10.1 Curves defined by parametric equations

1. parametric curve 參數曲線
2. cycloid 擺線 and conchoid 蚌線

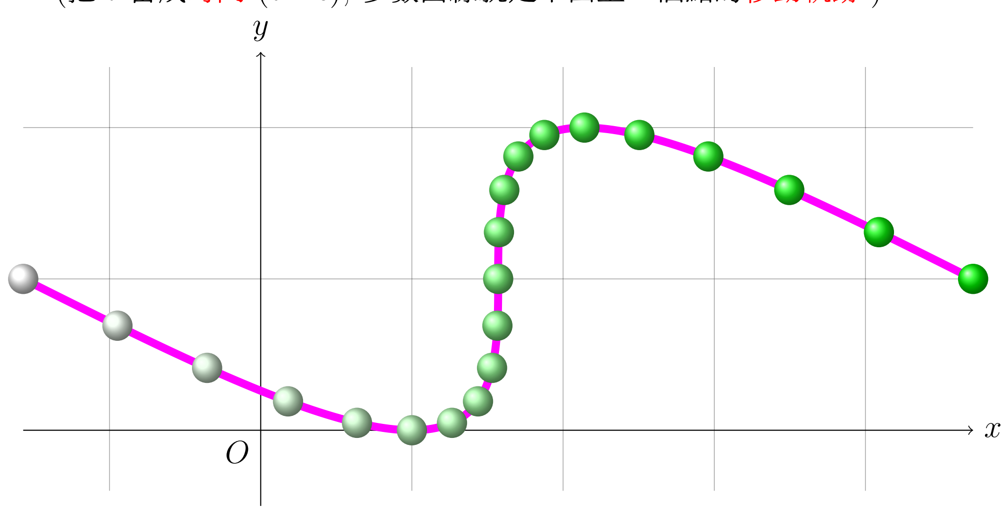
0.1 Parametric curve

愛心線 $(x^2+y^2-1)^3 = x^2y^3$ 是 rectangular (or Cartesian) coordinate system 直角 (或卡式) 坐標系方程式的曲線.

Define: ***parametric curve*** [ˌpɛrəˈmɛtrɪk] 參數曲線

1. ***parameter*** [pəˈræmətɚ] 參數: t.
2. ***parametric equations*** 參數方程式: $x = f(t)$, $y = g(t)$.
   x, y 看成是第三個變數 t 的函數.
3. ***parametric curve*** 參數曲線: $(x, y) = (f(t), g(t))$.
   每個 t 決定一個點 (x, y), 隨著 t 變化, $(x, y) = (f(t), g(t))$ 畫出的曲線.
4. ***initial point*** 起點: $(f(a), g(a))$, ***terminal point*** 終點: $(f(b), g(b))$.
   當 t 有給範圍時 $a \le t \le b$.

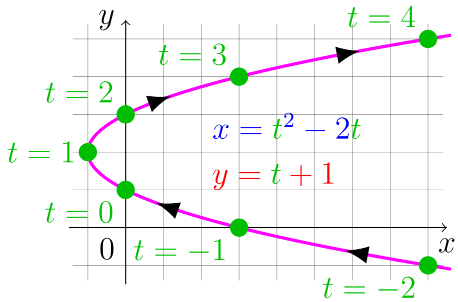
(把 t 當成時間 (time), 參數曲線就是平面上一個點的移動軌跡.)



Note: 相對於參數方程式, 只由 x, y 組成的叫做 ***Cartesian equation***.

Example 0.1 *Sketch and identify the curve defined by $x = t^2 - 2t$, $y = t + 1$. (沒特別指定, t 的範圍就是 $(-\infty, \infty)$.)*

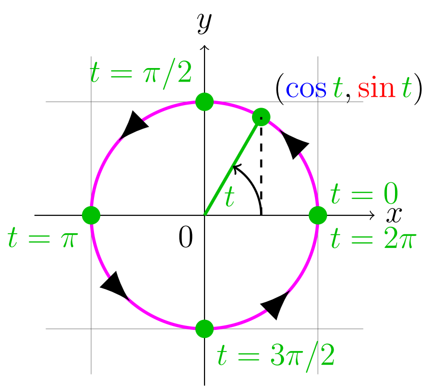
$t = y - 1$, $x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$, *a parabola.* ■

