3.9 Relative rates

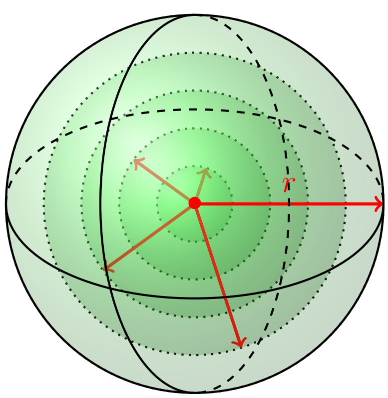
1. relative rates 相對率

Application of implicit differentiation.
用其他函數的變化率來表示某個函數的變化率. 跑千若飛 — $v_{跑} = v_{飛}$
Method: Implicit differentiation 隱微分.

0.1 Relative rates

Example 0.1 *Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?*
灌氣球, 體積(V)以 100 cm^3/s 增加. 半徑(r)在 25 cm 的增加率?

Given: $\dfrac{dV}{dt} = 100\ cm^3/s$. *Ask:* when $r = 25\ cm$, $\dfrac{dr}{dt} = ?$
The formula for the volume of a sphere: $V = \dfrac{4}{3}\pi r^3$.
Use implicit differentiation: $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$.
$\Longrightarrow \dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}\dfrac{dV}{dt} = \dfrac{1}{4\pi(25)^2}\cdot 100 = \dfrac{1}{25\pi}$.
Ans: $\underline{\dfrac{1}{25\pi}\ cm/s}$. ■



Note: 先微完再代 $f'(a) = \dfrac{d}{dx}f(x)\Big|_{x=a} \neq (f(a))'(=0)$, 答案別忘記單位.

Example 0.2 *A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?*
一梯長 5 m 依牆而立, 梯底(x)以 1 m/s 滑離, 梯頂(y)在底離 3 m 的下滑率?

Let the bottom of the ladder x m from from the wall and the top y m from the ground.

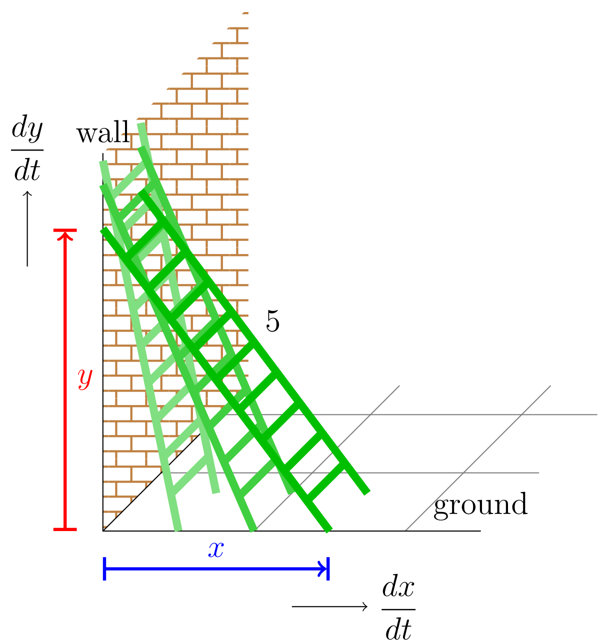
Given: $\dfrac{dx}{dt} = 1$ m/s. *Ask:* when $x = 3$ m, $-\dfrac{dy}{dt} = ?$
(注意方向, y 往上, 下滑率 $= -$ 增高率.)

The formula for the relation of x and y:
Pythagorean Theorem 畢氏定理: $x^2 + y^2 = 5^2 = 25$.

Use implicit differentiation: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.

$\implies y = 4$ *and* $-\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{3}{4} \cdot 1 = \dfrac{3}{4}$.

Ans: $\underline{\dfrac{3}{4}\ m/s}$. ■



Example 0.3 *A water tank has the shape of an inverted circular cone with base radius* 2 *m and height* 4 *m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.*
半徑 2 m 的倒圓錐的水塔以 *2* m^3/min 注入水, 找 3 m 深時的高度(h)變化率.

Given: $\frac{dV}{dt} = 2\ m^3/min$. *Ask:* when $h = 3\ m$, $\frac{dh}{dt} = ?$

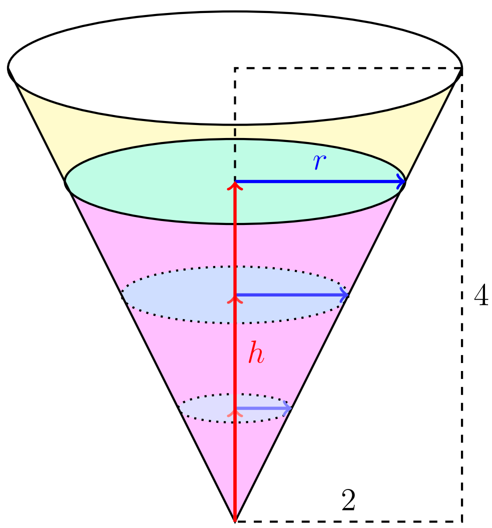
Let h *be the height and* r *be the radius of water, then* $h/r = 4/2$, $r = \frac{h}{2}$.

