3.10 Linear approximations and differentials

1. linear (tangent line) approximation 線性 (切線) 逼近
   $f(x) \approx f(a) + f'(a)(x-a)$
2. differentials 微分 dx, dy, Δx, Δy

0.1 Linear approximation

Recall: The tangent line equation of a curve $y = f(x)$ at $(a, f(a))$ is

$$y = f(a) + f'(a)(x-a)$$

Define: The ***linear (tangent line) approximation*** of f at a is

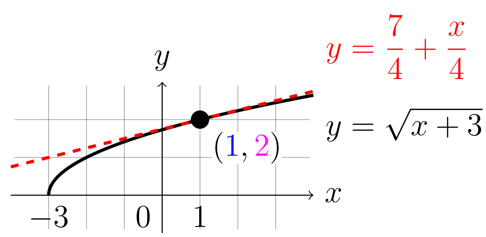
$$\boxed{\boldsymbol{f(x) \approx f(a) + f'(a)(x-a)}}.$$

Define: The ***linearization*** 線性化 (其實就是切線的函數) of f at a is

$$\boxed{\boldsymbol{L(x) = f(a) + f'(a)(x-a)}}.$$

Example 0.1 *Find the linearization of $f(x) = \sqrt{x+3}$ at 1, and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates* 高估 *or underestimates* 低估*?*

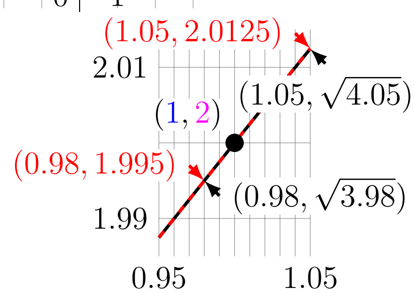
$f'(x) = \dfrac{1}{2\sqrt{x+3}}$, $f'(1) = \dfrac{1}{4}$, $f(1) = 2$.
$L(x) = f(1) + f'(1)(x-1)$
$= 2 + \dfrac{1}{4}(x-1) = \dfrac{7}{4} + \dfrac{x}{4}$,



$$\Longrightarrow \quad \sqrt{x+3} \quad \approx \quad \frac{7}{4} + \frac{x}{4}.$$

$$\sqrt{3.98} = \sqrt{0.98+3} \quad \approx \quad \frac{7}{4} + \frac{0.98}{4} = 1.995.$$

$$\sqrt{4.05} = \sqrt{1.05+3} \quad \approx \quad \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

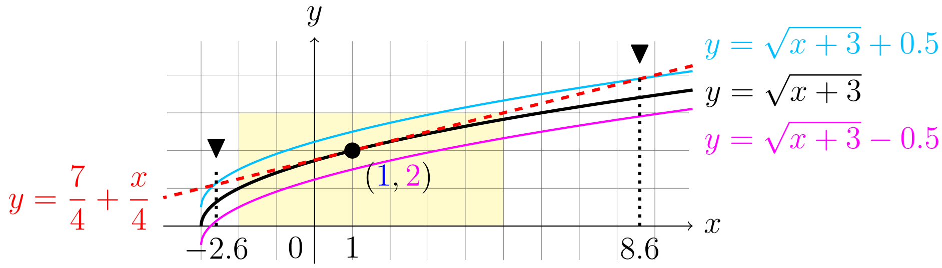


$\sqrt{3.98} = 1.99499373... < 1.995$,
$\sqrt{4.05} = 2.01246117... < 2.0125$.*(皆高估)*
(從圖形看, 切線在上面就會高估, 在下面就低估.)

Ans: $\underline{\dfrac{7}{4} + \dfrac{x}{4},\ \sqrt{3.98} \approx 1.995,\ \sqrt{4.05} \approx 2.0125,\ \textit{both overestimated.}}$ ■

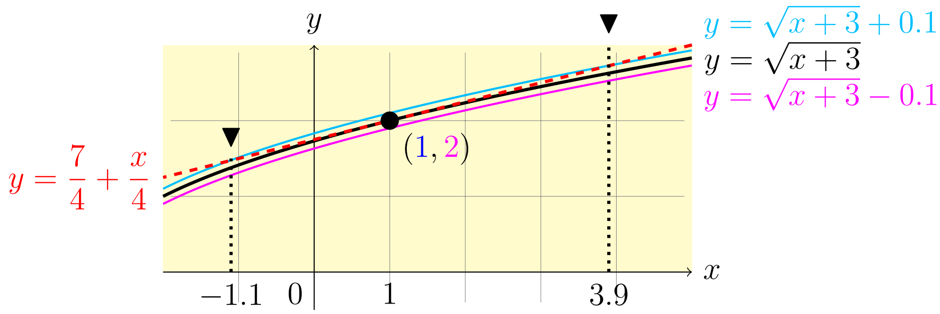
Example 0.2 *When does* $\sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$ *accurate to within 0.5? 0.1?*

