4.4 Indeterminate forms & ℓ'Hospital's rule

微分應用之三: 求未定型的極限. (★ 授課順序與 §4.3 調換.)

1. Indeterminate forms & ℓ'Hospital's rule 未定型與羅畢達法則
2. Indeterminate product, difference, power 變形的未定型

0.1 Indeterminate forms & ℓ'Hospital's rule

Define: A limit $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is called an ***indeterminate form*** 未定型

1. ***of type*** $\boxed{\dfrac{\mathbf{0}}{\mathbf{0}}}$ if $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$;
2. ***of type*** $\boxed{\dfrac{\boldsymbol{\infty}}{\boldsymbol{\infty}}}$ if $f(x)\to\pm\infty$ and $g(x)\to\pm\infty$ as $x\to a$.

Note: "$f(x)\to\pm\infty$ and $g(x)\to\pm\infty$", 都是指: (f and g 都有無限極限)

(1) $f(x)\to\infty$ and $g(x)\to\infty$,
(2) $f(x)\to-\infty$ and $g(x)\to\infty$,
(3) $f(x)\to\infty$ and $g(x)\to-\infty$,
(4) $f(x)\to-\infty$ and $g(x)\to-\infty$,

都算是 $\dfrac{\infty}{\infty}$ 的未定型.

Example 0.1

1. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$ *(*$=2$*, §2.3).* $(\dfrac{\mathbf{0}}{\mathbf{0}})$

2. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$ *(*$=1$*, §3.3).* $(\dfrac{\mathbf{0}}{\mathbf{0}})$

3. $\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}$ $=?$ $(\dfrac{\mathbf{0}}{\mathbf{0}})$

4. $\lim\limits_{x\to \infty}\dfrac{e^x}{x^2}$ $=?$ $(\dfrac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$

Theorem 1 (L'Hospital's Rule 羅畢達法則) *(求未定型極限)*
Suppose f and g are ***differentiable***, $\boldsymbol{g'(x) \neq 0}$ *near a, 可微, g' 近 a 非零 and $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an* ***indeterminate form*** *of $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\infty}{\infty}$, Then*

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right side ***exists*** *or is $\pm\infty$.*

Note: 未定型與羅畢達法則中, $x \to a$ 也可以是 $x \to a^+, a^-, \infty, -\infty$.
Note: 只要條件 (未定型) 滿足就可以重複使用. ♻
♦: 以法國侯爵羅畢達 (Guillaume François Antoine, Marquis de l'Hôpital) 命名, l'(=la) 句首大寫, H 必大寫, l'hôpital [lopitɑl][法] = the hospital [英].

Example 0.2 $\lim_{x\to 1} \frac{\ln x}{x-1} =?$

$\lim_{x\to 1} \ln x = \ln 1 = 0,\ \lim_{x\to 1}(x-1) = 0,\ (x-1)' = 1 \neq 0$ *near* $1.\ (\frac{\mathbf{0}}{\mathbf{0}})$

$\therefore \lim_{x\to 1} \frac{\ln x}{x-1} \overset{l'H}{=} \lim_{x\to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x\to 1} \frac{1/x}{1} = 1.$ ■

Example 0.3 *(twice)* $\lim_{x\to\infty} \frac{e^x}{x^2} =?$

$\lim_{x\to\infty} e^x = \infty,\ \lim_{x\to\infty} x^2 = \infty,\ (x^2)' = 2x \neq 0$ *as* $x \to \infty.\ (\frac{\infty}{\infty})$

$\therefore \lim_{x\to\infty} \frac{e^x}{x^2} \overset{l'H}{=} \lim_{x\to\infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to\infty} \frac{e^x}{2x}.$

$\lim_{x\to\infty} e^x = \infty,\ \lim_{x\to\infty} 2x = \infty,\ (2x)' = 2 \neq 0$ *as* $x \to \infty.\ (\frac{\infty}{\infty})$

