2.5 Continuity

1. continuous function 連續函數
2. combination of continuous functions 連續函數的組合
3. Intermediate Value Theorem 中間值定理

0.1 Continuous function

連續函數=沒有斷點, 而且具有傳遞極限的能力.

分別有: 單點連續, 左/右連續, 區段連續; 都是用極限來定義連續.

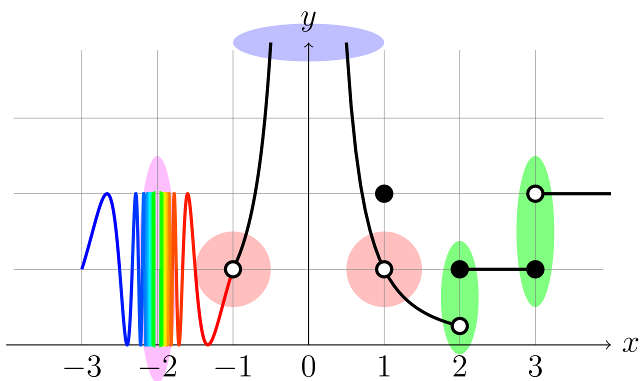
Define: 單點連續 A function $f(x)$ is ***continuous*** at a number a if

$$\boxed{\lim_{x\to a} f(x) = f(a)}.$$

$f(x)$ 在 a 連續, 代表三件事同時成立:
1. $x = a$ 有定義: $f(a)$; 2. $x = a$ 有極限: $\lim_{x\to a} f(x)$ 存在; 3. 極限等於函數值.

相反的, $f(x)$ 在 a 不連續的情形:

1. 極限存在, $f(x)$ undefined 或不相等: ***removable*** discontinuous.
2. 無限極限: ***infinite*** discontinuous.
3. 左右極限存在但不同: ***jump*** discontinuous.
4. 極限不存在: ***does not exist***. Ex: $\sin(1/x)$ at 0, 極限不存在.



$x = -1, 1$: removable; $x = 0$: infinite; $x = 2, 3$: jump.

Define: 左/右連續 A function $f(x)$ is continuous ***from the left*** at a number a if

$$\boxed{\lim_{x\to a^-} f(x) = f(a)}.$$

A function $f(x)$ is continuous ***from the right*** at a number a if

$$\boxed{\lim_{x\to a^+} f(x) = f(a)}.$$

上例中, 在 $x = 2$ 右連續, 在 $x = 3$ 左連續.

Ex: 在整數點 左連續 或 右連續 的函數:

$f(x) = [\![x]\!]$ (取整數).

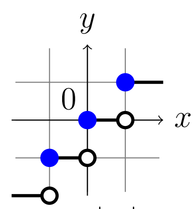
Gauss(高斯): bracket $[x](= [\![x]\!])$.

Iverson(艾佛森): floor $\lfloor x \rfloor (= [x])$, ceiling $\lceil x \rceil$.

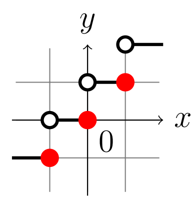
$(\lfloor x \rfloor \le x < \lfloor x \rfloor + 1,\ \lceil x \rceil - 1 < x \le \lceil x \rceil;$

$\lfloor e \rfloor = 2,\ \lceil e \rceil = 3,\ \lfloor -1.5 \rfloor = -2,\ \lceil -1.5 \rceil = -1.)$

補充: fractional part $\{x\} = x - \lfloor x \rfloor =]\!]x[\![=]x[$.



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

Define: 區段連續 A function $f(x)$ is continuous on an interval if it is continuous at every number in the interval.

(a, b): 在 (a, b) 中每個點都連續;

$[a, b)$: 在 (a, b) 中連續並且在 a 右連續;

$(a, b]$: 在 (a, b) 中連續並且在 b 左連續;

$[a, b]$: 在 (a, b) 中連續並且在 a 右連續, 在 b 左連續.

Example 0.1 *Show* $f(x) = 1 - \sqrt{1 - x^2}$ *is continuous on* $[-1, 1]$.

1. (中間連續) $-1 < a < 1$ $((-1, 1))$*:*

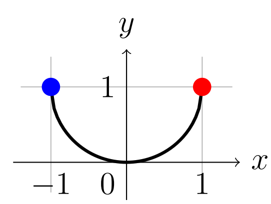
$\lim_{x\to a} f(x) = \lim_{x\to a}(1 - \sqrt{1 - x^2}) = 1 - \lim_{x\to a}\sqrt{1 - x^2}$

$= 1 - \sqrt{\lim_{x\to a}(1 - x^2)} = 1 - \sqrt{1 - a^2} = f(a)$.