(從 $(a, f(a))$ 沿切線找第一次跑出 $y = f(x) + \varepsilon$ 與 $y = f(x) - \varepsilon$ 包圍的 x.*)*



Solve $|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)| < 0.5$

$-2.66 \approx 3 - \sqrt{32} < x < 3 + \sqrt{32} \approx 8.66,$ *choose* $-2.6 < x < 8.6$.



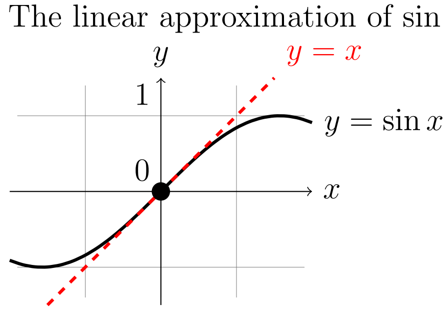
Solve $|\sqrt{x+3} - \left(\frac{7}{4} + \frac{x}{4}\right)| < 0.1$

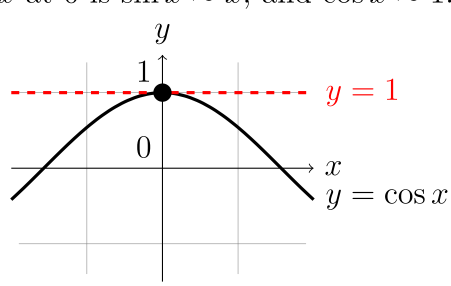
$-1.13 \approx 1.4 - \sqrt{6.4} < x < 1.4 + \sqrt{6.4} \approx 3.93,$ *choose* $-1.1 < x < 3.9$.

Ans: $\underline{-2.6 < x < 8.6,\ -1.1 < x < 3.9}$. ■

Application to physics:

The linear approximation of $\sin x$ and $\cos x$ at 0 is $\sin x \approx x$, and $\cos x \approx 1$.





Remark: $L(x)$ 與 a 有關, 切點$(a, f(a))$不同, 切線與 $L(x)$ 也不同.

0.2 Differential

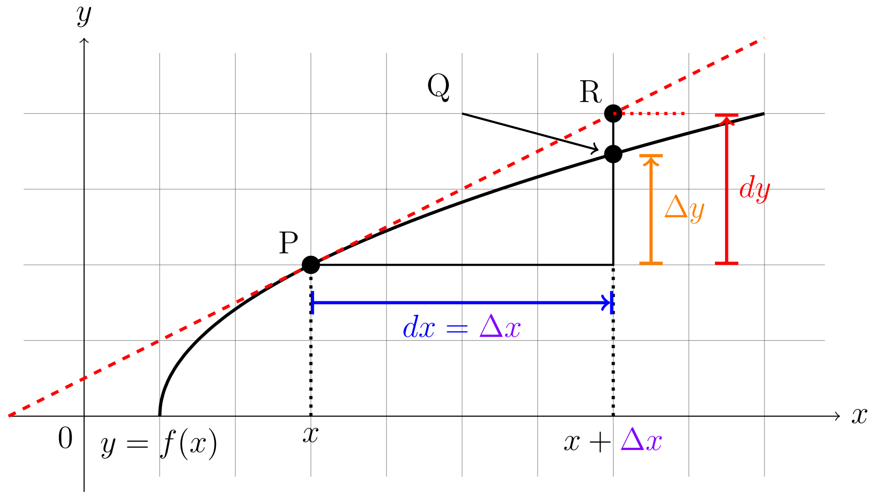
Define: If $y = f(x)$ and f is differentiable, then the ***differential*** 微分 dx is an independent variable, and the ***differential*** 微分 dy is defined by

$$\boxed{\boldsymbol{dy = f'(x)\ dx}}.$$

Let Δx be the change in (***increment*** 增量 of) x , then the change in y is

$$\boxed{\boldsymbol{\Delta y = f(x + \Delta x) - f(x)}}$$

The linear approximation: $f(a + dx) \approx f(a) + f'(a)\ dx = f(a) + dy$.



