

## 7.3 Trigonometric substitution

1.  $\int \sqrt{a^2 - x^2} dx \implies x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
2.  $\int \sqrt{a^2 + x^2} dx \implies x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3.  $\int \sqrt{x^2 - a^2} dx \implies x = a \sec \theta, 0 \leq \theta < \frac{\pi}{2}, \pi \leq \theta < \frac{3\pi}{2}$

變數變換之 — 三角變換法.

**Recall:**  $\int_0^1 \sqrt{1-x^2} dx = \frac{\pi}{4}$  看圖的, 用積分要怎麼算?

1. TFTC: 不認識反導數 (X);
2. 變數變換: 令  $u = 1 - x^2$ , 則  $du = -2x dx$ .

如果有一個  $x$  (或奇數次,  $x^2 = 1 - u$ ),  $\int x\sqrt{1-x^2} dx = \frac{-1}{2} \int \sqrt{u} du$ , 能算;

但是沒有  $x$  (或偶數次) 硬換:  $\int \sqrt{1-x^2} dx = \frac{-1}{2} \int \sqrt{\frac{u}{1-u}} du$ , 不能算 (X);

3. 分部積分:  $\int_0^1 \sqrt{1-x^2} dx = \left[ x\sqrt{1-x^2} \right]_0^1 + \int_0^1 \frac{x^2}{\sqrt{1-x^2}} dx$  更複雜 (X).

**How to compute?** 遇到有  $\sqrt{a^2 - x^2}$ ,  $\sqrt{a^2 + x^2}$ ,  $\sqrt{x^2 - a^2}$  的積分 ( $a > 0$ ), 如果不能變數變換, 可以用三角變換: 把  $x$  換成三角函數  $a \sin \theta$ ,  $a \tan \theta$ ,  $a \sec \theta$ .

### 0.1 $\int \sqrt{a^2 - x^2} dx$

Let  $\boxed{x = a \sin \theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}}$  (其實是 Let  $\theta = \sin^{-1} \frac{x}{a}$ .)

Then  $dx = a \cos \theta d\theta$  and  $\sqrt{a^2 - x^2} = a \cos \theta$ . (因為  $\theta$  範圍會是正的.)

$$\begin{aligned} \int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2(1 - \sin^2 \theta)} d(a \sin \theta) = \int a^2 \cos^2 \theta d\theta \\ &= \int a^2 \left( \frac{1 + \cos 2\theta}{2} \right) d\theta = \frac{a^2}{2} \theta + \frac{1}{4} a^2 \sin 2\theta + C \\ &= \frac{a^2}{2} \theta + \frac{1}{2} (a \sin \theta)(a \cos \theta) + C \quad (\text{換回 } x) \\ &= \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{a^2 - x^2} + C. \end{aligned}$$

$$a = 1 \implies \int_0^1 \sqrt{1-x^2} dx = \left[ \frac{1}{2} \sin^{-1} x + \frac{1}{2} x \sqrt{1-x^2} \right]_0^1 = \frac{\pi}{4}.$$

**0.2**  $\int \sqrt{a^2 + x^2} dx$

Let  $\boxed{x = a \tan \theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}.}$  (其實是 Let  $\theta = \tan^{-1} \frac{x}{a}$ .)

Then  $dx = a \sec^2 \theta d\theta$  and  $\sqrt{a^2 + x^2} = a \sec \theta$ . (因為  $\theta$  範圍是正的.)

$$\begin{aligned} \int \sqrt{a^2 + x^2} dx &= \int \sqrt{a^2(1 + \tan^2 \theta)} d(a \tan \theta) = \int a^2 \sec^3 \theta d\theta \\ &= \frac{a^2}{2} \sec \theta \tan \theta + \frac{a^2}{2} \ln |\sec \theta + \tan \theta| + C' \\ &= \frac{1}{2}(a \tan \theta)(a \sec \theta) + \frac{a^2}{2} \ln \left| \frac{a \tan \theta + a \sec \theta}{a} \right| + C' \\ &= \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C, \end{aligned}$$

where  $C = C' - \frac{a^2}{2} \ln a$ . ( $\because x + \sqrt{a^2 + x^2} > 0$ , 可以去絕對值.)

**0.3**  $\int \sqrt{x^2 - a^2} dx$

Let  $\boxed{x = a \sec \theta, 0 \leq \theta < \frac{\pi}{2}, \pi \leq \theta < \frac{3\pi}{2}.}$  (其實是 Let  $\theta = \sec^{-1} \frac{x}{a}$ .)