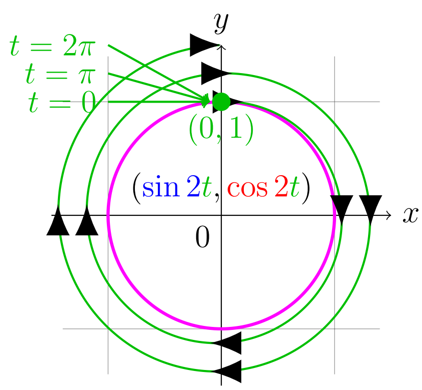


Note: 要標方向 (⟶▸), 從小往大.

Example 0.2 *What curve is represented by $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$?*

$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, *a unit circle.*
initial point $=$ *terminal point* $= (1, 0)$, *counterclockwise* 逆時針繞一圈. ■



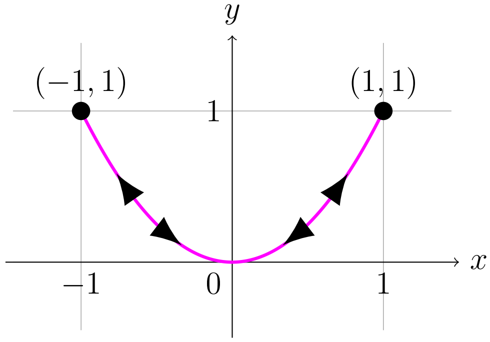


Example 0.3 *What curve is represented by $x = \sin 2t$, $y = \cos 2t$, $0 \le t \le 2\pi$?*

$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$, *a unit circle.*
initial point $=$ *terminal point* $= (0, 1)$, *clockwise* 順時針繞兩圈. ■

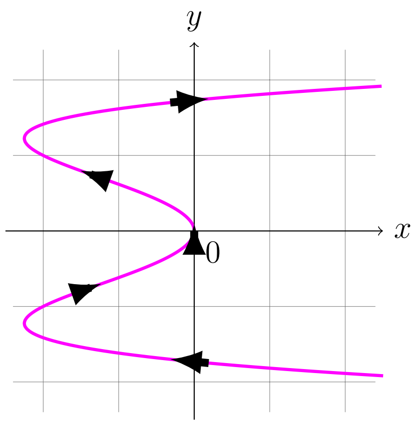
Example 0.4 *Sketch curve with parametric equations* $x = \sin t$, $y = \sin^2 t$.

$x^2 = \sin^2 t = y$, *a parabola.* $\because -1 \le \sin t \le 1$, $-1 \le x \le 1$. $\blacksquare$



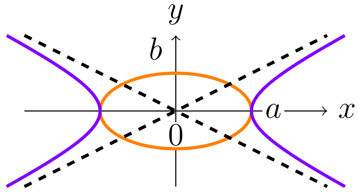
Example 0.5 *Give parametric equations for* $x = y^4 - 3y^2$.

