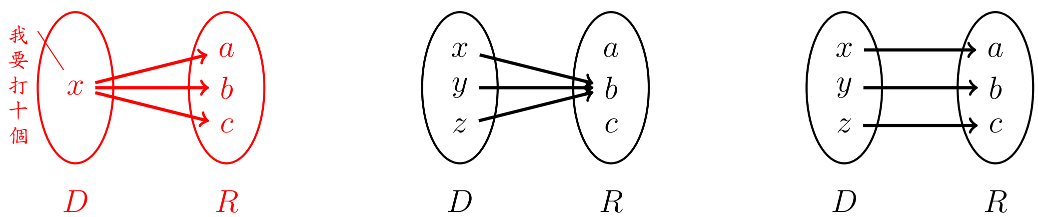
1.5 Inverse functions & logarithms

1. inverse function 反函數 $f^{-1}(x)$
2. logarithmic function 對數函數 $\log_a x$
3. inverse trigonometric function 反三角函數 arcsin or $\sin^{-1}$

0.1 Inverse function

A ***mapping*** 映射 f from ***domain*** 定義域 D to ***range*** 值域 R has three types:

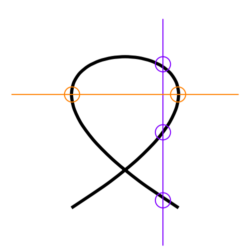
1. maps one to many 一對多 交大: 新竹, 西安, 上海, 北京, 西南。
2. maps many to one 多對一 0314/0351/0415微積分老師: 林武雄。
3. maps one to one 一對一 同時段(1GH4CD/2CD5EF/2IJ4IJ) 的教室。



A ***function*** 函數 f is a mapping without the type of one to many, and is

1. ***one-to-one*** (injective, an injection) 一對一 (單射)
   if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. $\implies |D| \leq |R|$. (人人點不同菜。)
2. ***onto*** (surjective, a surjection) 映成 (滿射)
   if $\forall\ y \in R, \exists\ x \in D, \ni f(x) = y. \implies |D| \geq |R|$. (道道菜有人點。)
3. ***one-to-one & onto*** (bijective, a bijection) 一對一且映成 (雙射)
   $\implies |D| = |R|$. (點好點滿。)

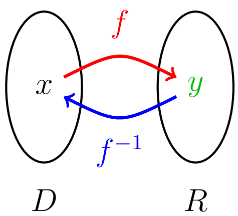
♦ : 證明正整數, 整數, 偶數, 有理數一樣多 ($|\mathbb{N}| = |\mathbb{Z}| = |2\mathbb{Z}| = |\mathbb{Q}|$): 找雙射。



Vertical line test: $f(x)$ is a function $\iff$ $y = f(x)$ intersects any vertical line $x = k$ at most one point.
(是函數 $\iff$ 任垂直線最多交一點。)
Horizontal line test: $f(x)$ is one-to-one $\iff$ $y = f(x)$ intersects any horizontal line $y = c$ at most one point.
(一對一 $\iff$ 任水平線最多交一點。)

Define: The ***inverse function*** 反函數 of a ***one-to-one*** function

$f: D \to R$ is $\boxed{\boldsymbol{f^{-1}}}$ $: R \to D$ s.t. $\boxed{f^{-1}(y) = x \iff f(x) = y}$.



1. $f^{-1}(f(x)) = x$, $\forall\, x \in D$.
2. $f(f^{-1}(y)) = y$, $\forall\, y \in R$.
3. f^{-1} is one-to-one.

Attention: $\boxed{\boldsymbol{f^{-1}(x) \neq \frac{1}{f(x)} = [f(x)]^{-1}}}$.

How to solve $f^{-1}(x)$: (A 先變再換, B 先換再變; 最好固定一種)
A1. write $y = f(x)$; A2. become $x = g(y)$; A3. exchange x, y to obtain.
B1. write $x = f(y)$; B2. become $y = g(x)$; B3. $f^{-1}(x) = g(x)$.