$\therefore \lim_{x\to\infty} \frac{e^x}{2x} \overset{l'H}{=} \lim_{x\to\infty} \frac{(e^x)'}{(2x)'} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$ *(每次都要檢查是不是未定型.)* ■

Example 0.4 $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} =?$

$\lim_{x\to\infty} \ln x = \infty,\ \lim_{x\to\infty} \sqrt[3]{x} = \infty,\ (\sqrt[3]{x})' = \frac{1}{3}x^{-2/3} \neq 0$ *as* $x \to \infty.\ (\frac{\infty}{\infty})$

$\therefore \lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} \overset{l'H}{=} \lim_{x\to\infty} \frac{1/x}{x^{-2/3}/3} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0.$ ■

Example 0.5 *(another method)* $\lim_{x\to 0}\frac{\tan x - x}{x^3}$ =?

$$\lim_{x\to 0}(\tan x - x)=0,\ \lim_{x\to 0}x^3=0,\ (x^3)'=3x^2\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore \lim_{x\to 0}\frac{\tan x - x}{x^3}\overset{l'H}{=}\lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

$$\lim_{x\to 0}(\sec^2 x - 1)=0,\ \lim_{x\to 0}3x^2=0,\ (3x^2)'=6x\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\overset{l'H}{=}\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}.$$

(路線分歧)

[Sol 1:] (繼續用羅畢達)

$$\lim_{x\to 0}2\sec^2 x\tan x=0,\ \lim_{x\to 0}6x=0,\ (6x)'=6\neq 0\ \textit{near}\ 0.\ (\frac{\mathbf{0}}{\mathbf{0}})$$

$$\therefore \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\overset{l'H}{=}\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x}{6}=\frac{4\cdot 1^2\cdot 0^2+2\cdot 1^4}{6}=\frac{1}{3}.$$

[Sol 2:] (變形用極限律)

$$\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\left(\frac{\sec^2 x}{3x}\frac{\sin x}{\cos x}\right)=\lim_{x\to 0}\left(\frac{\sec^3 x}{3}\frac{\sin x}{x}\right)$$

$$=\frac{1}{3}\left(\lim_{x\to 0}\sec x\right)^3\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{3}\cdot 1^3\cdot 1=\frac{1}{3}.\qquad\blacksquare$$

Example 0.6 *(If blindly using ℓ'Hospital's rule)* $\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}$ =?

$$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}\ \overset{l'H}{\cancel{=}}\ \lim_{x\to\pi^-}\frac{\cos x}{\sin x}=-\infty,\ (\cos x\to -1,\sin x\to 0)$$ *(**Wrong!**)*

$$\lim_{x\to\pi^-}\sin x=0,\ \lim_{x\to\pi^-}1-\cos x=2.$$ *(不是未定型, 不能用!)*

$$\therefore \lim_{x\to\pi^-}\frac{\sin x}{1-\cos x}=\frac{\lim_{x\to\pi^-}\sin x}{\lim_{x\to\pi^-}(1-\cos x)}=\frac{0}{2}=0.$$ *(不要瞎用!)* $\blacksquare$

Attention: 1. $\frac{f}{g}$ 要 $\frac{\mathbf{0}}{\mathbf{0}}$ 或 $\frac{\infty}{\infty}$ 未定型, 才可以使用羅畢達律改算 $\frac{f'}{g'}$ 的極限.

2. $\frac{f'}{g'}$ 的極限要存在或是 $\pm\infty$ 才能相等.

3. 注意! 不要把 $\frac{f'}{g'}$ 跟 $\left(\frac{f}{g}\right)'$ 搞錯.

