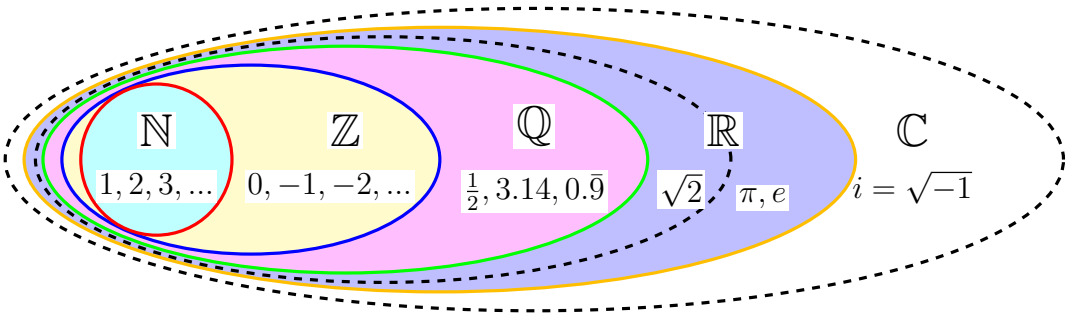
1.3 New functions from old functions

1. number system 數系
2. essential functions 基本函數
3. shifting, stretching and reflecting of functions 函數平移伸縮與反射
4. combination of functions 函數組合

0.1 Number system



$\mathbb{N}$: ***Natural*** number 自然數, positive integer 正整數: $1, 2, 3, \ldots$.

$\mathbb{Z}$: ***Integer*** 整數, Zahlen[德]"number": $0, \pm 1, \pm 2, \ldots$.

♦: Die ganzen Zahlen hat der liebe Gott gemacht,
alles andere ist Menschenwerk.
God made the integers, all else is the work of man.
整數神造, 其餘人爲。 — L. Kronecker

$\mathbb{Q}$: ***Rational*** number 有理數, Quoziente[義]"quotient": $\frac{p}{q}$, $p, q(\neq 0) \in \mathbb{Z}$.
(含有限小數 $3.14 = \frac{314}{100}$ 與循環小數 $0.\bar{9} = 0.999\cdots = 1$.)

$\mathbb{R}$: ***Real*** number 實數, 分成有理數與無理數 (***irrational*** numbers).
(無理數含 $\sqrt{2}, \pi, e, \cdots$, 不循環小數.)

♦: 第一次衝擊 500 B.C.: $\sqrt{2}$無理數爆誕 — Hippasus希帕索斯 v.s. Pythagoras畢達哥拉斯.

♦: 實數又可分成代數 (algebraic) 數/超越 (transcendental) 數:
是/否爲整係數多項式的根. 有理數 $\frac{p}{q}$, $\sqrt{2} \in$ 代數數; π, $e \in$ 超越數.

$\mathbb{C}$: ***Complex*** number 複數, 實數 $\cup$ 虛數 $(b \neq 0)$: $a + bi$, $a, b \in \mathbb{R}$,
$i := \sqrt{-1}$ imaginary unit 虛數單位.

♦: $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| \ll |\mathbb{R}|$, 但是存在無窮多的有理數逼近到任何實數.
3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, 3.1415926, ..., π.

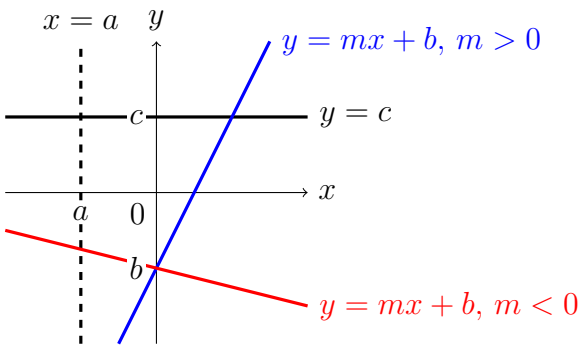
♦: 無限大的等級 — 阿列夫 (Aleph) 數 $\aleph_0 < \aleph_1 < \aleph_2 < ...$
$\aleph_0$(aleph-naught/zero/null[德]): (最小的) 可數 (countable) 無限大, $|\mathbb{N}| = \aleph_0$.
$\aleph_1$(aleph-one): 第二小的無限大.
連續統假設 (Continuum hypothesis): $\aleph_1 = 2^{\aleph_0} \iff |\mathbb{R}| = \aleph_1$.

