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A bounded dynamical network of curves and the stability of its steady states

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In this article, we study the dynamic behavior of a network that consists of curves that are in motion and bounded. We first focus on the construction of the model which is a system of nonlinear partial differential equations (PDEs). This system is subject to four conditions: angle and intersection conditions between the curves at the point that they meet and angle and intersection conditions between the curves and the boundary from which the network is bounded. Then, we define a linear operator and study the stability of the steady states of the corresponding boundary value problem (BVP).

KEYWORDS
angle conditions, boundary, curves, dynamical network, geometry, intersection conditions

MSC CLASSIFICATION
70K20, 70K70

1 | INTRODUCTION

In the last few decades, many authors have studied dynamical networks formed by curves in motion. The studies focus on the differential geometry of the problem; see earlier studies [1–5], numerical methods for dynamical systems; see previous works [6–11], and the stability of the network; see earlier research [12–16].

Focus has also been given on the mathematical modeling and applications in material science and engineering; see earlier studies [17–20].

In this article, we study a bounded network of curves that are in motion and intersect at a junction. We consider a parametric form of two variables for each curve and form a system of nonlinear partial differential equations (PDEs) that describes the motion of the network. We also consider the equation of the curve that describes the boundary of the domain that bounds the network. The system of nonlinear PDEs is subject to four conditions. Intersection and angle conditions at the junctions and at the boundary of the domain are formed using properties from differential geometry.

In Section 2, we form the boundary value problem (BVP), in Section 3, we linearize the nonlinear PDEs and reformulate efficiently the conditions of the problem, and in Section 4, we define the linear operator used for the linearization of the BVP, and by studying the eigenvalues of this operator, we conclude that the stability of the steady states of the network depends on the sign of the curvature of the boundary of the domain.

The results in this article aim to bring new ideas and insights that can be applied to other types of networks, such as hexagonal dynamical networks as they appear, for example, in soap bubbles, honeycomb, and grain growth, see other research [21–24], and also update geometrical properties that are used in applications in engineering such as the structure of electrical circuits and studies of elasticity and plasticity in material science; see earlier studies [25–27].

2 | PROBLEM FORMULATION

We will consider a dynamical network of three curves in motion in $\mathbb{R}^2$ that meet at a junction and are bounded from the boundary $\partial\Omega$ of a domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^2$.

Definition 2.1. With $T_i(x,t)$, we denote the unit tangent vector of curve i at x and time t and with $N_i(x,t)$ the unit normal vector of curve i at x and time t.

Let $u_i(x,t) = (u_{i1}(x,t), u_{i2}(x,t))$, $i = 1,2,3$, be a parametric form of curve i with

$$u_i : [0,1] \times [0,+\infty) \to \Omega, \; i = 1,2,3,$$

and

$$u_{ij} : [0,1] \times [0,+\infty) \to \Omega_j, \; i = 1,2,3, \; j = 1,2.$$

With $u_{ix} = \frac{\partial u_i}{\partial x}$, $u_{ixx} = \frac{\partial^2 u_i}{\partial x^2}$, we will denote the partial derivatives of first order and second order, respectively, of u_i in respect to x, while with $u_{it} = \frac{\partial u_i}{\partial t}$, we will denote the partial derivative of first order of u_i in respect to t.

Let $k_i(x,t)$, $k_i : \mathbb{R}^2 \to \mathbb{R}$ be curvature of curve i. At the steady states, $k_i = 0$. For these curves, by taking into account that normal velocity equals to curvature, we get the following:

$$\frac{\partial u_i}{\partial t} N_i = k_i, \; i = 1,2,3,$$

where dependencies on x and t have been omitted for simplicity in notation. Furthermore,

$$k_i N_i = \frac{|u_{ixx} \times u_{ix}|}{|u_{ix}|^3} \frac{1}{|u_{ixx}|} u_{ixx}.$$

Equivalently,

$$k_i N_i = \frac{1}{|u_{ix}|} u_{ixx},$$

because $|u_{ixx} \times u_{ix}| = |u_{ixx}||u_{ix}|$. Hence,

$$u_{it} = \frac{1}{|u_{ix}|} u_{ixx}, \; i = 1,2,3, \tag{1}$$

which is a nonlinear system of PDEs. We will now focus on the conditions.

The three curves meet at a junction. Thus, for each parametric form $u_i(x,t)$ of curve i, $i = 1,2,3$ and if we consider that each curve meets the other at one end at $x = 0$, the following relation holds:

$$u_1(0,t) = u_2(0,t) = u_3(0,t). \tag{2}$$

At $x = 0$, curve 1 forms an angle $\theta_{1,2}(t) = \theta_{1,2}$ with curve 2, and curve 2 forms an angle $\theta_{2,3}(t) = \theta_{2,3}$ with curve 3. Curve 3 forms an angle $2\pi - \theta_{1,2} - \theta_{2,3}$ rad with curve 1. Then

$$T_1(0,t)T_2(0,t) = \cos\theta_{1,2}, \; T_2(0,t)T_3(0,t) = \cos\theta_{2,3},$$

or equivalently,

$$\frac{u_{1x}(0,t)}{|u_{1x}(0,t)|}\frac{u_{2x}(0,t)}{|u_{2x}(0,t)|} = \cos\theta_{1,2}, \; \frac{u_{2x}(0,t)}{|u_{2x}(0,t)|}\frac{u_{3x}(0,t)}{|u_{3x}(0,t)|} = \cos\theta_{2,3},$$

or equivalently,

$$u_{1x}(0,t)u_{2x}(0,t) = s^t_{1x}(1)s^t_{2x}(1)\cos\theta_{1,2}, \; u_{2x}(0,t)u_{3x}(0,t) = s^t_{2x}(1)s^t_{3x}(1)\cos\theta_{2,3}. \tag{3}$$