Then  $dx = a \sec \theta \tan \theta d\theta$  and  $\sqrt{x^2 - a^2} = a \tan \theta$ . (因為  $\theta$  範圍是正的.)

$$\begin{aligned} \int \sqrt{x^2 - a^2} dx &= \int \sqrt{a^2(\sec^2 \theta - 1)} d(a \sec \theta) = \int a^2 \tan^2 \theta \sec \theta d\theta \\ &= \int a^2 (\sec^3 \theta - \sec \theta) d\theta \quad (\tan^2 x \sec x \text{ 不能變換.}) \\ &= \frac{a^2}{2} \sec \theta \tan \theta + \left(\frac{1}{2} - 1\right)a^2 \ln |\sec \theta + \tan \theta| + C' \\ &= \frac{1}{2}(a \sec \theta)(a \tan \theta) - \frac{a^2}{2} \ln \left| \frac{a \sec \theta + a \tan \theta}{a} \right| + C' \\ &= \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C, \end{aligned}$$

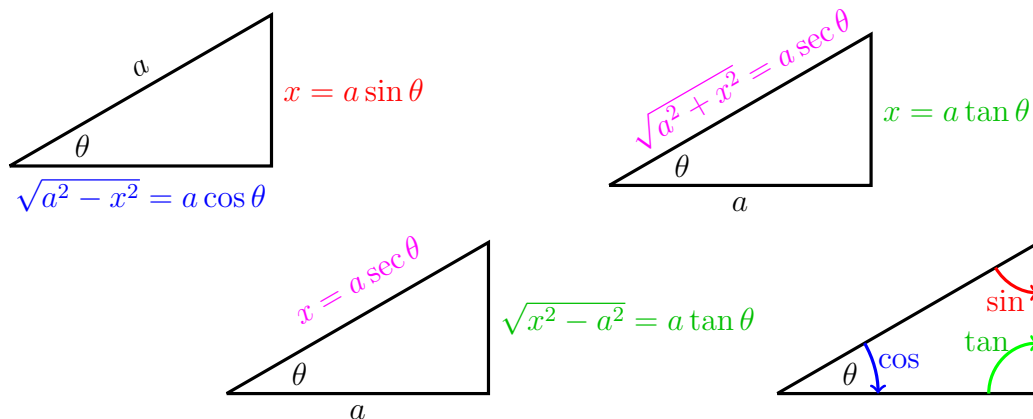
where  $C = C' + \frac{a^2}{2} \ln a$ . ( $\because x + \sqrt{x^2 - a^2} < 0$  when  $x < -a$ , 不可去絕對值.)

## 0.4 Remark

$$\begin{aligned}\int \sqrt{a^2 - x^2} dx &= \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C \\ \int \sqrt{a^2 + x^2} dx &= \frac{1}{2}x\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln(x + \sqrt{a^2 + x^2}) + C \\ \int \sqrt{x^2 - a^2} dx &= \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln |x + \sqrt{x^2 - a^2}| + C\end{aligned}$$

**Note:** 不要背公式, 會背錯. 應該記的是方法: 把  $x$  換成三角函數消去根號. 其實還是在做變數變換, 只是換個好記的方式, 所以要定  $\theta$  的範圍.

**Skill:** 畫圖有助於快速把  $\theta$  的三角函數換回  $x$  的函數. (夾角是  $\theta$ , 一邊是  $a$ , 一邊是  $x$ , 一邊是...)

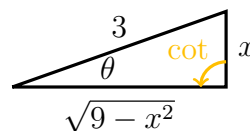


**Example 0.1**  $\int \frac{\sqrt{9 - x^2}}{x^2} dx.$

Let  $x = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .  $dx = 3 \cos \theta d\theta$  and  $\sqrt{9 - x^2} = 3 \cos \theta$ .

$$\begin{aligned}\int \frac{\sqrt{9 - x^2}}{x^2} dx &= \int \frac{3 \cos \theta}{9 \sin^2 \theta} \cdot 3 \cos \theta d\theta = \int \frac{\cos^2 \theta}{\sin^2 \theta} d\theta \\ &= \int \cot^2 \theta d\theta = \int (\csc^2 \theta - 1) d\theta = -\cot \theta - \theta + C,\end{aligned}$$

$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\sqrt{9 - x^2}}{x}$  or see diagram:



$$\int \frac{\sqrt{9 - x^2}}{x^2} dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1} \left( \frac{x}{3} \right) + C. \text{ (單項可以不加括號.)} \quad \blacksquare$$

**Example 0.2** Find the area enclosed by the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$ . Let  $x = a \sin \theta$ , (範圍...).

