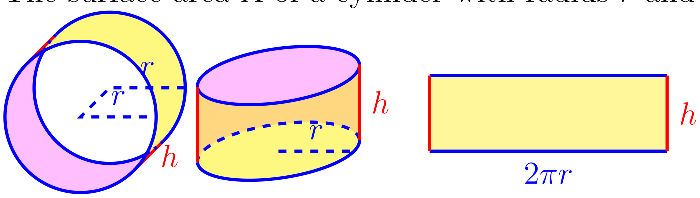
8.2 Area of a surface of revolution

1. surface area formula 表面公式 $S = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$

Cylinder 圓柱

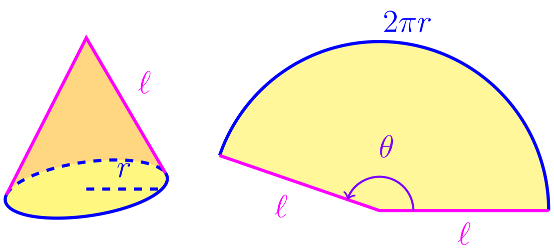
The surface area A of a cylinder with radius r and height h is $2\pi rh$.



$\boxed{A} = 2\pi rh$

Cone 圓錐

The surface area of a circular cone with base radius r and slant height ℓ is $\pi r\ell$.



$\ell\theta = 2\pi r,$

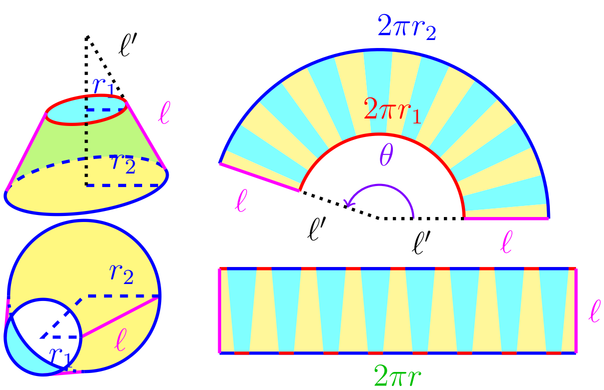
$A = \pi\ell^2\dfrac{\theta}{2\pi} = \dfrac{1}{2}\ell^2\theta = \pi r\ell.$

$\boxed{A} = \pi r\ell$

Note: 扇面積$=\dfrac{1}{2}$半徑$^2\times$夾角.

Band 帶

The surface area of a band (frustum 截頭 of a cone) with upper and lower radii r_1 and r_2 and slant height ℓ is $2\pi r\ell$, where $r = \dfrac{r_1+r_2}{2}$.



$$\frac{\ell'}{\ell'+\ell} = \frac{r_1}{r_2},\ (r_2-r_1)\ell' = r_1\ell,$$

$$\begin{aligned} A &= \pi r_2(\ell'+\ell) - \pi r_1\ell' \\ &= \pi(r_2-r_1)\ell' + \pi r_2\ell \\ &= \pi(r_1+r_2)\ell = 2\pi r\ell. \end{aligned}$$

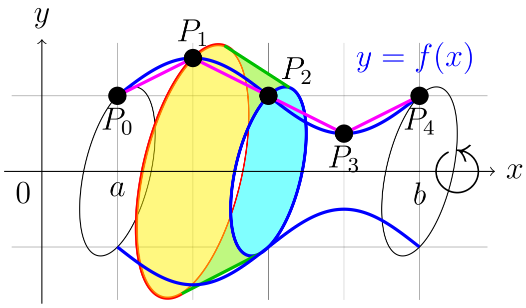
$$\boxed{\boldsymbol{A} = \boldsymbol{2\pi r\ell}}$$

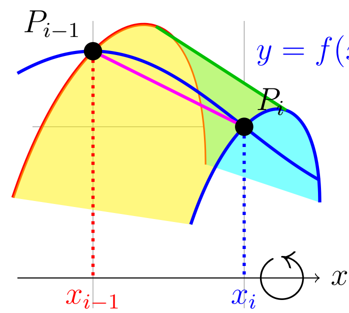
$\boxed{\text{帶面積} = [\text{均圓周長}] \times [\text{斜邊長}]}$

Note: 把圓錐想成 $r_1 = 0$ & $r_2 = r$ 的帶, 代入可得圓錐表面積公式.

Revolution 旋轉體

Rotating the curve of $y = f(x)$ from a to b about x-axis.
怎麼算? 切成 n 段用帶子 (band) 來估計 (不是用圓柱).