At the steady states, we will have $\theta_{1,2} = \theta_{2,3} = \frac{2\pi}{3}$ rad. Where s^t_i denotes the arc length parameter, with

$$s^t_i := s^t_i(x) = \int_0^x |u_{ix}(\xi,t)|d\xi,$$

and consequently,

$$s^t_{ix} = |u_{ix}(x,t)|.$$

Definition 2.2. With $f = 0$, $f : \mathbb{R}^2 \to \mathbb{R}$, we denote the equation of the boundary $\partial\Omega$ of Ω.

The network that the curves form is bounded by Ω, that is, each curve meets $\partial\Omega$ at $x = 1$. Hence, since $f = 0$ is the equation of $\partial\Omega$, we get the following:

$$f(u_i(1,t)) = 0,\ i = 1,2,3. \tag{4}$$

Definition 2.3. With $N^i_{\partial\Omega} := N_{\partial\Omega}(u_i(1,t)) = \frac{\nabla f(u_i(1,t))}{|\nabla f(u_i(1,t))|}$, we denote the unit normal vector of the boundary $\partial\Omega$ of Ω at $u_i(1,t)$, and $K^i_{\partial\Omega}$ is the curvature of the boundary $\partial\Omega$ of Ω at $u_i(1,t)$.

Each curve meets the boundary Ω with $\frac{\pi}{2}$ rad, that is, at the point that they meet, the unit tangent of the curve and the unit tangent of the boundary are orthogonal. Hence,

$$\frac{u_{ix}(1,t)}{|u_{ix}(1,t)|}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{\nabla f(u_i(1,t))}{|\nabla f(u_i(1,t))|} = 0,\ i = 1,2,3,$$

or equivalently,

$$u_{ix}(1,t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}N^i_{\partial\Omega} = 0,\ i = 1,2,3. \tag{5}$$

To sum up, we have the system of nonlinear PDEs (1) and its conditions (2)–(5).

3 | LINEARIZATION

The system of PDEs (1) is nonlinear. In this subsection, we will linearize effectively (1).

Theorem 3.1. *We consider the system of nonlinear PDEs (1) and its conditions (2)–(5). Then a linearization of this BVP consists of the linear system of PDEs:*

$$v_{it} = v_{ixx},\ i = 1,2,3, \tag{6}$$

and the conditions are as follows:

1. *Conditions for intersection at the junction at $x = 0$:*

$$v_1(0,t) = v_2(0,t) = v_3(0,t). \tag{7}$$

2. *Angle conditions at the junction at $x = 0$:*

$$v_{11x}(0,t) - v_{21x}(0,t) = 0,\ v_{21x}(0,t) - v_{31x}(0,t) = 0. \tag{8}$$

3. *Conditions for intersection at the boundary $\partial\Omega$ at $x = 1$:*

$$v_i(1,t)\nabla f = 0,\ i = 1,2,3. \tag{9}$$

4. *Angle conditions at the boundary $\partial\Omega$ at $x = 1$:*

$$[K^i_{\partial\Omega}v_i(1,t) - v_{ix}(1,t)]N_i = 0. \tag{10}$$

Where $\tilde{u}_i(x,t) := \hat{u}_i(s^t_i,t)$, $s^t_i = s^t_i(x)$, solution of (1). Where $\hat{u}_i(s^t_i,t)$ is parametric form of curve i with arc length parameter s^t_i, that is,

$$s^t_i := s^t_i(x) = \int_0^x |u_{ix}(\xi,t)|d\xi,\ s^t_{ix} = |u_{ix}(x,t)|.$$

In addition,

$$v_i(x,t) = v_{i1}(x,t)N_i + v_{i2}(x,t)T_i, \tag{11}$$

with

$$v_i : [0,1] \times [0,+\infty) \to \Omega, \ v_{ij} : [0,1] \times [0,+\infty) \to \Omega_j, \ i = 1,2,3, \ j = 1,2.$$

Furthermore, $T_i := T_i(s_i^t(x), t)$, $N_i := N_i(s_i^t(x), t)$, and $K^i_{\partial\Omega}$ the curvature of the boundary $\partial\Omega$ of Ω at $\tilde{u}_i(1,t)$.