0.2 Essential functions

Function 函數 $f(x)$ and the graph 圖形 $\{(x, y) : y = f(x)\}$ of $y = f(x)$.

1. Constant function 常數函數 $f(x) = c$. (圖形是水平線 $y = c$.)

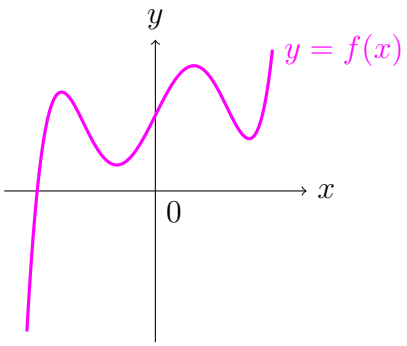
2. Linear function 線性函數 $f(x) = mx + b$.
   m: slope 斜率, b: y-intercept y-軸截距. (垂直線 $x = a$ 不是函數圖形.)



3. (n-th degree) Polynomial (n次) 多項式

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0.$$

   a_i: coefficient 係數, n: degree 次數.
   linear($n = 1$), quadratic($n = 2$), and cubic($n = 3$) function.

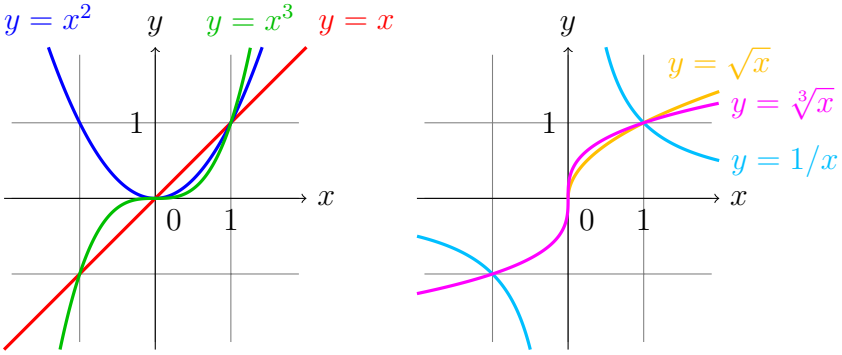


4. Power function 冪次函數 $f(x) = x^n$.

   polynomial $n = 1, 2, 3, \ldots$.

   root function 開根函數 $x^{1/n} = \sqrt[n]{x}$.

   reciprocal function 倒數函數 $x^{-1} = \dfrac{1}{x}$.



5. Algebraic function 代數函數 $+, -, \times, \div, \sqrt{\ }$ 組成

6. Exponential function 指數函數 (§1.4) a^x, $a > 0$.

7. Logarithmic function 對數函數 (§1.5) $\log_a x$, $a > 0$, $a \neq 1$.

8. Trigonometric function 三角函數 (§1.5)
   $\sin x, \cos x, \tan x, \cot x, \sec x, \csc x$.

9. Inverse trigonometric function 反三角函數 (§1.5)
   $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \cot^{-1} x, \sec^{-1} x, \csc^{-1} x$.

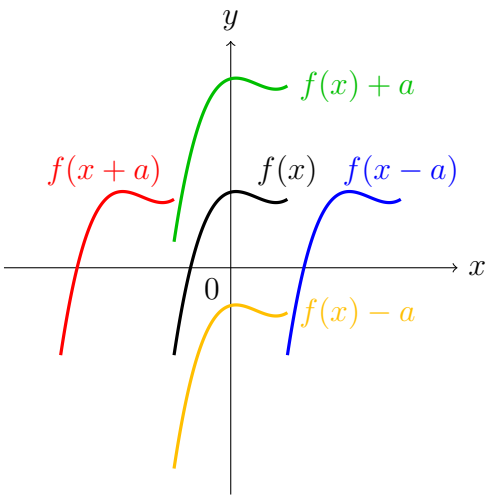
10. Hyperbolic trigonometric function 雙曲三角函數 (♦§3.11)
    $\sinh x, \cosh x, \tanh x, \coth x, \operatorname{sech} x, \operatorname{csch} x$.