Skill: 怎麼變? 加對減 乘對除 冪次對開根 x對y D對R 指數對對數
天對地 雨對風 大陸對長空 山花對海樹 赤日對蒼穹
雷隱隱 霧濛濛 日下對天中 風高秋月白 雨霽晚霞紅 —《笠翁對韻》

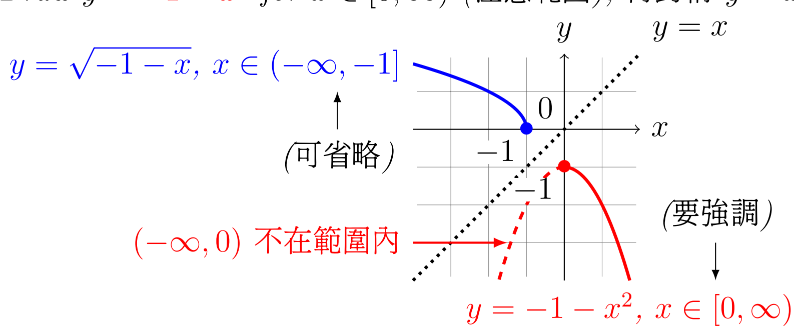
Example 0.1 $f(x) = x^3 + 2$, $f^{-1}(x) = ?$

$A.\ y = f(x) = x^3 + 2 \xrightarrow{(-2)} x^3 = y - 2 \xrightarrow{(\sqrt[3]{\ })} x = \sqrt[3]{y-2}$, 交換 x 與 y.
$B.\ x = f(y) = y^3 + 2 \longrightarrow y^3 = x - 2 \longrightarrow y = \sqrt[3]{x-2}$.
$\implies f^{-1}(x) = \sqrt[3]{x-2}$. ■

How to draw $f^{-1}(x)$: $(y, f^{-1}(y)) = (f(x), x)$,
$y = f(x)$ 與 $y = f^{-1}(x)$ 的圖形對稱於 $y = x$ (過原點 45° 直線).

Example 0.2 *Draw* $\sqrt{-1-x}$.

$\sqrt{-1-x} : (-\infty, -1] \to [0, \infty)$, 是 $-1 - x^2$ 的反函數 *(反函數是* $-1 - x^2$*)*.
Draw $y = -1 - x^2$ *for* $x \in [0, \infty)$ *(注意範圍), 再對稱* $y = x$ *畫.*



■

0.2 Logarithms

Define: The ***logarithmic function of base*** a (以 a 爲底的對數函數),

$$\boxed{\log_a x,\ a > 0,\ a \neq 1} : (0, \infty) \to (-\infty, \infty)$$

is the inverse function of $f(x) = a^x$. ("log(arithm) of x to the base a")

$$\boxed{\log_a y = x \iff a^x = y} \qquad 對_{底}真 = 指 \iff 底^{指} = 真$$

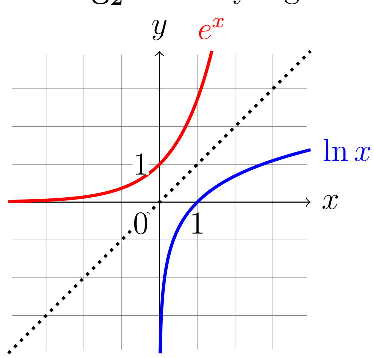
Note: a^x 要 $a > 0$, $\log_a x$ 不只要 $a > 0$, 還要 $a \neq 1$ 才有 one-to-one.
$\log_a(a^x) = x, \forall\, x \in (-\infty, \infty)$ (或 $x \in \mathbb{R}$); $a^{\log_a y} = y, \forall\, y \in (0, \infty)$ (或 $y > 0$).

Define: ***Natural logarithm*** 自然對數: ("natural log(arithm) of x")

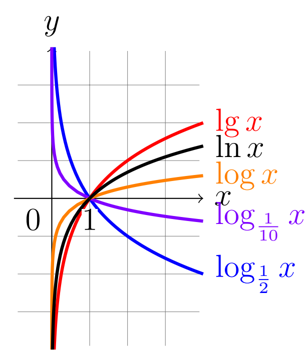
$$\boxed{f(x) = \ln x = \log_e x}$$

以 e 爲底的對數函數 (自然指數函數 e^x 的反函數: $\ln y = x \iff e^x = y$.)

