the frequency of a signal in transient conditions?
- How can the frequency be measured?

Intuitive Definition of Frequency

The intuitive definition of frequency is *the number of occurrences of a repeating event per unit of time*.

However, this definition is not adequate for transient phenomena unless one is able to define a "fraction" of an event.

Same conundrum is obtained if we define the (angular) frequency as the inverse of a period:

$$f = \frac{1}{T} \quad \text{or, equivalently,} \quad \omega = \frac{2\pi}{T}$$

If the signal is not periodic, it would seems to be impossible to define the frequency at all!

Definition of the IEEE Std. C37.118.1-2011

The IEEE Standard for Synchrophasor Measurements for Power Systems define the frequency of the voltage as follows.

Let the voltage signal be:

$$v(t) = V \cos\big(\theta(t)\big)$$

Then, the frequency is:

$$f(t) = \frac{1}{2\pi}\omega(t) = \frac{1}{2\pi}\,\frac{d\theta(t)}{dt}$$

Inconsistency of the Frequency Definition of the IEEE Standard - I

We have seen that the definition of frequency as the inverse of a period does not account for signals with time-varying frequencies.

But also the definition of the IEEE Std. C37.118.1-2011 has issues.

What is the frequency of the following signals?

$$v(t) = V_1 \cos\left(0.5\alpha(t)t^2\right) + \sum_{h=2}^{N} V_h \cos(h\omega(t)t + \theta_h)$$

$$v(t) = \sum_{h=1}^{N} V_h(t) \cos\left(\omega_h(t)\, t + \theta_h\right) + V_{\mathrm{dc}} \exp(-t/T_{\mathrm{dc}}) + \xi(t)$$

Inconsistency of the Frequency Definition of the IEEE Standard - II

Also the following simple signal leads to a somewhat unexpected result:

$$v(t) = V \cos\big(\theta(t)\big) = V \cos\big(\omega(t)t + \theta_o\big)$$

Then, the frequency is:

$$f(t) = \frac{1}{2\pi}\big(\omega(t) + \omega'(t)t\big)$$

One would probably have expected only the term $\omega(t)$.

In fact, in a previous version of the IEEE Std. C37.118, the term $\omega'(t)t$ was neglected!

More than one Definition of Frequency

There are, in effect two answers (both widely utilized in engineering applications) to the first question, namely *what is the frequency of a signal in transient conditions?*

The first is the concept of *Fourier frequencies*.

The other is the concept of *instantaneous frequency*.

Fourier Transform (FT) – I

Let us consider a scalar time-varying signal $v(t)$.

The FT (or *spectrum*) of the signal $v(t)$ is defined as:

$$\mathfrak{v}(\omega) = \mathcal{F}\left[v\right](\omega) = \frac{1}{\sqrt{2\pi}} \int v(t) e^{-\jmath\omega t} dt$$

where $\jmath$ is the imaginary unit and the integral has to be intended to be calculated in the range $t \in (-\infty, \infty)$.

The function $\mathfrak{v}(\omega)$ provides a representation of the signal $v(t)$ in the frequency domain or space.

Fourier Transform (FT) – II

The FT works the best, and in fact it was invented specifically for, stationary periodic signals.

For example, if $v(t) = V\cos(\omega_o t)$, one obtains:

$$\begin{aligned} \mathfrak{v}(\omega) &= \mathcal{F}\left[V\cos(\omega_o t)\right](\omega) \\ &= \sqrt{2\pi}\,\frac{V}{2}\left(\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right), \end{aligned}$$

which is a spectrum with non-null values only in $-\omega_o$ and ω_o, as the time-domain signal has only one frequency.

Fourier Transform (FT) – III

The application of the FT to power system analysis has found its natural field in the harmonic analysis (HA), i.e. the study of the effect of frequencies multiple of the fundamental one in stationary conditions.

In a HA, a signal can be represented as a sum of sinusoids and, possibly, a non-null constant term:

$$v(t) = \sum_h V_h \cos(h\omega_o t + \theta_h)\,, \quad h \in \{0, 1, \dots, n\}\,,$$

which can be conveniently studied in the frequency domain $\{0, \omega_o, \dots, h\omega_o, \dots, n\omega_o\}$ rather than in the time domain.

Fourier Transform (FT) – IV

Difficulties arise, however, when the signal is not periodic.

The spectrum becomes a continuum rather than a set of sharp frequency values.

Moreover, the integral of the FT has to be calculated for $t \in (-\infty, \infty)$, which is impractical for the vast majority of real-world applications.

Several *patches* have been proposed to overcome this issue, including a large variety of windowing techniques, more or less sophisticated, in order to compensate the inevitable approximations.

Instantaneous Frequency (IF) – I

For the definition of the IF, it is convenient to define a mathematical object called *analytic signal*.

This is a complex quantity, which is calculated from the signal $v(t)$ as follows:

$$\tilde{v}(t) = v(t) + \frac{\jmath}{\pi} \int \frac{v(r)}{t - r} dr \, .$$

The imaginary part of $\tilde{v}(t)$ is the Hilbert Transform (HT) of $v(t)$:

$$\mathcal{H}[v](t) = \hat{v}(t) = \frac{1}{\pi} \int \frac{v(r)}{t - r} dr$$

The analytic signal $\tilde{v}(t)$ can thus be written equivalently as:

$$\tilde{v}(t) = v(t) + \jmath \mathcal{H}[v](t) = v(t) + \jmath \hat{v}(t)$$

Instantaneous Frequency (IF) – II

The HT is often interpreted as a *rotation* of $-\pi/2$ of the signal to which it is applied.

This notion is justified from the fact that the HT of the sine and cosine functions are:

$$
\begin{aligned}
\mathcal{H}[\cos](\omega t) &= \sin(\omega t) \\
\mathcal{H}[\sin](\omega t) &= -\cos(\omega t)
\end{aligned}
$$

for $\omega > 0$ (analytic signals are defined only for $\omega > 0$).

