2.2 The limit of function; vertical asymptotes

1. concept of limit 極限的概念 $\lim_{x\to a} f(x) = L$ §2.4 會有正式的定義
2. one-side limit 單邊極限 $\lim_{x\to a^{\pm}} f(x) = L$
3. infinite limit 無限極限 $\lim_{x\to a} f(x) = \pm\infty$ (vertical asymptote 垂直漸近線)

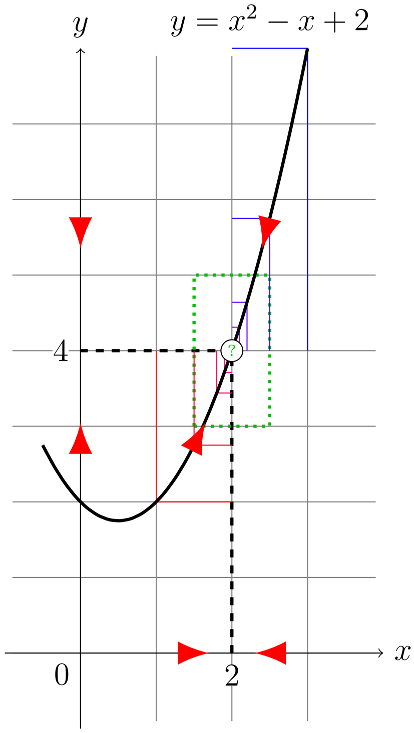
什麼是“極限”? 極限是種趨勢傾向; 一個函數在某個點的極限, 就是當你靠近這個點, 這個函數的趨勢傾向.

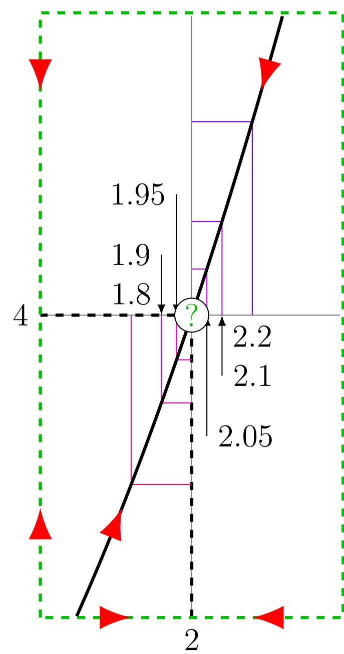
0.1 Concept of limit

Let $f(x) = x^2 - x + 2$. When x near 2, what's happened to $f(x)$?

$x(<2)$	$f(x)$
1	2
1.5	2.75
1.8	3.44
1.9	3.71
1.95	3.8525
1.99	3.9701
1.995	3.985025
1.999	3.997001

$x(>2)$	$f(x)$
3	8
2.5	5.75
2.2	4.64
2.1	4.31
2.05	4.1525
2.01	4.0301
2.005	4.015025
2.001	4.003001





Question: Where does $f(x)$ go when x go toward 2?
Answer: 4. (怎麼簡單表示? 用極限.)

Define: f is defined when x is <u>near</u> a. $[x \in (c,b)(\setminus\{a\}), c < a < b]$

$$\boxed{\lim_{x\to a} f(x) = L}$$

The ***limit*** 極限 of $f(x)$ is equals to L as x ***approaches*** 靠近 a.
(只要 x 靠近 a, $f(x)$ 的極限等於 L.)
或

$$\boxed{f(x) \to L \text{ as } x \to a}$$

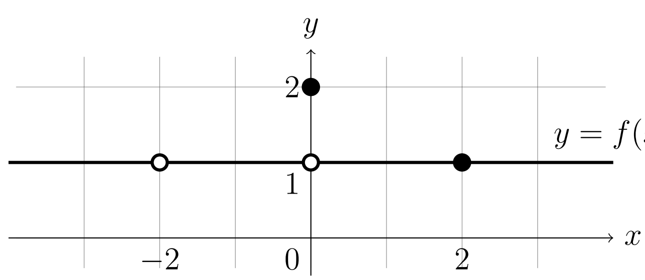
$f(x)$ approaches L as x approaches a. (只要 x 靠近 a, $f(x)$ 就會靠近 L.)