Note: $\boxed{\ln e^x = x}$, $\forall\, x \in \mathbb{R}$; $\boxed{e^{\ln y} = y}$, $\forall\, y > 0$.
$\log x = \log_{10} x$ common logarithm 常用對數, in science and engineering.
$\lg x = \log_2 x$ binary logarithm 二元對數, in computer science.



Attention:
ln / log 的 **l**
(我都寫 ℓ)
是小寫 **L**,
不是大寫 **I**,
也不是數字 **1**.



Law of logarithms 對數律: $a > 0$, $a \neq 1$, $x, y > 0$. ($a^b = x, a^c = y$)

1. 乘: $\log_a xy = \log_a x + \log_a y$. ($\iff a^b \times a^c = a^{b+c}$)
2. 除: $\log_a x/y = \log_a x - \log_a y$. ($\iff a^b/a^c = a^{b-c}$)
3. 冪次: $\log_a x^r = r \log_a x$ for $r \in \mathbb{R}$. ($\iff (a^b)^r = a^{rb}$)

◆ **History:**

• 200 B.C. Archimedes(阿基米德) 發現:

$$\begin{array}{ccccc} 1, & 10, & 100, & 1000, & \cdots \\ 0, & 1\,, & 2\,, & 3\,, & \ldots \end{array}$$

可以用第二列的加減表示第一列的乘除. .... $(\log_a x + \log_a y = \log_a xy)$

• 1544 Michael Stifel(斯基弗, 1487–1567) 在《Arithmetica Integra》中首次使用 exponent(指數) 這個字, 並寫道:

$$\begin{array}{ccccccccccc} \frac{1}{8}\,, & \frac{1}{4}\,, & \frac{1}{2}\,, & 1, & 2, & 4, & 8, & 16, & 32, & 64 \\ -3, & -2, & -1, & 0, & 1, & 2, & 3, & 4\,, & 5\,, & 6 \end{array}$$

還可以用第二列的乘除代替第一列的冪次與開根. .... $(\log_a x^r = r\log_a x)$

• 1614 John Napier (納皮爾, 1550–1617) 發表歷史上第一張對數表. 他花了 20 年解

$$N = 10^7(1-10^{-7})^L \text{ for } N = 5 \sim 10^7,$$

也就是解

$$L = \mathrm{Naplog}(N) = \log_{1-10^{-7}}(\frac{N}{10^7}) = 10^7 \log_{(1-10^{-7})^{10^7}}(\frac{N}{10^7})$$

其中的

$$(1-10^{-7})^{10^7} = 0.9999999^{10000000} \approx e^{-1},$$

所以

$$\mathrm{Naplog}(N) \approx -10^7 \ln(\frac{N}{10^7}).$$

後來他想到應該以 10 爲底, 可惜命不夠長. .... $(\log_x y = \dfrac{\log_a y}{\log_a x})$

• 1620 Jost Bürgi (比爾吉, 1552–1632) 發表《Progreß Tabulen》羅列

$$(1.0001)^n \text{ for } 0 \le n \le 23027.$$

• 1624 Henry Briggs (1555–1631), Napier 的朋友, 繼承其遺志發表 1–20,000 & 90,000–100,000 的14位數對數表 (of base 10); 1627 Ezechiel de Decker with Adriaan Valcq 補上 20,000–90,000.