$x = t^4 - 3t^2$, $y = t$. $\blacksquare$

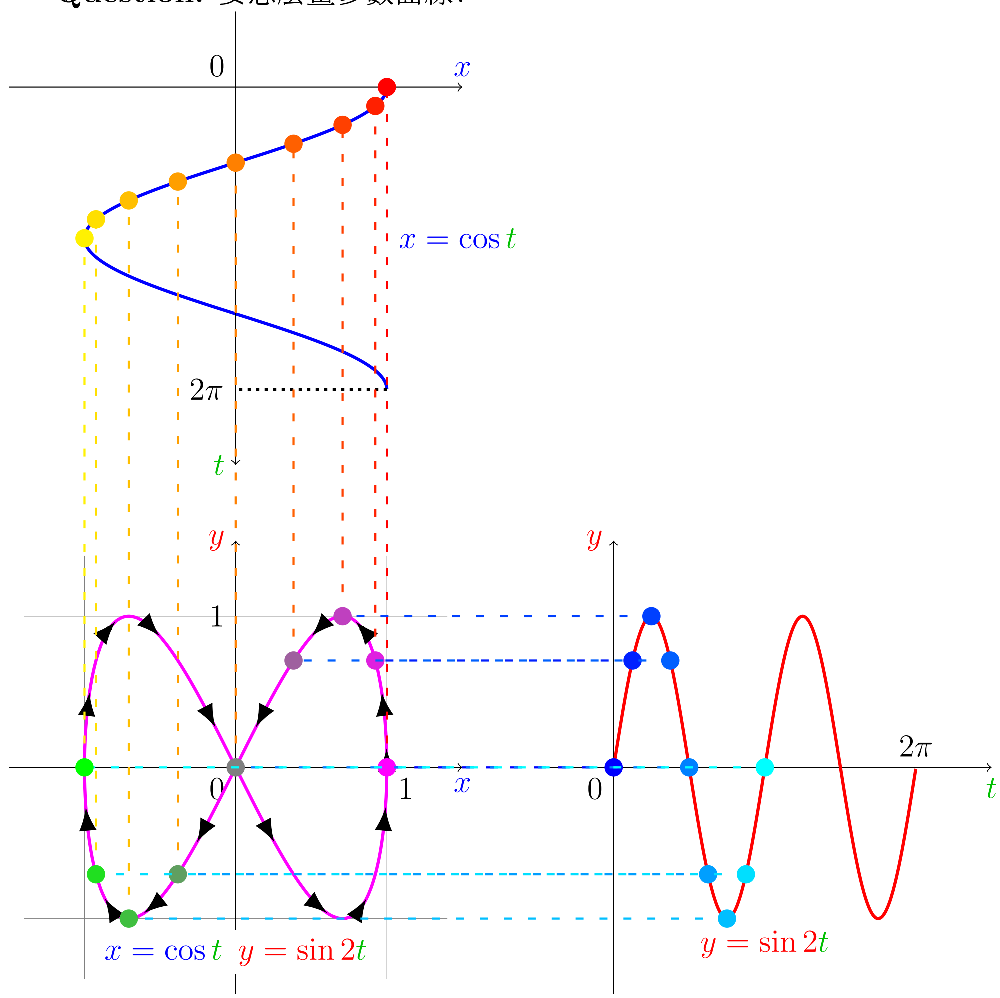


Additional: Hyperbola 雙曲線 $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \iff x = a\sec t,\ y = b\tan t$.

(Ex 10.1.34) Ellipse 橢圓 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \iff x = a\cos t,\ y = b\sin t$.



Question: 要怎麼畫參數曲線?

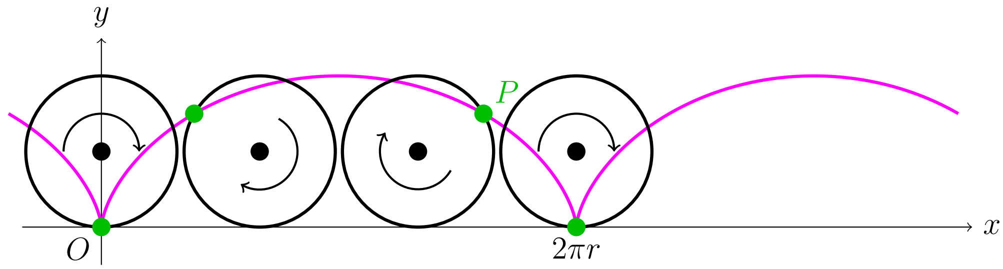


Remark:

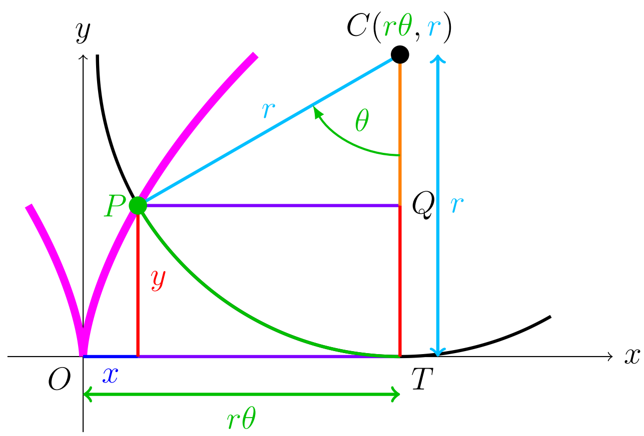
1. 畫圖要標示 t 從小到大的方向.
2. 從 parametric equations 得到 Cartesian(rectangular) equation: 消去 t.
3. 從 Cartesian equation 得到 parametric equations:
if $y = f(x)$, let $x = t$, $y = f(t)$, $t \in$ domain of f;
if $x = g(y)$, let $y = t$, $x = g(t)$, $t \in$ domain of g.
4. 不同 parametric equations 可得到方向速率不一樣的相同曲線.
5. parametric equations 可以表現比 rectilinear equations 更多的曲線.