$\overset{\lim_{x\to a} f(x)}{\text{A man}}$ is known by the company $\overset{a}{\text{he}}$ keeps. 觀其友, 知其人.

$\lim_{x\to 朱}$ 色$(x) =$赤, $\lim_{x\to 墨}$ 色$(x) =$黑. 近朱者赤, 近墨者黑. 朱有多赤? 墨有多黑?

Observation: 求極限 $\lim_{x\to a} f(x)$, 是研究 f 在 a 附近的行爲, 與 $f(a)$ 無關.
當 $\lim_{x\to a} f(x) = L$ (代表極限存在, 且爲一個確定值 L), $f(a)$ 會有三種情形:

1. $f(a)$ is not defined. (ex: when $a = -2$)
2. $f(a)$ is defined but $f(a) \neq L$. (ex: when $a = 0$)
3. $f(a) = L$. (ex: when $a = 2$) (這時候我們稱: f 在 a 點連續, see §2.5.)

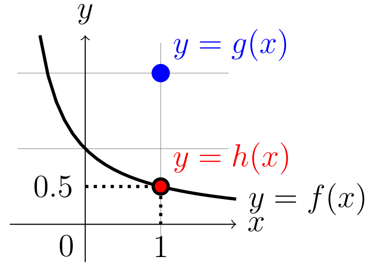


$\forall\, a$, $\lim_{x\to a} f(x) = 1$, $f(-2)$ 未定義, $f(0) = 2 \neq 1$, $f(2) = 1$.

Example 0.1 *Guess* $\lim_{x\to 1}\dfrac{x-1}{x^2-1}$ =?

Let $f(x)=\dfrac{x-1}{x^2-1}$. *(沒明說, 定義域:* $x\neq\pm 1$.*)*

$x(<1)$	$f(x)$	$x(>1)$	$f(x)$
0.5	0.666667	1.5	0.4
0.9	0.526316	1.1	0.47619
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975



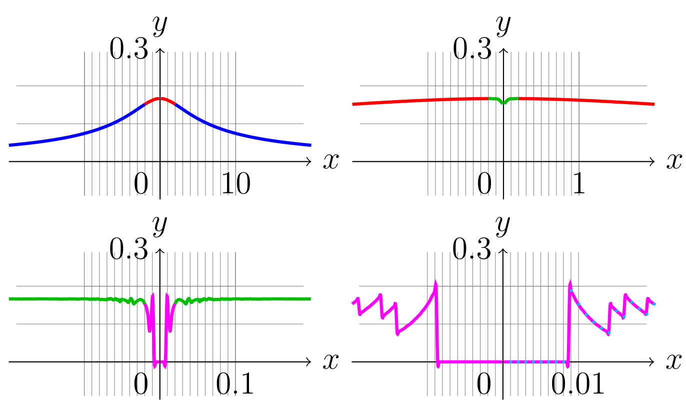
用計算機算應該是 0.5; 從繪圖軟體看應該也是 0.5. $\lim_{x\to 1}\dfrac{x-1}{x^2-1}=\dfrac{1}{2}$ ($\checkmark$). ■

(不管是: $\begin{cases} \textit{1. } f(1) \text{ 未定義}, \\ \textit{2. let } g(x)=f(x),\ x\neq 1 \textit{ and } g(1)=2, \\ \textit{3. let } h(x)=f(x),\ x\neq 1 \textit{ and } h(1)=0.5, \end{cases}$

都不會影響極限: $\lim_{x\to 1}f(x)=\lim_{x\to 1}g(x)=\lim_{x\to 1}h(x)=\dfrac{1}{2}$.*)*

Example 0.2 *Estimate* $\lim_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}$ =?

