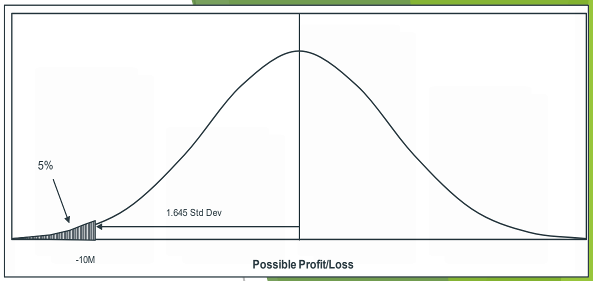


Quantitative Risk Management: market risk measurement methods

Svetlana Borovkova

Recall:

- Main risk measures for portfolios or risky assets are:
- VaR (Value-at-Risk) at horizon *h*, confidence level *a%*: highest loss with probability *a* when holding portfolio for *h* days
- ES or CVaR (Expected Shortfall) at horizon *h*, confidence level *a%*: average or expected loss when VaR is exceeded
- How to estimate these measures for a portfolio of risky assets, from historical data of prices/returns of these risky assets?
- These risk measures are uniquely determined by the distribution of losses, which we will call loss distribution

Portfolio Values and Losses

Consider a portfolio and let V_t denote its value at time t; we assume this random variable is observable at time t.

Suppose we look at risk from perspective of time t and we consider the time period $[t, t+1]$. The value V_{t+1} at the end of the time period is unknown to us.

The distribution of $(V_{t+1} - V_t)$ is known as the profit-and-loss or P&L distribution. We denote the loss by $L_{t+1} = -(V_{t+1} - V_t)$. By this convention, losses will be positive numbers and profits negative.

We refer to the distribution of L_{t+1} as the loss distribution.

So formally:

Let F_L denote the distribution function of loss

- Primary risk measure: Value at Risk defined as

$$\mathrm{VaR}_\alpha = q_\alpha(F_L) = F_L^{\leftarrow}(\alpha)\,, \tag{6}$$

  i.e. the α-quantile of F_L.

- Alternative risk measure: Expected shortfall defined as

$$\mathrm{ES}_\alpha = E\left(L \mid L > \mathrm{VaR}_\alpha\right)\,, \tag{7}$$

  i.e. the average loss when VaR is exceeded. ES_α gives information about frequency and size of large losses.

Unconditional risk estimation

- If we assume that losses are i.i.d. drawings from the (unknown) loss distribution F_L, then VaR and ES would not change from one day to the next
- This is called unconditional risk estimation
- In this approach, historical data can be used in its entirety and its raw form to estimate the loss distribution or only the risk measures
- This is quite typical approach in e.g., credit or operational risk

Conditional risk estimation

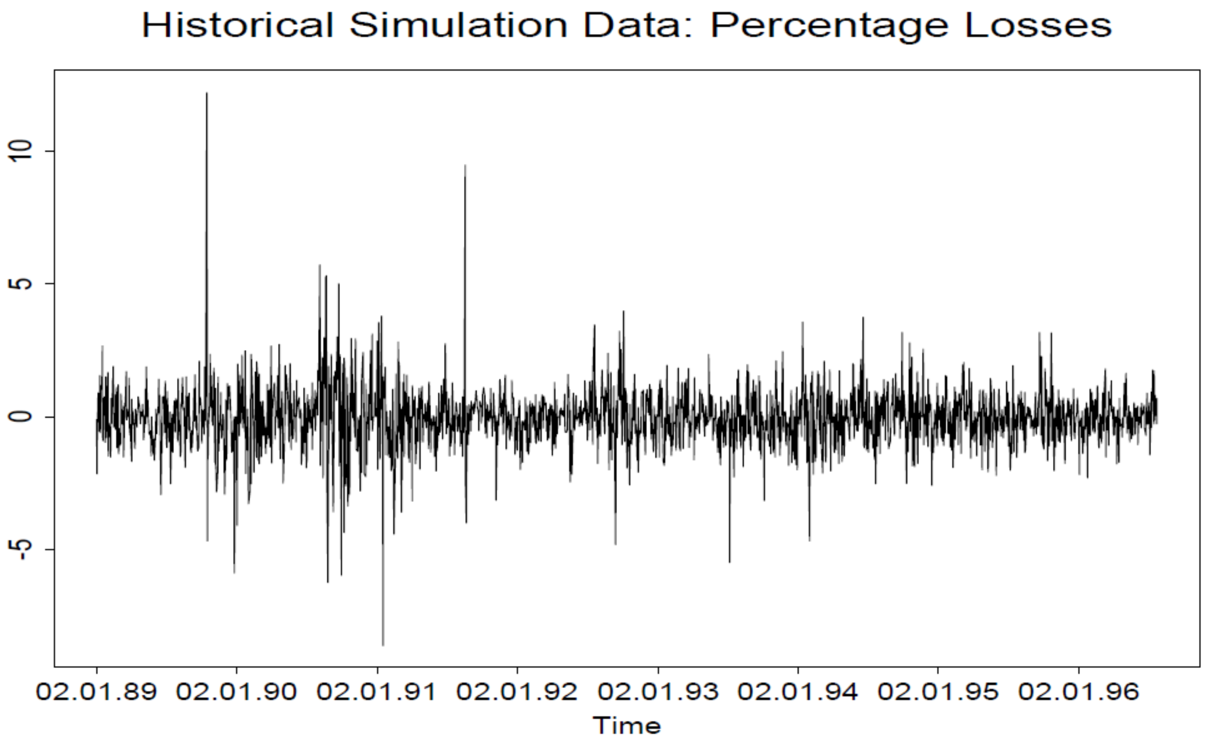
- However, if we assume that losses have time series dynamics, then loss distribution (and hence, VaR and ES) changes day to day, e.g., due to changing volatility
- This is called conditional risk estimation