0.2 Cycloid & conchoid

Define: The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a ***cycloid*** [ˈsaɪˌklɔɪd].
圓周上一點當圓沿一直線滾動所得的曲線稱爲擺線.



Example 0.6 *If circle has radius r and rolls along x-axis and if one position of P is the origin, find parametric equations.*



Suppose center is C, and $CT \perp x$-axis at T, $\theta = \angle TCP$.
Then $|OT| = r\theta$, $|CT| = r$, and $P(x, y)$ has
$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$,
$y = |CT| - |CQ| = r - r\cos\theta = r(1 - \cos\theta)$.

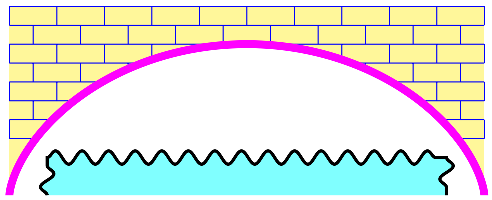
The parametric equations of cycloid are

$$\boxed{\boldsymbol{x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta), \quad \theta} \in \mathbb{R}.}$$

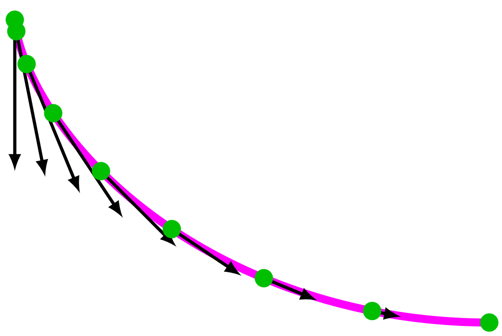
■

Story: Cycloid, the Helen of Geometers 幾何學的海倫 (眾人搶)

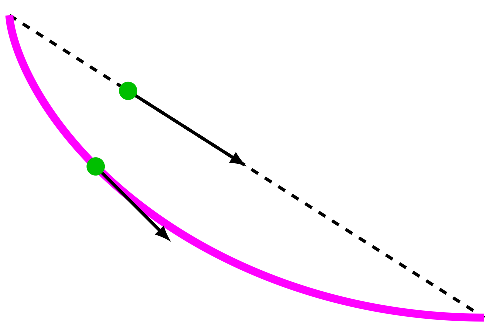
- 1599 Galileo 加利略, 橋要蓋成擺線.



- 1634 Roberval 羅伯歐, 橋下面面積 $3\pi r^2$ (10.2會教到).
- 1658 Wren 雷恩, 弧長 $8r$ (10.2會教到).
- 1673 Huygens 海更斯, **The Tautochrone Problem** [ˋtɔtəkron]
  等時曲線問題: 從哪滑到底都一樣快的曲線? 倒過來的 (inverted) 擺線 .