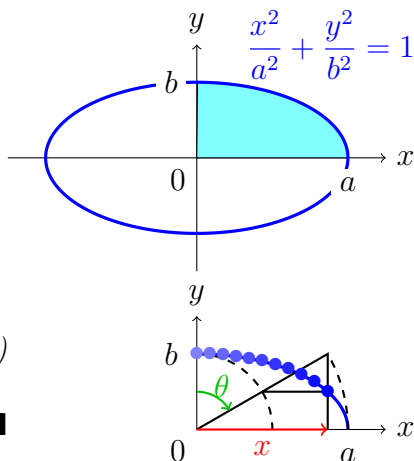
When  $x = 0$ ,  $\theta = 0$ , and when  $x = a$ ,  $\theta = \frac{\pi}{2}$ .

$$\text{橢圓面積} = 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$= 4 \int_0^{\pi/2} \frac{b}{a} \cdot a \cos \theta \cdot a \cos \theta d\theta$$

$$= 4ab \int_0^{\pi/2} \cos^2 \theta d\theta \quad (\cos^2 \theta = \frac{1 + \cos 2\theta}{2})$$

$$= 4ab \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = ab\pi. \quad \blacksquare$$



**Note:** 定積分版的變數變換, 可以上下界跟著換過去直接代入, 不用換回來.

**Additional:** 橢圓  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with  $a > b$ ,  $a$  與  $b$  分別稱為長軸與短軸, 當  $a = b = r$  就是半徑  $r$  的圓, 面積公式依然適用  $ab\pi = \pi r^2$ .

**Example 0.3**  $\int \frac{1}{x^2 \sqrt{x^2 + 4}} dx$ .

Let  $x = 2 \tan \theta$ , (範圍...).

$$\int \frac{1}{x^2 \sqrt{x^2 + 4}} dx = \int \frac{2 \sec^2 \theta}{4 \tan^2 \theta \cdot 2 \sec \theta} d\theta = \int \frac{\sec \theta}{4 \tan^2 \theta} d\theta$$

(路線分歧)

[Sol 1] 看出來了! (Recall: §5.4 Ex 3.)

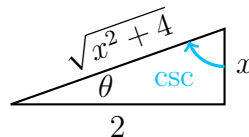
$$= \frac{1}{4} \int \csc \theta \cot \theta d\theta = -\frac{1}{4} \csc \theta + C = -\frac{\sqrt{x^2 + 4}}{4x} + C.$$

[Sol 2] 變數變換!

Let  $u = \sin \theta$ , then  $du = \cos \theta d\theta$ .

$$= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta = \frac{1}{4} \int \frac{du}{u^2} = -\frac{1}{4u} + C$$

$$= -\frac{1}{4 \sin \theta} + C = -\frac{\sqrt{x^2 + 4}}{4x} + C. \quad \blacksquare$$



**Example 0.4**  $\int \frac{x}{\sqrt{x^2+4}} dx.$

[Sol 1] (三角代換)

Let  $x = 2 \tan \theta$ , (範圍...).

$$\int \frac{x}{\sqrt{x^2+4}} dx = \int \frac{2 \tan \theta}{2 \sec \theta} 2 \sec^2 \theta d\theta = \int 2 \sec \theta \tan \theta d\theta = 2 \sec \theta + C$$

$$= \sqrt{x^2+4} + C.$$

[Sol 2] (變數變換 A)

Let  $u = x^2 + 4$ , then  $du = 2x dx$ .

$$\int \frac{x}{\sqrt{x^2+4}} dx = \int \frac{du}{2\sqrt{u}} = \sqrt{u} + C = \sqrt{x^2+4} + C.$$

[Sol 3] (變數變換 B)

Let  $v = \sqrt{x^2+4}$ , then  $dv = \frac{x}{\sqrt{x^2+4}} dx$ .

$$\int \frac{x}{\sqrt{x^2+4}} dx = \int dv = v + C = \sqrt{x^2+4} + C. \quad \blacksquare$$

**Note:** 先試變數變換, 不行再用三角代換 (較複雜).

**Example 0.5**  $\int \frac{dx}{\sqrt{x^2-a^2}}, a > 0.$

Let  $x = a \sec \theta$ , (範圍...).