Let $\mathcal{F}_t$ represent the history of the risk factors up to the present.

In the conditional problem we are interested in the distribution of $L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$ given $\mathcal{F}_t$, i.e. the conditional (or predictive) loss distribution for the next time interval given the history of risk factor developments up to present.

This problem forces us to model the dynamics of the risk factor time series and to be concerned in particular with predicting volatility. This seems the most suitable approach to market risk.

Historical method, also called historical simulation



Measurement of VaR: historical

- If you have historical portfolio returns, the task is easy:
- Calculate portfolio losses: loss(i) = - principal amount * return(i)
- Order them in increasing order: *l(1:n)* <= *l(2:n)* <=…<= *l(n:n)*
- Empirical quantile estimation: *VaR(1 day, a)* = *l(k:n)* such that *k=[na]*
- *For example*, if *a=0.95* and *n=1000*, then *VaR(a)* = *l(950:1000)*, i.e. 50th biggest loss.
- Historical ES: average of *[n*(1-a)]* highest losses
- For example, if *n=1000*, *a=95%*, then ES is the average of the 50 largest losses
- *Precision? Follows from formula for the standard error of sample quantiles, see e.g.* https://en.wikipedia.org/wiki/Quantile

When there are individual assets in portfolio:

- Collect a (large) dataset of historical (daily) asset returns
- Generate from that a (hypothetical) dataset of the portfolio losses: $l(1), l(2), \ldots, l(n)$
-
- This is not trivial→ see spreadsheet example
- And then do the same as for 1-asset portfolio (empirical quantile estimation - previous slide)

Measurement of VaR: historical

Advantages:

- easy to compute and to understand
- no need to know correlations between assets in the portfolio

Disadvantages:

- Forecast of *VaR* never bigger than biggest recorded loss:

  "Driving a car while looking through the rear view mirror"
- Not a problem if stress period is included
- Historical data might be unavailable for all assets in portfolio
- Risk measures do not adapt to the current situation, i.e., this is an unconditional approach

Var-Cov method, also called Normal VaR

- Assume that the distribution of daily returns of assets in the portfolio is multivariate normal
- Then, for linear portfolios, the portfolio returns are also normal
- Portfolio return and volatility *are calculated* via well-known portfolio mean return and portfolio variance formulae (see Lecture 1)
- Percentage-wise VaR(1,a) is calculated as:

$$VaR(1,\alpha) = \mu + z_\alpha \sigma$$

where μ is the average loss (so negative of the return) over one day, σ is daily volatility and z_α is the *ath* quantile of the standard normal distribution

- VaR in monetary units is calculated as: principal value * percentage VaR

Example

- Suppose a portfolio, which is worth 1M Euros, has mean return of 6% p/a and volatility 17% p/a (obtained by e.g., variance-covariance formula)
- What is VaR(1, 97.5%)?

- Mean loss over 1 day is approx. -6%/250=-0.024% (so almost zero)
- Volatility over 1 day is 17%/$\sqrt{250}$=1% per day

- VaR(1 day, 97.5%) = -0.024% + 1.96*1% = 1.936%
- In monetary units: VaR(1 day, 97.5%) = 19 360 Euros
- So in 2.5% of worst days, the loss of this portfolio will be 19360 Euros or more

Expected Shortall

- Expected shortfall for Normal loss distribution with mean loss μ and volatility σ:

$$ES_\alpha = \mu + \sigma \frac{f(F^{-1}(\alpha))}{1-\alpha}$$

- Here f is pdf and F cdf of standard normal distribution, and $F^{-1}(\alpha) = z_\alpha$ - exactly the z-value
- This is again in %. To obtain ES in monetary units, multiply by principal portfolio value

Remark. For a rv $Y \sim N(0,1)$ it can be shown that
$E(Y \mid Y > \Phi^{-1}(\alpha)) = \phi(\Phi^{-1}(\alpha))/(1-\alpha)$
where ϕ is standard normal density and Φ the df.

