8.1 Arc length

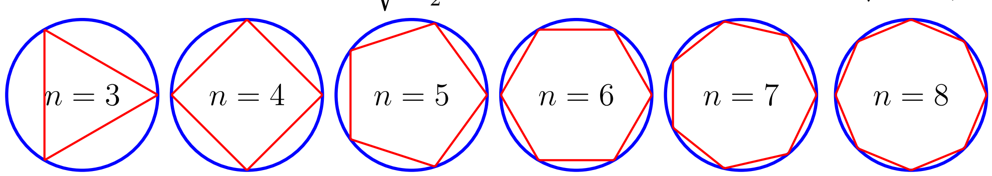
1. arc length formula 弧長公式 $L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx (=\int ds)$
2. arc length function 弧長函數 $s(x)=\int_a^x \sqrt{1+[f'(t)]^2}\,dt$

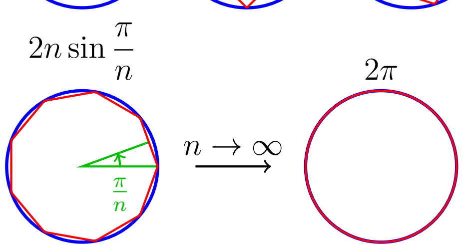
Recall: 積分的應用:

- 面積: $A=\displaystyle\int_a^b |f(x)-g(x)|\,dx=\int_c^d |f(y)-g(y)|\,dy;$
- 體積: $V=\displaystyle\int_a^b A(x)\,dx=\int_c^d A(y)\,dy;$
- 旋轉體: (逆紋切) disk, washer, (順紋切) cylindrical shell.
  圓盤: $V \overset{x\text{-axis}}{=} \displaystyle\int_a^b \pi[r(x)]^2\,dx \overset{y\text{-axis}}{=} \int_c^d \pi[r(y)]^2\,dy;$
  墊圈: $V \overset{x\text{-axis}}{=} \displaystyle\int_a^b \pi\{[R(x)]^2-[r(x)]^2\}\,dx \overset{y\text{-axis}}{=} \int_c^d \pi\{[R(y)]^2-[r(y)]^2\}\,dy;$
  柱殼: $V \overset{x\text{-axis}}{=} \displaystyle\int_c^d 2\pi r(y)h(y)\,dy \overset{y\text{-axis}}{=} \int_a^b 2\pi r(x)h(x)\,dx.$

弧長怎麼算? 用直線去估計曲線. Ex: 單位圓周長$=2\pi$.

$3\sqrt{3}$ $\quad 4\sqrt{2}$ $\quad 5\sqrt{\frac{5-\sqrt{5}}{2}}$ $\quad 6$ $\quad 8\sqrt{2-\sqrt{2}}$

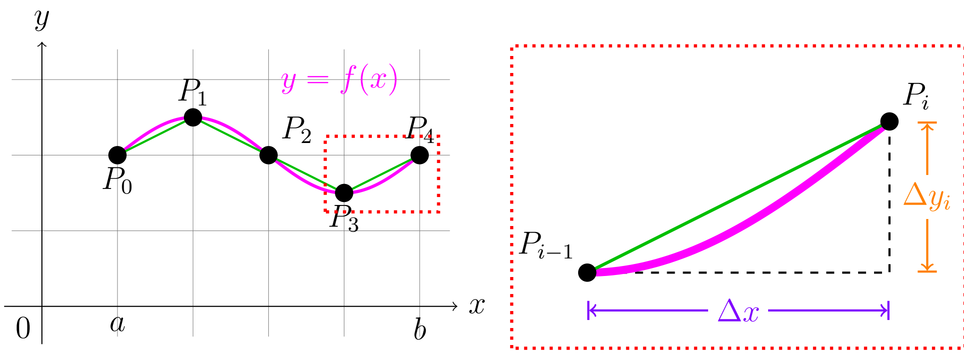




$$\lim_{n\to\infty} 2n\sin\frac{\pi}{n} = 2\pi \lim_{\frac{\pi}{n}\to 0^+} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 2\pi.$$

Idea: $\overbrace{n \text{ 等分} + \text{估計總和}}^{\text{黎曼和}} + \text{取極限} = \text{定積分}.$

0.1 Arc length formula



The curve of $y = f(x)$ from a to b.

把 $[a, b]$ 分成 n 等分, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, $P_i(x_i, f(x_i))$.