The notion of "rotation" given by the HT can be formalized observing that:

$$
\mathcal{F}\left[\hat{v}\right](\omega) = -\jmath\,\mathcal{F}\left[v\right](\omega)\,, \qquad \omega > 0
$$

Instantaneous Frequency (IF) – III

For example, consider the following signal:

$$v(t) = V\cos(\theta(t))$$

Then the analytic signal is:

$$\tilde{v}(t) = v(t) + \jmath\hat{v}(t) = V[\cos(\theta(t)) + \jmath\sin(\theta(t))] = V\, e^{\jmath\,\theta(t)}$$

Note that is $\theta(t) = \omega_o t + \phi$, the analytic signal of $v(t)$ coincides with that of phasors (without the frequency shift).

Instantaneous Frequency (IF) – IV

An analytic signal is a complex quantity, hence:

$$\tilde{v}(t) = \mathrm{v}(t)\, e^{\jmath\,\phi(t)}$$

where

$$\mathrm{v}(t) = |\tilde{v}(t)| = \sqrt{v^2(t) + \hat{v}^2(t)}$$

$$\phi(t) = \angle\tilde{v}(t) = \arctan\left(\frac{\hat{v}(t)}{v(t)}\right)$$

Then, the IF is defined as:

$$\boxed{\phi'(t) = \frac{v(t)\hat{v}'(t) - \hat{v}(t)v'(t)}{\mathrm{v}^2(t)}} \tag{1}$$

Instantaneous Frequency (IF) – V

Consider again the signal:

$$v(t) = V\cos(\theta(t))$$

Then the IF is:

$$\phi'(t) = \theta'(t)$$

Note also that, $\phi' = \frac{d}{dt}\ln(\tilde{v}(t))$ as V is constant.

If $\theta(t) = \omega_o t + \phi$, then, one obtains as expected $\phi' = \omega_o$.

Instantaneous Frequency (IF) – VI

So far, we have assumed a signal with constant magnitude.

What happens to the instantaneous frequency if the signal is:

$$v(t) = V(t)\cos(\theta(t))$$

This depends on the spectrum of $V(t)$ and $\theta(t)$.

Bedrosian Theorem – I

The Bedrosian theorem provides an important property of the HT.

This theorem proves that if two functions $\tilde{f}(t)$ and $\tilde{g}(t)$ are analytic and if $\mathfrak{f}(\omega) = \mathcal{F}\left[\tilde{f}\right](\omega)$ vanishes for $|\omega| > a$ and $\mathfrak{g}(\omega) = \mathcal{F}[\tilde{g}](\omega)$ vanishes for $|\omega| < a$, where a is a positive constant, then the following identity holds:

$$\mathcal{H}\left[\tilde{f}\,\tilde{g}\right](t) = \tilde{f}(t)\,\mathcal{H}[\tilde{g}](t)\,.$$

Bedrosian Theorem – II

The Bedrosian theorem has a special role in the estimation of frequency variations during electromechanical transients.

In electromechanical transients, the time-varying amplitude of the voltage has a low-frequency spectrum that does not overlap the high-frequency spectrum of the phase of the voltage itself.

Instantaneous Frequency (IF) – VII

Under the assumptions of the Bedrosian theorem, the HT of $v(t) = V(t)\cos(\theta(t))$ gives:

$$\begin{aligned}\mathcal{H}[v](t) = \mathcal{H}[V\,\cos(\theta)](t) &= V(t)\,\mathcal{H}[\cos(\theta)](t) \\ &= V(t)\,\sin(\theta(t))\,.\end{aligned}$$

And the analytic signal of the voltage can be written as:

$$\tilde{v}(t) = V(t)\,(\cos(\theta(t)) + \jmath\sin(\theta(t))) = V(t)\,e^{\jmath\,\theta(t)}\,,$$

which leads again to the IF as $\phi' = \theta'$.

Instantaneous Frequency or Fourier Frequencies?

At this point, the problem of the definition of the frequency in transient conditions seems to be resolved.

The HT and analytic signals seem to remove the limitations of the FT and lead to consistent results (for signals that satisfy the Bedrosian theorem).

Unfortunately, this is not so simple . . .

Paradoxes of the Instantaneous Frequency – I

Let us consider the following signal:

$$v(t) = V_1 \cos(\omega_1 t) + V_2 \cos(\omega_2 t)$$

with $\omega_1 > 0$ and $\omega_2 > 0$, which leads to the following instantaneous frequency:

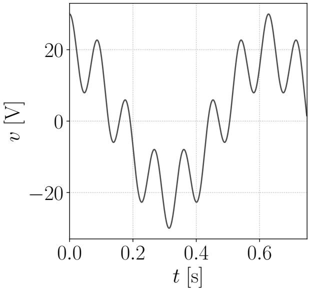
$$\phi'(t) = \frac{1}{2}(\omega_2 + \omega_1) + \frac{1}{2}\,\Delta\omega\,\frac{V_2^2 - V_1^2}{\mathrm{v}^2(t)}\,,$$

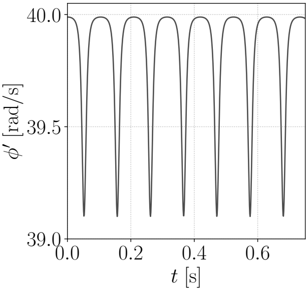
where $\Delta\omega = \omega_2 - \omega_1$, and:

$$\mathrm{v}^2(t) = V_1^2 + V_2^2 + 2V_1 V_2 \cos(\Delta\omega\ t)\,.$$

One the one hand, one would expect that the IF is, at least at certain instants, equal to ω_1 or ω_2. But this is not the case.

Let $\omega_1 = 10$, $\omega_2 = 70$, $V_1 = 20$, $V_2 = 10$.