Proof. Let

$$u_i(x,t) = \tilde{u}_i(x,t) + \epsilon_i v_i(x,t), \ 0 < \epsilon_i \ll 1, \tag{12}$$

with $\tilde{u}_i(x,t) := \hat{u}_i(s_i^t, t)$, $s_i^t = s_i^t(x)$, solution of (1). Where $\hat{u}_i(s_i^t, t)$ is parametric form of curve i with s_i^t arc length parameter, that is,

$$s_i^t := s_i^t(x) = \int_0^x |u_{ix}(\xi,t)| d\xi, \ s_{ix}^t = |u_{ix}(x,t)|.$$

Furthermore, let

$$v_i(x,t) = v_{i1}(x,t) N_i + v_{i2}(x,t) T_i,$$

with

$$v_i : [0,1] \times [0,+\infty) \to \Omega, \ v_{ij} : [0,1] \times [0,+\infty) \to \Omega_j, \ i = 1,2,3, \ j = 1,2.$$

By substituting (12) into (1), we get

$$\tilde{u}_{it} + \epsilon_i v_{it} = \frac{1}{|\tilde{u}_{ix} + \epsilon_i v_{ix}|^2} (\tilde{u}_{ixx} + \epsilon_i v_{ixx}), \ i = 1,2,3.$$

Let

$$F(\epsilon_i) = \frac{1}{|\tilde{u}_{ix} + \epsilon_i v_{ix}|^2} (\tilde{u}_{ixx} + \epsilon_i v_{ixx}), \ i = 1,2,3,$$

whereby

$$F(\epsilon_i) = F(0) + \epsilon_i F'(0) + \mathcal{O}(\epsilon_i^2), \ i = 1,2,3,$$

where

$$F(0) = \frac{1}{|\tilde{u}_{ix}|} \tilde{u}_{ixx},$$

and

$$F'(0) = \frac{v_{ixx} |\tilde{u}_{ix}|^2 - 2 v_{ix} \tilde{u}_{ix} \tilde{u}_{ixx}}{|\tilde{u}_{ix}|^4}.$$

Equivalently, since $\tilde{u}_i$ is defined as a parametric form of curve i, we have $|\tilde{u}_{ix}| = 1$, and $\tilde{u}_{ix} \tilde{u}_{ixx} = 0$; we have the following:

$$F'(0) = v_{ixx}.$$

Hence,

$$F(\epsilon_i) = \frac{1}{|\tilde{u}_{ix}|} \tilde{u}_{ixx} + \epsilon_i v_{ixx} + \mathcal{O}(\epsilon_i^2), \ i = 1,2,3,$$

and equivalently, a linearization of (1) is

$$\tilde{u}_{it} + \epsilon_i v_{it} = \frac{1}{|\tilde{u}_{ix}|} \tilde{u}_{ixx} + \epsilon_i v_{ixx}, \ i = 1,2,3,$$

or equivalently, since $\tilde{u}_i$ is assumed solution of (1),

$$v_{it} = v_{ixx}, \ i = 1,2,3.$$

We will now define the conditions. From (2) and by using (12), we get

$$\tilde{u}_1(0,t) + \epsilon_1 v_1(0,t) = \tilde{u}_2(0,t) + \epsilon_2 v_2(0,t) = \tilde{u}_3(0,t) + \epsilon_3 v_3(0,t),$$

whereby and since $\tilde{u}_i(0,t)$ is solution of (1), we arrive at

$$v_1(0,t) = v_2(0,t) = v_3(0,t).$$

From (3) and by using (12), we arrive at

$$[\tilde{u}_{1x}(0,t) + \epsilon_1 v_{1x}(0,t)][\tilde{u}_{2x}(0,t) + \epsilon_2 v_{2x}(0,t)] = s^t_{1x}(1) s^t_{2x}(1)\cos\theta_{1,2},$$

and

$$[\tilde{u}_{2x}(0,t) + \epsilon_2 v_{2x}(0,t)][\tilde{u}_{3x}(0,t) + \epsilon_3 v_{3x}(0,t)] = s^t_{2x}(1) s^t_{3x}(1)\cos\theta_{2,3}.$$

Equivalently, by ignoring the coefficients of $\epsilon_1\epsilon_2 \cong \epsilon^2$, $\epsilon_2\epsilon_3 \cong \epsilon^2$, with $0 < \epsilon_i \ll 1$ for $i = 1,2,3$, and by using $\tilde{u}_{1x}(0,t) = T_i$, for T_i at $x = 0$, $\forall i = 1,2,3$:

$$\left[1 + \sum_{i=1}^{2} \epsilon_i v_{i2x}(0,t)\right] \tilde{u}_{1x}(0,t)\tilde{u}_{2x}(0,t) + \epsilon_1 v_{11x}(0,t) T_2 N_1 + \epsilon_2 v_{21x}(0,t) T_1 N_2 = s^t_{1x}(1) s^t_{2x}(1)\cos\theta_{1,2},$$

and

$$\left[1 + \sum_{i=2}^{3} \epsilon_i v_{i2x}(0,t)\right] \tilde{u}_{2x}(0,t)\tilde{u}_{3x}(0,t) + \epsilon_2 v_{21x}(0,t) T_3 N_2 + \epsilon_3 v_{31x}(0,t) T_2 N_3 = s^t_{2x}(1) s^t_{3x}(1)\cos\theta_{2,3},$$

whereby taking into account that $\tilde{u}_i(x,t)$ is solution of (1) and $\epsilon_i \cong \epsilon$ with $0 < \epsilon \ll 1$, we have