Note: 給定 $dx = \Delta x$, Δy 是實際差值, dy 是估計差值.

Example 0.3 *Compare* Δy *and* dy *if* $y = f(x) = x^3 + x^2 - 2x + 1$ *and* x *changes (a) from 2 to 2.05 and (b) from 2 to 2.01.*

$f(2) = 9$, $f'(x) = 3x^2 + 2x - 2$, $f'(2) = 14$.

(a) $f(2.05) = 9.71765$,

$dx = \Delta x = 2.05 - 2 = 0.05$,

$\Delta y = f(2.05) - f(2) = 0.71765$,

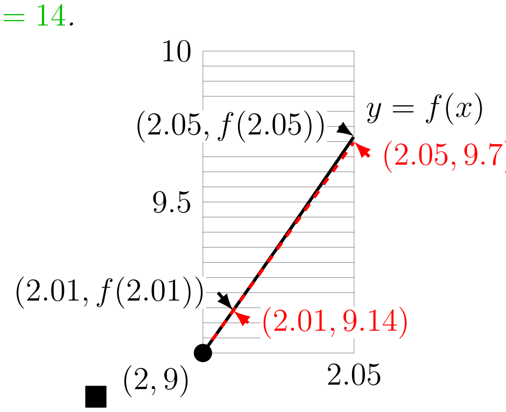
$dy = f'(2)\ dx = 14 \cdot 0.05 = 0.7$.

(b) $f(2.01) = 9.140701$,

$dx = \Delta x = 2.01 - 2 = 0.01$,

$\Delta y = f(2.01) - f(2) = 0.140701$,

$dy = f'(2)\ dx = 14 \cdot 0.01 = 0.14$. ■



Application: 用微分 (dy) 來估計誤差 (Δy): $\Delta y \approx dy$.

Example 0.4 *The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?*

$V = V(r) = \frac{4}{3}\pi r^3$. *(球體積公式)*
$\Delta V \approx dV = 4\pi r^2\ dr = 4\pi(21)^2(0.05) \approx 277.$

And: *The maximum error is about* $277\ cm^3$. ■

Errors 誤差: $y = f(x)$ at $x = a$,
the ***maximum error*** 最大誤差 is $\Delta y (\approx dy)$,
the ***relative error*** 相對誤差 is $\frac{\Delta y}{y} (\approx \frac{dy}{y})$ which can be expressed as
the ***percentage error*** 百分誤差 $\frac{\Delta y}{y} \times 100\%$.

Example 0.5 *(Continuous) relative error in* V *and* r.

$$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2\ dr}{\frac{4}{3}\pi r^3} = 3\frac{dr}{r}.$$