The N-day VaR

- The *h*-day VaR for market risk is usually assumed to be $\sqrt{h}$ times the one-day VaR:

$$VaR(h,\alpha) = \sqrt{h} \cdot VaR(1,\alpha)$$

- This assumption is only correct if daily returns are uncorrelated AND average daily return is (approximately) zero

- Otherwise we must use

$$VaR(h,\alpha) = \mu_h + z_\alpha \sigma_h$$

- What about *N*-day ES? Again, use *h*-day mean loss and *h*-day volatility and normal ES formula

Example cont'd

- Same example as before. Now calculate VaR(10 days, 97.5%)

- 10-day mean loss = -6%/25=-0.24%
- 10-day volatility = 1% * $\sqrt{10}$=3.16%

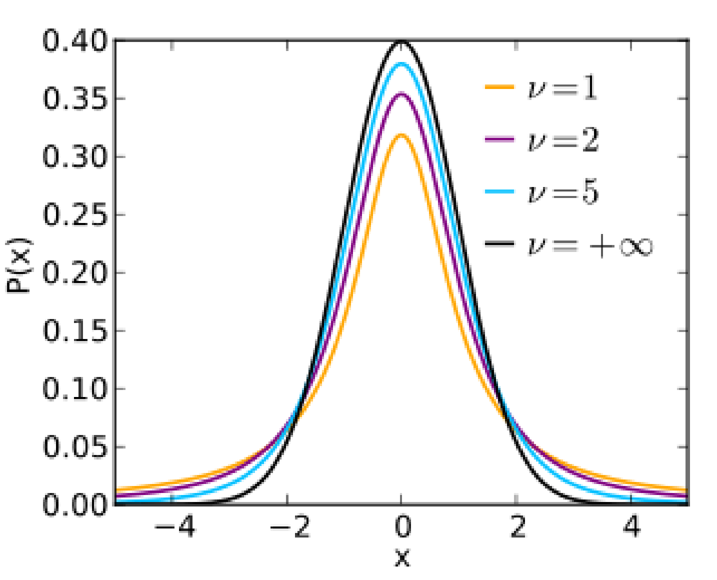
- VaR(10 days, 97.5%) = -0.24 + 1.96*3.16 = 5.95% or 59 500 euros

- If we assumed that daily return is zero, then

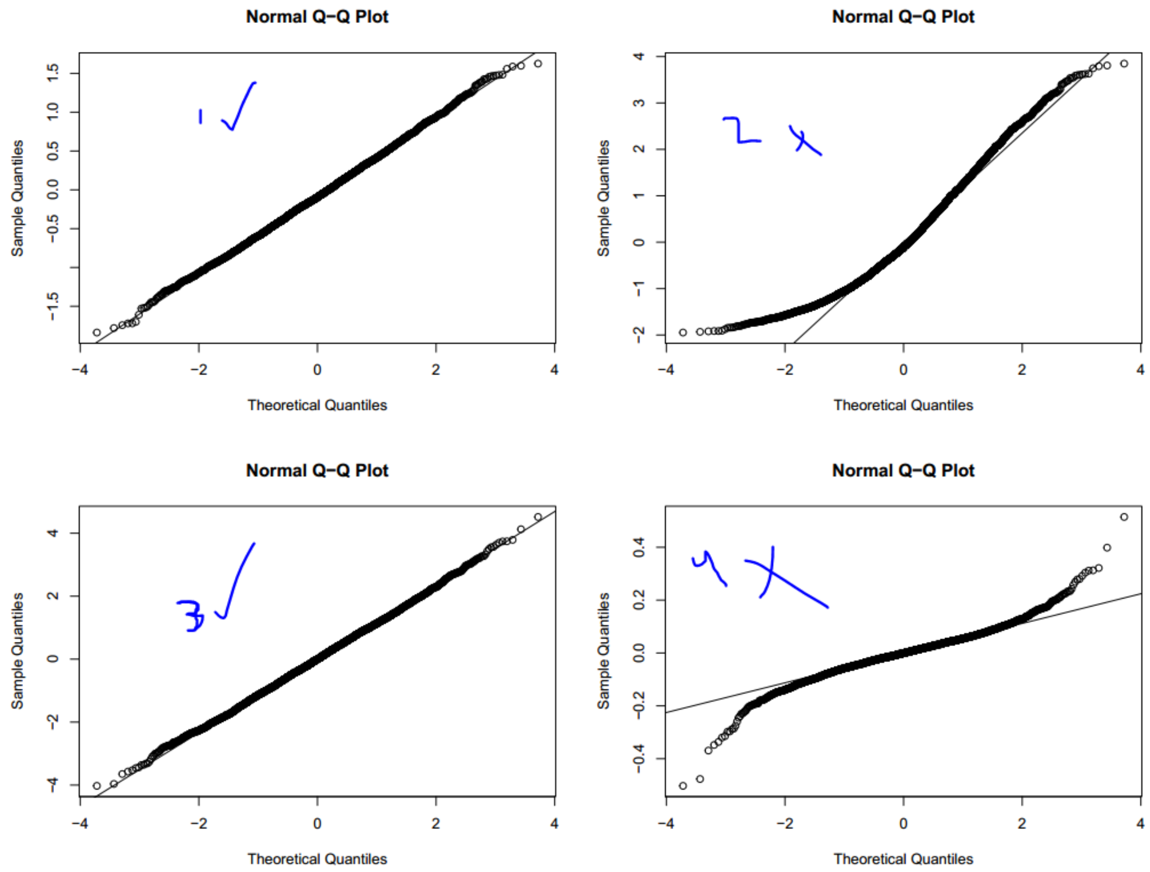
VaR(10 days, 97.5%) = 3.16*19360 = 61 178 euros