$$v_{11x}(0,t) T_2 N_1 + v_{21x}(0,t) T_1 N_2 = 0,$$

and

$$v_{21x}(0,t) T_3 N_2 + v_{31x}(0,t) T_2 N_3 = 0.$$

Equivalently, since $T_1 N_2 = T_2 N_3 = cos\left(\theta_{1,2} + \frac{\pi}{2}\right)$ and $T_2 N_1 = T_3 N_2 = cos\left(\theta_{1,2} - \frac{\pi}{2}\right)$:

$$v_{11x}(0,t) cos\left(\theta_{1,2} - \frac{\pi}{2}\right) + v_{21x}(0,t) cos\left(\theta_{1,2} + \frac{\pi}{2}\right) = 0,$$

and

$$v_{21x}(0,t) cos\left(\theta_{2,3} - \frac{\pi}{2}\right) + v_{31x}(0,t) cos\left(\theta_{2,3} + \frac{\pi}{2}\right) = 0.$$

Consequently,

$$v_{11x}(0,t) - v_{21x}(0,t) = 0, \;\; v_{21x}(0,t) - v_{31x}(0,t) = 0.$$

From (4) and by using (12), we arrive at

$$f(\tilde{u}_i(1,t) + \epsilon_i v_i(1,t)) = 0, \;\; i = 1,2,3,$$

or equivalently,

$$f(\tilde{u}_{i1}(1,t) + \epsilon_i v_{i1}(1,t), \tilde{u}_{i2}(1,t) + \epsilon_i v_{i2}(1,t)) = 0, \;\; i = 1,2,3,$$

or equivalently,

$$f(f^t_{i1}, f^t_{i2}) = 0, \;\; i = 1,2,3,$$

where

$$f_{i1}^t = \tilde{u}_{i1}(1,t) + \epsilon_i v_{i1}(1,t),\ \ f_{i2}^t = \tilde{u}_{i2}(1,t) + \epsilon_i v_{i2}(1,t).$$

We now differentiate in respect to ϵ_i and we get the following:

$$\frac{df}{d\epsilon_i} = 0,\ \ i = 1,2,3,$$

or equivalently,

$$\frac{\partial f}{\partial f_{i1}^t}\frac{df_{i1}^t}{d\epsilon_i} + \frac{df}{\partial f_{i2}^t}\frac{df_{i2}^t}{d\epsilon_i} = 0,\ \ i = 1,2,3,$$

or equivalently,

$$v_i(1,t)\nabla f = 0,\ \ i = 1,2,3.$$

From (5) and by using (12), we arrive at

$$[\tilde{u}_{ix}(1,t) + \epsilon_i v_{ix}(1,t)]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_{\partial\Omega}(\tilde{u}_i(1,t) + \epsilon_i v_i(1,t)) = 0,\ \ i = 1,2,3,$$

whereby using Taylor expansion at $\epsilon_i = 0$,

$$[\tilde{u}_{ix}(1,t) + \epsilon_i v_{ix}(1,t)]\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_{\partial\Omega}(\tilde{u}_i(1,t)) + \epsilon_i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\nabla N_{\partial\Omega}(\tilde{u}_i(1,t))v_i(1,t)\right] = 0.$$

Equivalently, if we ignore the coefficients of ϵ_i^2 and by using that $\tilde{u}_i(x,t)$ is a solution of (1), we arrive at the following:

$$\tilde{u}_{ix}(1,t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\nabla N_{\partial\Omega}^t v_i(1,t) + v_{ix}(1,t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N_{\partial\Omega}^t = 0,$$

or equivalently, and by using Goldman's formulas, see Goldman [3], and by considering N_i at $x = 1$, we arrive at the following:

$$K_{\partial\Omega}^i N_i v_i(1,t) - v_{ix}(1,t)N_i = 0,$$

or equivalently,

$$[K_{\partial\Omega}^i v_i(1,t) - v_{ix}(1,t)]N_i = 0.$$

The proof is completed. □

We now state the following corollary.

Corollary 3.1. *We consider the linear BVP that consists of the system of linear PDEs (6) and conditions (7)–(10). Then at the steady states ($k_i = 0$), the system will take the form:*

$$v_{i1t} = v_{i1xx},\ v_{i2t} = v_{i2xx},\ \ i = 1,2,3, \tag{13}$$

where $v_i : [0,1]\times[0,+\infty) \to \Omega$, $v_{ij} : [0,1]\times[0,+\infty) \to \Omega_j$, $i = 1,2,3,\ j = 1,2,$ *as defined in (11).*

Proof. By using the Frenet formulas, we get

$$v_{it} = [v_{i1t} + k_i v_{i2}]N_i + [v_{i2t} - k_i v_{i1}]T_i,\ \ v_{ix} = [v_{i1x} + k_i v_{i2}]N_i + [v_{i2x} - k_i v_{i1}]T_i,$$

and

$$v_{ixx} = [v_{i1xx} + 2k_i v_{i2x} + k_{ix}v_{i2} - k_i^2 v_{i1}]N_i + [v_{i2xx} - 2k_i v_{i1x} - k_{ix}v_{i1} - k_i^2 v_{i2}]T_i.$$

