5.3 The Fundamental Theorem of Calculus

Theorem 1 *Suppose f is* ***continuous on $[a,b]$***. 閉連續

$$(1) \qquad g(x)=\int_a^x f(t)\ dt \implies g'(x)=f(x)$$

$$(2) \qquad F'(x)=f(x) \implies \int_a^b f(x)\ dx = F(b)-F(a)$$

前言: 定積分是黎曼和的極限, 如果函數長得很複雜, 極限就會很難算. 有沒有一個好的方法可以用來算定積分?

0.1 TFTC (1)

Theorem 2 (The Fundamental Theorem of Calculus, Part 1)
If f is ***continuous on $[a,b]$*** 閉連續, *then*

$$\boxed{\boldsymbol{g(x)=\int_a^x f(t)\ dt}}$$

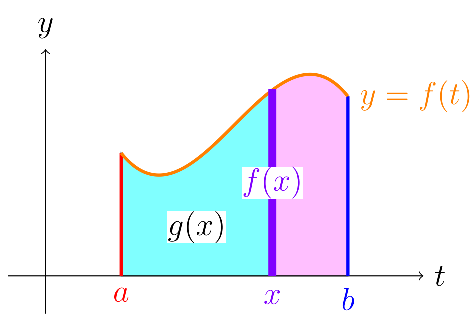
(1) is ***continuous on $[a,b]$*** 閉連續 *and*

(2) ***differentiable on (a,b)*** 開可微, *and*

(3)

$$\boxed{\boldsymbol{g'(x)=f(x)}}$$

$\because$ f 連續 (on $[a,x]$) 所以可積,
$\therefore$ g 是可以定義的 (黎曼和極限存在).
當 f 都是正的 (>0, 或非負 ≥ 0),
$g(x)$ 代表 f 以下從 a 到 x 的面積.
$g'(x)$ 代表在 x 下一個瞬間增加的面積 (改變率), 就是 $f(x)$ 的長度.



Leibniz notation: 先積一遍, 再微不變.

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

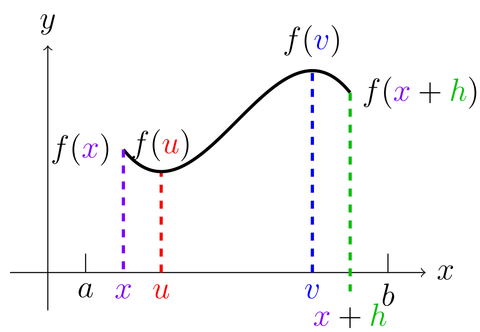
Note: f 要閉連續, x 要一致, t 也要一致.

Proof of TFTC (1).

(1) "g is differentiable on (a,b), and hence is continuous on (a,b)."
For $x \in (a,b)$,

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\left\{\frac{1}{h}\left[\int_a^{x+h} f(t)\ dt - \int_a^{x} f(t)\ dt\right]\right\} \\ &= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\ dt}{h} \end{aligned}$$

Assume $h>0$ with $x+h \le b$. $\because f$ 在 $[x, x+h] \subseteq [a,b]$ 連續,
by Extreme Value Theorem,



$\exists\ u, v \in [x, x+h]$

$\ni f$ 有最小值 $f(u)$ 與最大值 $f(v)$.

By property of definite integral,

$$\begin{aligned} f(u)h \le &\int_x^{x+h} f(t)\ dt \le\ f(v)h, \\ f(u)\ \le &\ \frac{\int_x^{x+h} f(t)\ dt}{h}\ \le\ f(v). \end{aligned}$$

As $h \to 0$, we have $u, v \to x$, and f is continuous, so $f(u), f(v) \to f(x)$.
$\lim\limits_{h\to 0} f(u) = \lim\limits_{u\to x} f(u) = f(x)$, $\lim\limits_{h\to 0} f(v) = \lim\limits_{v\to x} f(v) = f(x)$.

By Squeeze Theorem, $g'(x) = \lim\limits_{h\to 0}\dfrac{\int_x^{x+h} f(t)\ dt}{h} = f(x)$.

It is similar for $h<0$ $(x+h \ge a)$.

Therefore, g is differentiable on (a,b) and $g'(x) = f(x)$.

(2) "$g(x)$ is continuous from the right at a and from the left at b."
For $x \to a^+$, since f is continuous on $[a,x]$, by Extreme Value Theorem, $\exists\ u, v \in [a,x] \ni f(u)(x-a) \le \int_a^x f(t)\ dt \le f(v)(x-a)$, and by Squeeze Theorem,

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^+}\int_a^x f(t)\ dt = 0 = \int_a^a f(t)\ dt = g(a).$$

It is similar for $x \to b^-$ and $\displaystyle\int_x^b f(t)\ dt = \int_a^b f(t)\ dt - \int_a^x f(t)\ dt$.