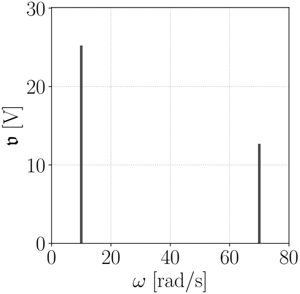


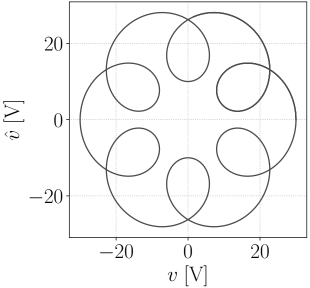


Paradoxes of the Instantaneous Frequency – III

Note that, in this case, which is stationary, the FT returns a results that is more consistent.

The IF describes the behavior of the analytic signal, not the harmonic content of the signal.





Paradoxes of the Instantaneous Frequency – IV

In the book “Time-Frequency Analysis”, Leon Cohen presents five paradoxes of the instantaneous frequency.

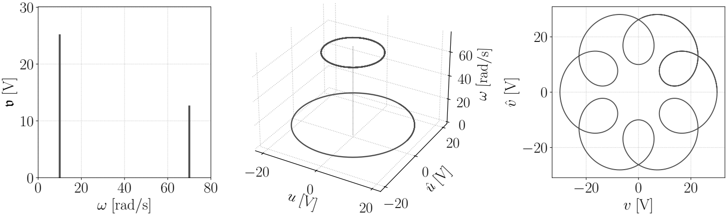
- *IF may not be one of the frequencies in the spectrum.*
- *If we have a line spectrum consisting of only a few sharp frequencies, then the IF may be continuous and range over an infinite number of values.*
- *Although the spectrum of the analytic signal is zero for negative frequencies, the IF may be negative.*
- *For a band-limited signal the IF may go outside the band.*
- *If the IF is an indication of the frequencies that exists at time t, one would presume that what the signal did a long time ago and is going to do in the future should be of no concern; only the present should count. However to calculate the analytic signal at time t we have to know the signal for all time.*

An Interpretation of the First Four Paradoxes

The first four paradoxes are, in effect, not a big issue.

The IF is a property of the signal, whereas the Fourier frequency is a coordinate. They have same units, but do not have to be correlated.

One can imagine a space of 3 coordinates $(v, \hat{v}, \omega)$:



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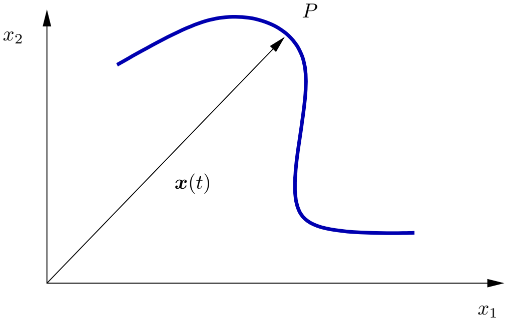
Differential Geometry - I

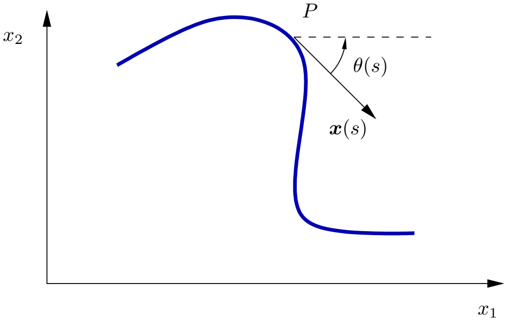
Let's consider a trajectory in space (for simplicity 2 dimensions are considered below):



The time derivative of the position vector $\boldsymbol{x}$ is:

$$\boldsymbol{x}' = \frac{d\boldsymbol{x}}{dt} = (x_1', x_2', \ldots, x_n') \, .$$

One can define the curve as a function of the *arc length* s:



Formally the arc length s is defined as:

$$s' = \frac{ds}{dt} = \sqrt{\mathbf{x}' \cdot \mathbf{x}'} = \sqrt{\sum_{i=1}^{n} (x_i')^2} = |\mathbf{x}'| \, .$$

Differential Geometry - III

The arc length allows defining a local system of coordinates

The *tangent* to the curve is given by:

$$\boldsymbol{T} = \dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{ds} = \frac{d\boldsymbol{x}}{dt}\frac{dt}{ds} = \frac{\boldsymbol{x}'}{|\boldsymbol{x}'|} \, .$$

The *normal* to the curve is given by:

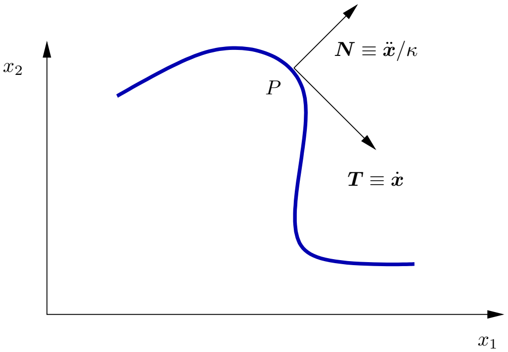
$$\boldsymbol{N} = \frac{\ddot{\boldsymbol{x}}}{|\ddot{\boldsymbol{x}}|} = \frac{\ddot{\boldsymbol{x}}}{\kappa} \, .$$

The *binormal* to the curve is given by:

$$\boldsymbol{B} = \boldsymbol{N} \times \boldsymbol{T} \, .$$

The basis can be defined for any dimension n.

The quantity κ is called *curvature*



and it is defined as:

$$\kappa = \frac{d\theta}{ds} = |\ddot{\mathbf{x}}| = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3} \,.$$

Differential Geometry - V

Let us consider now the second time derivative of $\boldsymbol{x}$. One can show that:

$$\boldsymbol{x}'' = \rho \boldsymbol{x}' + \boldsymbol{\omega} \times \boldsymbol{x}' ,$$

where

$$\rho = \frac{\boldsymbol{x}'}{x'} \cdot \frac{\boldsymbol{x}''}{x'} ,$$

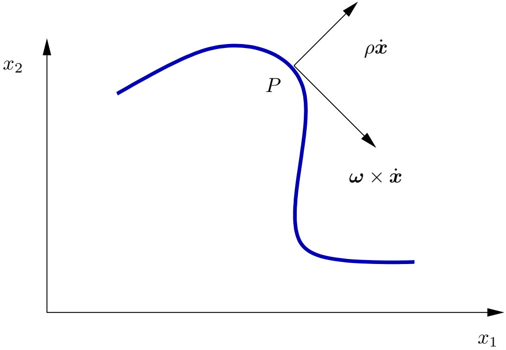
and

$$\boldsymbol{\omega} = \frac{\boldsymbol{x}'}{x'} \times \frac{\boldsymbol{x}''}{x'} .$$

Note that $\kappa = |\boldsymbol{\omega}||\boldsymbol{x}|' = \omega \, x'$.