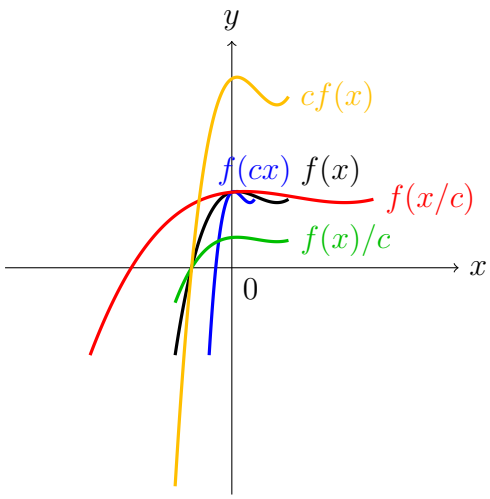
11. Inverse hyperbolic trigonometric function 反雙曲三角函數 (♦§3.11)
    $\sinh^{-1} x, \cosh^{-1} x, \tanh^{-1} x, \coth^{-1} x, \operatorname{sech}^{-1} x, \operatorname{csch}^{-1} x$.

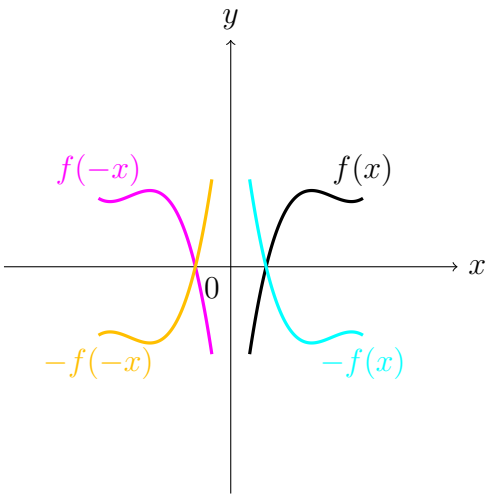
0.3 Shifting, stretching and reflecting of functions

Let $y = f(x)$ and $c > 1$.

1. Shift $y = f(x-a)$, $y = f(x+a)$, $y = f(x) - a$, $y = f(x) + a$.
   怎麼知道是往哪移? 代數字比較!
2. Stretch $y = f(cx)$, $y = f(x/c)$, $y = cf(x)$, $y = f(x)/c$.
3. Reflect (y-axis) $y = f(-x)$, (x-axis) $y = -f(x)$, (origin) $y = -f(-x)$.



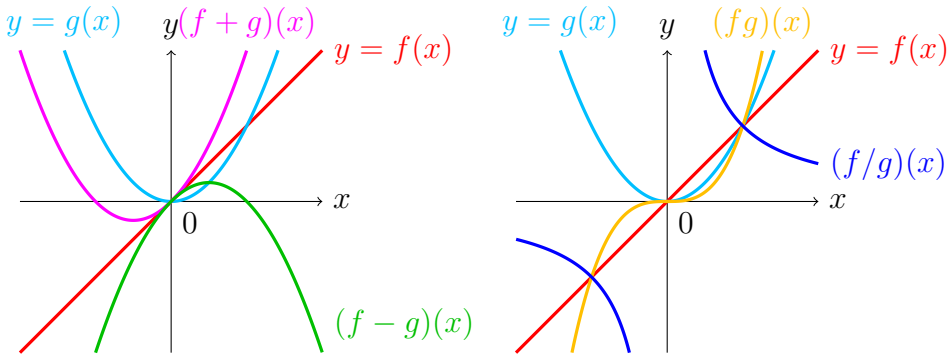




0.4 Combination of functions

1. Add, subtract, multiply and divide. 加減乘除

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x),$$
$$(fg)(x) = (f \cdot g)(x) = f(x) \times g(x) = f(x)g(x),$$
$$(f/g)(x) = \left(\frac{f}{g}\right)(x) = f(x) \div g(x) = \frac{f(x)}{g(x)}.$$



Rational function 有理函數 $\dfrac{P(x)}{Q(x)}$, $P(x), Q(x)$ are polynomials.

2. Composite function 合成函數 $(f \circ g)(x) = f(g(x))$.
Note: In general, $f \circ g \neq g \circ f$.

Example 0.1 $f(x) = \sin 2x$, $g(x) = x^2$;
$(f \circ g)(x) = f(g(x)) = \sin 2(x^2) = \sin 2x^2$,
$(g \circ f)(x) = g(f(x)) = (\sin 2x)^2 = \sin^2 2x$.