0.2 Indeterminate product, difference, power

Product: $\mathbf{0}\cdot\boldsymbol{\infty}$; Difference: $\boldsymbol{\infty}-\boldsymbol{\infty}$; Power: $\mathbf{0^0}, \boldsymbol{\infty^0}, \mathbf{1^\infty}$. ($\mathbf{0^\infty}=\mathbf{0}$ 不算)

(a) Type $\boxed{\mathbf{0}\cdot\boldsymbol{\infty}}$: $f\to 0$, $g\to\pm\infty$, 挑一個除到下面去

$$\begin{aligned}\lim_{x\to a} fg &= \lim_{x\to a}\frac{f}{1/g}, \qquad (\frac{\mathbf{0}}{\mathbf{0}})\\ \text{or} &= \lim_{x\to a}\frac{g}{1/f}. \qquad (\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}})\end{aligned}$$

Example 0.7 $\lim\limits_{x\to 0^+} x\ln x = ?$

$\lim\limits_{x\to 0^+} x = 0$, $\lim\limits_{x\to 0^+}\ln x = -\infty$, $(\frac{1}{x})' = -\frac{1}{x^2}\neq 0$ *near* 0. $(\mathbf{0}\cdot\boldsymbol{\infty})$

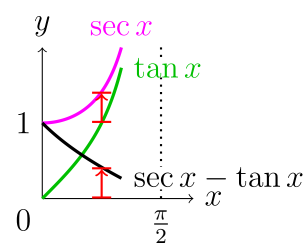
先試試 $(\frac{\mathbf{0}}{\mathbf{0}})$ $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+}\dfrac{x}{\frac{1}{\ln x}} \overset{l'H}{=} \lim\limits_{x\to 0^+}\dfrac{1}{\frac{-1}{(\ln x)^2}\frac{1}{x}}$ 變複雜, 但別放棄,

改用 $(\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}})$ $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x} \overset{l'H}{=} \lim\limits_{x\to 0^+}\dfrac{1/x}{-1/x^2} = \lim\limits_{x\to 0^+} -x = 0.$ ∎

Note: Type $\mathbf{0}\cdot\boldsymbol{\infty}$ 不一定都是 0. Ex: $\lim\limits_{x\to 0} x\cdot\frac{c}{x} = c$.

(b) Type $\boxed{\boldsymbol{\infty}-\boldsymbol{\infty}}$ (or $(-\boldsymbol{\infty})-(-\boldsymbol{\infty})$): $f-g$ 併成 $\frac{\mathbf{0}}{\mathbf{0}}$ or $\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$.

Example 0.8 $\lim\limits_{x\to\frac{\pi}{2}^-}(\sec x - \tan x) = ?$



$\lim\limits_{x\to\frac{\pi}{2}^-}\sec x = \lim\limits_{x\to\frac{\pi}{2}^-}\tan = \infty$. $(\boldsymbol{\infty}-\boldsymbol{\infty})$

變形: $\lim\limits_{x\to\frac{\pi}{2}^-}(\sec x - \tan x)$

$= \lim\limits_{x\to\frac{\pi}{2}^-}(\dfrac{1}{\cos x} - \dfrac{\sin x}{\cos x}) = \lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{1-\sin x}{\cos x}$

$\lim\limits_{x\to\frac{\pi}{2}^-}(1-\sin x) = 0$, $\lim\limits_{x\to\frac{\pi}{2}^-}\cos x = 0$, $(\cos x)' = -\sin x \neq 0$ *near* $\frac{\pi}{2}^-$. $(\frac{\mathbf{0}}{\mathbf{0}})$

$\Longrightarrow \overset{l'H}{=} \lim\limits_{x\to\frac{\pi}{2}^-}\dfrac{-\cos x}{-\sin x} = \dfrac{0}{1} = 0.$ ∎

Note: Type $\boldsymbol{\infty}-\boldsymbol{\infty}$ 不一定都是 0. Ex: $\lim\limits_{x\to\infty}[(x+c)-x] = c$.

(c) Type $\boxed{\mathbf{0^0, \infty^0, 1^\infty}}$:

f^g 取自然對數 $\ln$ 換成 $\mathbf{0\cdot\infty}$, 再取自然指數 e 換回來.