Let us call ρ *radial frequency* ;

and ω *azimuthal frequency* .



Differential Geometry - VII

If the curve is a 3-dimensional space and is not plane, then one can define another quantity, the *torsion* , as:

$$\tau = \frac{\boldsymbol{x}''' \cdot \boldsymbol{\omega}}{\kappa^2} \, .$$

And also a third frequency:

$$\xi = \tau \, x' \, ,$$

which we will call *torsional frequency* .

Differential Geometry - VIII

One can show that the time derivatives of the tangent, normal and binormal vectors are related through the azimuthal and torsional frequencies, as follows.

$$\begin{aligned} \mathbf{T}' &= \quad \omega \mathbf{N} \,, \\ \mathbf{N}' &= -\omega \mathbf{T} + \xi \mathbf{B} \,, \\ \mathbf{B}' &= -\xi \mathbf{N} \,. \end{aligned}$$

The equations above are based on the well-known Frenet-Serret formulas and can be extended to hyperspaces of dimensions higher than 3.

Why does this matter?

Let assume that the magnetic flux vector $\boldsymbol{\varphi}$ is "equivalent" to the *position vector* :

$$\boldsymbol{\varphi} = -\boldsymbol{x} \, .$$

Then, from the Faraday's law, the voltage vector $\boldsymbol{v}$ can be interpreted as a *velocity* :

$$\boldsymbol{v} = -\boldsymbol{\varphi}' = \boldsymbol{x}' \, .$$

Curvature as a function of the Voltage

Remembering the definition of radial, azimuthal and torsional frequencies we have:

$$\rho = \frac{v'}{v} = \frac{\boldsymbol{v}}{v} \cdot \frac{\boldsymbol{v}'}{v}\,,$$

$$\omega = \left| \frac{\boldsymbol{v}}{v} \times \frac{\boldsymbol{v}'}{v} \right| = |\boldsymbol{\omega}| = v\kappa\,,$$

$$\xi = \frac{\boldsymbol{v}'' \cdot \boldsymbol{\omega}}{\kappa^2/v} = v\tau\,.$$

Yet, these definitions do not seem to be particularly useful . . .

Let's do then some examples.

Stationary Single-phase Voltage – I

Let us consider a stationary single-phase voltage with constant angular frequency ω_o and magnitude V.
The voltage vector can be defined as:

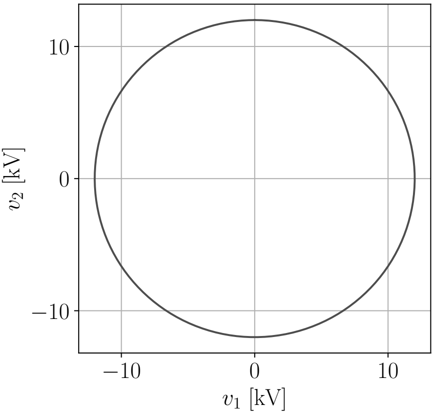
$$\boldsymbol{v} = V\cos(\theta)\mathbf{e}_1 + V\sin(\theta)\mathbf{e}_2 = v_1\mathbf{e}_1 + v_2\mathbf{e}_2\,,$$

Then, one can deduce:

$$\begin{aligned}
\rho &= \frac{1}{V^2}(v_1 v_1' + v_2 v_2') = \frac{\omega_o}{V^2}(-v_1 v_2 + v_2 v_1) = 0\,,\\
\omega &= \frac{1}{V^2}|v_1 v_2' - v_2 v_1'| = \frac{\omega_o}{V^2}|v_1^2 + v_2^2| = \omega_o\,,\\
\xi &= 0\,.
\end{aligned}$$

Stationary Single-phase Voltage – II

This results was to be expected as the voltage vector describes a circle, which is a plane curve with constant curvature!



Frequency of a DC Voltage?

In dc circuits, the voltage has only one component along the unique basis of the system, say $\mathbf{e}_{\rm dc}$, hence, $\boldsymbol{v} = v_{\rm dc}\,\mathbf{e}_{\rm dc}$ and $\boldsymbol{v}' = v'_{\rm dc}\,\mathbf{e}_{\rm dc}$. From the definitions of inner and wedge product one has:

$$\boldsymbol{v} \cdot \boldsymbol{v}' = v_{\rm dc}\, v'_{\rm dc}\,, \quad \boldsymbol{v} \times \boldsymbol{v}' = \mathbf{0}\,.$$

In dc, then, the radial frequency is equal to $\rho = v'_{\rm dc}/v_{\rm dc}$ and, as expected, the azimuthal and torsional frequencies are both null, i.e., $\omega = 0$ and $\xi = 0$.