把 $[a, b]$ 分成 n 等分, $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta$, $P_i(x_i, y_i = f(x_i))$.
The area of i-th band is

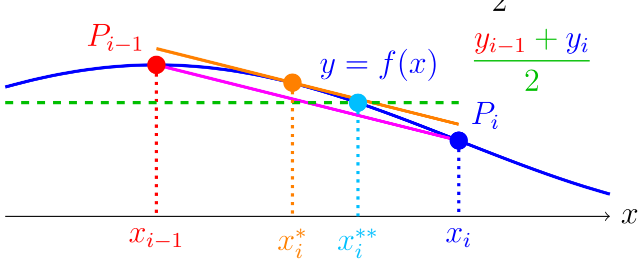
$$S_i = \underbrace{2\pi \frac{y_{i-1} + y_i}{2}}_{[\text{均圓周長}]} \underbrace{|P_{i-1}P_i|}_{[\text{斜邊長}]}$$

$\because |P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$ (by MVT, $\exists\ x_i^* \in [x_{i-1}, x_i]$),
and when Δx small, $y_{i-1} \approx f(x_i^*) \approx y_i$, $\dfrac{y_{i-1} + y_i}{2} \approx f(x_i^*)$. $\qquad (*)$

Then the surface area S of the revolution is

$$\begin{aligned} S \approx \sum_{i=1}^{n} S_i &= \sum_{i=1}^{n} 2\pi f(x_i^{**})\sqrt{1 + [f'(x_i^*)]^2}\Delta x \quad \boxed{\text{不是黎曼和}} \\ &\approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\sqrt{1 + [f'(x_i^{**})]^2}\Delta x. \quad \boxed{\text{是黎曼和}} \end{aligned}$$

Note: $(*)$ 更嚴僅的來說, $\because f$ is continuous, by Locating Root (勘根定理), $\exists\ x_i^{**} \in [x_{i-1}, x_i] \ni f(x_i^{**}) = \dfrac{f(x_{i-1}) + f(x_i)}{2} = \dfrac{y_{i-1} + y_i}{2}$. x_i^{**} 不一定等於 x_i^*.



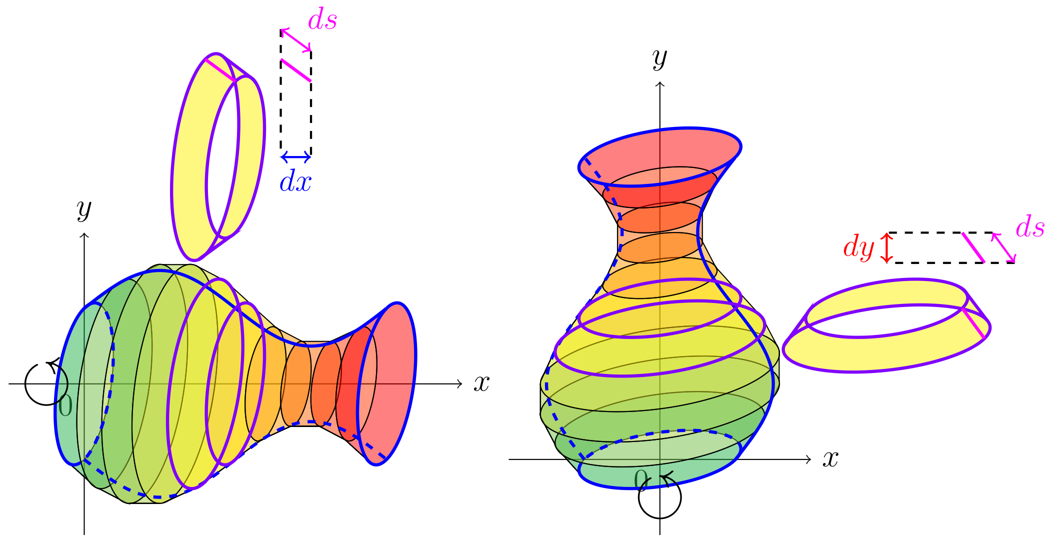
But $\because f'$ is continuous,
when $n \to \infty$,
$\Longrightarrow\ \Delta x \to 0$,
$\Longrightarrow\ x_i^* \to x_i^{**}$,
$\Longrightarrow\ f'(x_i^*) \to f'(x_i^{**})$.