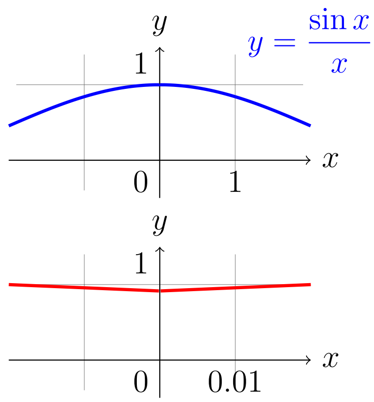
t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1	0.16228
± 0.5	0.16553
± 0.1	0.16662
± 0.05	0.16666
± 0.01	0.16667
± 0.0005	0.168
± 0.0001	0.2
± 0.00005	0
± 0.00001	0

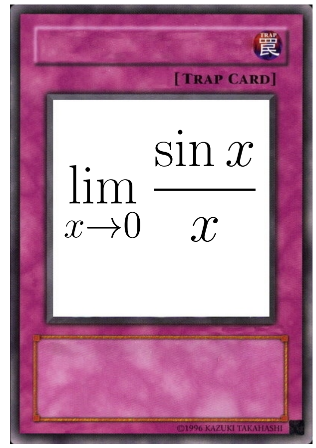


用計算機算應該是 0; 從繪圖軟體也看到 0. $\lim_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}=0$ **(Wrong!)** 因為 $\sqrt{t^2+9}\to 3$ *as* $t\to 0$, 計算機位數不足分子會變成 0, 再除以 t^2 還是 0, 所以會得到 0. 事實上 $\lim_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}=\dfrac{1}{6}$ *(所以要學微積分)*. ■

Example 0.3 *Guess* $\lim\limits_{t\to 0}\dfrac{\sin x}{x}$ =? *(陷阱卡發動!)*

x	$\dfrac{\sin x}{x}$
± 1	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983





用計算機跟繪圖軟體應該是 1*(?)*. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ *(✓, 之後 §3.2 會證明).* ■

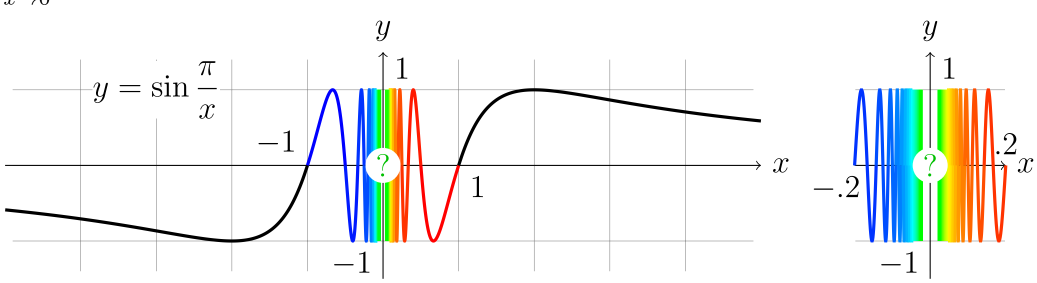
$\pm 10^{-4} \sim 10^{-5}$	0.99999999
$\pm 10^{-6} \sim 10^{-14}$	1
$\pm 10^{-15}$	0

(by Google, 其實計算機會算錯!)

Example 0.4 *Investigate* $\lim\limits_{x\to 0}\sin\dfrac{\pi}{x}$.

Let $f(x) = \sin\dfrac{\pi}{x}$. $f(x) = 0$ *when* $x = 1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dots, \dfrac{1}{10}, \dots, \dfrac{1}{100}$.

$\lim\limits_{x\to 0} f(x) = 0$ *? No,* 因爲還有很多 x *near* 0 使得 $f(x) = 1$. *(who?)*



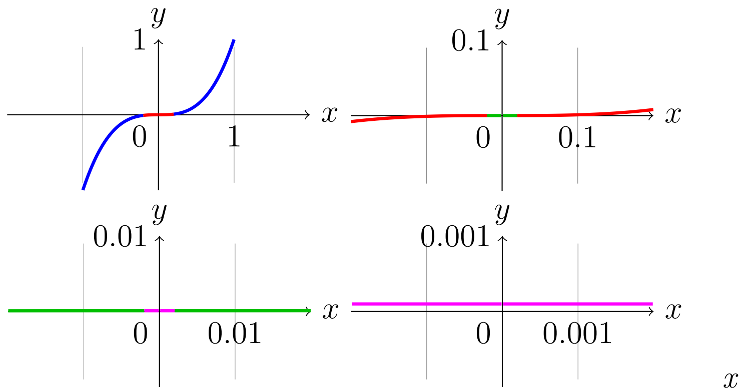
從圖中看出 $f(x) \not\to$ *some fixed number as* $x \to 0$, *so we say:*

$$\lim_{x\to 0}\sin\frac{\pi}{x}\ \boxed{\textbf{\textit{does not exist}}\ \text{不存在}}.$$ ■

$\lim\limits_{t\to \text{今晚}}$ 能 $\neq$ 我 — 能不能靠近我就在今晚.