Stationary Three-phase Voltage – I

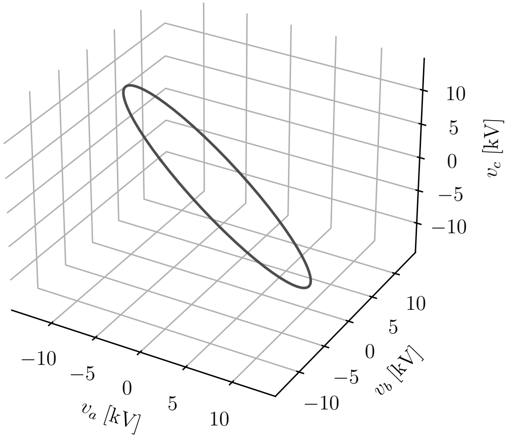
Let us consider a stationary three-phase voltage with constant angular frequency ω_o and magnitude V.
The voltage vector can be defined as:

$$
\begin{aligned}
\boldsymbol{v} &= V\sin(\theta_a)\mathbf{e}_a + V\sin(\theta_b)\mathbf{e}_b + V\sin(\theta_c)\mathbf{e}_c \\
&= v_a\mathbf{e}_a + v_c\mathbf{e}_b + v_c\mathbf{e}_c\,,
\end{aligned}
$$

$$
\begin{aligned}
\rho &= \frac{v_a v_a' + v_b v_b' + v_c v_c'}{v^2} = 0\,, \\
\omega &= \frac{1}{v^2}\sqrt{(v_a v_b' - v_b v_a')^2 + (v_b v_c' - v_c v_b')^2 + (v_c v_a' - v_a v_c')^2} \\
&= \frac{\sqrt{\omega_o^2 V^4 (2\sin^2(\alpha) + \sin^2(2\alpha))}}{\frac{3}{2}\,V^2} = \omega_o\,, \\
\xi &= 0\,.
\end{aligned}
$$

Stationary Three-phase Voltage – II

Also this results was to be expected as the voltage vector describes again a circle!



Clarke Transform – I

The previous result is not surprising also for another reason.

Let us consider the Clarke Transform (CT).

Let $\boldsymbol{v}_{abc}(t) = (v_a(t), v_b(t), v_c(t))$ be the voltage vector of a three-phase node.

The CT applied to this signal returns another vector, say $\boldsymbol{v}_{\alpha\beta\gamma}(t) = (v_\alpha(t), v_\beta(t), v_\gamma(t))$ calculated as:

$$
\begin{aligned}
\boldsymbol{v}_{\alpha\beta\gamma}(t) &= \mathbf{C}\,\boldsymbol{v}_{abc}(t) \\
&= \frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} v_a(t) \\ v_b(t) \\ v_c(t) \end{bmatrix} .
\end{aligned}
$$

Clarke Transform – II

If we apply the CT to the stationary balanced voltage of the previous example, we obtain:

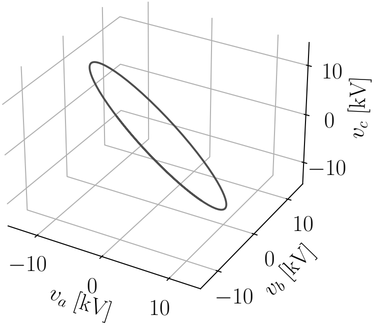
$$
\begin{aligned}
v_\alpha(t) &= V\cos(\omega_o t) \\
v_\beta(t) &= V\sin(\omega_o t) \\
v_\gamma(t) &= 0
\end{aligned}
$$

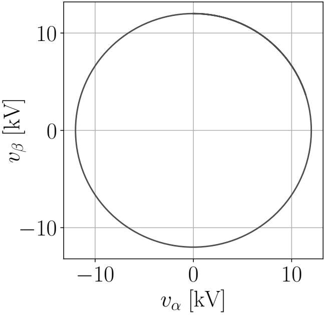
Note that $v_\alpha(t) + \jmath\, v_\beta(t)$ is the same as an analytic signal!

No surprise thus that $\omega = \omega_o$. In this case in fact, $\omega = \phi'$, i.e., coincides with the instantaneous frequency of the voltage on the $\alpha\beta$ axis.

Stationary Three-phase Voltage – III

The CT projects the circle in the space *abc* on the plane $\alpha\beta$





Clarke Transform – III

Note that, the same result is obtained also in non-stationary conditions, provided that the voltage $\boldsymbol{v}_{abc}(t)$ is balanced:

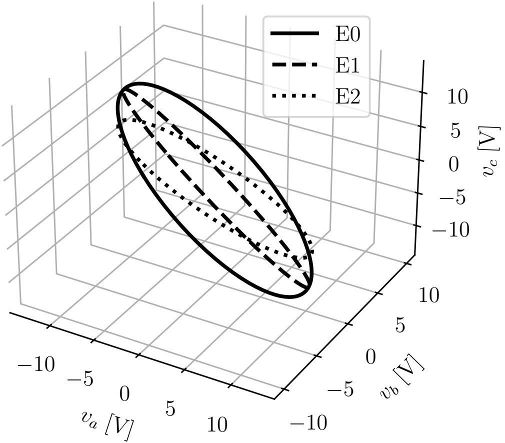
$$
\begin{aligned}
v_\alpha(t) &= V(t)\cos(\theta(t)) \\
v_\beta(t) &= V(t)\sin(\theta(t)) \\
v_\gamma(t) &= 0
\end{aligned}
$$

$v_\alpha(t) + \jmath\, v_\beta(t)$ is still an analytic signal and we have obtained it without having to calculate the Hilbert transform (i.e., without having to do the integral for $t \in (-\infty, \infty)$).

This seems to solve the fifth paradox of the instantaneous frequency!

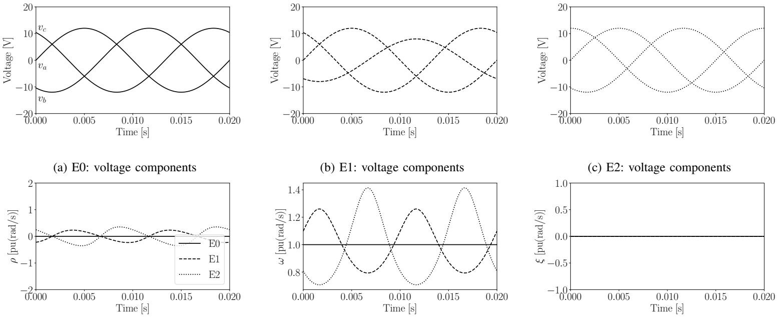
Unbalanced Tree-phase Voltage – I

Unbalanced transform the circle into an ellipse ...