Normal vs Student-t distribution

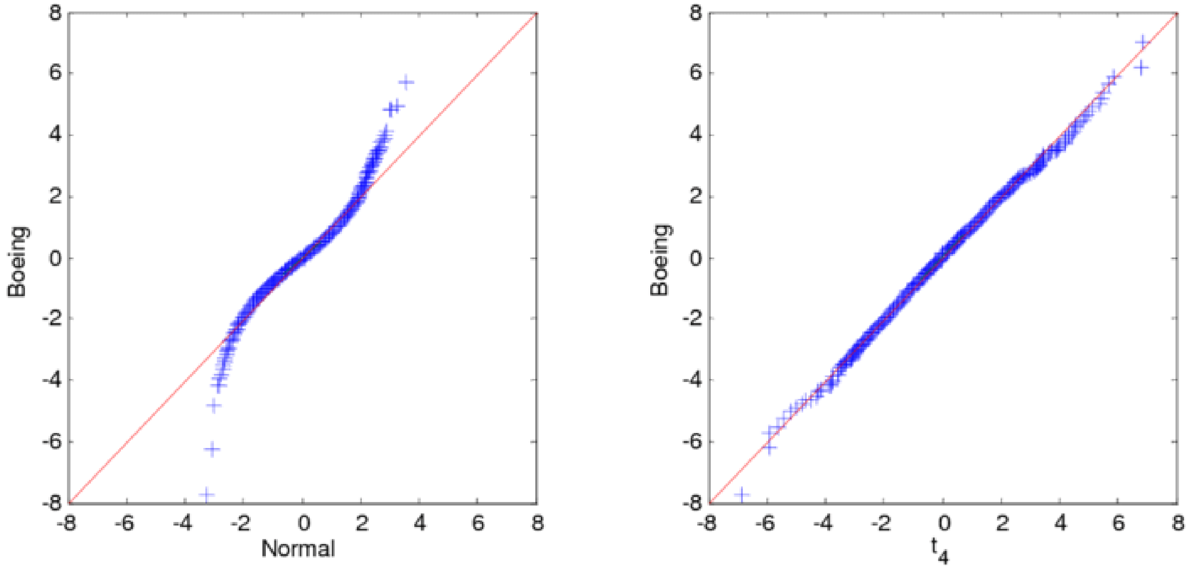
- Normal distribution is not a good model for financial asset returns
- Tails are more heavy than normal
- What to do? Use a variant of Variance-Covariance method with Student-t distribution with low number of d.f. instead of normal (i.e., replace normal quantiles (z-values) by Student-t quantiles)
- How to decide on degrees of freedom for Student-t distribution? QQ-plot
- But pay attention to normal vs Student-t variance/volatility!



Normal vs heavy tailed: QQ plots



Normal vs Student-t QQ plots



Volatility of Student-t distribution with *v* degrees of freedom

- In VaR formula you need to use σ parameter of Student-t distribution, but it is not its standard deviation
- The standard deviation of Student-t random variable with parameter σ and v degrees of freedom:

$$\text{StDev} = \sigma * \sqrt{v/(v-2)}.$$

- So suppose you estimated standard deviation of the returns from daily data, denote this estimate SD. Then σ you need to fill in VaR formula is

$$\sigma = \text{SD} \,/\, \sqrt{v/(v-2)}.$$

Conditional risk measurement

- Now everything we did needs to be repeated, but now the loss distribution of interest is not the unconditional distribution F_L, but the distribution of conditional loss $L_{t+1} \mid \mathcal{F}_t$.

- So we need to obtain conditional VaR and ES (which now depend on time t:

$$\begin{aligned} \mathrm{VaR}_\alpha^t &= F^{\leftarrow}_{[L_{t+1}|\mathcal{F}_t]}(\alpha), \\ \mathrm{ES}_\alpha^t &= E\left(L_{t+1} \mid L_{t+1} > \mathrm{VaR}_\alpha^t, \mathcal{F}_t\right). \end{aligned}$$