• 1727 Leonhard Euler (歐拉, 1707–1783) 命名 $e = 2.718281828...$, 1730 發表用極限定義自然指數與自然對數函數:

$$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n \qquad \& \qquad \ln x = \lim_{n\to\infty} n(x^{\frac{1}{n}}-1).$$

Change base formula 換底公式: $a > 0, a \neq 1,\ b > 0, b \neq 1,\ x > 0.$

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}.$$

Proof. Let $y = \log_a x \iff a^y = x$, then

$$\begin{aligned} \log_b x &= \log_b a^y \quad \text{(inverse function)} \\ &= y \log_b a \quad \text{(logarithmic law)} \\ &= \log_a x \log_b a, \\ \implies \log_a x &= \log_b x / \log_b a. \end{aligned}$$

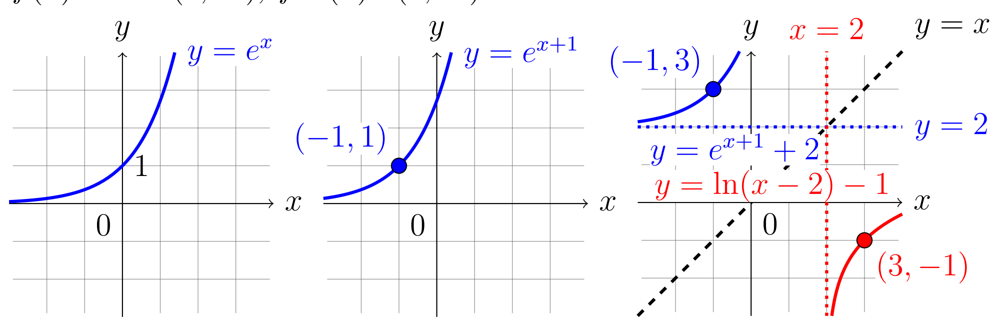
■

Note: 換底的好處 — 不用對每種底做對數表:
$\ln 2 \approx 0.7$, $\ln 10 \approx 2.3$. Then $\log 2 = \dfrac{\ln 2}{\ln 10} \approx \dfrac{7}{23}$, $\lg 10 = \dfrac{\ln 10}{\ln 2} \approx \dfrac{23}{7}$.

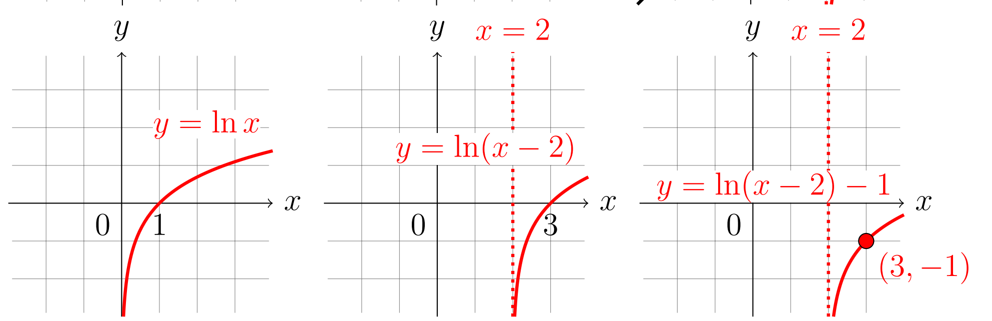
Example 0.3 *$f(x) = e^{x+1} + 2$, solve and draw $f^{-1}(x)$.*

Let $x = e^{y+1} + 2 \iff x - 2 = e^{y+1} \iff \ln(x-2) = y + 1$
$\iff f^{-1}(x) = y = \ln(x-2) - 1$. *(注意括號:* $\ln x - 2 \neq \ln(x-2)$*.)*
$f(x) : \mathbb{R} \to (2, \infty)$, $f^{-1}(x) : (2, \infty) \to \mathbb{R}$.