2. (左端右連) $a = -1$*:* $\lim_{x\to -1^+}(1 - \sqrt{1 - x^2}) = 1 = f(-1)$.

3. (右端左連) $a = 1$*:* $\lim_{x\to 1^-}(1 - \sqrt{1 - x^2}) = 1 = f(1)$.

($\because \lim_{1-x^2\to 0^+}\sqrt{1 - x^2} = \lim_{y\to 0^+}\sqrt{y} = 0.$*)*



$y = 1 - \sqrt{1 - x^2}$

Therefore, by the definition, $f(x)$ *is continuous on* $[-1, 1]$. ■

Recall: $\sqrt{\to 0} \ne 0$, $\sqrt{\to 0^+} = 0$.

0.2 Combination of continuous functions

用定義檢驗每個函數的連續性太耗時, 利用極限律 (加減乘除常數倍) 驗證.

Theorem 1 *If f and g are continuous at a ($\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$) and c is a constant, then:*

1. 加: $f+g$
2. 減: $f-g$
3. 乘: $f\times g$
4. 除: $f\div g$, *if* $g(a)\neq 0$
5. 常數倍: cf

are continuous at a.

Proof. (只證明加法)

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}[f(x)+g(x)] && \text{(極限加法)}\\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) && \text{(連續定義)}\\ &= f(a)+g(a) = (f+g)(a).\end{aligned}$$ ■

Observation: 在哪連續:
常數函數 $f(x)=c$ 跟 $f(x)=x$ are continuous on everywhere ($\mathbb{R}=(-\infty,\infty)$).
Any polynomial 多項式 $f(x)$ is continuous on $\mathbb{R}$ (its domain).
Any ***rational function*** 有理函數 $f(x)=\dfrac{P(x)}{Q(x)}$, $P(x), Q(x)$ are polynomials, is continuous on its domain $D=\{x : Q(x)\neq 0\}$ (分母不爲零處).

List of functions which are continuous on their domains:

1. 多項式 polynomials
2. 有理函數 ration functions (分母不爲 0)
3. 開根函數 root functions (開偶次根裡面要 ≥ 0)
4. 三角函數 trigonometric function
5. 反三角函數 inverse trigonometric function
6. 指數函數 exponential functions ($\mathbb{R}$)
7. 對數函數 logarithmic functions ($(0,\infty)$)

Composed function 合成函數

$$\boxed{\boldsymbol{f \circ g(x) = f(g(x))}}$$

Example 0.2 $f(x) = e^x$, $g(x) = x^2$, *then* $(f \circ g)(x) = f(g(x)) = e^{x^2}$, *and* $(g \circ f)(x) = g(f(x)) = (e^x)^2 = e^{2x}$.

Lemma 2 *If* f *is continuous at* b *and* $\lim\limits_{x \to a} g(x) = b$, *then* $\lim\limits_{x \to a} f(g(x)) = f(b)$.

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).$$

(連續函數可以傳遞極限 (存在且等於 b), 就算 g 在 a 不連續也可以.)

Note: $x \to a \implies g(x) \to b$, $y \to b \implies f(y) \to f(b)$.
Replace y by $g(x)$, we have $x \to a \implies f(g(x)) \to f(b)$.

Theorem 3 *If* g *is continuous at* a *and* f *is continuous at* $g(a)$, *then* $f \circ g$ *is continuous at* a. *(*$\lim\limits_{x \to a} f(g(x)) = f(\lim\limits_{x \to a} g(x)) = f(g(a))$.*)*

A continues function of a continuous function is a continuous function.
連續函數的連續函數是連續函數.

0.3 Intermediate Value Theorem

Theorem 4 (Intermediate Value Theorem 中間值定理)
If f *is continuous on the closed interval* $[a, b]$ *with* $f(a) \neq f(b)$, *and* N *is any number between* $f(a)$ *and* $f(b)$. *Then there exists a number* c *in* (a, b) *such that* $f(c) = N$. *(*頭尾異, 閉連續, 中間值 *(*N*)* 有中間解 *(*c*).)*

Note: N between $f(a)$ and $f(b)$ $\iff$ $(f(a) - N)(f(b) - N) < 0$.
Application: 勘根定理 ($N = 0$)

Corollary 5 (Locating roots of equation) *If* f *is continuous on* $[a, b]$ *and* $f(a) \cdot f(b) < 0$, *then* $\exists\, c \in (a, b) \ni f(c) = 0$.

Remark: 連續函數的極限等於代入函數後的值,
所以求連續函數 (定義域裡) 的極限就是代進去算.
已知的七種函數: 開根有理多項式, 指對三角反三角, 經過: 加減乘除常數倍, 冪次開根 (later) 與組合 (連續函數的連續函數), 都是連續函數.