Therefore, g is continuous on $[a,b]$. ■

Example 0.1 *Find the derivative of* $g(x) = \int_0^x \sqrt{1+t^2}\ dt$.

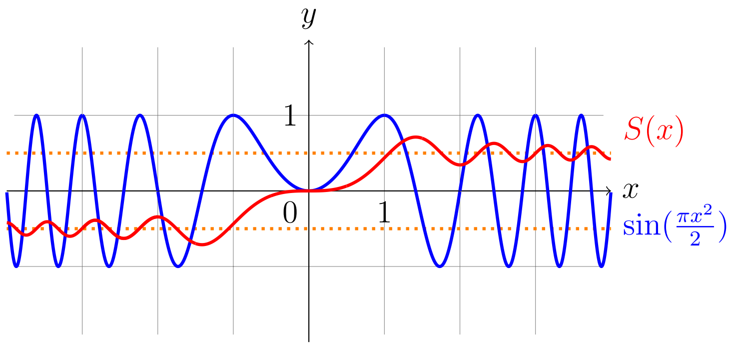
$\because \sqrt{1+x^2}$ *is continuous on* $\mathbb{R}$*, by TFTC part 1,* $g'(x) = \sqrt{1+x^2}$. ■

Example 0.2 *Find the derivative of Fresnel (Sine) Function*

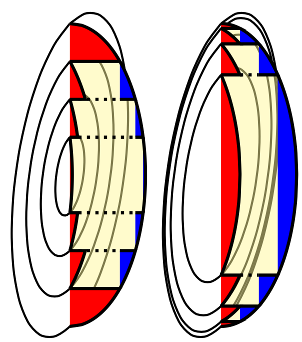
$$S(x) = \int_0^x \sin(\frac{\pi t^2}{2})\ dt.$$

French physicist Augustin Fresnel [freɪˋnɛl] 菲涅耳*: theory of diffraction of light wave* 光波繞射理論*. (Other is* $C(x) = \int_0^x \cos(\frac{\pi t^2}{2})\ dt$*.)*

$\because \sin(\frac{\pi x^2}{2})$ *is continuous on* $\mathbb{R}$*, by TFTC part 1,* $S'(x) = \sin(\frac{\pi x^2}{2})$. ■



Frensel's len



Example 0.3 *Find* $\frac{d}{dx}\int_1^{x^4} \sec t\ dt$.

*(*長得不太一樣, 要用 *Chain rule.)*
Let $u = x^4$,

$$\begin{aligned}
\frac{d}{dx}\int_1^{x^4} \sec t\ dt &= \frac{d}{dx}\int_1^{u} \sec t\ dt \qquad (\text{對 } u \text{ 的函數微 } x) \\
&= \frac{d}{du}\left[\int_1^{u} \sec t\ dt\right]\cdot\frac{du}{dx} \\
&= \sec u\cdot(x^4)' \qquad (\text{別忘代回 } u = x^4) \\
&= 4x^3\sec(x^4).
\end{aligned}$$

■

Skill: 不一致, 就要設新變數讓他一致, 再用 Chain Rule.

0.2 TFTC (2)

Theorem 3 (The Fundamental Theorem of Calculus, Part 2)
If f is ***continuous on*** $[a,b]$ 閉連續, *then*

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where F is any ***antiderivative*** *of f, i.e. $F' = f$.*

算積分雖可以用 Riemann sum 逼近, 但太複雜, 極限也不好求.
但是, 用 TFTC (2) 只要找反導數代上界減代下界.
Leibniz notation: 先微再積, 代上減下.

$$\int_a^b \frac{d}{dx}F(x)\,dx = F(b) - F(a)$$

Attentin: 不是隨便的 F 都可以, F 要是個<u>閉連續函數的反導數</u>.

Note: 其他寫法:

$$F(b) - F(a) = \boxed{F(x)\Big|_a^b} = \boxed{\Big[F(x)\Big]_a^b} = \boxed{F(x)\Big]_a^b}.$$

(不推薦單邊中括號的寫法.)

Proof of TFTC (2).
Let $g(x) = \displaystyle\int_a^x f(t)\,dt$. By TFTC part 1, $g'(x) = f(x)$.
So $F(x) = g(x) + C$, as $g(x)$, is continuous on $[a,b]$.
Therefore,

$$\begin{aligned} F(b) - F(a) &= (g(b) + \cancel{C}) - (g(a) + \cancel{C}) \\ &= \int_a^b f(t)\,dt - \cancelto{0}{\int_a^a f(t)\,dt} \\ &= \int_a^b f(t)\,dt. \end{aligned}$$

■

Example 0.4 $\int_1^3 e^x\ dx = ?$

$\because$ *e^x is an antiderivative of e^x ($e^x + C$ is the most general one),*