Unbalanced Tree-phase Voltage – II

. . . and show non-null radial frequency and non-constant azimuthal frequency.



(a) E0: voltage components (b) E1: voltage components (c) E2: voltage components

(d) E0-E2: ρ (e) E0-E2: ω (f) E0-E2: ξ

Unbalanced Tree-phase Voltage – II

The reason why ω is not constant for unbalanced voltage can be, again, interpreted in terms of the Clarke transform and analytic signals.

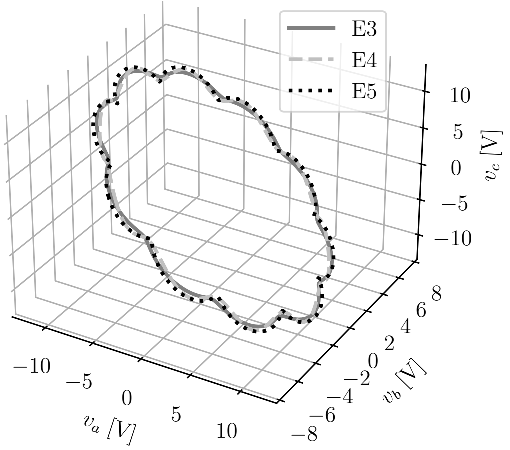
An unbalance implies that there are a positive $(+\omega_o)$ and a negative sequence $(-\omega_o)$, and possibly also a zero sequence.

In this case thus $v_\alpha(t) + \jmath\, v_\beta(t)$ is not an analytic signal and ω is not the instantaneous frequency.

If we filter the negative and zero sequences, then $\phi' = \omega$.

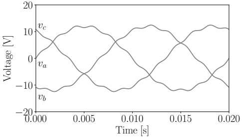
Stationary Tree-phase Voltage with Harmonics – I

Balanced harmonics leads to a ripple . . .

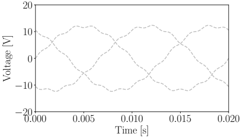


Stationary Tree-phase Voltage with Harmonics – II

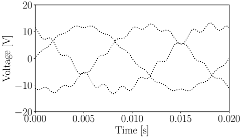
. . . and show non-null radial, azimuthal and torsional frequency (the latter is not null only if the harmonics are unbalanced).



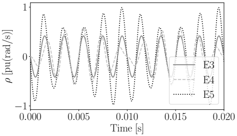
(a) E3: voltage components



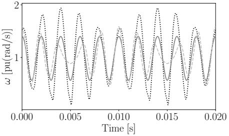
(b) E4: voltage components



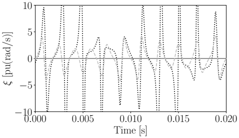
(c) E5: voltage components



(d) E3-E5: ρ



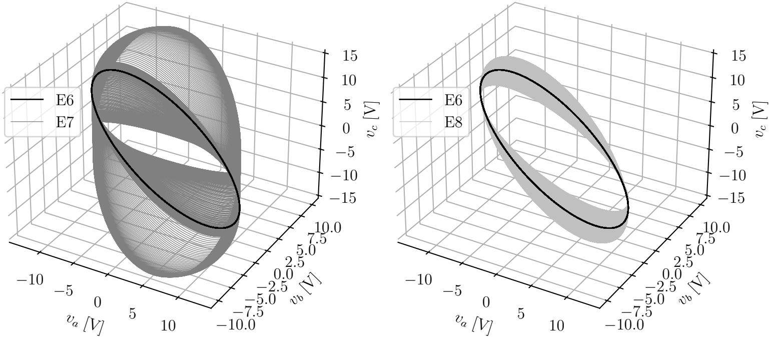
(e) E3-E5: ω



(f) E3-E5: ξ

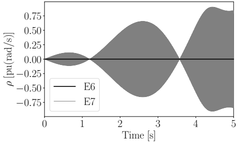
Time-varying Tree-phase Voltage – I

Voltages with time-varying magnitude and angular frequency leads to interesting shapes . . .

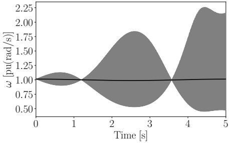


Time-varying Three-phase Voltage – II

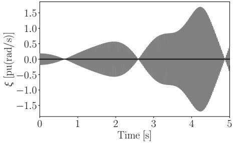
. . . and show frequencies with complex behaviors.



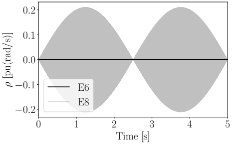
(a) E6 and E7: ρ.



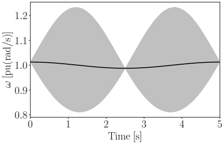
(b) E6 and E7: ω.



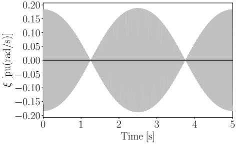
(c) E6 and E7: ξ.



(d) E6 and E8: ρ.



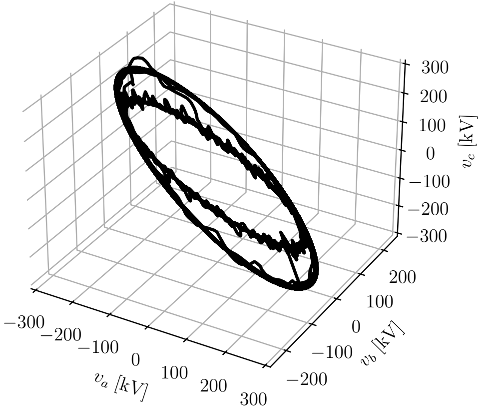
(e) E6 and E8: ω.



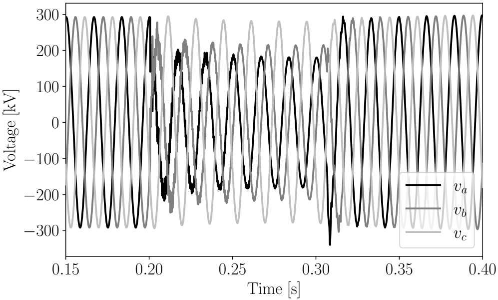
(f) E6 and E8: ξ.