Then the length L of the curve is

$$L \approx \sum_{i=1}^{n} |P_{i-1}P_i|,$$

where $|P_{i-1}P_i|$ is the length of segment $P_{i-1}P_i$.

Define: The ***length*** 弧長 L of the curve $y = f(x)$ from a to b is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|.$$

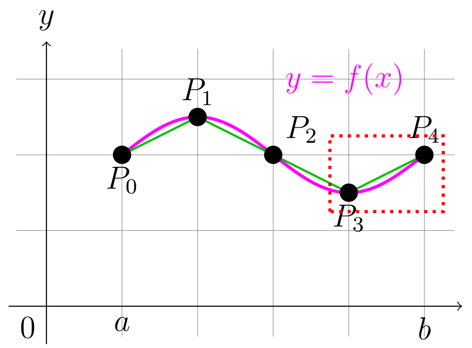
Define: A function f is ***smooth*** 平滑 if f' is continuous (at a point, on an interval, on its domain).

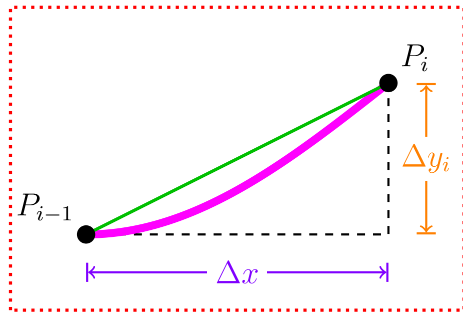
Theorem 1 *If f' is* ***continuous*** *on $[a, b]$ (f is smooth), then the length of the curve $y = f(x)$, $a \le x \le b$, is*

$$\boxed{L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx}$$

(先微分, 再平方, 後加一, 開根號, 做積分.)

怨言嘆語: 課本定義了 “smooth” (沒斷沒折沒尖沒角), 可是它又很不喜歡用。





Proof. Let $\Delta y_i := f(x_i) - f(x_{i-1})$, by Mean value theorem,
$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1}) = f'(x_i^*)\Delta x$,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y_i)^2} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + [f'(x_i^*)\Delta x]^2} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \end{aligned}$$

($\because$ f' and hence $\sqrt{1 + [f'(x)]^2}$ is continuous, limit exists, integrable.) ■

Note: Leibniz notation: $f'(x) = \dfrac{dy}{dx}$,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Note: 以 y 的觀點版本 $g'(y)$ is continuous on $[c, d]$, $x = g(y)$, $c \le y \le d$,

$$\boxed{\boldsymbol{L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy}} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Skill: 如果直的不好切 (dx), 可以切横的 (dy).
對 x 積分: 切成寬度一樣的線段; 對 y 積分: 切成高度一樣的線段.

Example 0.1 *Find the length of the arc of the* ***semicubical*** 半三次 *parabola* $y^2 = x^3$ *between point* $(1,1)$ *and* $(4,8)$.

$y = \pm x^{3/2}$, $(1,1)$ 到 $(4,8)$ 是在 $x \geq 0$ 的部分, $y = x^{3/2}$ *and* $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2}$.

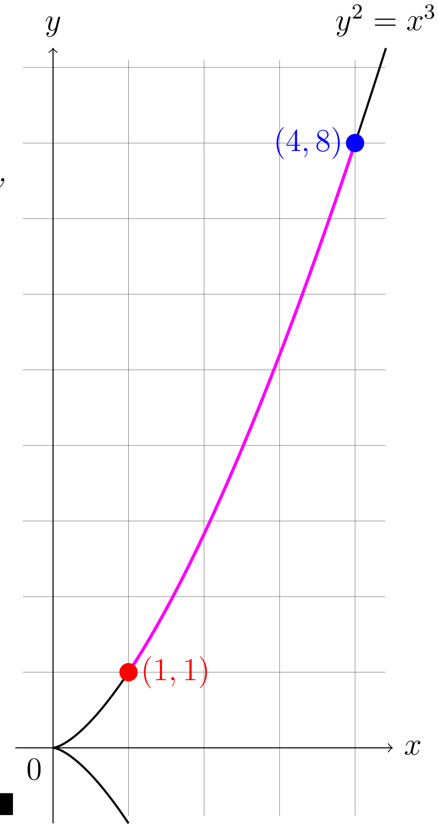
(先算算看函數好不好積)

$\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \sqrt{1+(\frac{3}{2}\sqrt{x})^2} = \sqrt{1+\frac{9}{4}x}$.

