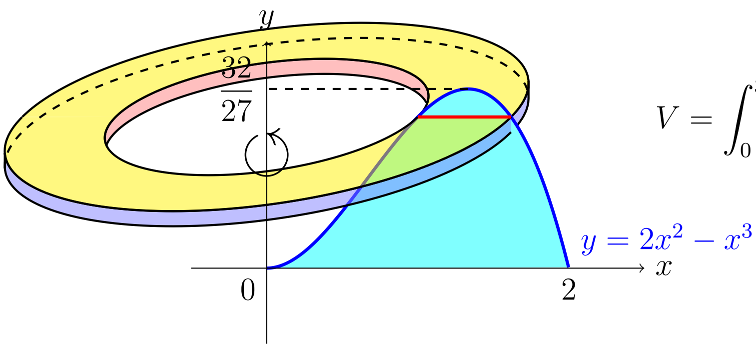
6.3 Volumes by cylindrical shells

另一種求體積法: 剝殼法 (洋蔥)

英語教室: cylindrical [səˈlɪdrɪkl] 柱狀的, shell [ʃɛl] 殼.

當 $y = f(x)$ 繞著 y-軸 or 垂直線 $x = a$, 體積用 disk/washer 對 y 積分: 圓盤法 $V = \int_c^d A(y)\, dy$ 但是有時候很難去算出 $x = f^{-1}(y)$ 來得到內/外半徑.

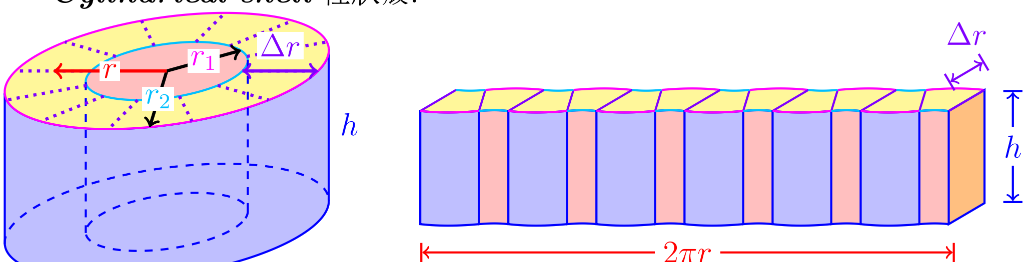
Example 0.1 *Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.*



$$V = \int_0^{32/27} \pi(?^2 - ?^2)\, dy$$

Note: 算出 $x = f^{-1}(y)$ 要解三次方程, 還要算出上下界 (極值 0 and $\frac{32}{27}$), 判斷左右的函數, 是非常的複雜.

Cylindrical shell 柱狀殼:



$$V = \pi {r_2}^2 h - \pi {r_1}^2 h = 2\pi \frac{r_2 + r_1}{2}\, h(r_2 - r_1) = 2\pi r h \Delta r.$$

where $r = \frac{r_2 + r_1}{2}$ the average radius of the shell 平均半徑

and $\Delta r = r_2 - r_1$ the thickness of the shell 厚度.

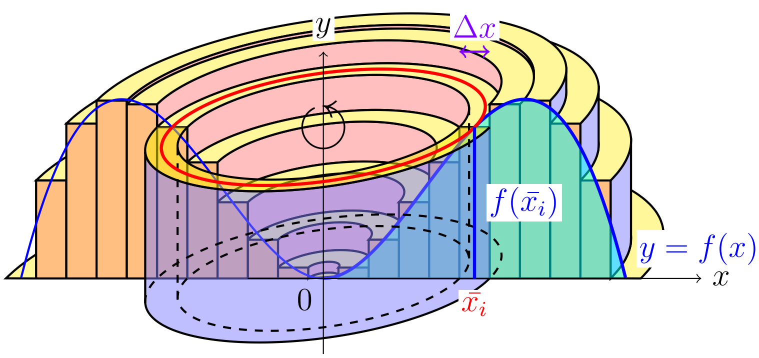
體積	(平均) 圓周長		高度		厚度
Volume =	Circumference	×	Height	×	Thickness

The ***method of cylindrical shells*** 剝殼法:

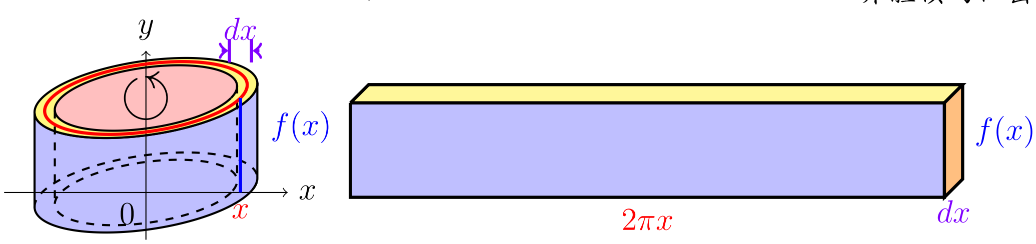
Let S 是由 $y=f(x)(\geq 0)$, $y=0$, $x=a$, $x=b(>a\geq 0)$ 所圍區域, 繞 y-軸所成.

把 $[a,b]$ 分成 n 等分, $\Delta x=\dfrac{b-a}{n}$, $x_i=a+i\Delta x$. 考慮中點 $\bar{x}_i=\dfrac{x_{i-1}+x_i}{2}$. Let V_i 是 $f(\bar{x}_i)\Delta x$ 繞 y-軸的體積, then

$$\begin{aligned} V_i &= 2\pi\bar{x}_i f(\bar{x}_i)\Delta x, \\ V \approx \sum_{i=1}^{n} V_i &= \sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\Delta x, \\ V &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi\bar{x}_i f(\bar{x}_i)\Delta x. \end{aligned}$$



如果你願意
一層一層一層
的剝開我的心
你會發現
你會訝異
你　　　最壓抑
這是我剝殼法
最深處
算體積的秘密



Theorem 1 (Method of cylindrical shells) *The volume of the solid obtained by rotating about the y-axis the region bounded under the curve $y=f(x)$ from a to b, where $b>a\geq 0$, is*

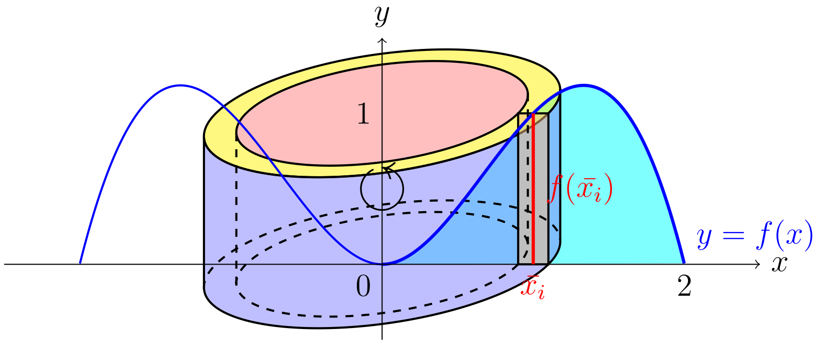
$$\boxed{\boldsymbol{V=\int_a^b 2\pi x f(x)\ dx}}$$

Note:

$y=f(x)$ 繞 y-軸, $V=\int 2\pi x f(x)\ dx$. 橫著剝, 對 x 積。
$x=f(y)$ 繞 x-軸, $V=\int 2\pi y f(y)\ dy$. 縱著剝, 對 y 積。

Example 0.2 *(Continuous)* $f(x) = 2x^2 - x^3$.