Thus, we can write (6) in the following form:

$$v_{i1t} + k_i v_{i2} = v_{i1xx} + 2k_i v_{i2x} + k_{ix}v_{i2} - k_i^2 v_{i1},\ \ i = 1,2,3,$$

and

$$v_{i2t} - k_i v_{i1} = v_{i2xx} - 2k_i v_{i1x} - k_{ix} v_{i1} - k_i^2 v_{i2}, \ i = 1, 2, 3.$$

At the steady states ($k_i = 0$), the system will take the form:

$$v_{i1t} = v_{i1xx}, \ v_{i2t} = v_{i2xx}, \ i = 1, 2, 3.$$

The proof is completed. □

4 | STABILITY

In this section, we will study the stability of the steady states of (6) with conditions (7)–(10). System (6) at the steady states takes the form of (13) which is separable, and the general solution can be expressed as follows:

$$v_{ij}(x, t) = X_{ij}(x) Y_{ij}(t), \ i = 1, 2, 3, \ j = 1, 2,$$

where

$$X_{ij} : [0, 1] \to \Omega_j, \ Y_{ij} : [0, +\infty) \to \Omega_j, \ i = 1, 2, 3, \ j = 1, 2.$$

For the eigenvalue problem that appears, we are interested in studying the stability and existence of eigenvalues with $\lambda \leq 0$. If $\lambda < 0$, we have the following:

$$v_{ij} = \left[c_{ij} \cosh\left(\sqrt{-\lambda} x\right) + d_{ij} \sinh\left(\sqrt{-\lambda} x\right) \right] e^{\lambda t}, \ i = 1, 2, 3, \ j = 1, 2,$$

while if $\lambda = 0$, we have the following:

$$v_{ij} = c_{ij} + d_{ij} x, \ i = 1, 2, 3, \ j = 1, 2.$$

If there do not exist eigenvalues with $\lambda \leq 0$, then $c_{ij} = d_{ij} = 0, \forall i = 1, 2, 3, \forall j = 1, 2$.

Theorem 4.1. *We consider the linear BVP that consists of the system of linear PDEs (6) and conditions (7)–(10). Then,*

1. *If $K^i_{\partial\Omega} > 0, \forall i = 1, 2, 3$, then there exist negative eigenvalues and the steady states are unstable. The eigenvalues are given from the solutions of the algebraic equations:*

$$K^i_{\partial\Omega} - \sqrt{-\lambda} \tanh\left(\sqrt{-\lambda}\right) = 0, \ i = 1, 2, 3.$$

   The eigenfunctions are then given by the following:

$$v_{i1} = c_{i1} \cosh\left(\sqrt{-\lambda} x\right) e^{\lambda t}, \ i = 1, 2, 3,$$

   and

$$v_{i2} = -d_{i2} \left[\tanh\left(\sqrt{-\lambda}\right) \cosh\left(\sqrt{-\lambda} x\right) \right] e^{\lambda t}, \ i = 1, 2, 3,$$

   where c_{i1}, d_{i2} are constant.

2. *If $K^i_{\partial\Omega} = 0$ or $K^i_{\partial\Omega} = 1, \forall i = 1, 2, 3$, then there exists the zero eigenvalue and the steady states are neutral stable. The eigenfunctions are then given by the following:*

   - *If $K^i_{\partial\Omega} = 0$,*

$$v_{i1} = c_{i1}, \ i = 1, 2, 3,$$

     and

$$v_{i2} = -d_{i2}(1 + x), \ i = 1, 2, 3,$$

     where c_{i1}, d_{i2} are constants.

- *If* $K^i_{\partial\Omega} = 1$,

$$v_{i1} = d,\ i = 1,2,3,$$

and

$$v_{i2} = 0,\ i = 1,2,3,$$

where d is constant.

Proof. We consider the linear system of PDEs (6) and conditions (7)–(10). Then if $\lambda < 0$, we have the following:

$$v_{ij} = \left[c_{ij}\cosh\left(\sqrt{-\lambda}x\right) + d_{ij}\sinh\left(\sqrt{-\lambda}x\right)\right]e^{\lambda t},\ i = 1,2,3,\ j = 1,2.$$

From (7) and (11), we have

$$v_1(0,t) = v_2(0,t) = v_3(0,t),$$

or equivalently,

$$c_{11}N_1 + c_{12}T_1 = c_{21}N_2 + c_{22}T_2 = c_{31}N_1 + c_{32}T_3,$$

whereby we can get

$$c_{11} = c_{21}N_1N_2 + c_{22}N_1T_2,\ c_{21} = c_{31}N_2N_3 + c_{32}N_2T_3,\ c_{31} = c_{11}N_3N_1 + c_{12}N_3T_1,$$

and

$$c_{12} = c_{21}T_1N_2 + c_{22}T_1T_2,\ c_{22} = c_{31}T_2N_3 + c_{32}T_2T_3\ c_{32} = c_{11}T_3N_1 + c_{12}T_3T_1.$$