Example 0.5 *Guess* $\lim_{x\to 0}\left(x^3+\dfrac{\cos 5x}{10000}\right)=?$

x	$x^3+\dfrac{\cos 5x}{10000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101
0.005	0.00010009
0.001	0.00010000



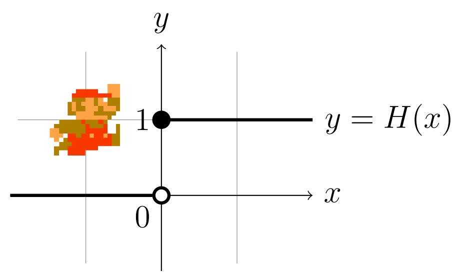
計算機位數不足 *(<5)* 會算出 0, 其實答案是 0.0001. ■

Note: 圖不一定畫得出來, 用看的不一定看得出來, 用猜的不一定會猜對, 用計算機算不一定會算出來, 可能算出錯的答案. 要用定義(ε-δ) 或其他工具來證明.

Example 0.6 *The* ***Heaviside*** 黑維賽 *(step* 階躍*) function*

$$H(t)=\begin{cases}1, & t\ge 0\\ 0, & t<0\end{cases}$$

(今晚的我, 沒有極限!)



$\lim_{x\to 0} H(x)$ *does not exist, but there are something to say.*

♦: 有些書上定義 $H(x)=\dfrac{1}{2}[1+\operatorname{sgn}(x)]$, $H(0)$ 有些不定義, 有些定爲 $\dfrac{1}{2}$.

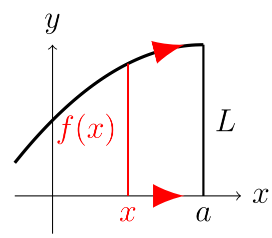
0.2 One-side limit

有時候雖然沒有 (雙邊) 極限, 但是這個函數還是有一些很好的性質 — 單邊極限.

左極限:

$$\boxed{\lim_{x\to a^-} f(x) = L} \quad \text{or}$$

$$\boxed{f(x) \to L \text{ as } x \to a^-}$$

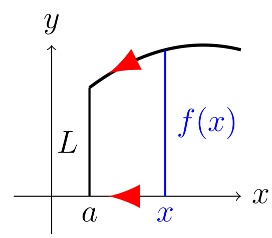


The ***left-hand limit*** of $f(x)$ is equals to L as x approaches a.
$f(x)$ approaches L as x approaches a from the left.

右極限:

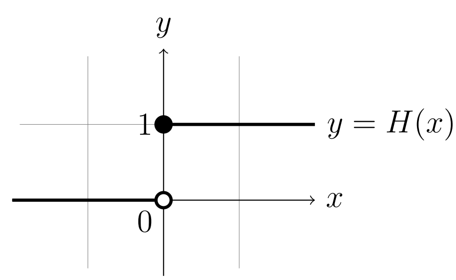
$$\boxed{\lim_{x\to a^+} f(x) = L} \quad \text{or}$$

$$\boxed{f(x) \to L \text{ as } x \to a^+}$$



The ***right-hand limit*** of $f(x)$ is equals to L as x approaches a.
$f(x)$ approaches L as x approaches a from the right.

Recall Heaviside function $H(x)$,
$\lim_{x\to 0^-} H(x) = 0$ and $\lim_{x\to 0^+} H(x) = 1$.



Fact: 由極限, 左極限, 右極限的 (概念) 定義可以得到一個事實:

$$\boxed{\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.}$$

($\Rightarrow$) Trivial.
($\Leftarrow$) 要有左極限, 要有右極限, 這兩個極限要一樣, 就會有極限 (等於共同的極限).