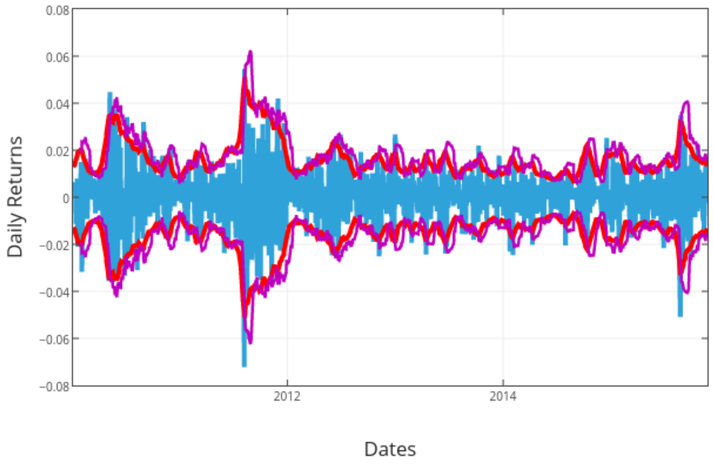
- In Var-Cov method, this amounts to keeping the parametric form of the loss distribution the same (e.g., normal) but using time-changing volatility in all formulae

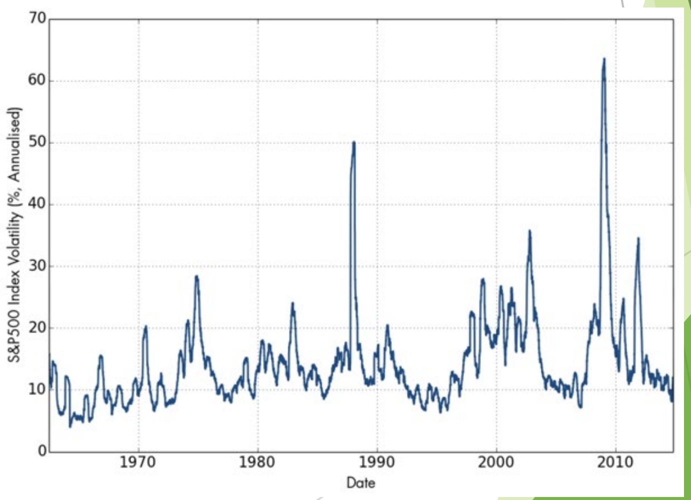
- There is also a conditional variant of historical simulation method, called Filtered Historical Simulation. It is very popular now but quite complicated, and we will discuss it first thing on next lecture.

Non-constant volatility

- Both historical and Variance-Covariance methods just described do not consider different volatility regimes

S&P 500 Daily Returns and Stochastic vs Historical Volatility bands





Weighting Scheme for variance

- We assume that the returns satisfy: $u_n = \sigma_n z_n$, where z_n is Normal (0,1) (or Student-t) random variable

To calculate current variance or volatility, instead of assigning equal weights to past squared returns, we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

and where weights decrease for more "past" observations

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1-\lambda) u_{n-1}^2$$

- It is usually taken λ = 0.94 (RiskMetrics uses λ = 0.94 for daily volatility forecasting)
- Sometimes 0.97 is used to reduce cyclicality of risk measures

GARCH (1,1)

- In GARCH (1,1) (Bollerslev, 1986) we assign some weight to the long-run average variance rate (in comparison to EWMA):

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

- All coefficients are again estimated by ML-like methods from historical data
- Weights must be positive and sum to 1:

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and the long-term variance is given by:

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

Stationarity condition? Weak stationarity = existence of long-term variance. When is this guaranteed?

Example: same portfolio as before; suppose we estimated GARCH(1,1) model, assume mean daily loss is zero

- GARCH(1,1): $\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$
- The long-run variance is 0.0002 so that the long-run volatility per day is 1.4%
- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

  $$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

  The new volatility is 1.53% per day
- Old VaR(1 day, 97.5%) = 0.016*1.96*1M = 31360 Euros
- New VaR(1 day, 97.5%) = 0.0153*1.96*1M = 29988 Euros
- Required capital decreases by 4116 Euros

Higher order GARCH(p,q)

- We can take into account more lagged variables, of squared returns as well as past variances
- This leads to higher order GARCH:

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

- However, this is not used in practice, only GARCH(1,1).
- This is because nothing beats it in terms of fit and low number of parameters.

What about (dynamic) covariances and correlations?

- We can try to use similar models to those for volatilities
- For example, EWMA: $\text{cov}_n = \lambda\, \text{cov}_{n-1} + (1-\lambda) x_{n-1} y_{n-1}$
- But then we will quickly run into a problem: variance-covariance matrix Ω must be positive semidefinite: for all vectors $\boldsymbol{w}$ we must have:

$$\mathbf{w}^{\mathbf{T}} \Omega \mathbf{w} \geq 0$$

- For example, this matrix cannot be a covariance matrix:

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

Simple solution: CCC model (constant conditional correlation)

- Use GARCH for all individual assets' variances
- Estimate long-run correlation matrix of returns and ***keep it constant***
- Update covariances with time-changing volatilities and constant correlations: $Cov(t; i, j) = \rho_{ij}\sigma_{t,i}\sigma_{t,j}$
- Then use usual Var-Cov methodology for VaR and ES estimation
- Problem with this nice and easy method: correlations between asset returns go up in times of stress!
- More complex model: DCC (Dynamic Conditional Correlation)