Equivalently,

$$c_{11} = -\frac{1}{2}c_{21} - \frac{\sqrt{3}}{2}c_{22},\ c_{21} = -\frac{1}{2}c_{31} - \frac{\sqrt{3}}{2}c_{32},\ c_{31} = -\frac{1}{2}c_{11} - \frac{\sqrt{3}}{2}c_{12},$$

and

$$c_{12} = -\frac{\sqrt{3}}{2}c_{21} - \frac{1}{2}c_{22},\ c_{22} = -\frac{\sqrt{3}}{2}c_{31} - \frac{1}{2}c_{32},\ c_{32} = -\frac{\sqrt{3}}{2}c_{11} - \frac{1}{2}c_{12},$$

or equivalently,

$$3\left[\sum_{i=1}^{3} c_{i1}\right] + \sqrt{3}\left[\sum_{i=1}^{3} c_{i2}\right] = 0,\ \sqrt{3}\left[\sum_{i=1}^{3} c_{i1}\right] + 3\left[\sum_{i=1}^{3} c_{i2}\right] = 0,$$

and hence,

$$\sum_{i=1}^{3} c_{i1} = \sum_{i=1}^{3} c_{i2} = 0. \tag{14}$$

From (8),

$$d_{11} = d_{21} = d_{31}. \tag{15}$$

From (9) at $x = 1$,

$$v_i(1,t)T_i = 0,\ i = 1,2,3,$$

or equivalently, using (11),

$$v_{i2}(1,t) = 0,\ i = 1,2,3,$$

or equivalently,

$$c_{i2}\cosh\left(\sqrt{-\lambda}\right) + d_{i2}\sinh\left(\sqrt{-\lambda}\right) = 0,\ i = 1,2,3,$$

or equivalently,

$$c_{i2} + d_{i2} \tanh\left(\sqrt{-\lambda}\right) = 0, \;\; i = 1, 2, 3. \tag{16}$$

From (10) at $x = 1$, we have the following:

$$\left[K^i_{\partial\Omega} v_i(1,t) - v_{ix}(1,t)\right] N_i = 0,$$

or equivalently, using (11),

$$K^i_{\partial\Omega} v_{i1}(1,t) - v_{i1x}(1,t) = 0,$$

or equivalently,

$$K^i_{\partial\Omega}\left[c_{i1}\cosh\left(\sqrt{-\lambda}\right) + d_{i1}\sinh\left(\sqrt{-\lambda}\right)\right] - \sqrt{-\lambda}\left[c_{i1}\sinh\left(\sqrt{-\lambda}\right) + d_{i1}\cosh\left(\sqrt{-\lambda}\right)\right] = 0,$$

or equivalently,

$$K^i_{\partial\Omega}\left[c_{i1} + d_{i1}\tanh\left(\sqrt{-\lambda}\right)\right] - \sqrt{-\lambda}\left[c_{i1}\tanh\left(\sqrt{-\lambda}\right) + d_{i1}\right] = 0,$$

or equivalently,

$$\left[K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\right)\right] c_{i1} + \left[K^i_{\partial\Omega}\tanh\left(\sqrt{-\lambda}\right) - \sqrt{-\lambda}\right] d_{i1} = 0. \tag{17}$$

The eigenvalue λ exists in Equations (16) and (17), but it is only (17) that affects it.

Since $\lambda < 0$, in (17), we have that

$$K^i_{\partial\Omega}\tanh\left(\sqrt{-\lambda}\right) - \sqrt{-\lambda} \neq 0, \;\; i = 1, 2, 3.$$

For $K^i_{\partial\Omega} \leq 0$, we have that

$$K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\right) < 0, \;\; i = 1, 2, 3.$$

Consequently, if we use (15) and set $d_{i1} = d$, $\forall i = 1, 2, 3$, we conclude to $d = c_{i1} = 0$. If this would not hold, c_{i1} would have the opposite sign from d which is not possible from (14). Hence, from the algebraic Equations (14)–(17), we have $d_i = c_{ij} = 0$, $i = 1, 2, 3$, $j = 1, 2$. This means that for $K^i_{\partial\Omega} \leq 0$, there do not exist negative eigenvalues.

For $K^i_{\partial\Omega} > 0$, however, there exist eigenvalues λ such that

$$K^i_{\partial\Omega} - \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\right) = 0.$$

In this case, $d_{i1} = 0$, $\forall i = 1, 2, 3$, and the algebraic system of Equations (14)–(16) is underdetermined. This means that for $K^i_{\partial\Omega} \leq 0$, there exist negative eigenvalues that are the solutions of the above equation, and the eigenfunctions are given by

$$v_{i1} = c_{i1}\cosh\left(\sqrt{-\lambda}x\right)e^{\lambda t}, \;\; i = 1, 2, 3,$$

and

$$v_{i2} = -d_{i2}\left[\tanh\left(\sqrt{-\lambda}\right)\cosh\left(\sqrt{-\lambda}x\right)\right]e^{\lambda t}, \;\; i = 1, 2, 3.$$