0.1 Surface area formula

Define: Let S denote the ***surface area*** of the surface obtained by rotating the curve $y = f(x)$ from a to b, assuming f is positive and has a continuous derivative (smooth) on $[a, b]$, about the ***x-axis***, is

$$\boxed{\boldsymbol{S} = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx} \quad \boxed{\text{不建議背}}$$

$$\boxed{= \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi y\, ds}$$



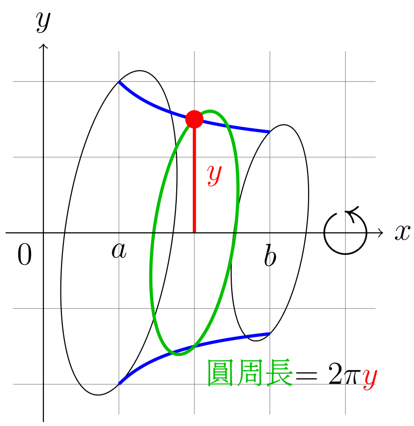
Note: If $x = g(y)$ from c to d about ***y-axis***, it is

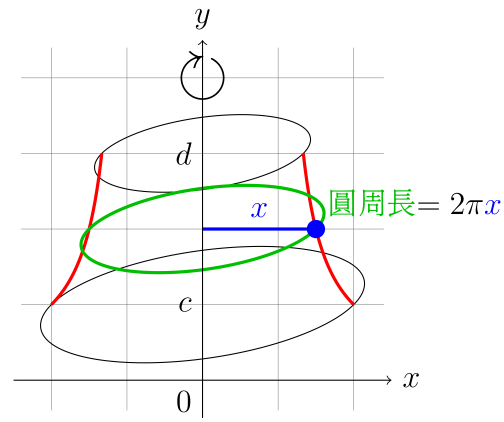
$$\boxed{\boldsymbol{S} = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy} \quad \boxed{\text{不建議背}}$$

$$\boxed{= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi x\, ds}$$

Recall: $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$

Attention: 不管繞 x-axis 或 y-axis, 跟弧長一樣, 可以對 x 積分, 也可以對 y 積分, 重點在半徑: 繞 x-axis, 半徑是 y; 繞 y-axis, 半徑是 x.
課本的公式 $\int 2\pi f(x)\sqrt{\cdots}\, dx$ 只針對繞 x-axis, 繞其他線就不對; 所以不建議背.





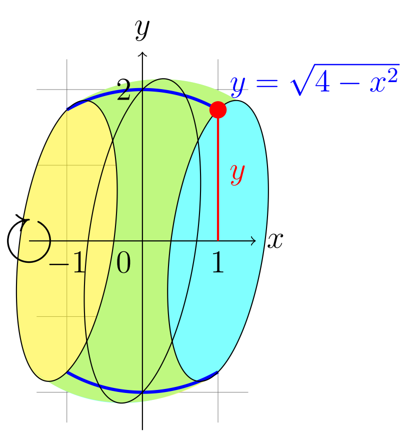
Example 0.1 *Find the area of surface obtained by rotating the curve* $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ *about the* x*-axis.*

$$\boxed{S} = \int 2\pi y \; ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(繞 } x\text{-axis, 半徑是 } y.\text{)}$$

(對 x 積分, 要寫成 $y = f(x)$, ds 變出 dx)

$$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}},\; ds = \sqrt{1+(\frac{dy}{dx})^2}\; dx = \sqrt{1+\frac{x^2}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}\; dx,$$

$$\therefore S = \int_{-1}^{1} 2\pi\sqrt{4-x^2}\frac{2}{\sqrt{4-x^2}}\; dx = 4\pi\int_{-1}^{1}\; dx = 4\pi(2) = 8\pi. \qquad \blacksquare$$



Example 0.2 *Find the area of surface obtained by rotating the parabola* $y = x^2$ *from* $(1, 1)$ *to* $(2, 4)$ *about the* y*-axis.*

$$\boxed{S} = \int 2\pi x \; ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(繞 } y\text{-axis, 半徑是 } x\text{)}$$

[Sol 1] (對 y 積分: $y = x^2$ 不是 $x = g(y)$ 型式, 要解反函數; ds 變出 dy.)

$x = \sqrt{y}$, $1 \le y \le 4$, $\dfrac{dx}{dy} = \dfrac{1}{2\sqrt{y}}$, $ds = \sqrt{1 + (\dfrac{dx}{dy})^2}\; dy = \sqrt{1 + \dfrac{1}{4y}}\; dy$.

$$\therefore S = \int_1^4 2\pi\sqrt{y}\sqrt{1 + \frac{1}{4y}}\; dy = \pi \int_1^4 \sqrt{4y + 1}\; dy$$

(變數變換 Let $u = 4y + 1$, $5 \le u \le 17$, $du = 4\; dy$.)