Backtesting VaR and ES models

- In-sample vs out-of-sample? Preferably out-of-sample (moving window approach), but in-sample can also give a good indication of model performance

- How to test VaR model?
- What do we want from a good VaR model? Two things:

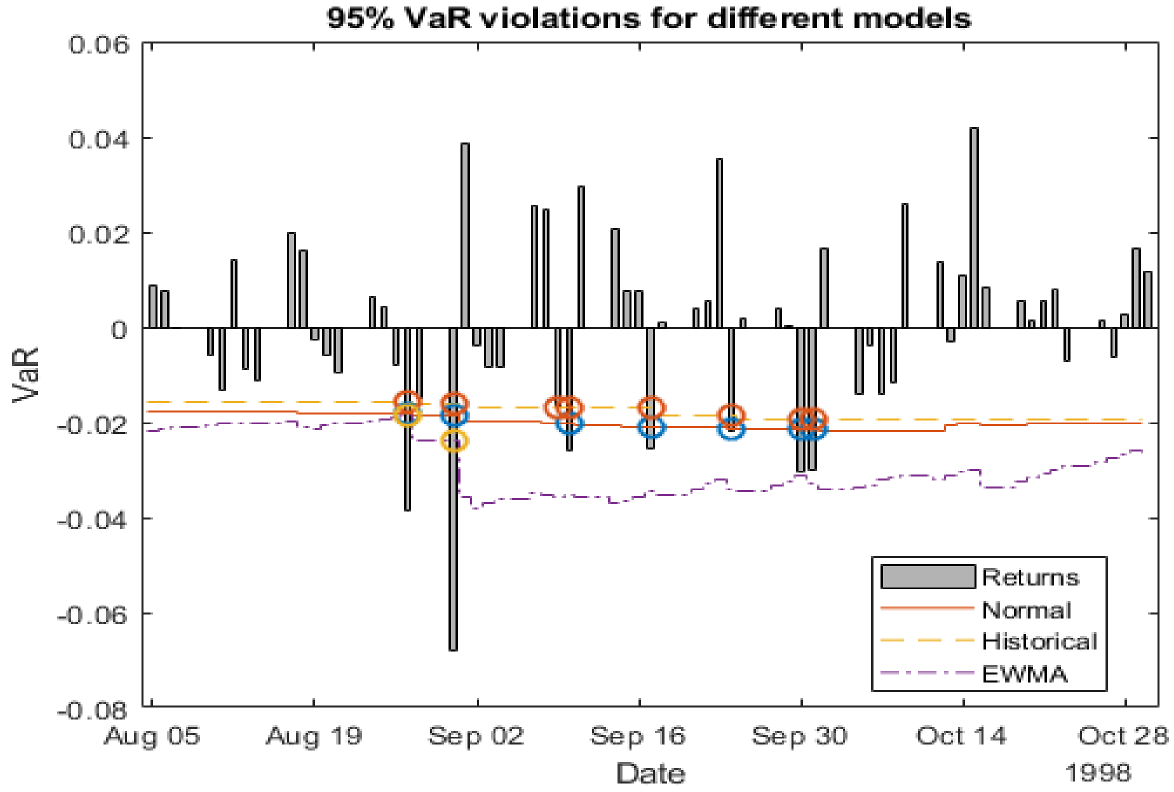
→ that losses exceed VaR(a%) approximately 1-a% of the time

