7.1 Integration by parts

1. indefinite integration version $\int fg'\,dx = fg - \int f'g\,dx$
2. definite integration version $\int_a^b fg'\,dx = fg|_a^b - \int_a^b f'g\,dx$

相愛容易相處難, 微分容易積分難。

$$\boxed{\text{Chain Rule}} \longleftrightarrow \boxed{\text{Substitute Rule}}$$
$$\boxed{\text{Product Rule}} \longleftrightarrow \boxed{\text{Integration by Parts}}$$

SOP—積分123:

1. 積分公式 (Antiderivative) 有沒有?
   如果是基本函數 x^n, e^x, $\ln x$, $\sin x$, $\sin^{-1} x$,... 的導函數,
   by TFTC: $F' = f \implies \int f\,dx = F + C$.
   如果是他們的加減常數倍, $\int (cf \pm g)\,dx = c\int f\,dx \pm \int g\,dx$.
2. 變數變換 (Substitution Rule) 換不換?
   如果是剛好可以換乾淨&變簡單, $\int f'(g)g'\,dx = f(g) + C$.
3. 分部積分 (Integration by Part) 分一分? $\int fg'\,dx = fg - \int f'g\,dx$.

0.1 Indefinite integral version

$$\boxed{\int \boldsymbol{f(x)g'(x)\,dx} = \boldsymbol{f(x)g(x)} - \int \boldsymbol{g(x)f'(x)\,dx}}$$

Proof. Recall Product Rule: $(fg)' = f'g + fg'$.

By TFTC, $fg = \int (fg)'\,dx = \int (f'g + fg')\,dx = \int f'g\,dx + \int fg'\,dx$,

$$\int fg'\,dx = fg - \int f'g\,dx.$$

(不用$+C$, 不定積分本身就是最一般的反導數 (有C).) ∎

Skill: 記憶法: Let $u = f(x)$ and $v = g(x)$, then differentials $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. (把 du, dv 當作 differential 微分.) By Substitution Rule:

$$\boxed{\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du}}$$

Example 0.1 $\int x \sin x \, dx = ?$

[Ver 1: 正式]
Let $f(x) = x$ *and* $g'(x) = \sin x$*, then* $f'(x) = 1$ *and* $g(x) = -\cos x$*.*

$$\int \underbrace{x}_{f(x)} \underbrace{\sin x}_{g'(x)} \, dx = \underbrace{x}_{f(x)} \underbrace{(-\cos x)}_{g(x)} - \int \underbrace{-\cos x}_{g(x)} \cdot \underbrace{1}_{f'(x)} \, dx$$

$$= -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + C.$$

[Ver 2: 非正式] 令 u 是其中一個函數, 剩下 *(*含 dx*)* 令爲 dv, 找出 du *&* v*.*
Let $u = x$ *and* $dv = \sin x \, dx$*, then* $du = dx$ *and* $v = -\cos x$*.*

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} \boxed{= \int \underbrace{x}_{u} \underbrace{d(-\cos x)}_{dv}} = \underbrace{x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{-\cos x}_{v} \underbrace{dx}_{du}$$

$$= -x\cos x + \sin x + C. \qquad \blacksquare$$

Attention: 1. [Ver 2] 中 "$d(-\cos x)$" 是 $\boxed{\text{非正式}}$ 的寫法, 但是推薦使用.
2. 下括號 "$\underbrace{\cdots}_{\cdots}$" 是注釋, 不用寫.
3. 非證明題可以省略寫 "Let $u = \cdots$" 節省時間.

Note: 1. 別忘了 $+C$;
2. 怎麼檢查對不對? 還是一樣用微分! (這時候會用上乘積律)
$(-x\cos x + \sin x + C)' = -\cos x - x(-\sin x) + \cos x + 0 = x \sin x$;
3. 換人積積看?

if let $u = \sin x$ and $dv = x \, dx$, then $du = \cos x \, dx$ and $v = \dfrac{x^2}{2}$.

$$\int \underbrace{\sin x}_{u} \cdot \underbrace{x \, dx}_{dv} \boxed{= \int \underbrace{\sin x}_{u} \underbrace{d(\frac{x^2}{2})}_{dv} = \underbrace{\sin x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{d \sin x}_{du}}$$

$$= \underbrace{\sin x}_{u} \cdot \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\cos x \, dx}_{du},$$ 變得更複雜難積 (X).

Attention: 不保證一定算得出來, 只是換個函數積分.