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\; du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

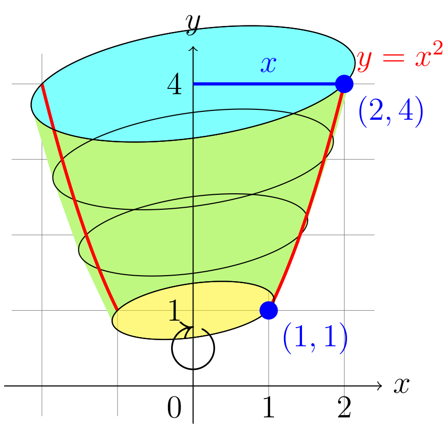
[Sol 2] (對 x 積分: ds 變出 dx.)

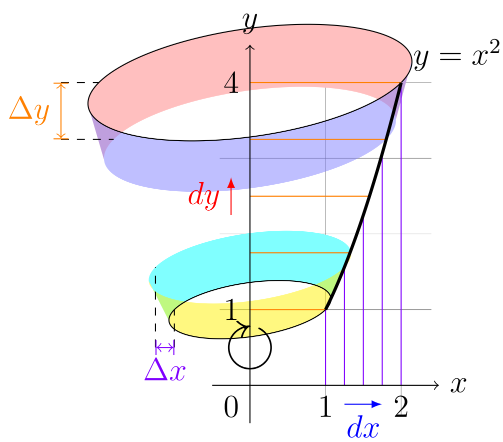
$y = x^2$, $1 \le x \le 2$, $\dfrac{dy}{dx} = 2x$, $ds = \sqrt{1 + (\dfrac{dy}{dx})^2}\; dx = \sqrt{1 + 4x^2}\; dx$.

$$\therefore S = \int_1^2 2\pi x\sqrt{1 + 4x^2}\; dx$$

(變數變換 Let $u = 1 + 4x^2$, $5 \le u \le 17$, $du = 8x\; dx$.)

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\; du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}). \qquad \blacksquare$$





Example 0.3 *Find the area of surface obtained by rotating the curve $y = e^x$, $0 \le x \le 1$ about the x-axis.*

$$\boxed{S} = \int 2\pi y \; ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(繞 } x\text{-axis, 半徑 } y\text{, 對 } x \text{ 積分.)}$$

$\dfrac{dy}{dx} = e^x$, $dy = e^x\ dx$, $ds = \sqrt{1 + (\dfrac{dy}{dx})^2}\ dx = \sqrt{1+e^{2x}}\ dx$,

$S = \displaystyle\int_0^1 2\pi e^x \sqrt{1+e^{2x}}\ dx \overset{y\,=\,e^x}{=} 2\pi \int_1^e \sqrt{1+y^2}\ dy \quad \ldots\ldots\ldots\ldots$ *(變數變換)*

Let $y = \tan\theta$, $\theta \in (-\dfrac{\pi}{2}, \dfrac{\pi}{2})$,

then $\dfrac{\pi}{4} \le \theta \le \tan^{-1} e$, $dy = \sec^2\theta\ d\theta$, $\sqrt{1+y^2} = \sec\theta$. $\ldots\ldots\ldots$ *(三角變換)*

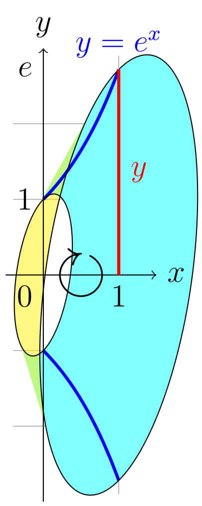
$S = 2\pi \displaystyle\int_{\pi/4}^{\tan^{-1} e} \sec^3\theta\ d\theta$

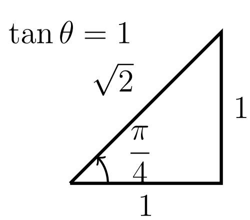
$= 2\pi \cdot \dfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_{\pi/4}^{\tan^{-1} e} \quad \ldots\ldots\ldots\ldots$ *(可以利用圖直接代)*

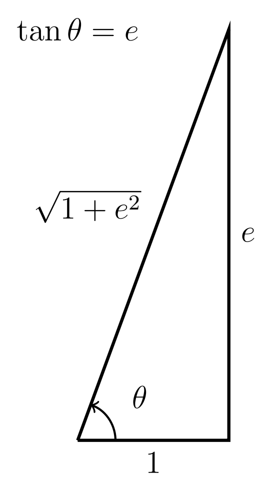
$\Big(= \pi\Big[y\sqrt{1+y^2} + \ln|y + \sqrt{1+y^2}|\Big]_1^e\Big) \quad \ldots\ldots\ldots\ldots\ldots\ldots$ *(換回 y 再代)*