■



A:



B:

0.3 Inverse trigonometric function

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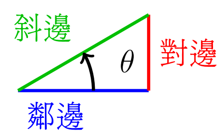
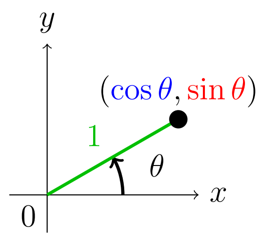
狹義



廣義



$$\begin{array}{ll}
\sin\theta = \dfrac{\text{對邊}}{\text{斜邊}} & \cos\theta = \dfrac{\text{鄰邊}}{\text{斜邊}} \\
\tan\theta = \dfrac{\text{對邊}}{\text{鄰邊}} = \dfrac{\sin\theta}{\cos\theta} & \cot\theta = \dfrac{\text{鄰邊}}{\text{對邊}} = \dfrac{\cos\theta}{\sin\theta} \\
\sec\theta = \dfrac{\text{斜邊}}{\text{鄰邊}} = \dfrac{1}{\cos\theta} & \csc\theta = \dfrac{\text{斜邊}}{\text{對邊}} = \dfrac{1}{\sin\theta}
\end{array}$$

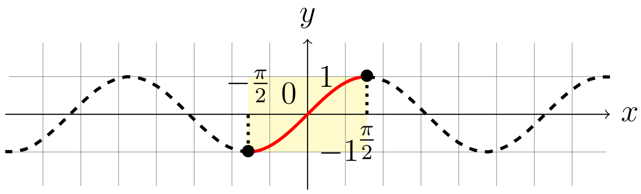
Trigonometric 三角函數 $f(x) \in \{\sin x, \cos x, \tan x, \cot x, \sec x, \csc x\}$ 是 periodic 週期函數 $(f(x+2\pi) = f(x))$, $x \in \mathbb{R}$, 不是 one-to-one, 所以要限制定義域, 使 $f(x)$ 變成 one-to-one, 才能考慮反函數.

Define: The inverse function of restricted sine function is called the ***inverse sine function***, $\boxed{\boldsymbol{\sin^{-1} x}}$, or the ***arcsine function***, $\boxed{\boldsymbol{\arcsin x}}$. (受限制的正弦函數的反函數=反正弦函數)

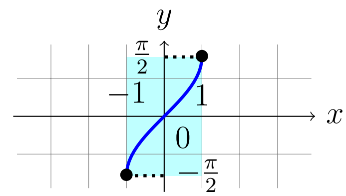
function	restricted domain	range	inverse
$\sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\left[-1, 1\right]$	$\sin^{-1} x = \arcsin x$
$\cos x$	$\left[0, \pi\right]$	$\left[-1, 1\right]$	$\cos^{-1} x = \arccos x$
$\tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left(-\infty, \infty\right)$	$\tan^{-1} x = \arctan x$
$\cot x$	$\left(0, \pi\right)$	$\left(-\infty, \infty\right)$	$\cot^{-1} x$
$\sec x$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\left(-\infty, -1\right] \cup \left[1, \infty\right)$	$\sec^{-1} x$
$\csc x$	$\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$	$\left(-\infty, -1\right] \cup \left[1, \infty\right)$	$\csc^{-1} x$

Attention: $\sin^n x = (\sin x)^n$ for $n \in \mathbb{N}$, $\boxed{\boldsymbol{\sin^{-1} x \neq \dfrac{1}{\sin x} = (\sin x)^{-1}}}$.

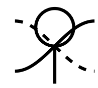
1. **Sine** 正弦 $\sin x : \mathbb{R} \to [-1, 1]$ (廣義).



$\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

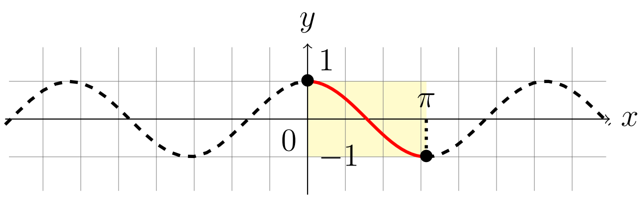


$\sin^{-1} x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

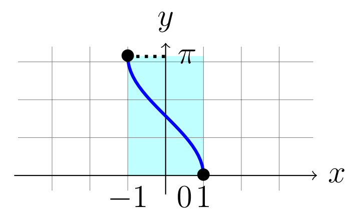


$\sin^{-1}(\sin x) = x, \forall\, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin(\sin^{-1} y) = y, \forall\, y \in [-1, 1]$.