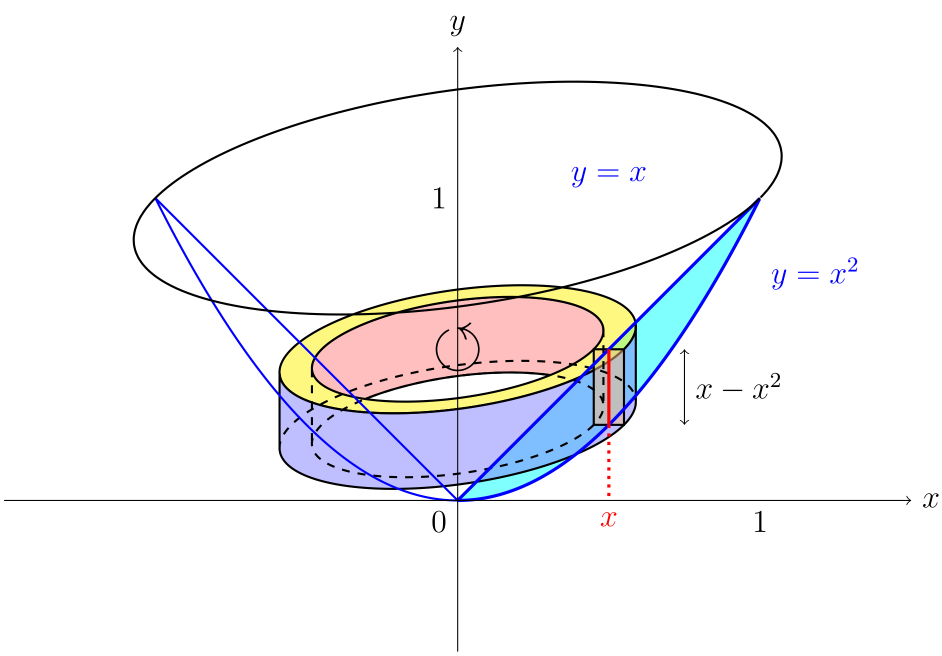
$$V = \int_0^2 2\pi x f(x)\; dx = 2\pi \int_0^2 2x^3 - x^4\; dx = 2\pi \left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{16\pi}{5}. \qquad \blacksquare$$



Example 0.3 *Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.*

When radius x from 0 to 1, the circumference $2\pi x$ and height $x - x^2$.

$$V = \int_0^1 2\pi x(x - x^2)\; dx = 2\pi \int_0^1 x^2 - x^3\; dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}. \qquad \blacksquare$$

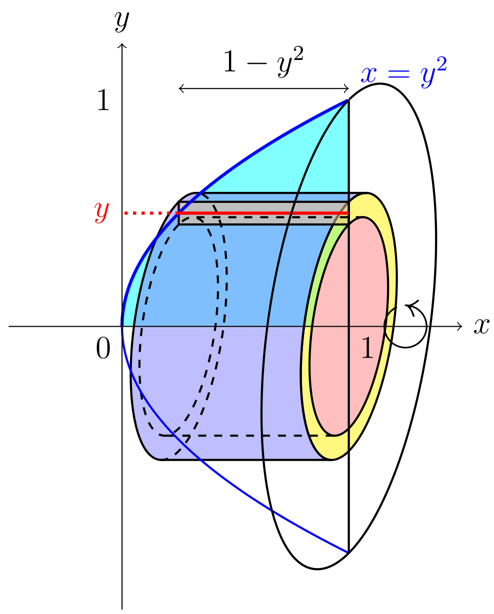


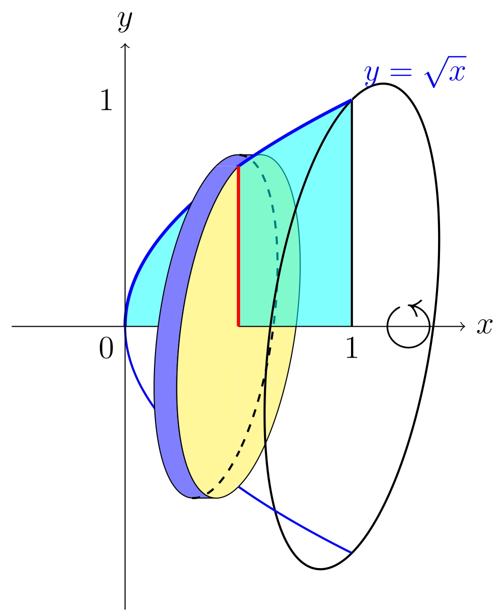
Example 0.4 *Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.*

$x = y^2$, *circumference* $2\pi y$, *height* $1 - y^2$.

$V = \displaystyle\int_0^1 2\pi y(1 - y^2)\, dy = 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{\pi}{2}.$ ■

Recall (6.2.ex2) disk: $V = \displaystyle\int_0^1 \pi x\, dx = \frac{\pi}{2}$ is simpler.





(比較剝殼法 (順紋切) 與切片法 (逆紋切).)