$\Big(= \pi\Big[e^x\sqrt{1+e^{2x}} + \ln|e^x + \sqrt{1+e^{2x}}|\Big]_0^1\Big) \quad \ldots\ldots\ldots\ldots\ldots\ldots$ *(換回 x 再代)*

$= \pi[e\sqrt{1+e^2} + \ln(\sqrt{1+e^2} + e) - \sqrt{2} - \ln(\sqrt{2}+1)]$. ■

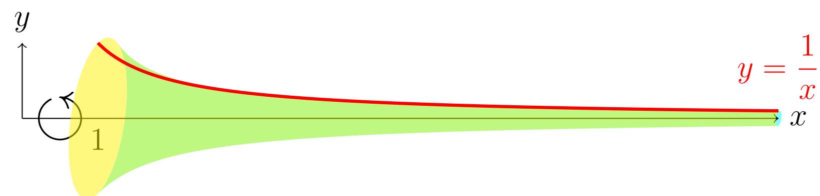






Skill: 畫圖找半徑列式 $S = \int 2\pi y\ ds$ (直繞) 或 $S = \int 2\pi x\ ds$ (平繞), 再看要積誰把 ds 變出 dx (要把 y 變成 x 的函數) 或 dy (要把 x 變成 y 的函數).

♦ Additional: Gabriel's Horn



曲線 $\boldsymbol{y=\frac{1}{x},\ x\geq 1}$ 繞 x-軸的旋轉體稱爲加百列的號角/托里拆利小號 (***Gabriel's Horn***/***Torricelli's trumpet***), 由義大利物理&數學家托里拆利 (Evangelista Torricelli) 所發明 (也發明氣壓計)。

♠《啟示錄 (Revelation)》中寫到: 大天使加百列 (Archangel Gabriel) 吹響號角宣告審判日 (Judgment Day) 的到來。

♡ 有限體積: (Exercise 7.8.63)

$$V=\int_1^\infty \frac{\pi}{x^2}\,dx=\lim_{t\to\infty}\int_1^t\frac{\pi}{x^2}\,dx=\lim_{t\to\infty}\Big[-\frac{\pi}{x}\Big]_1^t=\pi-\lim_{t\to\infty}\frac{\pi}{t}=\pi.$$

♢ 無限面積: (Exercise 8.2.27)

$$S=\int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx=\int_1^\infty 2\pi\frac{\sqrt{1+x^4}}{x^3}\,dx$$

$$=\int_1^\infty \pi\frac{\sqrt{1+(x^2)^2}}{(x^2)^2}\cdot 2x\,dx=\int_1^\infty\pi\frac{\sqrt{1+v^2}}{v^2}\,dv \qquad (v=x^2,\ dv=2x\,dx.)$$

$$=\int_{\pi/4}^{\pi/2}\pi\frac{\sec\theta}{(\tan\theta)^2}\cdot\sec^2\theta\,d\theta \qquad (v=\tan\theta, \sqrt{1+v^2}=\sec\theta,\ dv=\sec^2\theta\,d\theta)$$

$$=\int_{\pi/4}^{\pi/2}\pi\frac{\sec\theta}{\tan^2\theta}(\tan^2\theta+1)\,d\theta=\int_{\pi/4}^{\pi/2}\pi(\sec\theta+\frac{\sec\theta}{\tan^2\theta})\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}\pi(\sec\theta+\csc\theta\cot\theta)\,d\theta \qquad (\frac{\sec\theta}{\tan^2\theta}=\frac{\cos\theta}{\sin^2\theta}=\csc\theta\cot\theta)$$

$$=\lim_{t\to\pi/2}\int_{\pi/4}^t\pi(\sec\theta+\csc\theta\cot\theta)\,d\theta=\lim_{t\to\pi/2}\pi\Big[\ln|\sec\theta+\tan\theta|-\csc\theta\Big]_{\pi/4}^t$$

$$=\lim_{t\to\pi/2}\pi\Big(\underbrace{\ln|\sec t+\tan t|}_{\to\infty}-\underbrace{\csc t}_{\to 1}\Big)-\pi\Big(\ln|\sqrt{2}+1|-\sqrt{2}\Big)=\infty.$$

♣ 漆匠的矛盾 (***Painter's paradox***): 裝得滿 Gabriel's horn 的油漆卻塗不滿它的內壁表面。 — 那五顆檸?!