2.4 The precise definition of a limit

1. definition of limit 極限定義
2. one-side limit 單邊極限
3. infinite limit 無限極限

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什麼叫靠近 (approach)? 一公分? 一公尺? 一公里?
你問我靠你有多近? 我挨你有幾分? 你去想一想, 你去看一看, ε-δ 我的近.

0.1 Definition of limit

Recall: $\lim_{x\to a} f(x) = L \iff f(x) \to L$ as $x \to a$. 怎麼說明靠近 (approach "$\rightarrow$")?
要用 ε-δ 語言: 以 ε & δ 代表距離, 用來描述靠近.

Define: $f(x)$ is defined on (b, c) with $b < a < c$ (except a possibly).

$$\boxed{\lim_{x\to a} f(x) = L}$$

$$\text{if } \boxed{\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \ni 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.}$$

如果對所有 $\varepsilon > 0$, 都存在 $\delta > 0$, 使得只要 $0 < |x-a| < \delta$, 就會 $|f(x)-L| < \varepsilon$.

Notation:

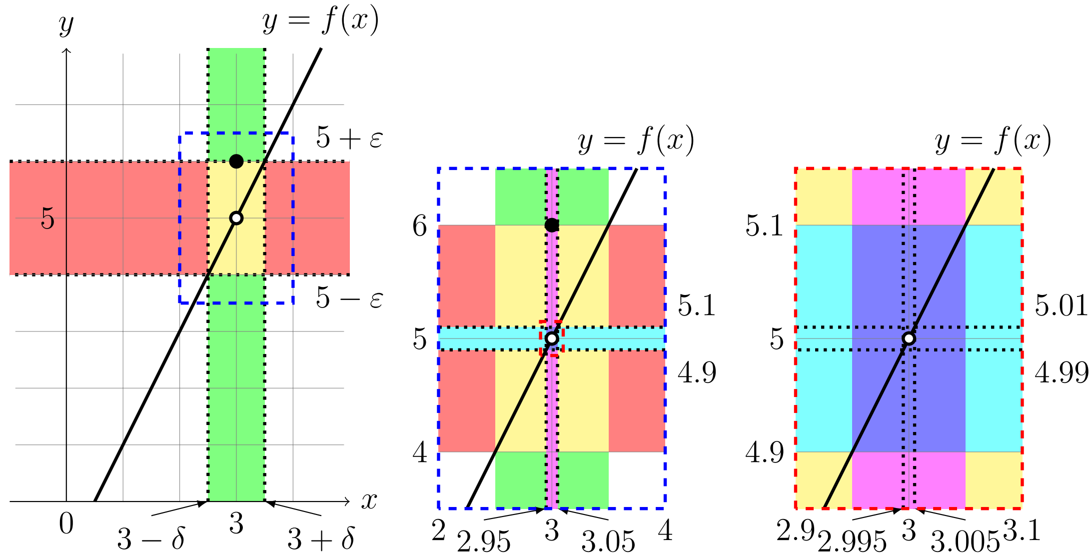
- $\boxed{\forall}$: for all 對所有;
- $\boxed{\exists}$: exists 存在;
- $\boxed{\ni}$: such that (s.t.) 使得;
- $\boxed{\Longrightarrow \ (\Rightarrow, \rightarrow)}$: implies 若 (前者爲真) 則 (後者爲真).

$\lim_{x\to a} f(x) = L$ 代表: 如果你要 $f(x)$ 以 (你要的) ε 的距離靠近 L, 它能保證, 只要 x 是以 (保證會有) δ 的距離靠近 a 就有。

反過來 (腳色互換), 要證明 $\lim_{x\to a} f(x) = L$, 就要對任意給定的 ε 找出 δ, 保證只要 x 是以 δ 的距離 (或更小) 靠近 a, $f(x)$ 就會以 (至少有) ε 的距離靠近 L。

Ex: $f(x) = \begin{cases} 2x-1, & x \neq 3 \\ 6, & x = 3 \end{cases}$, $\lim\limits_{x\to 3} f(x) = 5$ (polynomial).

How to prove $f(x) \to 5$ as $x \to 3$?



Case 1. $\varepsilon = 0.1$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < 0.1 \iff |x-3| < 0.05$.

所以只要 x 以 0.05 的距離靠近 3, $f(x)$ 就會以 0.1 的距離靠近 5.

Case 2. $\varepsilon = 0.01$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < 0.01 \iff |x-3| < 0.005$.

所以只要 x 以 0.005 的距離靠近 3, $f(x)$ 就會以 0.01 的距離靠近 5.

Case 3. $\varepsilon > 0$.