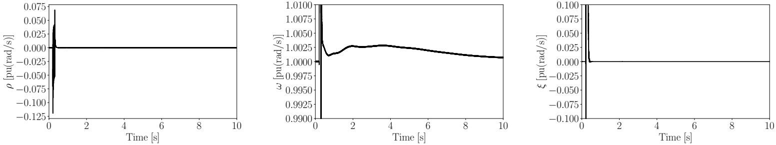
Real-world example – I

Let us consider the voltage of a bus of the New England 39-bus system following a three-phase short-circuit.



Real-world example – II

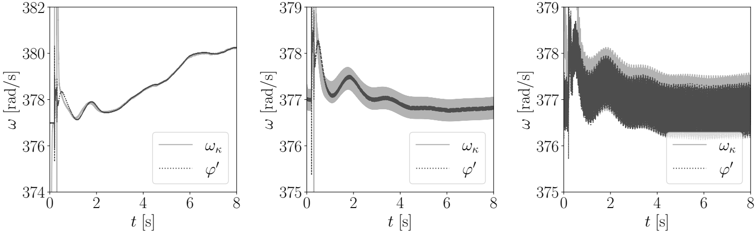




(a) ρ (b) ω (c) ξ

Real-world example – III

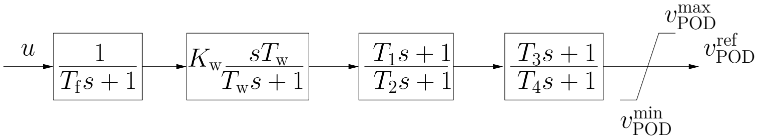
Effect of noise, unbalance and harmonics (ϕ' is obtained with a PLL and ω_κ is the azimuthal frequency)



Example: Curvature Control – I

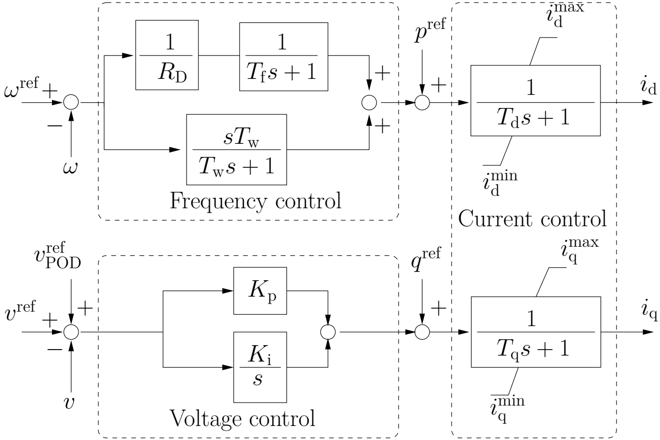
The concept of *curvature* appears to provide a relevant information on the local transient behavior of electrical quantities.

Let us consider the curvature as an input signal for an auxiliary power oscillator damper (POD) connected DERs



Example: Curvature Control – II

The DER control the frequency and the voltage.



Example: Curvature Control – III

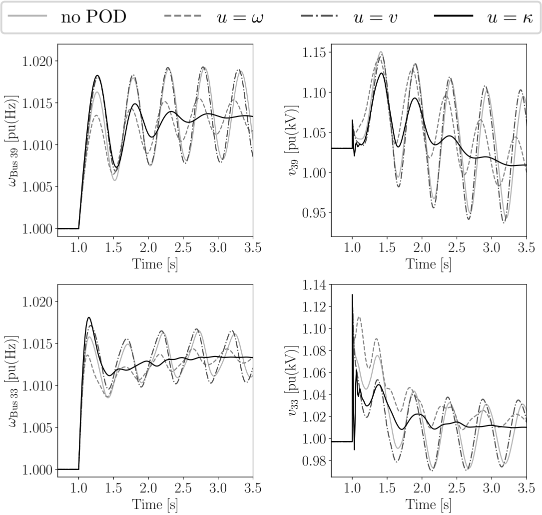
Let us consider the New England 39-bus system with 70% of DER penetration.

Let compare first the participation factors to dominant modes of different input signals of the POD.

Mode	x-dom.	p		
		ω_h	v_h	$\kappa_h = \omega_h / v_h$
$-1.169 \pm j12.573$	$\omega_{\text{Syn 1}}$	0.87e-5	-0.22e-5	**1.05e-5**
$-1.447 \pm j24.050$	$\omega_{\text{Syn 3}}$	2.62e-3	-2.93e-3	**5.70e-3**
$-1.947 \pm j27.404$	$\omega_{\text{Syn 4}}$	1.02e-4	-2.49e-4	**3.52e-4**
$-1.951 \pm j29.345$	$\omega_{\text{Syn 8}}$	1.54e-4	-0.61e-4	**2.07e-4**
$-1.998 \pm j23.490$	$\omega_{\text{Syn 9}}$	6.01e-2	-0.52e-2	**6.35e-2**

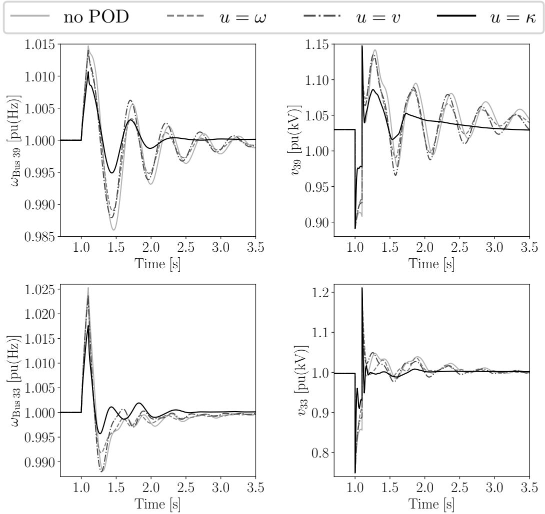
Example: Curvature Control – IV

Transient following the outage of 24% of the load.