Let $u = 1+\dfrac{9}{4}x$, *then* $du = \dfrac{9}{4}\,dx$, $dx = \dfrac{4}{9}\,du$,

when $x = 1$, $u = \dfrac{13}{4}$, *when* $x = 4$, $u = 10$.

$$\begin{aligned} L &= \int_1^4 \sqrt{1+\frac{9}{4}x}\,dx \\ &= \int_{13/4}^{10} \frac{4}{9}\sqrt{u}\,du \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_{13/4}^{10} \\ &= \frac{8}{27}\left[10^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] \\ &= \frac{1}{27}(80\sqrt{10} - 13\sqrt{13}). \end{aligned}$$



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Question: 如果不順怎麼辦? Ex: $y^2 = x^3$ from $(1,-1)$ to $(1,1)$.
Answer: 切成順的分段算.

閒言閒語: 一個長得簡單的函數在一個美麗的區間中, 經過微分平方加一根號積分, 弧長很醜陋很複雜是很自然很常見的; 反之, 弧長長得很簡單的, 就很可能函數長得很複雜, 或是區間長得很醜陋。

Example 0.2 *Find the length of the arc of the parabola $y^2 = x$ between point* $(0,0)$ *and* $(1,1)$.

用 $y = \sqrt{x}$ *(*負不合*)*, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$,

$L = \displaystyle\int_0^1 \sqrt{1 + \frac{1}{4x}}\ dx$ …瑕積分*!* 可以用變數變換算, 不過很複雜. *(try yourself.)*

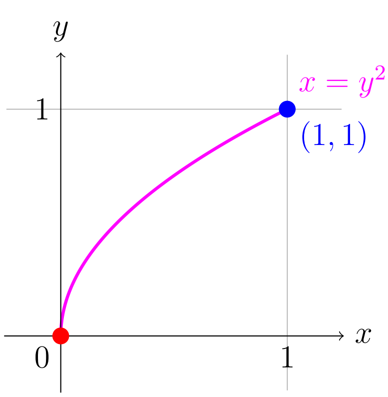
改用 $x = y^2$, $\dfrac{dx}{dy} = 2y$.

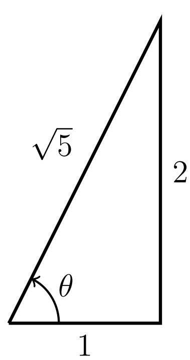
$L = \displaystyle\int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy = \int_0^1 \sqrt{1+4y^2}\ dy$

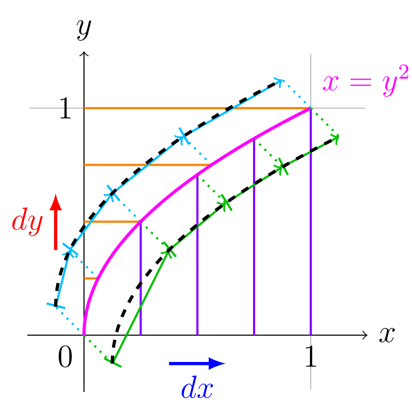
*(*用三角變換*)* *Let* $y = \dfrac{1}{2}\tan\theta$, $\theta \in (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$, *then* $\sqrt{1+4y^2} = \sec\theta$, $dy = \dfrac{1}{2}\sec^2\theta\ d\theta$, *when* $y = 0$, $\theta = \tan^{-1} 0 = 0$, *when* $y = 1$, $\theta = \tan^{-1} 2$.

$$\begin{aligned}
L &= \int_0^1 \sqrt{1+4y^2}\ dy = \int_0^{\tan^{-1}2} \frac{1}{2}\sec^3\theta\ d\theta \\
&= \frac{1}{2}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\tan^{-1}2} \\
&= \frac{1}{4}(2\sec(\tan^{-1}2) + \ln|\sec(\tan^{-1}2) + 2|) \\
&= \frac{1}{4}(2\sqrt{5} + \ln(\sqrt{5}+2)) \qquad \textit{(用看圖)} \\
&= \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5}+2)}{4}.
\end{aligned}$$

*(*可以把 $\sec\theta$ 代回 $\sqrt{1+4y^2}$, $\tan\theta$ 代回 $2y$, 上下界 0 *to* 1.*)* ■







Example 0.3 *(a) Set up an integral for the length of the arc of the hyperbola $xy = 1$ from the point $(1, 1)$ to the point $(2, \frac{1}{2})$.*
(b) Use Simpson's Rule with $n = 10$ to estimate the arc length.