$|f(x) - L| = |(2x-1) - 5| = 2|x-3| < \varepsilon \iff |x-3| < \dfrac{\varepsilon}{2}$.

所以$\overset{\boxed{\text{as}}}{\text{只要}}$ x 以 $\delta \overset{(\leq)}{=} \dfrac{\varepsilon}{2}$ (或更小) 的距離$\overset{\boxed{\to 3}}{\text{靠近 3}}$, $f(x)$ 就會以 (至少有) ε 的距離$\overset{\boxed{\to 5}}{\text{靠近}}$ 5. $\therefore$ By the definition of limit, $\lim\limits_{x\to 3} f(x) = 5$.

Note: 不要搞反了: $|f(x) - L| < \varepsilon \cancel{\Longrightarrow} 0 < |x-a| < \delta$.

$f(x)$ 靠近 L 的地方可能很多, 可能 x 離 a 很遠, 但是 $f(x)$ 還是很靠近 L.

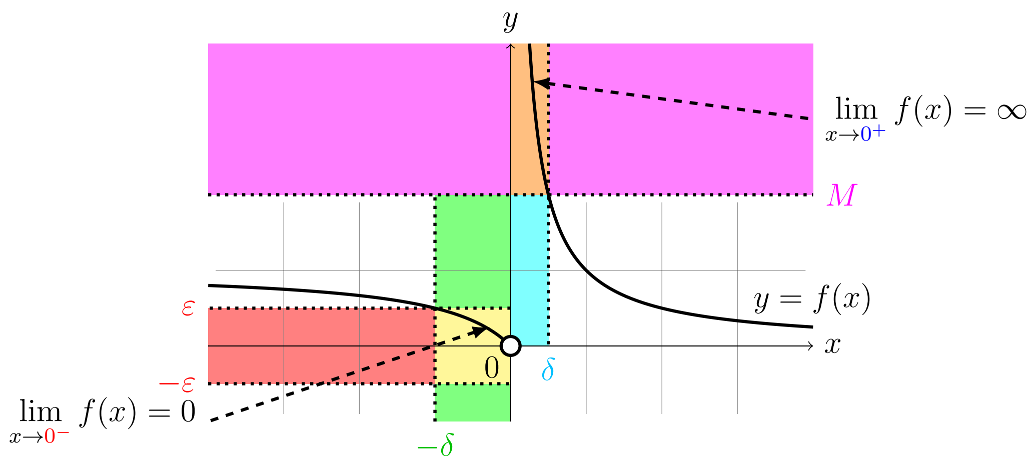
0.2 One-side limit

Define:

$$\boxed{\lim_{x\to \substack{a^- \\ a^+}} f(x) = L}$$

if $\boxed{\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \ni \begin{matrix} a-\delta < x < a \\ a < x < a+\delta \end{matrix} \implies |f(x) - L| < \varepsilon.}$

(Prove by definition "$\lim\limits_{x\to a} f(x) = L \iff \lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$".)



0.3 Infinite limit

Define:

$$\boxed{\lim_{x\to \substack{a \\ a^- \\ a^+}} f(x) = \begin{matrix} \infty \\ -\infty \end{matrix}}$$

if $\boxed{\forall\ \begin{matrix} M > 0 \\ N < 0 \end{matrix},\ \exists\, \delta > 0,\ \ni \begin{matrix} 0 < |x-a| < \delta \\ a-\delta < x < a \\ a < x < a+\delta \end{matrix} \implies \begin{matrix} f(x) > M. \\ f(x) < N \end{matrix}}$

怎麼描述任意大/小? 任何 (至少比零)大/小的 M/N, 都能找到 δ, 保證只要 x 以 δ 的距離 (從左/右邊) 靠近 a, $f(x)$ 就會比 M/N 還大/小。

How to prove limit: (標準流程)

Step 1. Guessing a value for δ ($\delta = \delta(\varepsilon)$).

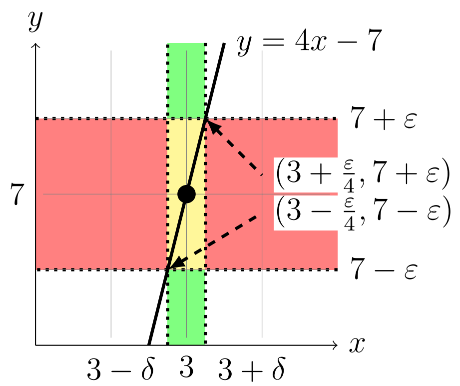
Step 2. Showing this δ works.

