2.3 Calculating limits using the limit laws

1. limit laws 極限律
2. left/right-hand limit 左右極限
3. Squeeze Theorem 夾擠定理

不是每個極限都能明顯的看出來或是算出來猜對, 但是可以利用已知的極限來算一些的極限.

0.1 Limit laws

Limit laws 極限律: $\boxed{\lim_{x\to a} f(x) = L,\ \lim_{x\to a} g(x) = M,}$ (極限要存在) constant c.

1. 加: $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.
2. 減: $\lim_{x\to a}[f(x)-g(x)] = L - M$.
3. 乘: $\lim_{x\to a}[f(x)\times g(x)] = L \times M$.
4. 除: $\lim_{x\to a}[f(x)\div g(x)] = L \div M$, if $\boxed{M \neq 0}$ (分母極限不爲零).
5. 常數倍: $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$.

Extended: (§2.5 會證)

6. 冪次: $\lim_{x\to a}[f(x)]^n = L^n$.
7. 開根: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$, $L > 0$ when n is even. (開偶次根要正.)

Obvious results:

8. $\lim_{x\to a} c = c$. (毫無反應, 只是個常數 c.)
9. $\lim_{x\to a} x = a$.

Note: One-side limit ($x \to a^-/a^+$) 也適用 limit laws. (要同一邊)

Attention: Infinite limit <u>不適用</u> limit laws. ($\because$ 極限不存在.)

Example 0.1 **(使用極限律)** *a)* $\lim_{x\to 5}(2x^2-3x+4)=?$ *(多項相加要加括號.)*
$=2(\lim_{x\to 5}x)^2-3\lim_{x\to 5}x+\lim_{x\to 5}4=2(5)^2-3(5)+4=39.$

b) $\lim_{x\to -2}\dfrac{x^3+2x^2-1}{5-3x}=?$ *(一個分式可以不用括號.)*

$$=\frac{(\lim_{x\to -2}x)^3+2(\lim_{x\to -2}x)^2-\lim_{x\to -2}1}{\lim_{x\to -2}5-3\lim_{x\to -2}x}=\frac{(-2)^3+2(-2)^2-1}{5-3(-2)}=-\frac{1}{11}.$$

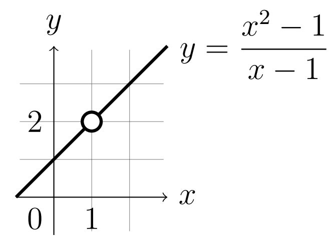
用了: 加, 減, 乘 (冪次), 常數倍, c, x, 除$(\lim_{x\to -2}(5-3x)=11\neq 0)$. ■

★ ***polynomial of degree n*** n-次多項式:

$$f(x)=a_nx^n+\ldots+a_1x+a_0,\qquad a_i\in\mathbb{R},\ a_n\neq 0,$$

$\boxed{\lim_{x\to a}f(x)=f(a)}$(求極限等於直接代入 a.)

Example 0.2 **(同約)** $\lim_{x\to 1}\dfrac{x^2-1}{x-1}=?$



1. 不能直接代 1, 因爲不是多項式.

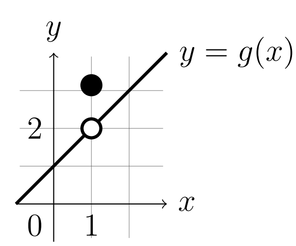
2. 不能用極限律, $\because \lim_{x\to 1}\dfrac{x^2-1}{x-1}=\dfrac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x-1)}$,

分母 $\lim_{x\to 1}(x-1)=0$.

要用代數的方法: $\lim_{x\to 1}\dfrac{x^2-1}{x-1}=\lim_{x\to 1}\dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=\lim_{x\to 1}(x+1)=2.$

(爲什麼可以約掉 $x-1$? x 靠近 1 但不是 1, $x-1$ 靠近 0 但不是 0. 約!) ■

Example 0.3 **(換人算)** $g(x)=\begin{cases}x+1, & x\neq 1\\ \pi, & x=1\end{cases}$, $\lim_{x\to 1}g(x)=?$



極限只看附近, 不管 $g(1)=2$, π, *or undefined*,
都不會影響極限. 可以用好算的函數代替不好算的.
$\therefore \lim_{x\to 1}g(x)=\lim_{x\to 1}(x+1)=2.$ ■

★ if $f(x)=g(x)$, $\forall\ x$ near a, and $\lim_{x\to a}f(x)=L$, then $\lim_{x\to a}g(x)=L$.

Example 0.4 **(同乘)** *(Recall)* $\lim\limits_{t\to 0}\dfrac{\sqrt{t^2+9}-3}{t^2}=?$

$\because \lim\limits_{t\to 0}(\sqrt{t^2+9}+3)=\sqrt{(\lim\limits_{t\to 0}t)^2+9}+3=6\neq 0$, *(不是零才能同乘)*

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}=\lim_{t\to 0}\frac{(\sqrt{t^2+9}-3)(\sqrt{t^2+9}+3)}{t^2(\sqrt{t^2+9}+3)}\quad \textit{(上下同乘 } \sqrt{t^2+9}+3\textit{)}$$

$$=\lim_{t\to 0}\frac{\cancel{t^2}+\cancel{9}-\cancel{9}}{\cancel{t^2}(\sqrt{t^2+9}+3)}=\lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}=\frac{\lim\limits_{t\to 0}1}{\lim\limits_{t\to 0}(\sqrt{t^2+9}+3)}=\frac{1}{6}.\qquad\blacksquare$$

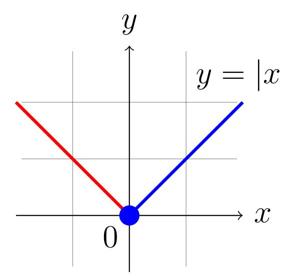
Note: 利用 $(a+b)(a-b)=a^2-b^2$, $\sqrt{\cdots}-\cdots$ 同乘 $\sqrt{\cdots}+\cdots$.
(試試利用 $(a\pm b)(a^2\mp ab+b^2)=a^3\pm b^3$, $\sqrt[3]{\cdots}\pm\cdots$ 同乘? $\sqrt[n]{\cdots}\pm\cdots$?)