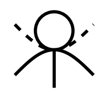
2. **Cosine** 餘弦 $\cos x : \mathbb{R} \to [-1, 1]$ (廣義).



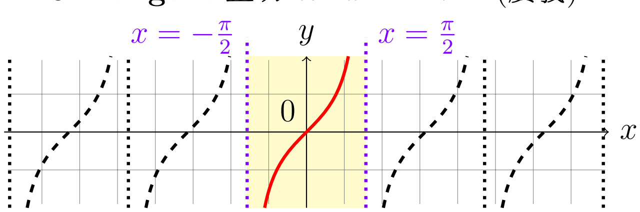
$\cos x : [0, \pi] \to [-1, 1]$



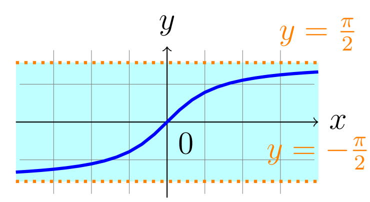
$\cos^{-1} x : [-1, 1] \to [0, \pi]$



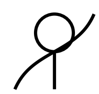
3. **Tangent** 正切 $\tan x : \mathbb{R} \to \mathbb{R}$ (廣義).



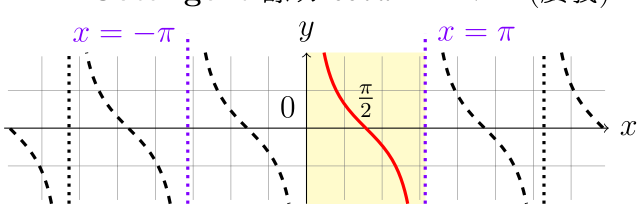
$\tan x : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$



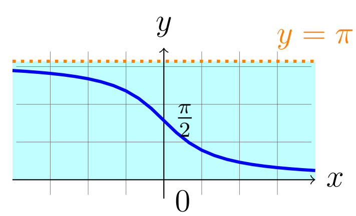
$\tan^{-1} x : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$



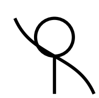
4. **Cotangent** 餘切 $\cot x : \mathbb{R} \to \mathbb{R}$ (廣義).



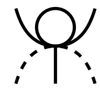
$\cot x : (0, \pi) \to \mathbb{R}$

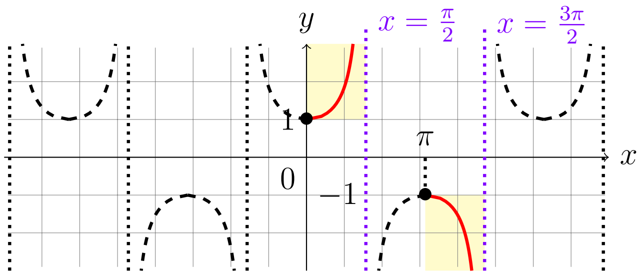


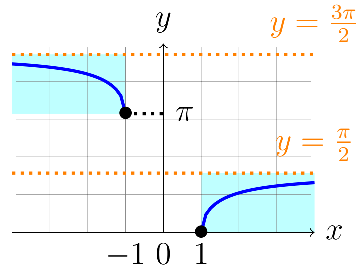
$\cot^{-1} x : \mathbb{R} \to (0, \pi)$



5. **Secant** 正割 $\sec x : \mathbb{R} \to (-\infty, -1] \cup [1, \infty)$ (廣義).



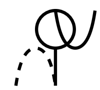


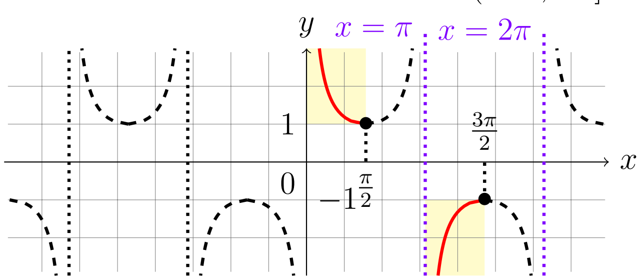


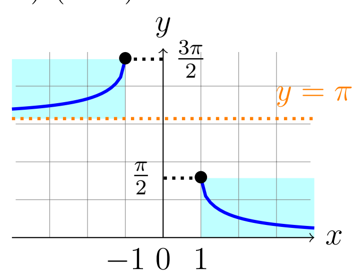
$\sec x : [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2}) \to (-\infty, -1] \cup [1, \infty)$