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a \sec \theta \tan \theta}{a \tan \theta} d\theta = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$$

$$\left( = \ln \left| \frac{x + \sqrt{x^2-a^2}}{a} \right| + C = \ln |x + \sqrt{x^2-a^2}| - \ln a + C \right)$$

$$= \ln |x + \sqrt{x^2-a^2}| + C. \quad (-\ln a \text{ 被 } C \text{ 吃掉了.}) \quad \blacksquare$$

[♦ Optional solution]

$\because \cosh^2 \theta - \sinh^2 \theta = 1$ , let  $x = a \cosh \theta$ , then  $dx = a \sinh \theta d\theta$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a \sinh \theta}{a \sinh \theta} d\theta = \int d\theta = \theta + C = \cosh^{-1} \frac{x}{a} + C. \quad \blacksquare$$

**Question:** 一定要用  $x = a \sin \theta / a \tan \theta / a \sec \theta$ ?

**Answer:** 可以用  $x = a \cos \theta / a \cot \theta / a \csc \theta$ , 但是  $dx$  會有負號.

**Example 0.6** (三角變換→變數變換)  $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx$ .

$$(4x^2+9)^{3/2} = 8\sqrt{x^2+(3/2)^2}^3, \text{ let } x = \frac{3}{2}\tan\theta, \text{ then } \sqrt{4x^2+9} = 3\sec\theta.$$

When  $x = 0$ ,  $\theta = 0$ , and when  $x = \frac{3\sqrt{3}}{2}$ ,  $\tan\theta = \sqrt{3}$ ,  $\theta = \frac{\pi}{3}$ .

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx = \int_0^{\pi/3} \left(\frac{3}{2}\tan\theta\right)^3 \frac{3}{2}\sec^2\theta d\theta = \frac{3}{16} \int_0^{\pi/3} \frac{\sin^3\theta}{\cos^2\theta} d\theta$$

Let  $u = \cos\theta$ , then  $du = -\sin\theta d\theta$ . ( $\cos$  偶數次,  $\sin$  奇數次, 令  $u = \cos$ )

When  $\theta = 0$ ,  $u = 1$ , and when  $\theta = \frac{\pi}{3}$ ,  $u = \frac{1}{2}$ .

$$\begin{aligned} &= \frac{3}{16} \int_0^{\pi/3} \frac{\cos^2\theta - 1}{\cos^2\theta} (-\sin\theta) d\theta = \frac{3}{16} \int_1^{1/2} \frac{u^2 - 1}{u^2} du = \frac{3}{16} \int_1^{1/2} 1 - \frac{1}{u^2} du \\ &= \frac{3}{16} \left[ u + \frac{1}{u} \right]_1^{1/2} = \frac{3}{16} \left[ \left( \frac{1}{2} + 2 \right) - (1 + 1) \right] = \frac{3}{32}. \quad \blacksquare \end{aligned}$$

[Another](其實用變數變換比較簡單: Try  $w = 4x^2 + 9$ .) .....

Let  $v = \sqrt{4x^2+9}$ , then  $dv = \frac{4x}{\sqrt{4x^2+9}} dx$ ,  $x^2 = \frac{v^2-9}{4}$ .

$$\begin{aligned} \int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} dx &= \int_0^{3\sqrt{3}/2} \frac{x^2}{4(4x^2+9)} \frac{4x}{\sqrt{4x^2+9}} dx = \int_3^6 \frac{v^2-9}{16v^2} dv \\ &= \frac{1}{16} \int_3^6 1 - \frac{9}{v^2} dv = \frac{1}{16} \left[ v + \frac{9}{v} \right]_3^6 = \frac{1}{16} \left[ (6 + \frac{3}{2}) - (3 + 3) \right] = \frac{3}{32}. \quad \blacksquare \end{aligned}$$

**Note:** 不要放棄嘗試變數變換, 可以換了又換.

**Example 0.7** (配方→變數變換→三角變換)  $\int \frac{x}{\sqrt{3-2x-x^2}} dx$ .

先配方變換:  $3-2x-x^2 = 4-(x+1)^2$ .

Let  $u = x+1$ , then  $du = dx$ ,  $x = u-1$  and  $3-2x-x^2 = 4-u^2$ .

再三角變換 (已試過變數變換): Let  $u = 2\sin\theta$ , then  $\sqrt{4-u^2} = 2\cos\theta$ .

$$\begin{aligned} &\int \frac{x}{\sqrt{3-2x-x^2}} dx \xrightarrow{\boxed{\rightarrow u}} \int \frac{u-1}{\sqrt{4-u^2}} du \xrightarrow{\boxed{\rightarrow \theta}} \int \frac{2\sin\theta-1}{2\cos\theta} 2\cos\theta d\theta \\ &= \int 2\sin\theta - 1 d\theta = -2\cos\theta - \theta + C \quad (\text{換回 } \theta \rightarrow u \rightarrow x.) \\ &\xrightarrow{\boxed{\rightarrow u}} -\sqrt{4-u^2} - \sin^{-1} \frac{u}{2} + C \xrightarrow{\boxed{\rightarrow x}} -\sqrt{3-2x-x^2} - \sin^{-1} \frac{x+1}{2} + C. \quad \blacksquare \end{aligned}$$

**Note:** 相當於變換  $x(=u-1) = 2\sin\theta - 1$ , 但是不容易看出來.