韓愈《祭十二郎文》: 彼蒼者天, 曷其有極! — 若且唯若, 左右有極, 極極相及。

0.3 Infinite limit (& vertical asymptote)

無限極限:

$$\boxed{\lim_{x\to a} f(x) = \infty}, \quad \boxed{\lim_{x\to a} f(x) = -\infty},$$

$$\boxed{\lim_{x\to a^-} f(x) = \infty}, \quad \boxed{\lim_{x\to a^-} f(x) = -\infty},$$

$$\boxed{\lim_{x\to a^+} f(x) = \infty}, \quad \boxed{\lim_{x\to a^+} f(x) = -\infty} \quad \text{or}$$

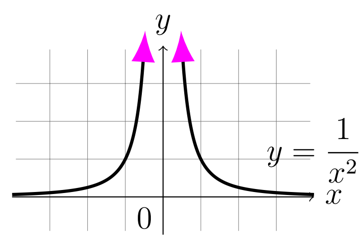
$$\boxed{f(x) \to \infty/-\infty \text{ as } x \to a/a^-/a^+}$$

$f(x)$ can be arbitrarily 任意 **large** (**negative**) as x approaches $a/a^-/a^+$.

Attention: ∞: infinity 無限大; $-\infty$: negative infinity 負無限大 (無限小). 都是符號, 並不是一個數字, 所以這種時候極限是不存在.

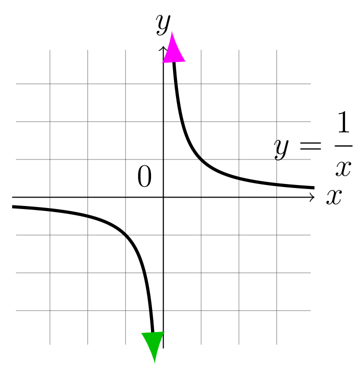
Example 0.7 *Does* $\lim_{x\to 0} \frac{1}{x^2}$ *exist?*

No, but $\lim_{x\to 0} \frac{1}{x^2} = \infty$. ■



Example 0.8 *Does* $\lim_{x\to 0} \frac{1}{x}$ *exist?*

No, but $\lim_{x\to 0^+} \frac{1}{x} = \infty$ *&* $\lim_{x\to 0^-} \frac{1}{x} = -\infty$. ■



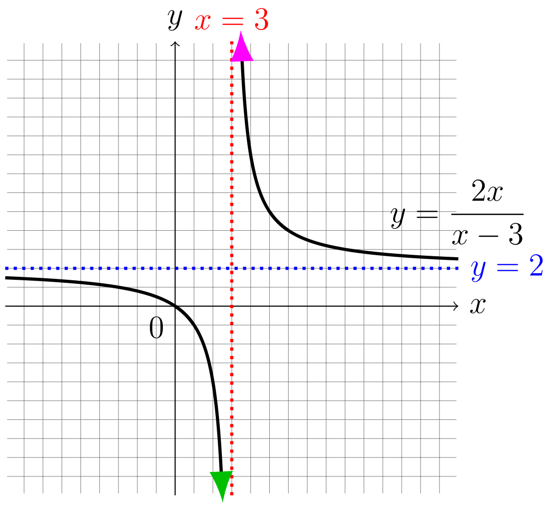
Question: 選項有 does not exist 與 $\infty/-\infty$ 要選誰?
Answer: 有無限極限最好是選 $\infty/-\infty$.

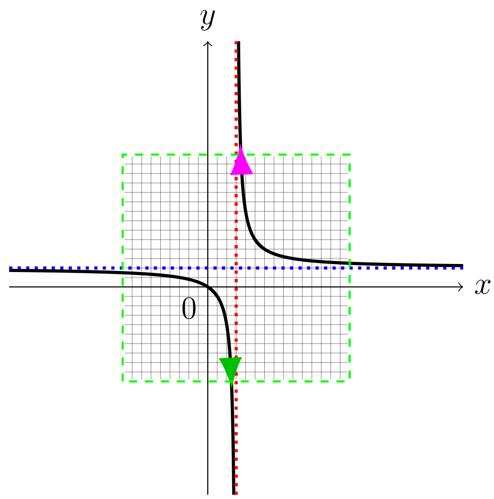
Define: $\boxed{\boldsymbol{x=a}}$ is a ***vertical asymptote*** 垂直漸近線 of $y = f(x)$ if infinite limit ($\infty/-\infty$) occurs at $a/a^-/a^+$. 當無限極限的6種情形之一發生時。