Let $y = f^g$, $\ln y = g \ln f$, $\lim\limits_{x\to a} \ln y \overset{l'H}{=} L/\infty/-\infty$,

$\lim\limits_{x\to a} y = \lim\limits_{x\to a} e^{\ln y} \overset{*}{=} e^{\lim\limits_{x\to a} \ln y} = e^L/\infty/0$. (不可以寫 $\cancel{e^\infty = \infty}$, $\cancel{e^{-\infty} = 0}$.)

($*$: e^x 處處連續 $\Longrightarrow$ 可以傳遞極限.)

f^g	$f\to$	$g\to$	$\ln f\to$	$g\cdot\ln f$
$\mathbf{0^0}$	0^+	0	$-\infty$	$\mathbf{0\cdot\infty}$
$\mathbf{\infty^0}$	∞	0	∞	$\mathbf{0\cdot\infty}$
$\mathbf{1^\infty}$	1	$\pm\infty$	0	$\mathbf{\infty\cdot 0}$
$\mathbf{0^\infty}$	0^+	$\pm\infty$	$-\infty$	$\mp\infty$

這型的極限是 0 或 ∞, 用不著羅畢達.

Example 0.9 $\lim\limits_{x\to 0^+}(1+\sin 4x)^{\cot x} = ?$

$\lim\limits_{x\to 0^+}(1+\sin 4x) = 1$, $\lim\limits_{x\to 0^+}\cot x = \infty$. $\mathbf{(1^\infty)}$

Let $y = (1+\sin 4x)^{\cot x}$, $\ln y = \cot x \ln(1+\sin 4x) = \dfrac{\ln(1+\sin 4x)}{\tan x}$.

$\lim\limits_{x\to 0^+}\ln(1+\sin 4x) = 0$, $\lim\limits_{x\to 0^+}\tan x = 0$, $(\tan x)' = \sec^2 x \neq 0$ *near* 0^+. $(\mathbf{\frac{0}{0}})$

$$\therefore \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x}$$

$$\overset{l'H}{=} \lim_{x\to 0^+}\left(\frac{4\cos 4x}{1+\sin 4x}\frac{1}{\sec^2 x}\right) = \frac{4\cdot 1}{1+0}\cdot\frac{1}{1^2} = 4. \text{ (還沒完!)}$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+}\ln y} = e^4.$$ ■

Example 0.10 $\lim\limits_{x\to 0^+} x^x = ?$

$\lim\limits_{x\to 0^+} x = 0$. $\mathbf{(0^0)}$

Let $y = x^x$, $\ln y = x\ln x = \dfrac{\ln x}{1/x}$. $(\mathbf{0^0 \to 0\cdot\infty \to \frac{\infty}{\infty}})$

$$\therefore \lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{l'H}{=} \ldots \text{(前面剛講過)} = 0. \text{ (還沒完!)}$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim\limits_{x\to 0^+}\ln y} = e^0 = 1.$$ ■

Attention: 注意! 取了自然對數 $\ln$ 別忘記最後還要用自然指數 e 變回去.

Example 0.11 *(If ℓ'Hospital's rule fails)* $\lim\limits_{x\to\infty}\dfrac{x+\cos x}{x}$ =?

Indeterminate type of $\dfrac{\infty}{\infty}$ *(自己檢查).*

$\lim\limits_{x\to\infty}\dfrac{x+\cos x}{x}\overset{l'H}{=}\lim\limits_{x\to\infty}\dfrac{1-\sin x}{1}$ *does not exist nor infinite limit.*

$\therefore\lim\limits_{x\to\infty}\dfrac{x+\cos x}{x}$ *does not exist. (**Wrong!**)*

這題要用 *Squeeze Theorem:*

Consider $x>0$ *since* $x\to\infty$*, then* $1-\dfrac{1}{x}\le\dfrac{x+\cos x}{x}\le 1+\dfrac{1}{x}$,

$\lim\limits_{x\to\infty}(1-\dfrac{1}{x})=\lim\limits_{x\to\infty}(1+\dfrac{1}{x})=1,\implies\lim\limits_{x\to\infty}\dfrac{x+\cos x}{x}=1.$ ■

Attention: 如果 $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ 不存在也不是 $\pm\infty$, 則 ℓ'Hospital's rule 不能用 ($\overset{l'H}{=}$ 不成立); 但並不代表極限就不存在! 這時候要改用其他方法.