Skill 1: 通常 $u = f(x)$ 會選擇 <u>$f'(x)$(導數) 變簡單</u>的.
推薦: 多項式 (x^{-n} 不算), $\ln x$.

Example 0.2 $\boxed{\int \ln x \; dx} = ?$

Let $u = \ln x$ *and* $dv = dx$, *then* $du = \frac{1}{x}\; dx$ *and* $v = x$.

$$\int \underbrace{\ln x}_{u}\underbrace{dx}_{dv} = \underbrace{\ln x}_{u}\cdot\underbrace{x}_{v} - \int \underbrace{x}_{v}\underbrace{d\ln x}_{du} = \underbrace{\ln x}_{u}\cdot\underbrace{x}_{v} - \int \underbrace{x}_{v}\cdot\underbrace{\frac{1}{x}\; dx}_{du}$$

$= x\ln x - \int \; dx = \boxed{x\ln x - x + C}$. *(加入你的不定積分表)* ■

Note: 1. $\boxed{\int \; dx = \int 1 \; dx} = x + C$, 不是漏打, 是積 1.

2. x 乘在 $\sin x$, $\ln x$ ... 等後面要加"·"區隔, 乘前面可以省略.
Ex: $\ln x \cdot x = x\ln x = \ln x^x \neq \ln(x\cdot x) = \ln x^2 = 2\ln x \neq (\ln x)^2$.

Example 0.3 $\int t^2 e^t \; dt = ?$

Let $u = t^2$ *and* $dv = e^t \; dt$, *then* $du = 2t \; dt$ *and* $v = e^t$.

$$\int \underbrace{t^2}_{u}\underbrace{e^t\; dt}_{dv} = \int \underbrace{t^2}_{u}\underbrace{de^t}_{dv} = \underbrace{t^2}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{dt^2}_{du}$$

$= \underbrace{t^2}_{u}\underbrace{e^t}_{v} - \int \underbrace{e^t}_{v}\underbrace{2t\; dt}_{du} = t^2 e^t - 2\int te^t \; dt$ *(雖然沒解決, 但是函數變簡單.)*

再對 $\int te^t \; dt$ 用一次分部積分法: $(\because u, v$ 用過了, *let* $U = t$, $dV = e^t \; dt$.)

$$\int \underbrace{t}_{U}\underbrace{e^t\; dt}_{dV} = \int \underbrace{t}_{U}\underbrace{de^t}_{dV} = \underbrace{t}_{U}\underbrace{e^t}_{V} - \int \underbrace{e^t}_{V}\underbrace{dt}_{dU} = te^t - e^t + C.$$

$$\int t^2 e^t \; dt = t^2 e^t - 2\int te^t \; dt = t^2 e^t - 2te^t + 2e^t + C_1, \text{ where } C_1 = -2C.$$ ■

Note: 分部積分可以用了再用. — 一次分不夠, 你可以分第二次。
不用加那麼多種 C, 最後的答案 $+C$ 就好: $\int t^2 e^t \; dt = t^2 e^t - 2te^t + 2e^t + C$.

Skill 2: 通常 $dv = g'(x) \; dx$ 會選擇 $g(x)$ (反導數) 不變難的.
推薦: x^{-n}, e^x, $\sin x$, $\cos x$, $\sec^2 x$, $\sec x \tan x$.

Example 0.4 $\int e^x \sin x \; dx = ?$

(熟練後可以不用 *Let* $f(x)/u = ...,\ g'(x)/dv = ...$)

$$\int \underbrace{e^x}_{u} \underbrace{\sin x \; dx}_{dv} \boxed{= \int \underbrace{e^x}_{u} \underbrace{d(-\cos x)}_{dv} = \underbrace{e^x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{de^x}_{du}}$$

$$= \underbrace{e^x}_{u} \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \underbrace{e^x \; dx}_{du} = -e^x \cos x + \int e^x \cos x \; dx,$$

(和原積分相似, 變成 $\cos x$, 再做一次.)

$$\int \underbrace{e^x}_{U} \underbrace{\cos x \; dx}_{dV} \boxed{= \int \underbrace{e^x}_{U} \underbrace{d \sin x}_{dV} = \underbrace{e^x}_{U} \underbrace{\sin x}_{V} - \int \underbrace{\sin x}_{V} \underbrace{de^x}_{dU}}$$

$$= \underbrace{e^x}_{U} \underbrace{\sin x}_{V} - \int \underbrace{\sin x}_{V} \cdot \underbrace{e^x \; dx}_{dU} = e^x \sin x - \int e^x \sin x \; dx,$$