Note: 曲線 $y = f(x)$ 離開原點越遠就會越靠近的直線稱爲它的漸近線。

Example 0.9 *Find* $\lim\limits_{x\to 3^+} \dfrac{2x}{x-3}$ *and* $\lim\limits_{x\to 3^-} \dfrac{2x}{x-3}$.

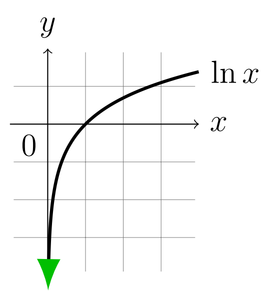
$\lim\limits_{x\to 3^+} \dfrac{2x}{x-3} = \infty$ *and* $\lim\limits_{x\to 3^-} \dfrac{2x}{x-3} = -\infty$. ■





左極限是 $-\infty$, 右極限是 ∞, 不只不存在, 還不相同。
這時候 $\lim\limits_{x\to 3} \dfrac{2x}{x-3}$ 不存在 (does not exist), 但是有垂直漸近線 $x = 3$.
(What is $y = 2$ called?)

Example 0.10 $\lim\limits_{x\to 0^+} \ln x = ?$



$\lim\limits_{x\to 0^+} \ln x = -\infty$. ■

Attention: 垂直漸近線是 $x = 0$, <u>不是 $x = 0^+$</u>!

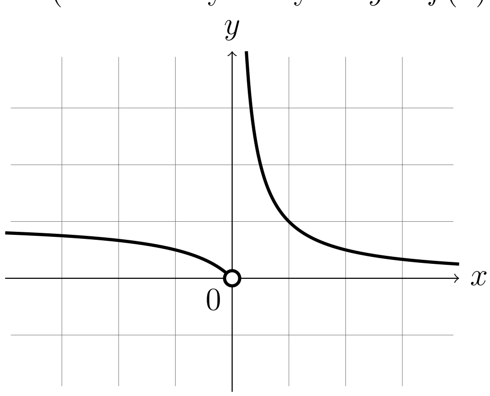
(左極限 $\lim\limits_{x\to 0^-} \ln x = ?$ *極限* $\lim\limits_{x\to 0} \ln x = ?$ *)*

Note: $x = 0$ (y-axis) y-軸, 是 $y = \ln x$ 的垂直漸近線。
也是所有對數函數圖形 $y = \log_a x$ ($a > 0$, $a \neq 1$, $x > 0$) 的垂直漸近線。
When $a > 1$, $\log_a x \to -\infty$; when $0 < a < 1$, $\log_a x \to \infty$.

Remark: When ask $\lim_{x\to a} f(x) =?$

1. $\exists$, $\lim_{x\to a} f(x) = L$.
2. $\nexists$, does not exist. but
   (a) $\exists$ one-side limit
      i. right-hand limit $\lim_{x\to a^+} f(x) = L$.
      ii. left-hand limit $\lim_{x\to a^-} f(x) = L$.
   (b) $\exists$ infinite limit (with V.A. $x = a$)
      i. $\lim_{x\to a} f(x) = \infty$.
      ii. $\lim_{x\to a^+} f(x) = \infty$.
      iii. $\lim_{x\to a^-} f(x) = \infty$.
      iv. $\lim_{x\to a} f(x) = -\infty$.
      v. $\lim_{x\to a^+} f(x) = -\infty$.
      vi. $\lim_{x\to a^-} f(x) = -\infty$.
   (c) just does not exist.

(What can you say for $y = f(x)$ about $x = 0$?)



$\lim_{x\to 0} f(x) =?$

$\lim_{x\to 0^+} f(x) =?$

$\lim_{x\to 0^-} f(x) =?$