

3.4 The chain rule

1. chain rule 連鎖律 $(f(g))' = f'(g)g'$

連鎖律 – 微分的超必殺技! 必學!

微積至尊, 寶刀連鎖, 微遍天下, 莫敢不微! 分部不出, 誰與爭鋒?

連鎖用的好, 微分沒煩惱; 微分學得好, 積分不苦惱.

0.1 Chain Rule

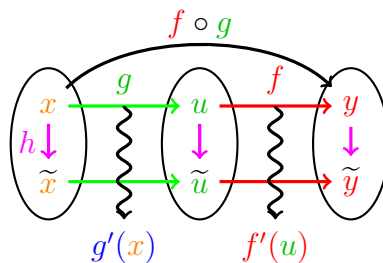
Recall: 合成函數: $f(g(x)) = (f \circ g)(x)$ & $g(f(x)) = (g \circ f)(x)$ are composite functions (composition) of functions $f(x)$ & $g(x)$.

The Chain Rule: 連鎖律, 鏈鎖律, 鏈鎖法則
If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, $y = f(u)$ and $u = g(x)$

are both differentiable, then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.



Proof. (跟書上使用 Leibniz 的寫法不同, 但精神上是一樣的.)

$\because g$ is differentiable at x , by Theorem, $\implies g$ is continuous at x

$$\iff g(x+h) \rightarrow g(x) \text{ as } h \rightarrow 0, \lim_{h \rightarrow 0} \implies \lim_{g(x+h) \rightarrow g(x)}.$$

$$\begin{aligned} [f(g(x))]' &= \lim_{h \rightarrow 0} \frac{f(g(x+h)) - f(g(x))}{h} \quad (\text{一乘一除 } g(x+h) - g(x)) \\ &= \lim_{h \rightarrow 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \right] \\ &= \lim_{g(x+h) \rightarrow g(x)} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(g(x)) \cdot g'(x). \quad \blacksquare \end{aligned}$$

Note: $f'(g(x))$ 是先把 $f(x)$ 微分得到導函數 $f'(x)$, 再把 $g(x)$ 代入 $f'(x)$.

◆ **Additional:** 如果 $g(x+h) - g(x) = 0$ 怎麼辦? 這時候 $g(x) = c$ 常數.

How to use: Let $u = u(x)$, then $f = f(u)$ and

$$f'(x) = f'(u)u'(x) = f'(u(x))u'(x).$$

Step 1. 計算 $f'(u)$: 把 f 看成 u 的函數, 對 u 微分;

Step 2. 計算 $u'(x)$: 把 u 看成 x 的函數, 對 x 微分;

Step 3. 把結果相乘, 把 u 代回 x 的函數 $u(x)$, 就是 f 對 x 的微分。

Example 0.1 $f(x) = \sqrt{x^2 + 1}$, $f'(x) = ?$

Let $u = u(x) = x^2 + 1$, then $f(u) = \sqrt{u}$ (變成 u 的函數) and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}(\sqrt{u}) \cdot \frac{d}{dx}(x^2 + 1) = \frac{1}{2\sqrt{u}} \cdot 2x \\ (\text{代回 } u = u(x)) &= \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}}. \quad \blacksquare \end{aligned}$$

Example 0.2 (a) $f(x) = \sin(x^2)$, $f'(x) = ?$ (b) $f(x) = \sin^2 x$, $f'(x) = ?$

(a) Let $u = u(x) = x^2$, then $f(u) = \sin u$ and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}(\sin u) \cdot \frac{d}{dx}(x^2) = \cos u \cdot 2x \\ (\text{代回 } u = u(x)) &= \cos(x^2) \cdot 2x = 2x \cos(x^2). \end{aligned}$$

(如果不會弄錯, $\frac{d}{du}f(u) \cdot \frac{d}{dx}u(x)$ 可以省略 “.” 成 $\frac{d}{du}f(u)\frac{d}{dx}u(x)$.)

(b) Let $u = u(x) = \sin x$, then $f(u) = u^2$ and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du}(u^2) \frac{d}{dx}(\sin x) = 2u \cdot \cos x \\ (\text{代回 } u = u(x)) &= 2 \sin x \cdot \cos x = 2 \sin x \cos x. \quad \blacksquare \end{aligned}$$

(如果 $f(x)$ 只有一項而且不會弄錯, $\frac{d}{dx}(\dots)$ 可以省略括號 “(,)”.)