(還是沒解決, 但是變出了負的$(-)$原式, 可以做!)

$$\begin{aligned}
\int e^x \sin x \; dx &= -e^x \cos x + \int e^x \cos x \; dx \\
&= -e^x \cos x + e^x \sin x - \int e^x \sin x \; dx, \\
2\int e^x \sin x \; dx &= e^x(\sin x - \cos x), \\
\int e^x \sin x \; dx &= \frac{1}{2} e^x(\sin x - \cos x) + C.
\end{aligned}$$

■

Attention: 選擇要一致, 如果第二次用 $\int \cos x \; de^x$ 就會變回原題目.

Skill 3: 分部積分完又<u>出現原式</u>, 可以<u>移項合併</u>, <u>最後</u>再一起<u>$+C$</u>.

Question: 這一題可以挑 $u = \sin x,\ v = e^x$ (換人積) 嗎?

Answer: 可以, 請務必試試: $\int \sin x \cdot e^x \; dx = \int \sin x \; de^x = ...$

Question: Who is u and who is v?

Answer: 積不下去就換人積積看. $\int$經驗 d作業

Example 0.5 *Prove the reduction formula*

$$\int \sin^n x \; dx \;=\; -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \; dx$$

where $n \geq 2$ is an integer.

Proof. *Let $u = \sin^{n-1} x$ and $dv = \sin x \; dx$,*
then $du = (n-1)\sin^{n-2} x \cos x \; dx$ and $v = (-\cos x)$.

$$\begin{aligned}
\int \sin^n x \; dx \;&=\; \int \sin^{n-1} x \sin x \; dx \;=\; \boxed{\int \sin^{n-1} x \; d(-\cos x)} \\
&= \boxed{\sin^{n-1} x\,(-\cos x) - \int (-\cos x)\; d\sin^{n-1} x} \\
&= \sin^{n-1} x\,(-\cos x) - \int (-\cos x)\,(n-1)\sin^{n-2} x \cos x \; dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \; dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x \; dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \; dx - (n-1)\int \sin^n x \; dx, \\
n\int \sin^n x \; dx \;&=\; -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \; dx, \\
\int \sin^n x \; dx \;&=\; -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x \; dx.
\end{aligned}$$

(不用加C, 因爲還有不定積分.) ■

Note: 減化公式說明 $\int \sin^n x \; dx$ 最後可以變成 $\int \sin x \; dx$ (if n is odd) 或是 $\int \; dx$ (if n is even) 與 $\sin x, \cos x$ 的組合.

補充: (Exercise 7.1.48.) integer $n \geq 2$,

$$\int \cos^n x \; dx \;=\; \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x \; dx$$

(More reduction formula see Exercise 7.1.51–54.)

0.2 Definite integral version

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx}$$

Note: 差別在代入上下界, 沒有$+C$.

Example 0.6 $\int_0^1 \tan^{-1} x\,dx = ?$

$$\int_0^1 \tan^{-1} x\,dx \boxed{= \tan^{-1} x \cdot x\Big|_0^1 - \int_0^1 x\,d\tan^{-1} x}$$

$$= \tan^{-1} x \cdot x\Big|_0^1 - \int_0^1 x\,\frac{1}{1+x^2}\,dx \qquad \left((\tan^{-1} x)' = \frac{1}{1+x^2}.\right)$$

$$= x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= (1\cdot\tan^{-1} 1 - 0\cdot\tan^{-1} 0) - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx.$$

Use Substitution Rule: let $t = 1+x^2$ *(*$\because u, v$ *are used),*
then $dt = 2x\,dx$*,* $x\,dx = \frac{1}{2}\,dt$*, when* $x=0$*,* $t=1$*, and when* $x=1$*,* $t=2$*.*

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\Big[\ln|t|\Big]_1^2 \qquad \left(\int \frac{dt}{t} = \ln|t| + C, \text{ 因爲 } t>0, \text{ 這裡可以 } \ln t.\right)$$

$$= \frac{1}{2}(\ln 2 - \ln 1) \qquad (\ln 1 = 0, \text{ 不要沒事寫一堆.}) \text{ 成堂步 } \ln 1\text{: 異議阿里!}$$

$$= \frac{\ln 2}{2}.$$

$$\therefore \int_0^1 \tan^{-1} x\,dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{\ln 2}{2}\left(= \frac{\pi}{4} - \ln\sqrt{2}\right). \qquad \blacksquare$$