$\sec^{-1} x$
$: |x| \geq 1 \to [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

6. **Cosecant** 餘割 $\csc x : \mathbb{R} \to (-\infty, -1] \cup [1, \infty)$ (廣義).



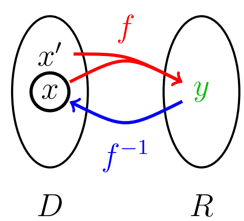




$\csc x : (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}] \to (-\infty, -1] \cup [1, \infty)$

$\csc^{-1} x$
$: |x| \geq 1 \to (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

Question: 一定要限制在這些區間嗎?
Answer: 不一定, 只要能 one-to-one 就好.
Question: 爲什麼要限制在這些區間?
Answer: see §3.5, §7.3.



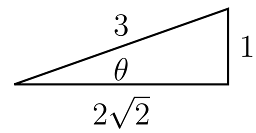
(Fill by yourself:

n	1	2	3	4	5	6	7	8
$\sin^{-1}(\sin n)$	1							
$\cos^{-1}(\cos n)$	1							
$\tan^{-1}(\tan n)$	1							
$\cot^{-1}(\cot n)$	1							
$\sec^{-1}(\sec n)$	1							
$\csc^{-1}(\csc n)$	1							

hint: $\sin(\pi - \theta) = \sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(\pi + \theta) = \tan\theta$.)

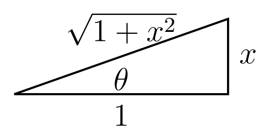
Example 0.4 *(a)* $\sin^{-1}\frac{1}{2}=?$ *(b)* $\tan(\arcsin\frac{1}{3})=?$

(a) Let $x=\sin^{-1}\frac{1}{2} \iff \sin x=\frac{1}{2}$,
$x=(2k+\frac{1}{6})\pi$ *or* $(2k+\frac{5}{6})\pi$, *only* $\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.
(b) Let $\theta=\arcsin\frac{1}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \iff \sin\theta=\frac{1}{3}$,
$\tan\theta=\frac{1}{\sqrt{3^2-1^2}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$. ■



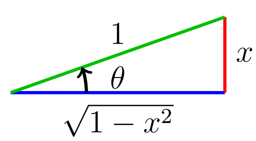
Example 0.5 *Simplify* $\cos(\tan^{-1}x)$.

Let $\theta=\tan^{-1}x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \iff \tan\theta=x$.
$\sec^2\theta=1+\tan^2\theta$, $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+x^2}$.
*(*負不合*,* $\because\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\sec\theta\geq 1>0$.*)*
$\therefore\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{1+x^2}}$.
[Another method]: See diagram. ■

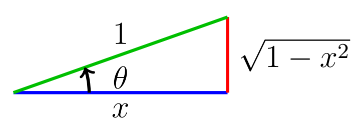


Skill: Diagram: Inverse $\left\{\begin{array}{c}\text{sine / tangent / secant}\\ \text{cosine/cotangent/cosecant}\end{array}\right\}$ function