$\therefore \int_1^3 e^x\ dx = e^x\Big|_1^3 = e^3 - e^1 = e^3 - e.$ ■

Example 0.5 *Find the area under $y = x^2$ from 0 to 1. ($\int_0^1 x^2\ dx = ?$)*

$\because \dfrac{x^3}{3}$ *is an antiderivative of x^2,* $\therefore \int_0^1 x^2\ dx = \dfrac{x^3}{3}\Big|_0^1 = \dfrac{1^3}{3} - \dfrac{0^3}{3} = \dfrac{1}{3}.$ ■

Example 0.6 $\int_3^6 \dfrac{dx}{x} (= \int_3^6 \dfrac{1}{x}\ dx) = ?$

$\because \ln x$ *is an antiderivative of* $\dfrac{1}{x}$ *for* $x > 0$,

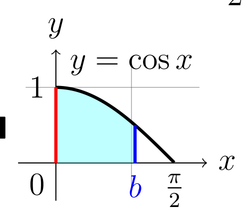
$\therefore \int_3^6 \dfrac{1}{x}\ dx = \ln x\Big|_3^6 = \ln 6 - \ln 3 = \ln \dfrac{6}{3} = \ln 2.$ ■

Note: $\boxed{\int_a^b \frac{dx}{x} = \int_a^b \frac{1}{x}\ dx}$ 是習慣的寫法, 不可以約掉 x; $\ln|x| + C$ is the most general one for $x \neq 0$, 因爲 $[3, 6] \subseteq (0, \infty)$, 這裡只要 $\ln x$ 就好.

Example 0.7 *Find the area of cosine curve from 0 to b, where $0 \leq b \leq \dfrac{\pi}{2}$.*

$\because \sin x$ *is an antiderivative of* $\cos x$,

$\therefore \int_0^b \cos x\ dx = \sin x\Big|_0^b = \sin b - \sin 0 = \sin b.$ ■



Example 0.8 *What wrong with the calculation?*

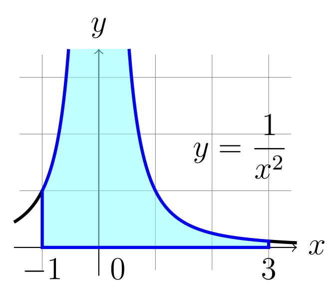
$$\int_{-1}^3 \frac{1}{x^2}\ dx = -\frac{1}{x}\Big]_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

應該是正的.

$\because \dfrac{1}{x^2}$ *is* *NOT* *continuous on* $[-1, 3]$,

$\therefore$ *TFTC does not hold.*

In fact, $\int_{-1}^3 \dfrac{1}{x^2}\ dx$ *does not exist.* ■



0.3 TFTC

Theorem 4 *Suppose f is* ***continues on $[a, b]$***. 閉連續

$$g(x) = \int_a^x f(t)\ dt \implies g'(x) = f(x)$$
$$F'(x) = f(x) \implies \int_a^b f(x)\ dx = F(b) - F(a)$$

Recall:
Leibniz notation: 先積一遍, 再微不變. (f 要在 $[a, b]$ 連續)

$$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$$

Leibniz notation: 先微再積, 代上減下. (F' 要在 $[a, b]$ 連續)

$$\int_a^b \frac{d}{dx}F(x)\ dx = F(b) - F(a)$$

Additional: $f(x)$ 閉連續 $\implies g(x) = \displaystyle\int_a^x f(t)\ dt$ 是閉連續開可微. 而且 $f(x)$ 的反導數 $F(x) = g(x) + C$, 所以也是閉連續開可微.

但是, 一個閉連續開可微的函數 $G(x)$, $G'(x)$ 不一定是閉連續, 就是說, $G(x)$ 不一定是個閉連續函數的反導數. 所以 (不適用 TFTC)

$$\int_a^b \frac{d}{dx}G(x)\ dx \cancel{=} G(b) - G(a)$$

例如: $G(x) = \sqrt{x}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$, but $G'(x) = \dfrac{1}{2\sqrt{x}}$ is continuous on $(0, 1]$, 不能用 TFTC: $\displaystyle\int_0^1 \frac{dx}{2\sqrt{x}} \cancel{=} \sqrt{x}\Big|_0^1$.
(此例要用 §7.8 瑕積分:
$\displaystyle\int_0^1 \frac{dx}{2\sqrt{x}} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{2\sqrt{x}} = \lim_{t\to 0^+}\sqrt{x}\Big|_t^1 = \sqrt{1} - \lim_{t\to 0^+}\sqrt{t} = 1 - 0 = 1.$

後言: Barrow (Newton's teacher) 發現:
求切線 (微分) 與 求面積 (積分) 互為 inverse process 逆程序.
Newton & Leibniz 發展出一套有系統的方法, 並建立學說 (極限, 黎曼和, ... 等).
積分可以用 $\lim\sum$ 來算, 但是用反導數會更好算; 我們將會介紹其他的技巧. (§ 7)