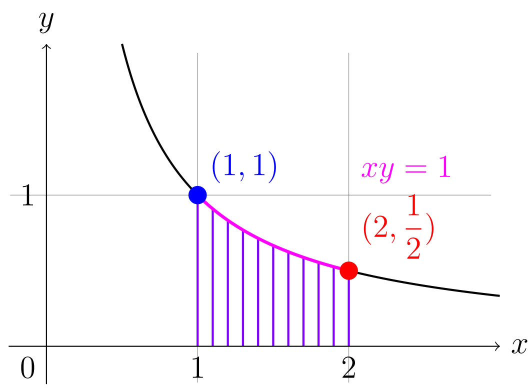
(a) $y = \frac{1}{x}$, $\frac{dy}{dx} = -\frac{1}{x^2}$.

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx \left(= \int_1^2 \frac{\sqrt{x^4+1}}{x^2}\, dx \right).$$

(b) $a = 1$, $b = 2$, $\Delta x = 0.1$, $x_i = 1+0.1i$, (Simpson: $\frac{\Delta x}{3}[1+4+(2+4)+1]$.)

$$L \approx S_{10} = \frac{\Delta x}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + \cdots + 4f(1.9) + f(2)]$$

≈ 1.1321. (這個不好積, 只能用估計. 這裡的 $f(x)$ 是 $\sqrt{1+\frac{1}{x^4}}$.) ■



Skill: 先把要積的函數整理好.

對 x 積分:
1. $y = f(x)$(把 y 寫成 x 的函數),
2. $f'(x)$(微分),
3. $[f'(x)]^2$(平方),
4. $1 + [f'(x)]^2$(加一),
5. $\sqrt{1 + [f'(x)]^2}$(開根).

看看好不好積, 不好積改對 y 積分:
$x = g(y)$(把 x 寫成 y 的函數), $\sqrt{1 + [g'(y)]^2}$(微分, 平方, 加一, 開根).

0.2 Arc length function

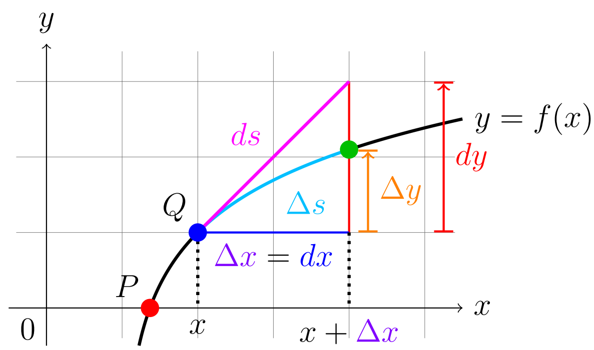
The length of the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is

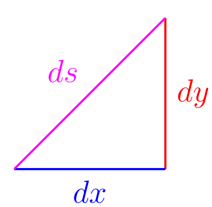
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

Define: The ***arc length function*** 弧長函數

$$\boxed{s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt}$$

is the length from the initial point $P_0(a, f(a))$ to $Q(x, f(x))$ along the curve $y = f(x)$.





Remind: $\frac{dy}{dx} = \frac{d}{dx} f(x) = y' = f'(x)$ 是 f 的導函數 (derivative), ds, dx, dy 是變數, 叫微分 (differential), $dy = f'(x)\, dx = \frac{dy}{dx}\, dx$, $\Delta x = dx$, $\Delta y = f(x + \Delta x) - f(x)$ 是改變量 (increment), $\Delta y \approx dy$.

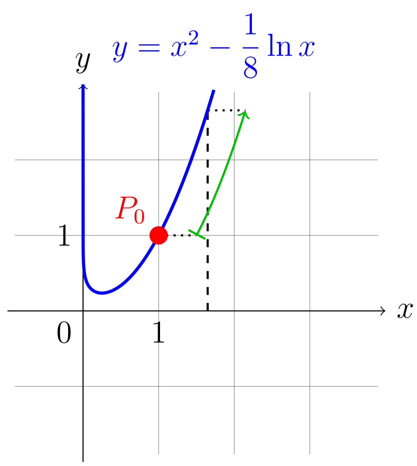
Attention: 可以用 differential 幫忙記公式, 但是不可約分