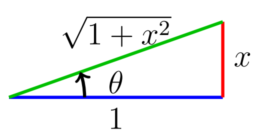
$\sin^{-1}x$



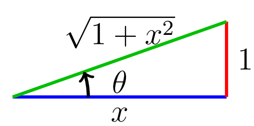
$\cos^{-1}x$



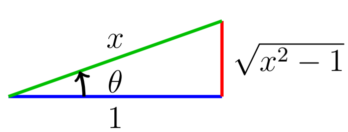
$\tan^{-1}x$



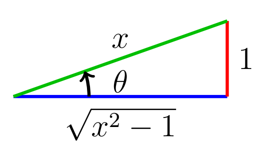
$\cot^{-1}x$



$\sec^{-1}x$



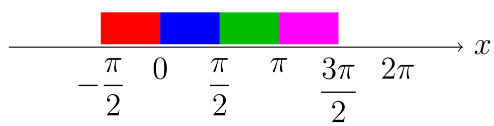
$\csc^{-1}x$



♦ Additional: Answer

A	n	1	2	3	4	5	6	7	8
B	$\pi - n$	2.14	1.14	0.14	−0.86	−1.86	−2.86	−3.86	−4.86
C	$n - \pi$	−2.14	−1.14	−0.14	0.86	1.86	2.86	3.86	4.86
D	$2\pi - n$	5.28	4.28	3.28	2.28	1.28	0.28	−0.72	−1.72
E	$n - 2\pi$	−5.28	−4.28	−3.28	−2.28	−1.28	−0.28	0.72	1.72
F	$3\pi - n$	8.42	7.42	6.42	5.42	4.42	3.42	2.42	1.42
G	$n - 3\pi$	−8.42	−7.42	−6.42	−5.42	−4.42	−3.42	−2.42	−1.42
H	$4\pi - n$	11.56	10.56	9.56	8.56	7.56	6.56	5.56	4.56

	Domain	Range
$\sin^{-1}(\sin n)$	AB EF	
$\cos^{-1}(\cos n)$	A DE H	
$\tan^{-1}(\tan n)$	A C E G	
$\cot^{-1}(\cot n)$	A C E G	
$\sec^{-1}(\sec n)$	A DE H	
$\csc^{-1}(\csc n)$	AB EF	



How to read tables:

1. Find $\cos^{-1}(\cos 5)$: Look the column of $n = 5$ in the 1st table.
2. In the 2nd table $\cos^{-1}(\cos n)$ domain ADEH
($\because \cos 5 = \cos(2\pi - 5) = \cos(5 - 2\pi) = \cos(4\pi - 5)$):
Look numbers in rows ADEH $\{5, 1.28, -1.28, 7.56\}$.
3. $\cos^{-1}(\cos n)$ range blue$(0 \sim \pi/2)$ and green$(\pi/2 \sim \pi)$:
Find the number of color blue or green 1.28 in the row D $(2\pi - n)$.
4. $\cos^{-1}(\cos 5) = 2\pi - 5$.

n	1	2	3	4	5	6	7	8
$\sin^{-1}(\sin n)$	1	$\pi - 2$	$\pi - 3$	$\pi - 4$	$5 - 2\pi$	$6 - 2\pi$	$7 - 2\pi$	$3\pi - 8$
$\cos^{-1}(\cos n)$	1	2	3	$2\pi - 4$	$2\pi - 5$	$2\pi - 6$	$7 - 2\pi$	$8 - 2\pi$
$\tan^{-1}(\tan n)$	1	$2 - \pi$	$3 - \pi$	$4 - \pi$	$5 - 2\pi$	$6 - 2\pi$	$7 - 2\pi$	$8 - 3\pi$
$\cot^{-1}(\cot n)$	1	2	3	$4 - \pi$	$5 - \pi$	$6 - \pi$	$7 - 2\pi$	$8 - 2\pi$
$\sec^{-1}(\sec n)$	1	$2\pi - 2$	$2\pi - 3$	4	$2\pi - 5$	$2\pi - 6$	$7 - 2\pi$	$4\pi - 8$
$\csc^{-1}(\csc n)$	1	$\pi - 2$	$\pi - 3$	4	$3\pi - 5$	$3\pi - 6$	$7 - 2\pi$	$3\pi - 8$

(Find by yourself:
$\sin(\sin^{-1} 1)$, $\cos(\cos^{-1} 1)$, $\tan(\tan^{-1} 1)$, $\cot(\cot^{-1} 1)$, $\sec(\sec^{-1} 1)$, $\csc(\csc^{-1} 1)$=?)