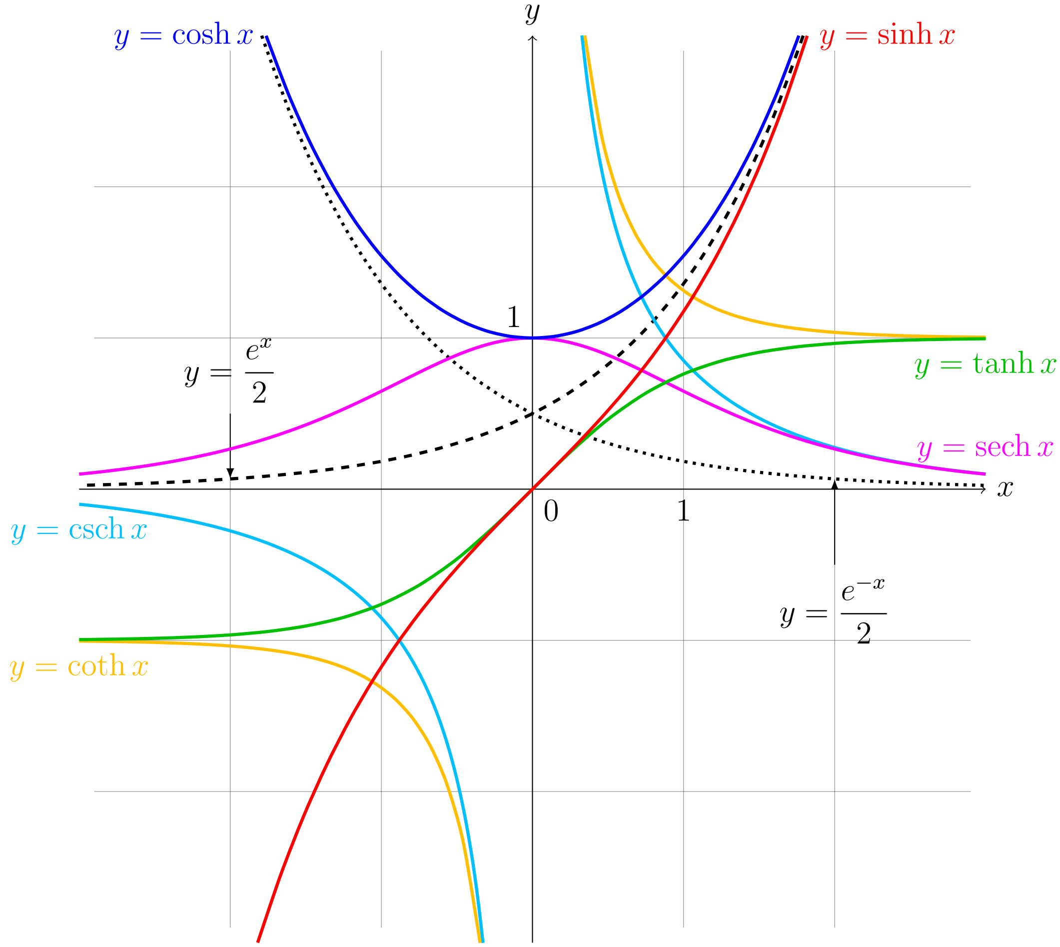
(The relative error in V *is about 3 times the one in* r.*)*

$$\frac{dr}{r} = \frac{0.05}{21} \approx 0.0024, \text{ and hence } \frac{dV}{V} \approx 3 \times 0.0024 \approx 0.007.$$

Ans: *The percentage errors are* 0.24% *in radius and* 0.7% *in volume.* ■

⧫ 3.11 Hyperbolic functions (optional)

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x},$$
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \coth x = \frac{\cosh x}{\sinh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}.$$



⧫: Euler's formula: $e^{ix} = \cos x - i \sin x$, $i = \sqrt{-1}$.

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$
$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

- **Identity:**
  $\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$.
  $\cosh^2 x - \sinh^2 x = 1$, $1 - \tanh^2 x = \operatorname{sech}^2 x$, $\coth^2 x - 1 = \operatorname{csch}^2 x$,
  $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$,
  $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

- **Derivative:**
  $(\sinh x)' = \cosh x$;
  $(\cosh x)' = \sinh x$;
  $(\tanh x)' = \operatorname{sech}^2 x$;
  $(\coth x)' = -\operatorname{csch}^2 x$;
  $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$;
  $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$.

- **Antiderivative:**
  $\int \sinh x \; dx = \cosh x + C$;
  $\int \cosh x \; dx = \sinh x + C$;
  $\int \tanh x \; dx = -\ln|\operatorname{sech} x| + C$;
  $\int \coth x \; dx = -\ln|\operatorname{csch} x| + C$;
  $\int \operatorname{sech} x \; dx = \tan^{-1}(\sinh x) + C$;
  $\int \operatorname{csch} x \; dx = \ln|\coth x - \operatorname{csch} x| + C$.

- **Inverse:**
  $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, $x \in \mathbb{R}$;
  $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$, $x \geq 1$ (limited);
  $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$, $-1 < x < 1$;
  $\coth^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{x-1}{x+1}\right)$, $|x| > 1$;
  $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$, $0 < x \leq 1$ (limited);
  $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$, $x \neq 0$.

- **Derivative of inverse:**
  $(\sinh^{-1} x)' = \dfrac{1}{\sqrt{x^2+1}}$;
  $(\cosh^{-1} x)' = \dfrac{1}{\sqrt{x^2-1}}$;
  $(\tanh^{-1} x)' = \dfrac{1}{1-x^2}$;
  $(\coth^{-1} x)' = -\dfrac{1}{x^2-1}$;
  $(\operatorname{sech}^{-1} x)' = -\dfrac{1}{x\sqrt{1-x^2}}$;
  $(\operatorname{csch}^{-1} x)' = -\dfrac{1}{|x|\sqrt{x^2+1}}$.