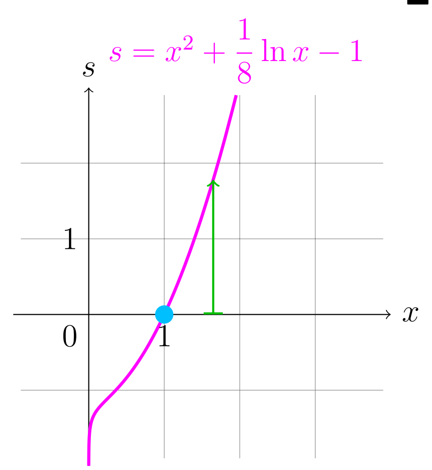
$$dy = \frac{dy}{dx} dx \overset{!}{=} \frac{dy}{\cancel{dx}} \cancel{dx}$$

Example 0.4 *Find the length function for the curve* $y = x^2 - \dfrac{1}{8}\ln x$ *taking* $P_0(1,1)$ *as the starting point.*

$$
\begin{aligned}
f(x) &= y = x^2 - \frac{1}{8}\ln x, \\
f'(x) &= 2x - \frac{1}{8x}, \\
1 + [f'(t)]^2 &= 1 + \left(2t - \frac{1}{8t}\right)^2 = 1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2} \\
&= 4t^2 + \frac{1}{2} + \frac{1}{64t^2} = \left(2t + \frac{1}{8t}\right)^2, \quad \text{(能配方)} \\
s(x) &= \int_1^x \sqrt{1 + [f'(t)]^2}\, dt = \int_1^x \left(2t + \frac{1}{8t}\right) dt \\
&= \left[t^2 + \frac{1}{8}\ln t\right]_1^x = x^2 + \frac{1}{8}\ln x - 1.
\end{aligned}
$$

■





Note: 微分平方加一根號能積的不多, 都在 sample & exercise, 記得要練習. 例如: 根號一次式: $\int \sqrt{ax+b}\,dx$ (變數變換); 根號二次式: $\int \sqrt{ax^2+bx+c}\,dx$ (三角變換), 根號平方: $\int \sqrt{[f(x)]^2}\; dx = \int |f(x)|\; dx =$ (配平方), ...etc.

Skill: 怎麼記? 畢氏定理 & 弧長=積斜邊.

$$\boxed{(\boldsymbol{ds})^2 = (\boldsymbol{dx})^2 + (\boldsymbol{dy})^2} \quad \& \quad \boxed{\boldsymbol{L} = \int \boldsymbol{ds}}$$

$$= \int \sqrt{(dx)^2 + (dy)^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

If $y = f(x) \implies dy = f'(x)\, dx$, $ds = \sqrt{1 + [f'(x)]^2}\, dx$.
If $x = g(y) \implies dx = g'(y)\, dy$, $ds = \sqrt{1 + [g'(y)]^2}\, dy$.
(哪個好算用哪個.)

Proof. When $y = f(x)$, $dy = f'(x)\, dx = \dfrac{dy}{dx} dx$.

$\because s(x) = \displaystyle\int_a^x \sqrt{1 + [f'(t)]^2}\, dt$, by TFTC,

$$\frac{ds}{dx} = s'(x) = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

$\implies \dfrac{ds}{dx} \geq 1$ and $\dfrac{ds}{dx} = 1$ when $f'(x) = 0$,

Let $s = s(x)$, then $ds = s'(x)\, dx = \dfrac{ds}{dx} dx$, and

$(ds)^2 = \left(\dfrac{ds}{dx}\right)^2 (dx)^2 = \left[1 + \left(\dfrac{dy}{dx}\right)^2\right] (dx)^2 = (dx)^2 + (dy)^2$, and

$L = \displaystyle\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int ds$.

When $x = g(y)$, $dx = g'(y)\, dy = \dfrac{dx}{dy} dy$, and $s(y) = \displaystyle\int_c^y \sqrt{1 + [g'(t)]^2}\, dt$,

$$\frac{ds}{dy} = s'(y) = \sqrt{1 + [g'(y)]^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2},$$

$\implies \dfrac{ds}{dy} \geq 1$ and $\dfrac{ds}{dy} = 1$ when $g'(y) = 0$,

Similarly, $(ds)^2 = (dx)^2 + (dy)^2$ and $L = \displaystyle\int ds$. ■