We consider again the linear system of PDEs (6) and conditions (7)–(10). Then if $\lambda = 0$, we have the following:

$$v_{ij} = c_{ij} + d_{ij}x, \;\; i = 1, 2, 3, \;\; j = 1, 2.$$

From (7), we have

$$v_1(0,t) = v_2(0,t) = v_3(0,t),$$

or equivalently,

$$c_{11}N_1 + c_{12}T_1 = c_{21}N_2 + c_{22}T_2 = c_{31}N_1 + c_{32}T_3,$$

and consequently, (14) holds. From (8), we have that (15) also holds. From (9), at $x = 1$, we have the following:

$$v_i(1,t)T_i = 0, \;\; i = 1,2,3,$$

or equivalently, using (11),

$$v_{i2}(1,t) = 0, \;\; i = 1,2,3,$$

or equivalently,

$$c_{i2} + d_{i2} = 0, \;\; i = 1,2,3. \tag{18}$$

From (10) at $x = 1$, we have the following:

$$[K^i_{\partial\Omega} v_i(1,t) - v_{ix}(1,t)]N_i = 0,$$

or equivalently, using (11),

$$K^i_{\partial\Omega} v_{i1}(1,t) - v_{i1x}(1,t) = 0,$$

or equivalently,

$$K^i_{\partial\Omega}[c_{i1} + d_{i1}] - d_{i1} = 0,$$

or equivalently,

$$K^i_{\partial\Omega} c_{i1} + [K^i_{\partial\Omega} - 1]d_{i1} = 0. \tag{19}$$

If $K^i_{\partial\Omega} \neq 0, 1$, then from the algebraic Equations (14), (15), (18), and (19), we have $d_{ij} = c_{ij} = 0$, $i = 1,2,3$, $j = 1,2$.

For $K^i_{\partial\Omega} = 0$, however, from (19), we have that $d_{i1} = 0$, $\forall i = 1,2,3$. Then the algebraic system of Equations (14)–(16) is underdetermined. This means that for $K^i_{\partial\Omega} = 0$, the eigenvalue $\lambda = 0$ exists. The eigenfunctions will then be given by

$$v_{i1} = c_{i1}, \;\; i = 1,2,3,$$

and

$$v_{i2} = -d_{i2}(1 + x), \;\; i = 1,2,3.$$

For $K^i_{\partial\Omega} = 1$ in (19), we have that $c_{i1} = 0$, $i = 1,2,3$. Then from the algebraic Equations (14) and (18), we get $d_{i2} = c_{ij} = 0$, $i = 1,2,3$, $j = 1,2$. Consequently, for $K^i_{\partial\Omega} = 1$, the eigenvalue $\lambda = 0$ exists. If we use (15) and set $d_{i1} = d$, $i = 1,2,3$, the eigenfunctions will be given by the following:

$$v_{i1} = d, \;\; i = 1,2,3,$$

and

$$v_{i2} = 0, \;\; i = 1,2,3.$$

The proof is completed. □

4.1 | Numerical example

For the sake of illustration, we consider the simple example of a dynamical network of three curves with boundary the unit circle, as shown in Figure 1. In this case, we have that $K^i_{\partial\Omega} = 1 > 0$, and thus, from Theorem 4.1, there exist negative eigenvalues of the corresponding BVP and the steady state of the network is unstable. Indeed, calculation of the eigenvalues from the solution of the following:

$$1 - \sqrt{-\lambda}\tanh\left(\sqrt{-\lambda}\right) = 0, \;\; i = 1,2,3,$$

gives that $\lambda = -1.4392 < 0$, with multiplicity equal to three.

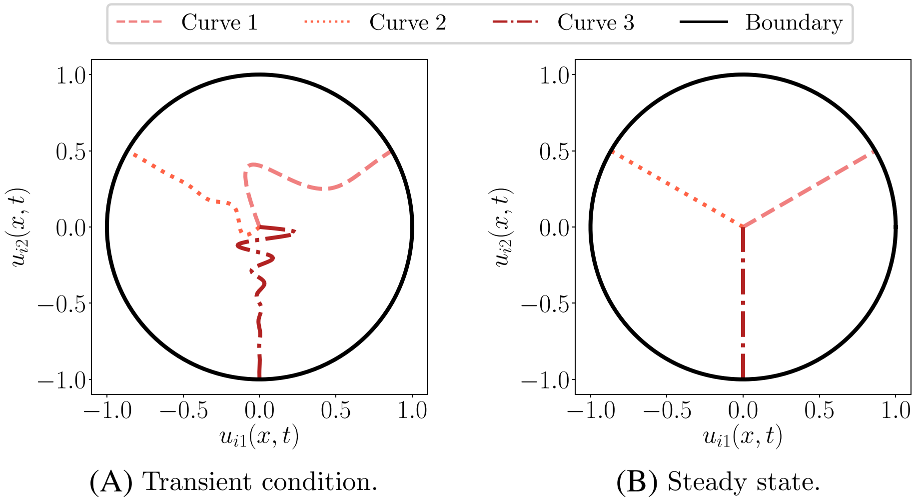

FIGURE 1 (A, B) Illustration of simple network of three curves with circular boundary. [Colour figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSIONS

In this article, we studied a network of curves that are in motion, meet at a junction, and are bounded. We first focused on the construction of the model which is a BVP that consists of the system of nonlinear PDEs (1) and conditions (2)–(5). We then linearized the PDEs, reformulated the conditions, and for the new BVP that appeared, the system of linear PDEs (6), and conditions (7)–(10), we studied the stability of the steady states.

We concluded that stability depends on the sign of the curvature of the boundary of the domain at the points that the boundary meets each curve. We also provided the negative eigenvalues and their eigenfunctions, as well as the eigenfunctions of the zero eigenvalue.