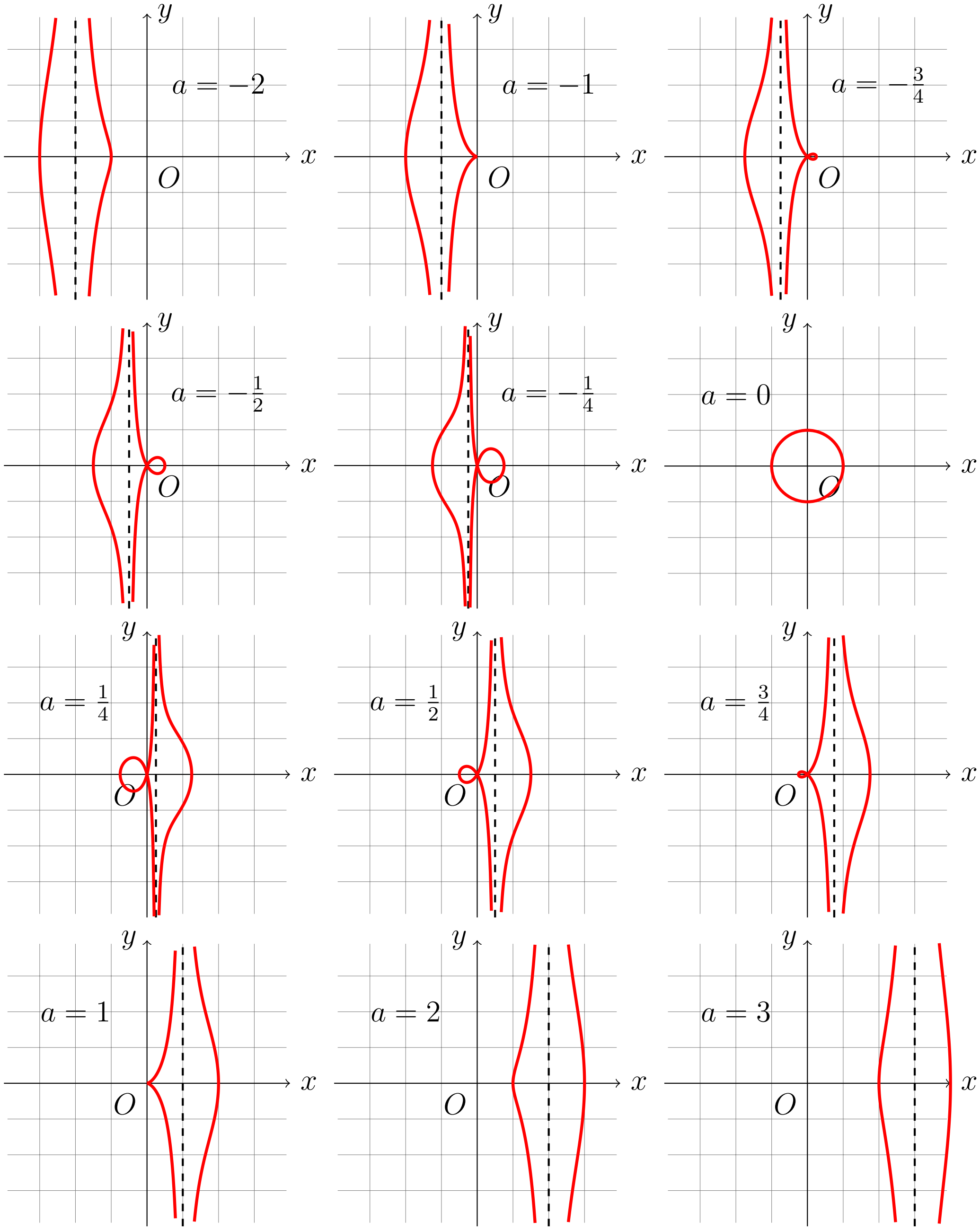
- 1696 Bernoulli 白努力, **The Brachistochrone Problem** [brəˋkɪstəˏkron]
  最速降線問題: 滑 (斜下方) 下來最快的曲線? 倒過來的擺線.



Conchoids of Nicomedes: [ˈkɑŋkɔɪd]

$$x = a + \cos t, \quad y = a \tan t + \sin t.$$

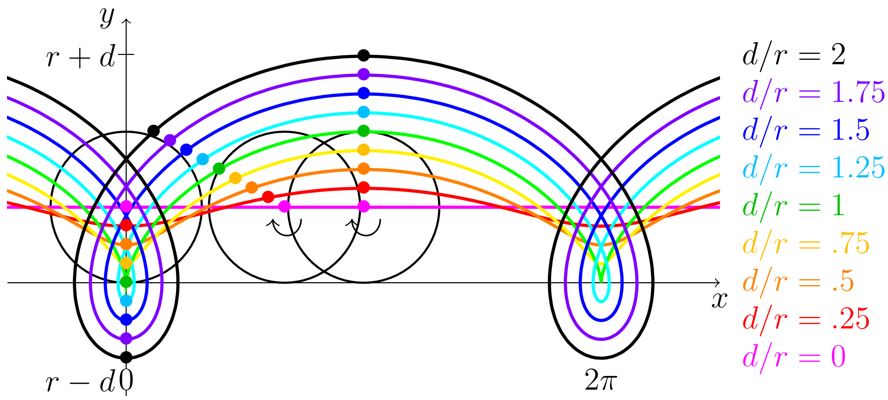
尼科梅德斯 (西元前希臘數學家) 的蚌線 (conch[康殼]: 蚌).