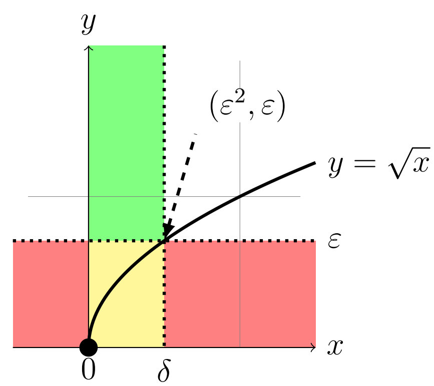
Example 0.1 *Prove* $\lim_{x\to 3}(4x-5) = 7$.

1. $|(4x-5)-7| < \varepsilon \iff 4|x-3| < \varepsilon \iff |x-3| < \dfrac{\varepsilon}{4}$, *guess* $\delta = \dfrac{\varepsilon}{4}$.
2. *Given* $\varepsilon > 0$, *choose* $\delta = \dfrac{\varepsilon}{4}$.

If $0 < |x-3| < \delta$, *then* $|f(x)-7| = |(4x-5)-7| = 4|x-3| < 4\cdot\delta = 4\cdot\dfrac{\varepsilon}{4} = \varepsilon$.
Therefore, by the definition (of the limit), $\lim_{x\to 3}(4x-5) = 7$. ■

Skill 1: 用 $|f(x)-L| < \varepsilon$ 推出 $|x-a| < \delta(\varepsilon)$, 猜 $\delta = \delta(\varepsilon)$.





Example 0.2 *Prove* $\lim_{x\to 0^+}\sqrt{x} = 0$.

1. $|\sqrt{x}-0| = \sqrt{x} < \varepsilon \iff x < \varepsilon^2$, *guess* $\delta = \varepsilon^2$.
2. *Given* $\varepsilon > 0$, *choose* $\delta = \varepsilon^2$.

If $0 < x < \delta$, *then* $|f(x)-0| = |\sqrt{x}-0| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$.
Therefore, by the definition (of the right-hand limit), $\lim_{x\to 0^+}\sqrt{x} = 0$. ■

Attention: $\lim_{x\to 0}\sqrt{x} \neq 0$. (Can you explain why?)

Example 0.3 *Prove* $\lim_{x\to a} c = c$. *(Choose* $\delta = 1$.*)*

Example 0.4 *Prove* $\lim_{x\to a} x = a$. *(Choose* $\delta = \varepsilon$.*)*

Example 0.5 *Prove* $\lim\limits_{x\to 3} x^2 = 9$.

1. $|x^2-9| = |x+3||x-3|$. *(*$|x-3|$ *很靠近零, 但是* $|x+3|$ *呢?)*

idea: *if* $|x+3| < C$ *and* $|x-3| < \dfrac{\varepsilon}{C}$ *for some* $C > 0$,

then $|x^2-9| < C \cdot \dfrac{\varepsilon}{C} = \varepsilon$.

When $|x-3| < 1$, $|x+3| < 7$*; so let* $C = 7$ *and guess* $\delta = \min\left\{1, \dfrac{\varepsilon}{7}\right\}$.

2. Given $\varepsilon > 0$*, choose* $\delta = \min\left\{1, \dfrac{\varepsilon}{7}\right\}$. *(選最小才能保證* $<, <$ *都成立.)*

If $0 < |x-3| < \delta$*, then* $0 < |x-3| < 1 \implies |x+3| < 7$*, and* $0 < |x-3| < \dfrac{\varepsilon}{7}$,

so $|f(x)-9| = |x^2-9| = |x+3||x-3| < 7 \cdot \dfrac{\varepsilon}{7} = \varepsilon$.

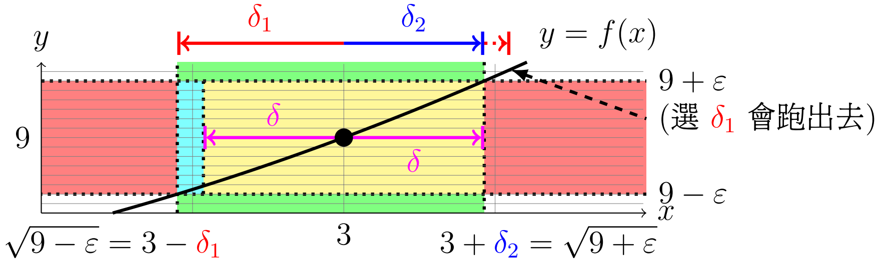
Therefore, by the definition, $\lim\limits_{x\to 3} x^2 = 9$. ■

Skill 2: δ 可以嘗試一些數字 (like 1) 夾住其他乘積項, 再讓 δ 取最小值.