Example: Curvature Control – V

Transient following a fault at bus 12 cleared after 0.1 s.



Park Transform – I

Let us assume balanced conditions with no harmonics from now on.

This means that the zero-sequence is null and also that the torsional frequency is null.

Then, let us apply the Park transform, say $\mathbf{P}(t)$, which can be interpreted as the Clarke transform shifted by a frequency $\omega_{\rm p}$:

$$\mathbf{P}(t) = \begin{bmatrix} \cos(\theta_{\rm p}(t)) & \sin(\theta_{\rm p}(t)) & 0 \\ -\sin(\theta_{\rm p}(t)) & \cos(\theta_{\rm p}(t)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{C} \, ,$$

where $\theta_{\rm p}(t) = \omega_{\rm p} t + \phi_{\rm p}$.

Park Transform – II

The Park transform of a balanced three-phase voltage leads to the voltage vector:

$$\boldsymbol{v} = v_d \mathbf{e}_d + v_q \mathbf{e}_q + 0 \mathbf{e}_o \, .$$

The time derivative becomes:

$$\boldsymbol{v}' = (v_d' - \omega_o v_q)\mathbf{e}_d + (v_q' + \omega_o v_d)\mathbf{e}_q = \tilde{v}_d' \mathbf{e}_d + \tilde{v}_q' \mathbf{e}_q \, ,$$

where:

$$\mathbf{e}_d' = \omega_o \mathbf{e}_q \, , \quad \mathbf{e}_q' = -\omega_o \mathbf{e}_d \, .$$

Park Transform – III

Then the components of the generalized frequency are:

$$
\begin{aligned}
\rho &= \frac{v_d v_d' + \omega_o v_d v_q + v_q v_q' - \omega_o v_d v_q}{v^2} = \frac{v_d v_d' + v_q v_q'}{v^2} \, , \\
\boldsymbol{\omega} &= \begin{bmatrix} 0 & v_d \tilde{v}_q' - v_q \tilde{v}_d' \\ v_q \tilde{v}_d' - v_d \tilde{v}_q' & 0 \end{bmatrix} , \\
\omega &= \frac{v_q' v_d + \omega_o v_d^2 - v_d' v_q + \omega_o v_q^2}{v^2} = \omega_o + \frac{v_q' v_d - v_d' v_q}{v^2} \, , \\
\xi &= 0 \, .
\end{aligned}
$$

Complex Frequency

Let us observe the following.
If $\bar{v} = v_d + \jmath v_q = v\angle\theta$, then one has:

$$\omega = \theta' = \frac{v_d' v_q - v_q' v_d}{v^2} \, ,$$

and

$$\rho = \frac{v'}{v} = \frac{v_d' v_d + v_q' v_q}{v^2} \, .$$

Let us define $u = \ln(v)$. Then one has:

$$\rho = \frac{d}{dt}\ln(v) = \frac{v'}{v} = u'$$

Hence:

$$\bar{\eta} = \frac{d}{dt}(u + \jmath\theta) = u' + \jmath\theta' = \rho + \jmath\omega$$

Analytic Signal and Complex Frequency

We derive now an important relationship. Let:

$$\bar{v} = v_d + j v_q \,,$$

be the Park-transformed voltage vector, or simply Park vector, expressed as a complex time-dependent quantity.

The azimuthal frequency of the Park vector is the instantaneous frequency of an analytic signal shifted by $\omega_{\rm p}$:

and, based on the definition of $\bar{\eta}$, one has:

$$\bar{v}' = (\rho + \jmath\omega)\bar{v} = \bar{\eta}\,\bar{v} \,.$$

Geometric Frequency – I

The latter result can be generalized to three dimensions, in fact:

$$\boldsymbol{v}' = [\rho + \boldsymbol{\omega}\times]\boldsymbol{v} \, .$$

We can also extend the definition of frequency as a multivector:

$$\rho + \boldsymbol{\omega} = \frac{\boldsymbol{v} \cdot \boldsymbol{v}'}{|v|^2} + \frac{\boldsymbol{v} \times \boldsymbol{v}'}{|v|^2} \tag{2}$$

where Log is the analytic continuation of the natural logarithm in the multivector space.

Geometric Frequency – II

And, in fact, we can also go beyond three dimensions:

$$\rho + \boldsymbol{\omega} = \frac{\boldsymbol{v} \cdot \boldsymbol{v}'}{|v|^2} + \frac{\boldsymbol{v} \wedge \boldsymbol{v}'}{|v|^2}$$

where $\wedge$ is the wedge product defined as:

$$\boldsymbol{v} \wedge \boldsymbol{v}' = \boldsymbol{v} \otimes \boldsymbol{v}' - \boldsymbol{v}' \otimes \boldsymbol{v}$$

where $\otimes$ is the outer product:

$$\boldsymbol{x} \otimes \boldsymbol{y} = \begin{bmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & \ddots & \vdots \\ x_n y_1 & \dots & x_n y_n \end{bmatrix}$$

Geometric Frequency and Complex Frequency

We define:

The multivector $\boldsymbol{\Omega} = \rho + \boldsymbol{\omega}$ as *geometric frequency*

The complex value $\bar{\eta} = \rho + \jmath\omega$ as *complex frequency*

The complex frequency is a special case (plane curve) of the geometric one, which is valid for space curves (and higher dimensional spaces too!)

Note also that, formally one can define $\jmath = \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$

Table of Contents

Conclusions

Need of New Approaches

Differential geometry provides a consistent (and visual) interpretation of the instantaneous frequency of the voltage

Many Frequencies

Based on a geometric interpretation, one can see that there are "many" different frequencies. The correct understanding of the physical meaning of each frequency is crucial.

Applications?

The theory discussed here can be applied to develop efficient controllers for converter-interfaced generation in transmission systems. But this is matter for another seminar ...

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Thank you!