→ that losses exceed VaR in a “random” fashion, and not in big clusters

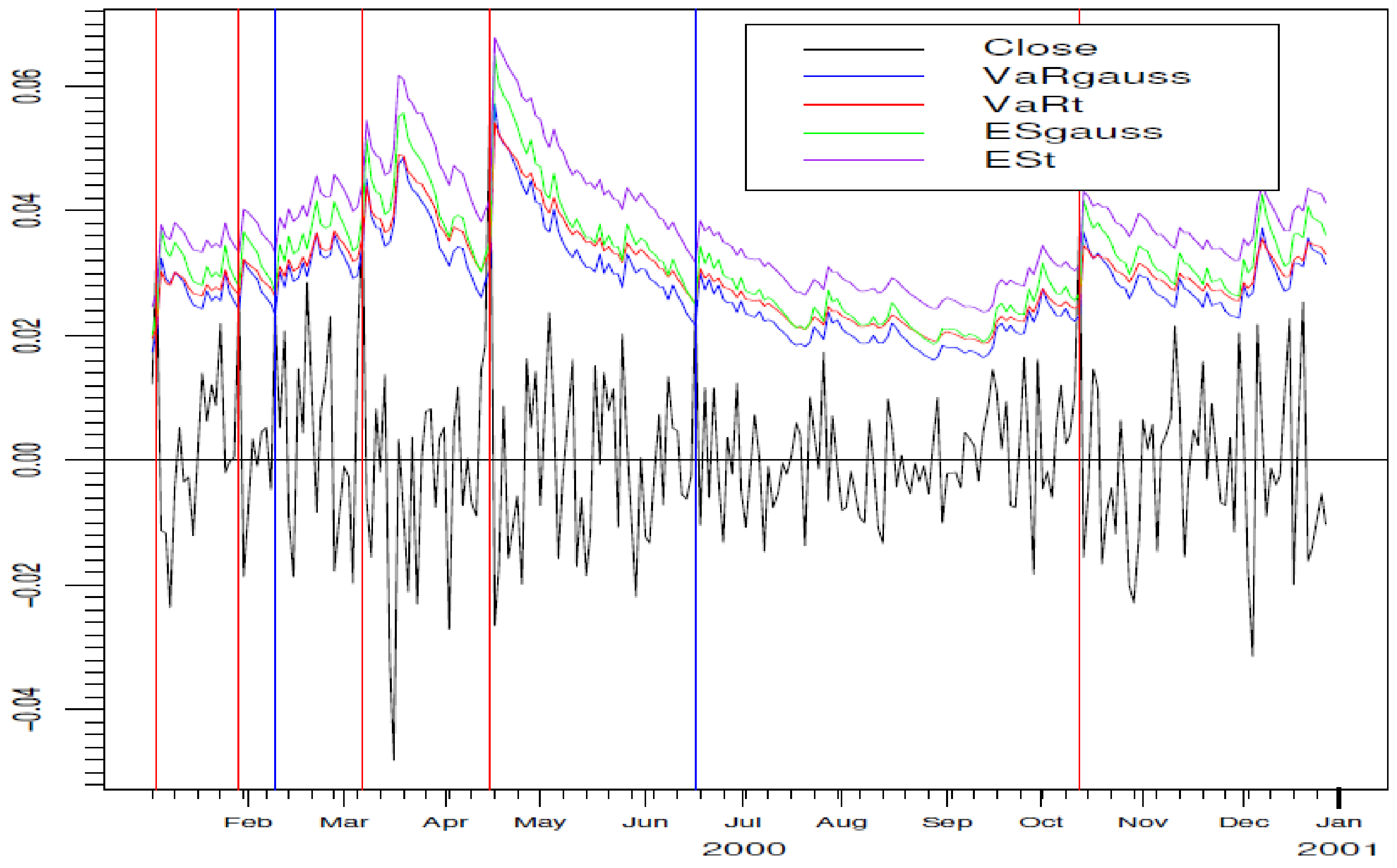
Ad-hoc testing:

- The first wish can be “tested” by counting days in historical dataset when loss exceeds VaR (this is called VaR violation) and comparing this number with expected number of violations: N*(1-a)
- The second wish can be “tested” by looking at graphs of losses and instances of VaR violations, and observing whether there are clusters of VaR violations

Example of VaR violations



VaR violations for conditional RM



How to statistically test VaR?

- Let $I_t = 1_{\{L^{\Delta}_{t+1} > \mathrm{VaR}^t_\alpha\}}$ be the indicator of VaR violation on day t+1

(i.e., it is equal to 1 if t -day loss is greater than t-day VaR and zero otherwise)

- Under the null hypothesis that VaR model is correct, these indicators are (i.i.d.) Bernoulli random variables:

$$I_t = 1_{\{Z_{t+1} > q_\alpha(F_Z)\}} \sim \mathrm{Be}(1-\alpha);$$

- Of course we do not know VaR^t_α and will in practice look at the violation indicators $\widehat{I}_t = 1_{\left\{L^{\Delta}_{t+1} > \widehat{\mathrm{VaR}}^t_\alpha\right\}}$. We expect these to be roughly iid Bernoulli$(1-\alpha)$.

How to construct statistical test?

- The total number of violations in the test set of size *N* has Binomial distribution under the null hypothesis: $\sum_{t=1}^{N} I_t \sim Bin(N, 1-\alpha)$

- To conclude whether the observed number of violations significantly differs from the expected number: $N * (1-\alpha)$, the binomial test statistics can be used:

$$T(N) = \frac{\sum_{t=1}^{N} I_t - N(1-\alpha)}{\sqrt{N\alpha(1-\alpha)}} \sim Normal(0,1)$$

  and its p-value can be computed.

- For example: suppose historical dataset is 4 years (1000 observations). Suppose that we are testing 99% daily VaR and we observed 15 violations (and we expect 10). Is the difference statistically significant?

- We calculate T(1000)= (15-10)/ $\sqrt{1000 * 0.01 * 0.99}$=1.59. p-value (two-sided) corresponding to this value of T statistics is 0.11 so the difference is NOT significant at 5%. One-sided p-value is 0.055 so the difference is STILL not significant at 5%.

Violation Count Tables and Binomial Tests

Quantile	Method	S&P $n = 7414$	DAX $n = 5146$
95%	Expected	371	257
95%	GARCH (Normal)	384 (0.25)	238 (0.11)
95%	GARCH (t)	404 (0.04)	253 (0.41)
99%	Expected	74	51
99%	GARCH (Normal)	104 (0.00)	74 (0.00)
99%	GARCH (t)	78 (0.34)	61 (0.11)

Expected and observed violation counts for VaR estimates for two market indices obtained from GARCH modelling (Gaussian and scaled t innovations). Methods use last 1000 data values for each forecast. p-value for binomial test given in brackets.