[Another method]: (用 **Skill 1**)

$$\begin{array}{llclcll}
\because & & & |x^2-9| & < & \varepsilon & \\
\iff & -\varepsilon & < & x^2-9 & < & \varepsilon & \\
\iff & 9-\varepsilon & < & x^2 & < & 9+\varepsilon & \\
\implies & \sqrt{9-\varepsilon} & < & x & < & \sqrt{9+\varepsilon} & (\text{when } \varepsilon < 9) \\
\iff & \sqrt{9-\varepsilon}-3 & < & x-3 & < & \sqrt{9+\varepsilon}-3 & (\sqrt{9-\varepsilon}-3 < 0) \\
\implies & 3-\sqrt{9-\varepsilon} & > & |x-3| & < & \sqrt{9+\varepsilon}-3 &
\end{array}$$

Choose $\delta = \min\{3-\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon}-3\}$ when $\varepsilon < 9$, and choose $\delta = \sqrt{9+\varepsilon}-3$ when $\varepsilon \geq 9$.



從 (a, L) 沿 $y = f(x)$ 找第一次跑出 $y = L+\varepsilon$ 與 $y = L-\varepsilon$ 包圍的 x, 選 $\delta = \min\{|x-a|\}$ (要取最小, 這也是最大可能的 δ), 但是有時候不好算.

♡考: 給定 ε 找最大/可用的 δ. (100,101,102會考)

Example 0.6 *Prove limit law: (addition)*

$$\boxed{\lim_{x\to a} f(x) = L \,\&\, \lim_{x\to a} g(x) = M \implies \lim_{x\to a}[f(x)+g(x)] = L+M.}$$

Proof. *Given* $\varepsilon > 0$. $|[f(x)+g(x)]-(L+M)| = |(f(x)-L)+(g(x)-M)| \le |f(x)-L| + |g(x)-M|$. $(\because |a+b| \le |a|+|b|.)$

$\because \lim_{x\to a} f(x) = L$, $\exists\ \delta_1 > 0$, $\ni 0 < |x-a| < \delta_1 \implies |f(x)-L| < \frac{\varepsilon}{2}$.

$\because \lim_{x\to a} g(x) = M$, $\exists\ \delta_2 > 0$, $\ni 0 < |x-a| < \delta_2 \implies |g(x)-M| < \frac{\varepsilon}{2}$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

If $0 < |x-a| < \delta$, *then* $0 < |x-a| < \delta_1$ *and* $0 < |x-a| < \delta_2$,

and so $|f(x)-L| < \frac{\varepsilon}{2}$ *and* $|g(x)-M| < \frac{\varepsilon}{2}$,

$\implies |[f(x)+g(x)]-(L+M)| \le |f(x)-L| + |g(x)-M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Therefore, by the definition, $\lim_{x\to a}[f(x)+g(x)] = L+M$. ■

Skill 3: 用 triangle inequality 三角不等式 ($|a+b| \le |a|+|b|$, $|a+b+c| \le |a|+|b|+|c|$, ...) 分成總和爲 ε 的多項 ($\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$, $\frac{\varepsilon}{3}+\frac{2\varepsilon}{3}$, $\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$, ...), 找出個別的 δ, 最後再取最小值 (保證每項不等式都成立).

Example 0.7 (Extended) *(continuous)* $\implies \lim_{x\to a} f(x)g(x) = LM$.

Proof. $|fg-LM| = |(fg-Lg)+(Lg-LM)| \le |f-L||g| + |L||g-M|$.

1. $\exists\ \delta_1 \ni |x-a| < \delta_1 \implies |g-M| < 1 \iff |g| < |M|+1$;

2. $\exists\ \delta_2 \ni |x-a| < \delta_2 \implies |f-L| < \dfrac{\varepsilon}{2(|M|+1)}$;

3. $\exists\ \delta_3 \ni |x-a| < \delta_3 \implies |g-M| < \dfrac{\varepsilon}{2(|L|+1)}$. *(*避開 $L=0$*)*

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. *If* $0 < |x-a| < \delta$, *then (*略*) and*

$$|fg-LM| < \frac{\varepsilon}{2(|M|+1)} \cdot (|M|+1) + |L| \cdot \frac{\varepsilon}{2(|L|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ ■

*[Another proof:] (*時間夠再說*)*

$|fg-LM| \le |f-L||g-M| + |L||g-M| + |f-L||M|$

$$< \frac{\overset{(\delta_1)}{\varepsilon}}{3\max\{1,|M|\}} \cdot \overset{(\delta_3)}{1} + |L| \cdot \frac{\overset{(\delta_1)}{\varepsilon}}{3\max\{1,|L|\}} + \frac{\overset{(\delta_2)}{\varepsilon}}{3\max\{1,|M|\}} \cdot |M|$$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ *when choose* $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. ■