Question: $\overset{l'H}{=}$ 是啥? 一定要寫嗎?
Answer: 只是解釋這一步是用 ℓ'Hospital's rule, 證明才要寫.

Question: 每次都要這麼麻煩嗎?
Answer: 不用!

1. 先檢查是不是未定型$(\mathbf{0^0},\ \mathbf{\infty^0},\ \mathbf{1^\infty}\to\mathbf{0\cdot\infty},\ \mathbf{\infty-\infty}\to\dfrac{\mathbf{0}}{\mathbf{0}},\ \dfrac{\mathbf{\infty}}{\mathbf{\infty}})$.
2. 直接 $\overset{l'H}{=}\lim\limits_{x\to a}\dfrac{f'}{g'}$.
3. 還是未定型: goto 2.
4. 極限存在 或是 $\infty/-\infty$ $\implies$ (如果有取 $\ln x$ 要再取 e^x) 答案.
5. 否則 $\implies$ 劃掉並找別的方法 (換另一型或用夾擠定理).

Question: 老師你沒檢查 $g'(x)\neq 0$ near a!
Answer: 不用! 如果 $g(x)$ 可微分且 $g'(x)=0$ near a,
則 $g(x)$ 是常數函數, $g(x)\not\to 0$ 除非 $g(x)=0$ (完全不能求極限).
所以在檢查是不是未定型時就會排除.

⧫ Additional: Proof of L'Hospital Rule

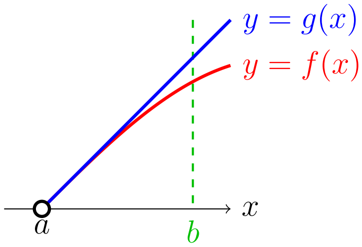
For $\frac{\mathbf{0}}{\mathbf{0}}$: $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$.

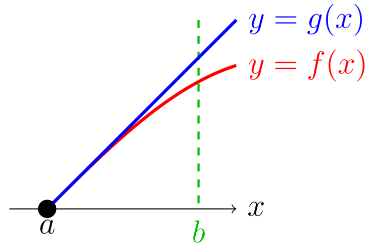
$\because$ f and g differentiable, $\exists\, b > a$, $\ni$ f and g are continuous on $(a, b]$.

Assume $f(a) = g(a) = 0$, (otherwise, consider and replace by:

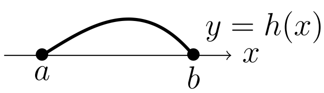
$F(x) = \begin{cases} f(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$ and $G(x) = \begin{cases} g(x) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$)

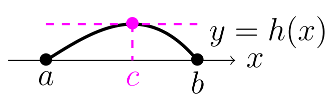
then f and g are continuous on $[a, b]$ and differentiable on (a, b).

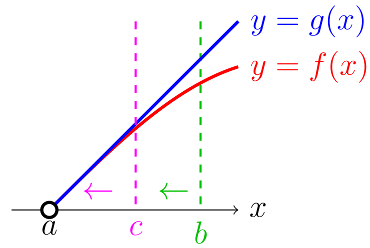




Let $h(x) = f(x) - \frac{f(b)}{g(b)} g(x)$, ($\because g(b) \neq 0$) then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. By Rolle's Theorem, $\exists\, c \in (a, b)$, $\ni h'(c) = f'(c) - \frac{f(b)}{g(b)} g'(c) = 0$, ($\because g'(c) \neq 0$) $\implies \frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)}$.







When $x = b \to a^+ \implies y = c \to a^+$,

$$\implies \lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(y)}{g'(y)} = \lim_{y\to a^+} \frac{f'(y)}{g'(y)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

For $\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$, consider $\frac{1/g}{1/f}$ $(\frac{\mathbf{0}}{\mathbf{0}})$. ■