How to test ES?

- Similar to VaR: compute average loss on the days that VaR is exceeded and compare it to the estimated ES
- Significance in difference between the two can be established by a regular t-test, because under null hypothesis it has mean zero.

Stressed VaR, stress testing

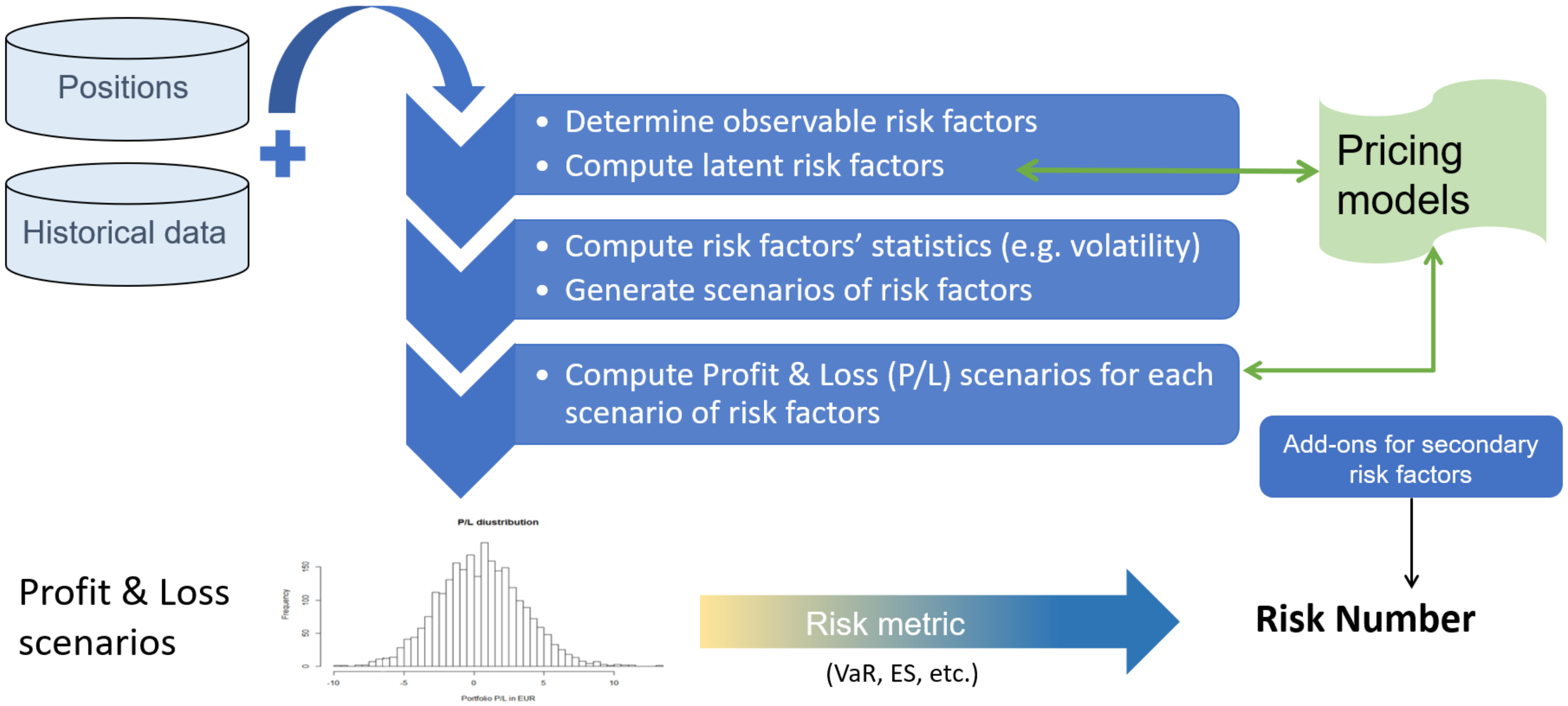
- Nowadays VaR and ES must be augmented with stress scenarios
- This can be done in several ways, for example:

$$VaR(total) = 0.75\ VaR + 0.25\ VaR_{stress}$$

- VaR(stress) could be obtained from stressed historical and/or hypothetical scenarios, e.g. using historical prices/returns from stress periods (2007-2008, 2011, Feb-May 2020).

- Stress testing: examining potential losses in hypothetical extreme scenarios, without giving them probabilities
- Example of stress scenarios: daily loss of all equity markets of 10% (or more), or daily shock to exchange rate 20%.

General setup of MR model



Read the following chapters/sections of the book:

- Chapter 4, section 4.2
- Chapter 9:

9.2 (except EVT-based estimators, pp 348-349)

9.3: subsections 9.3.1, 9.3.2, 9.3.4

- Section 9.3, and especially subsection 9.3.4 will serve you as a guidance for your first assignment: comparison of various RM models