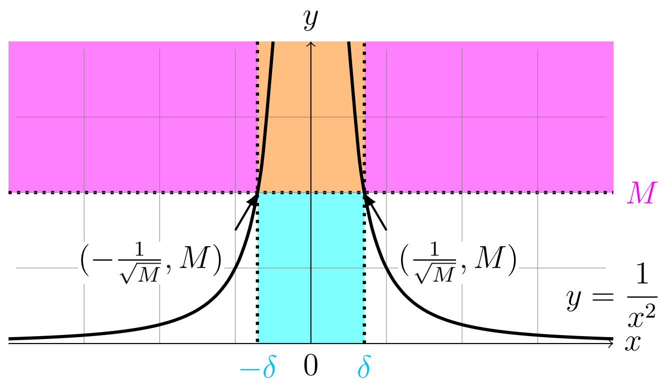
($\dfrac{\varepsilon}{3\max\{1,|L|\}} = \min\{\dfrac{\varepsilon}{3}, \dfrac{\varepsilon}{3|L|}\}$. 分割 ε 與選擇 δ 的方法都不是唯一.*)*

Example 0.8 *(infinite limit) Prove* $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

1. $\frac{1}{x^2} > M \iff |x| < \frac{1}{\sqrt{M}}$, *guess* $\delta = \frac{1}{\sqrt{M}}$.
2. *Given* $M > 0$, *choose* $\delta = \frac{1}{\sqrt{M}}$.

If $0 < |x - 0| < \delta$, *then* $\frac{1}{x^2} > \frac{1}{\delta^2} = \frac{1}{(\frac{1}{\sqrt{M}})^2} = M$.

Therefore, by the definition, $\lim_{x\to 0} \frac{1}{x^2} = \infty$. ($\frac{1}{x^2} \to \infty$ *as* $x \to 0$.) ■



Remind: $\lim\limits_{x\to a} f(x) = L$ or $f(x) \to L$ as $x \to a$

$\qquad a^- \qquad \infty \qquad\qquad \infty \qquad a^-$

$\qquad a^+ \qquad -\infty \qquad\qquad -\infty \qquad a^+$

if $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$, $\ni 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.

$\qquad M > 0 \qquad\qquad a - \delta < x < a \qquad\qquad f(x) > M$

$\qquad N < 0 \qquad\qquad a < x < a + \delta \qquad\qquad f(x) < N$

When proving

- limit: $0 \boxed{<} |x - a| < \delta$ 避開 $x = a$.
- one-side limit: $a - \delta < x < a$ & $a < x < a + \delta$ 左右邊不同.
- infinite limit: $f(x) > M$ & $f(x) < N$ 沒有絕對值.

Remark: 計算極限的方法: 極限律, 左右極限, 夾擠定理, 都可用 ε-δ 證明. (Try to prove by ε-δ: limit laws, left/right-hand limits, Squeeze Theorem.)

♦ Additional: Proof of left/right-hand limits

"$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$"

Proof.

$(\Rightarrow)$ $\forall\ \varepsilon > 0$,

$\because \lim_{x\to a} f(x) = L$, $\exists\ \delta > 0 \ni 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

If $a - \delta < x < a$, then $0 < a - x = |x-a| < \delta$, $\implies |f(x) - L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a^-} f(x) = L$.

If $a < x < a + \delta$, then $0 < x - a = |x-a| < \delta$, $\implies |f(x) - L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a^+} f(x) = L$.

$(\Leftarrow)$ $\forall\ \varepsilon > 0$,

$\because \lim_{x\to a^-} f(x) = L$, $\exists\ \delta_1 > 0 \ni a - \delta_1 < x < a \implies |f(x) - L| < \varepsilon$;

$\because \lim_{x\to a^+} f(x) = L$, $\exists\ \delta_2 > 0 \ni a < x < a + \delta_2 \implies |f(x) - L| < \varepsilon$.

Choose $\delta = \min\{\delta_1, \delta_2\}$.

If $0 < |x-a| < \delta$, then $\left\{ \begin{array}{ccc} \text{either} & -\delta < x - a < 0, & a - \delta_1 < a - \delta < x < a \\ \text{or} & 0 < x - a < \delta, & a < x < a + \delta < a + \delta_2 \end{array} \right\}$

$\implies |f(x) - L| < \varepsilon$.

$\therefore$ by the definition, $\lim_{x\to a} f(x) = L$. ∎