⧫ Additional: Stories in Exercises

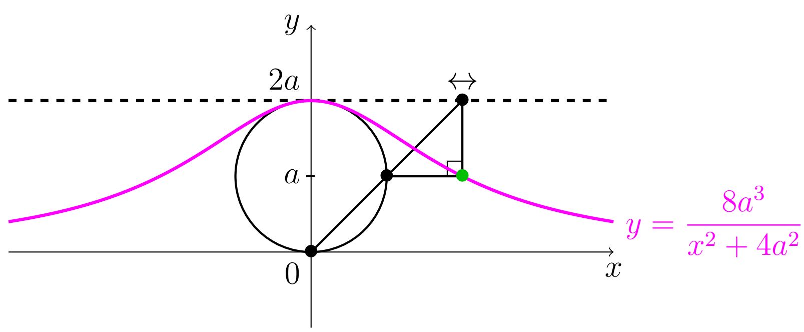
♠ ***trochoid*** 次擺線 (Exercise 10.1.40)
Roberval 創字源於 “trochos” 希臘文的 wheel 輪子。

$$x = r\theta - d\sin\theta \qquad y = r - d\cos\theta$$



♡ ***witch of Maria Agnesi*** 箕舌線 (Exercise 10.1.43), see also §3.2。

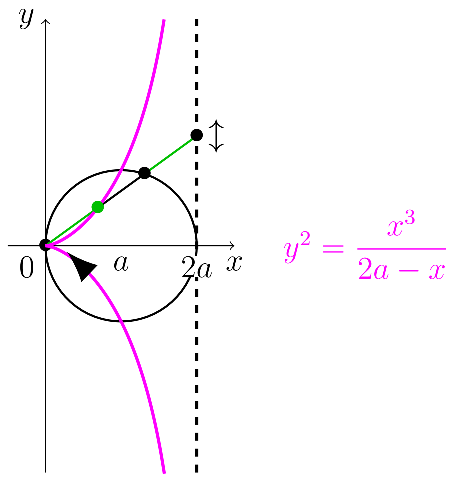
$$x = 2a\cot\theta \qquad y = 2a\sin^2\theta$$



◇ ***cissoid of Diocles*** 戴可利斯的蔓葉線 (Exercise 10.1.44)

"cissoid"字意爲"像長春藤的", B.C. 180 希臘幾何學家 Diocles 試圖解決倍立方問題 (Delian problem) 時發現。

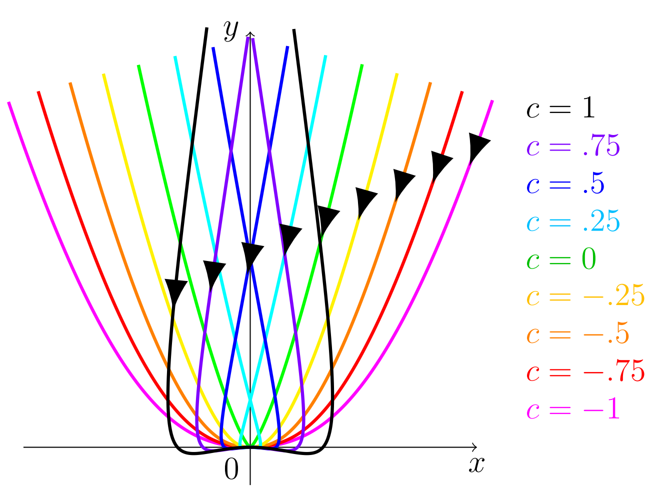
$$x = 2a\sin^2\theta \qquad y = 2a\tan\theta\sin^2\theta$$



♣ ***swallowtail catastrophe curves*** 燕尾型突變曲線 (Exercise 10.1.48)