The formula for the volume of a circular cone: $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$.

Use implicit differentiation: $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$.

$\implies \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} = \frac{4}{\pi(3)^2} \cdot 2 = \frac{8}{9\pi}$.

Ans: $\underline{\frac{8}{9\pi}\ m/min.}$ ■



Example 0.4 *Car A is traveling west at 90 km/h and car B is traveling north at 100 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 60 m and car B is 80 m from the intersection?*
A車西行 90 km/h, B 車北行 100 km/h, 當 A 離交會點 60 m, B 離 80 m, 兩車接近率?

Let x and y be the distance from the intersection to A and B, respectively, and let z be the distance from A to B.

Given: $-\dfrac{dx}{dt} = 90$ *km/h*, $-\dfrac{dy}{dt} = 100$ *km/h. (注意方向)*

Ask: when $x = 0.06$ *km and* $y = 0.08$ *km,* $-\dfrac{dz}{dt} =?$ *(注意單位)*

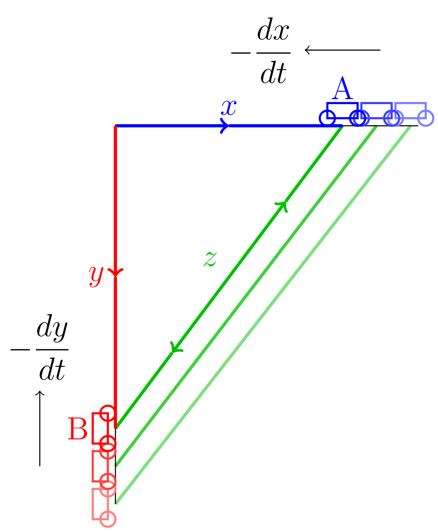
The formula for the relation of x, y and z:
Pythagorean Theorem 畢氏定理: $x^2 + y^2 = z^2$.

Use implicit differentiation: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$.

When $x = 0.06$ *and* $y = 0.08$, $z = 0.1$.

$$\Longrightarrow \quad -\frac{dz}{dt} = \frac{1}{z}\left(-x\frac{dx}{dt} - y\frac{dy}{dt}\right) = \frac{1}{0.1}(0.06 \cdot 90 + 0.08 \cdot 100) = 134.$$

Ans: <u>134 *km/h*</u>. ■



Example 0.5 *A man walks along a straight path at a speed of* 1.5 m/s*. A searchlight is located on the ground* 6 m *from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is* 8 m *from the point on the path closest to the searchlight?*
一人以 1.5 m/s 行直路, 一燈離路 6 m 追人, 當人離最近點 8 m 的燈轉率?

Let x *be the distance form the man to the point on the path closest to the light, and let* θ *be the angel between the beam and the perpendicular* 垂線 *to the path.*

Given: $\dfrac{dx}{dt} = 1.5$ m/s. *Ask:* *when* $x = 8$ m, $\dfrac{d\theta}{dt} = ?$

From graph: $\tan\theta = \dfrac{x}{6}$.

[Sol 1] Use implicit differentiation: $\sec^2\theta \dfrac{d\theta}{dt} = \dfrac{1}{6}\dfrac{dx}{dt}$.

When $x = 8$, $\cos\theta = \dfrac{6}{\sqrt{8^2+6^2}} = \dfrac{3}{5}$.

$$\Longrightarrow \frac{d\theta}{dt} = \frac{1}{6}\cos^2\theta \frac{dx}{dt} = \frac{1}{6}\cdot\left(\frac{3}{5}\right)^2 \cdot 1.5 = 0.09.$$

[Sol 2] $\theta = \tan^{-1}\dfrac{x}{6}$, $u = \dfrac{x}{6}$, $\dfrac{d}{du}\tan^{-1}u = \dfrac{1}{1+u^2}$.

Use chain rule: $\dfrac{d\theta}{dt} = \dfrac{1}{1+(x/6)^2}\dfrac{1}{6}\dfrac{dx}{dt} = \dfrac{1}{1+(8/6)^2}\dfrac{1}{6}(1.5) = 0.09.$

Ans: $\underline{0.09\ rad/s}$. ■