0.2 Left/right-hand limit

$$\boxed{\lim_{x\to a}f(x)=L \iff \lim_{x\to a^-}f(x)=\lim_{x\to a^+}f(x)=L.}$$

Skill: 使用時機: 分段定義的函數 $f(x)=\begin{cases}\cdots, & \text{if } x\cdots;\\ \cdots, & \text{if } x\cdots.\end{cases}$

Example 0.5 $\lim\limits_{x\to 0}|x|=?$



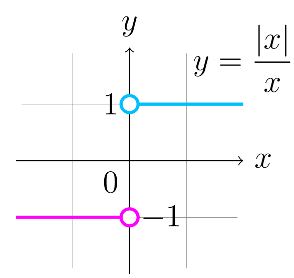
$\because |x|=\begin{cases}x, & x\geq 0;\\ -x, & x<0.\end{cases}$

(左) $\lim\limits_{x\to 0^-}|x|=\lim\limits_{x\to 0^-}(-x)=0,$

(右) $\lim\limits_{x\to 0^+}|x|=\lim\limits_{x\to 0^+}x=0.$

$\therefore \lim\limits_{x\to 0}|x|=\lim\limits_{x\to 0^-}|x|=\lim\limits_{x\to 0^+}|x|=0.$ $\blacksquare$

Example 0.6 $\lim\limits_{x\to 0}\dfrac{|x|}{x}=?$



$\because \dfrac{|x|}{x}=\begin{cases}1, & x>0;\\ -1, & x<0.\end{cases}$

$\lim\limits_{x\to 0^+}\dfrac{|x|}{x}=\lim\limits_{x\to 0^+}1=1,\ \lim\limits_{x\to 0^-}\dfrac{|x|}{x}=\lim\limits_{x\to 0^-}-1=-1.$

$\lim\limits_{x\to 0^-}\dfrac{|x|}{x}=-1\neq 1=\lim\limits_{x\to 0^+}\dfrac{|x|}{x}$, *(左右不同)*

$\therefore \lim\limits_{x\to 0}\dfrac{|x|}{x}$ *does not exist.* $\blacksquare$

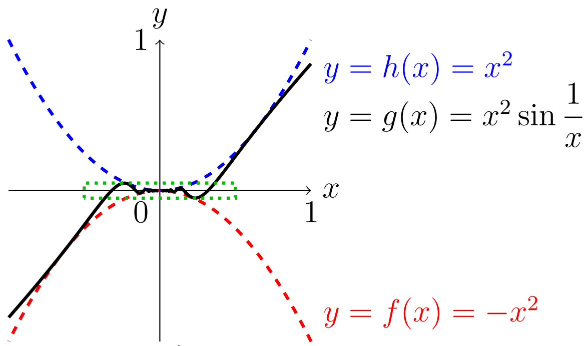
0.3 Squeeze Theorem

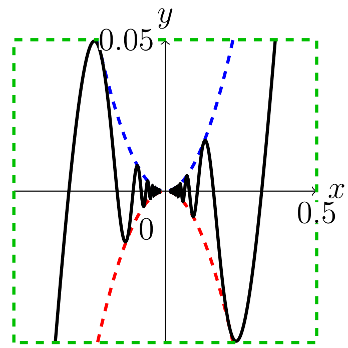
Lemma 1 *If $f(x) \le g(x)$ when x near a, and $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist, then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$.*

Theorem 2 (Squeeze/Sandwich/Pinching Theorem 夾擠定理)
If $\boxed{\boldsymbol{f(x) \le g(x) \le h(x)}}$ *when x near a,* (∗)(三個函數排成一列)
and $\boxed{\lim\limits_{x\to a} \boldsymbol{f(x)} = \lim\limits_{x\to a} \boldsymbol{h(x)} = \boldsymbol{L}}$, (∗∗) (前後極限存在相等於 L)
then $\boxed{\lim\limits_{x\to a} \boldsymbol{g(x)} = \boldsymbol{L}}$. (極限存在並等於 L)

Example 0.7 $\lim\limits_{x\to 0} x^2 \sin \dfrac{1}{x} =?$

1. 不能乘, $\lim\limits_{x\to 0} \sin \dfrac{1}{x}$ 不存在; *2.* 不能約分; *3.* 不能分左右. ....用夾擠!





$\because -1 \le \sin \dfrac{1}{x} \le 1$ *(range of $\sin x$ is $[-1, 1]$),*
let $f(x) = -x^2$, $g(x) = x^2 \sin \dfrac{1}{x}$ and $h(x) = x^2$.

Then $f(x) \le g(x) \le h(x)$ when x near 0, ... (∗)
and $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x) = 0$. .... (∗∗)
By the Squeeze Theorem, $\lim\limits_{x\to 0} g(x) = 0$. ■

Remark: Compute limit:
1. 極限律: 加減乘除常數倍, 冪次開根 $c\&x$;
2. 代數方法: 同乘同除非零項, 或換成好算的函數算;
3. 左右極限: 分段函數看左右;
4. 夾擠定理. (很強大, 但是難在找到極限好算又相同的兩個函數來夾.)