爲混沌理論 (Chaos Theory) 中突變理論 (Catastrophe Theory) 裡的七種基本型突變 (catastrophe): 摺疊 (fold), 尖點 (cusp), 燕尾 (swallowtail), 蝴蝶 (butterfly), 雙曲臍 (hyperbolic umbilic), 橢圓臍 (elliptic umbilic), 拋物臍 (parabolic umbilic) 之一。

$$x = 2ct - t^3 \qquad y = -ct^2 + 3t^4$$

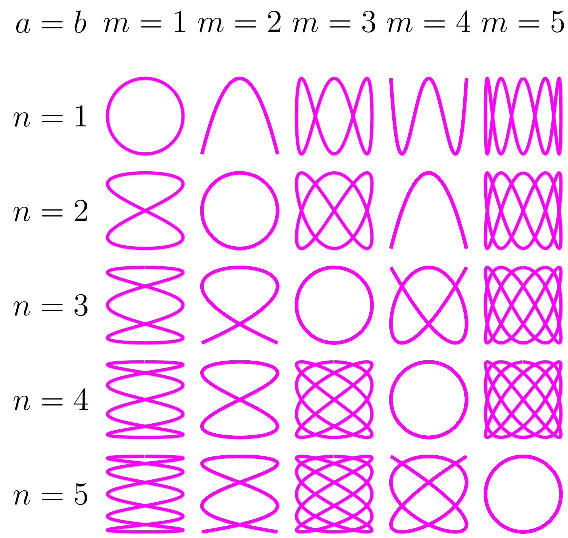


Lissajous figures/curves 利薩茹圖形/曲線 (Exercise 10.1.51)

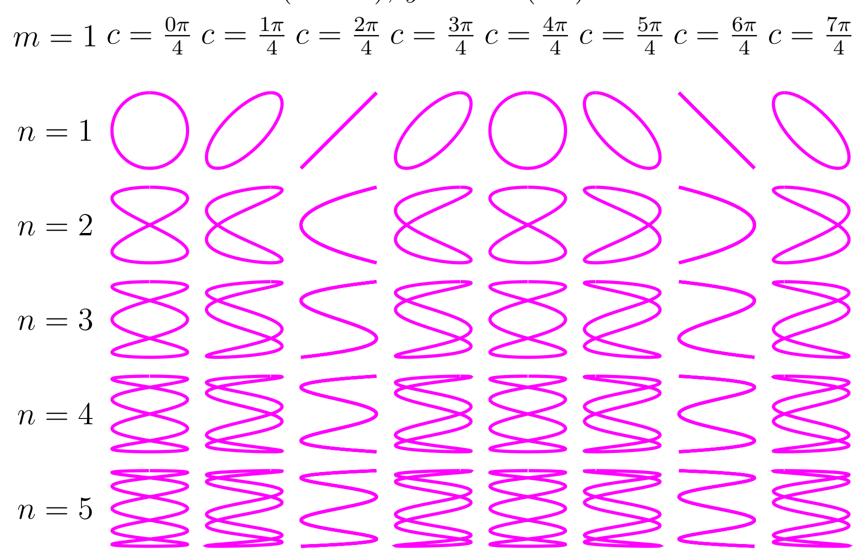
1815 美國數學家 Bowditch 首先研究, 故又稱鮑迪奇曲線 (Bowditch's curves)。
1857 法國數學家 Lissajous 做更詳細的研究。其應用在示波器 (oscilloscopes) 上。

$$x = a\sin nt \qquad y = b\cos mt$$

(長 $2a$ 寬 $2b$ 的矩形 $[-a,a]\times[-b,b]$ 內。)



更一般的是 $x = a\sin(nt+c)$, $y = b\cos(mt)$ 版本.



Exam 10.1

Matching parametric equations with figures:

(A) $x = t(\sin t + \cos t)$, $y = t(\sin t - \cos t)$, $0 \leq t \leq 4\pi$.

(B) $x = t(\sin t - \cos t)$, $y = t(\sin t + \cos t)$, $0 \leq t \leq 4\pi$.

(C) $x = t(\cos t + \sin t)$, $y = t(\cos t - \sin t)$, $0 \leq t \leq 4\pi$.

(D) $x = t(\cos t - \sin t)$, $y = t(\cos t + \sin t)$, $0 \leq t \leq 4\pi$.