As a further extension of this article, we aim to apply the results to hexagonal networks and study the stability of this type of networks having also in mind applications in soap bubbles, honeycomb, and grain growth. Additionally, we aim to use the techniques and properties proved in the main results of this article to update the geometrical properties that are used in electrical circuit theory and elasticity, plasticity problems in material science. For all these, there is already some ongoing research.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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REFERENCES

1. A. Freire, *Mean curvature motion of triple junctions of graphs in two dimensions*, Commun. Partial Differ. Equ. **35** (2010), no. 2, 302–327.
2. D. Garlaschelli, F. Ruzzenenti, and R. Basosi, *Complex networks and symmetry I: a review*, Symmetry **2** (2010), 1683–1709, DOI 10.3390/sym2031683.
3. R. Goldman, *Curvature formulas for implicit curves and surfaces*, Comput. Aided Geom. Des. **22** (2005), no. 7, 632–658.
4. E. Lenes, E. Mallea-Zepeda, M. Robbiano, and J. Rodríguez, *On the diameter and incidence energy of iterated total graphs*, Symmetry **10** (2018), no. 7, 252.

5. P. Pozzi and B. Stinner, *On motion by curvature of a network with a triple junction*, The SMAI J. Comput. Math. **7** (2021), 27–55.
6. L. Bronsard and B. T. R. Wetton, *A numerical method for tracking curve networks moving with curvature motion*, J. Computat. Phys. **120** (1995), no. 1, 66–87.
7. C. M. Elliott and H. Fritz, *On approximations of the curve shortening flow and of the mean curvature flow based on the DeTurck trick*, IMA J. Numerical Anal. **37** (2017), no. 2, 543–603.
8. L. Hou, M. Small, and S. Lao, *Dynamical systems induced on networks constructed from time series*, Entropy **17** (2015), 6433–6446, DOI 10.3390/e17096433.
9. H. B. Li, M. Y. Song, E. J. Zhong, and X. M. Gu, *Numerical gradient schemes for heat equations based on the collocation polynomial and Hermite interpolation*, Mathematics **7** (2019), no. 1, 93.
10. C. Tortorelli, M. Mantegazza, and V. M. Novaga, *Motion by curvature of planar networks*, 2003. arXiv preprint math/0302164.
11. X. Wang, G.-P. Jiang, and X. Wu, *State estimation for general complex dynamical networks with incompletely measured information*, Entropy **20** (2018), 5, DOI 10.3390/e20010005.
12. B. Boutarfa and I. Dassios, *A stability result for a network of two triple junctions on the plane*, Math. Methods Appl. Sci. **40** (2017), no. 17, 6076–6084.
13. I. Dassios, *Stability of basic steady states of networks in bounded domains*, Comput. Math. Appl. **70** (2015), no. 9, 2177–2196.
14. I. Dassios, *Stability of bounded dynamical networks with symmetry*, Symmetry **10** (2018), no. 4, 121.
15. E. Yanagida and R. Ikota, *A stability criterion for stationary curves to the curvature driven-motion with a triple junction*, Differ. Integr. Equ. **16** (2003), no. 6, 707–726.
16. R. Ikota and E. Yanagida, *Stability of stationary interfaces of binary-tree type*, Calc. Var. **22** (2004), no. 4, 375–389.
17. G. Bellettini, M. Chermisi, and M. Novaga, *Crystalline curvature flow of planar networks*, Interfaces Free Bound. **8** (2006), no. 4, 481–521.
18. E. Fried and M. E. Gurtin, *Gradient nano-scale polycrystalline elasticity: inter grain interactions and triple-junction conditions*, J.. Mech. Phys. Solids. **57** (2009), no. 10, 1749–1779.
19. M. E. Gurtin and L. Anand, *Nano-crystalline grain boundaries that slip and separate: a gradient theory that accounts for grain-boundary stress and conditions at a triple-junction*, J. Mech. Phys. Solids. **56** (2008), no. 1, 184–199.
20. F. Ruzzenenti, D. Garlaschelli, and R. Basosi, *Complex networks and symmetry II: reciprocity and evolution of world trade*, Symmetry **2** (2010), 1710–1744, DOI 10.3390/sym2031710.
21. Corematerials, *Growth of a two-dimensional grain structure*, 2009. https://www.youtube.com/watch?v%3DJ_2FdkRqmCA
22. H. Graham, *An introduction to hexagonal geometry*, 2010. https://www.hexnet.org
23. J. Humphreys, *Grain growth simulation*, 2012. https://www.youtube.com/watch?v%3DHsVnswfHGXM.
24. T. Pappas, Hexagons in nature, *The joy of mathematics*, Wide World Publ./Tetra, San Carlos, CA, 1989, pp. 74–75.
25. I. Dassios, A. Jivkov, A. Abu–Muharib, and P. James, *A mathematical model for plasticity and damage: a discrete calculus formulation*, J. Comput. Appl. Math. Elsevier **312** (2017), 27–38.
26. I. Dassios, G. O'Keeffe, and A. Jivkov, *A mathematical model for elasticity using calculus on discrete manifolds*, Math. Methods Appl. Sci. **41** (2018), no. 18, 9057–9070.
27. F. Milano, G. Tzounas, I. Dassios, and T. Kerci, *Applications of the Frenet frame to electric circuits*, IEEE Trans. Circuits Syst. I **69** (2022), no. 4, 1668–1680.

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