The Power Rule Combined with the Chain Rule:

$n \in \mathbb{R}$, $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}.$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x). \quad (\text{不建議背})$$

Example 0.3 $y = (x^3 - 1)^{100}$, $y' = ?$

Let $u = u(x) = x^3 - 1$, then $f(u) = u^{100}$ and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du} u^{100} \frac{d}{dx} (x^3 - 1) = 100u^{99} \cdot 3x^2 \\ (\text{代回 } u = u(x)) &= 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}. \quad \blacksquare \end{aligned}$$

Note: $(f/g)' = [f \cdot (g)^{-1}]' = f'(g)^{-1} + (-1)f(g)^{-2}g' = (f'g - fg')/(g)^2$.
(書上 Examples(4-6): $\frac{1}{\sqrt[3]{x^2 + x + 1}}$, $\left(\frac{t-2}{2t+1}\right)^9$, $(2x+1)^5(x^3-1+1)^4$ 略.)

Example 0.4 $f(x) = e^{\sin x}$, $f'(x) = ?$

Let $u = u(x) = \sin x$, then $f(u) = e^u$ and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du} e^u \frac{d}{dx} \sin x = e^u \cdot \cos x \\ (\text{代回 } u = u(x)) &= e^{\sin x} \cdot \cos x = e^{\sin x} \cos x. \quad \blacksquare \end{aligned}$$

Example 0.5 $f(x) = a^x$, $a > 0$, $f'(x) = ?$ (用定義證過了)

$a^x = e^{\ln a^x} = e^{x \ln a}$. Let $u = u(x) = x \ln a$, then $f(u) = e^u$ and

$$\begin{aligned} f'(x) &= f'(u)u'(x) \\ &= \frac{d}{du} e^u \frac{d}{dx} (x \ln a) = e^u \cdot \ln a \\ (\text{代回 } u = u(x)) &= e^{x \ln a} \cdot \ln a = a^x \ln a. \quad \blacksquare \end{aligned}$$

Question: 怎麼選擇 $u = u(x)$?

Answer: 寫習題賺經驗值.

Note: 多重連鎖律: $f = f(u)$, $u = u(v)$, $v = v(x)$,

$$\boxed{\frac{df}{dx} = \frac{df}{du} \frac{du}{dv} \frac{dv}{dx}}.$$

(四重、五重可不可以? 可以! 九九重陽都沒問題.)

Example 0.6 $f(x) = \sin(\cos(\tan x))$, $f'(x) = ?$

Let $v = v(x) = \tan x$ and $u = u(v) = \cos v$, then $f(u) = \sin u$ and

$$\begin{aligned} f'(x) &= f'(u)u'(v)v'(x) \\ &= \frac{d}{du} \sin u \frac{d}{dv} \cos v \frac{d}{dx} \tan x \\ &= \cos u \cdot (-\sin v) \cdot \sec^2 x \\ (\text{全部代回}) &= \cos(\cos(\tan x)) \cdot (-\sin(\tan x)) \cdot \sec^2 x \\ &= -\cos(\cos(\tan x)) \sin(\tan x) \sec^2 x. \end{aligned}$$

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Example 0.7 $f(\theta) = e^{\sec 3\theta}$, $f'(\theta) = ?$

Let $v = v(\theta) = 3\theta$ and $u = u(v) = \sec v$, then $f(u) = e^u$ and

$$\begin{aligned} f'(\theta) &= f'(u)u'(v)v'(\theta) \\ &= \frac{d}{du} e^u \frac{d}{dv} \sec v \frac{d}{d\theta} 3\theta \\ &= e^u \cdot \sec v \tan v \cdot 3 \\ (\text{全部代回}) &= e^{\sec 3\theta} \cdot \sec 3\theta \tan 3\theta \cdot 3 \\ &= 3e^{\sec 3\theta} \sec 3\theta \tan 3\theta. \end{aligned}$$

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Skill: 目標: 簡化函數. $e^{\sec 3\theta} = e^{\sec v} = e^u \rightarrow \text{Let } v = 3\theta, u = \sec v.$

Question: 為什麼要令 $u = u(x)$ (變數名 = 函數名)?

Answer: 代回時比較不會找錯.

Question: 一定要 Let ..., $f'(x) = f'(u)u'(x)$? 可不可以直接算出來?

Answer: 當然可以直接算! 如果是證明題, 為了部分分數還是要寫. 但是因為常常... 常會忘記 $f'(u)$ 還要乘 $u'(x)$, 寫出來可以避免漏掉.