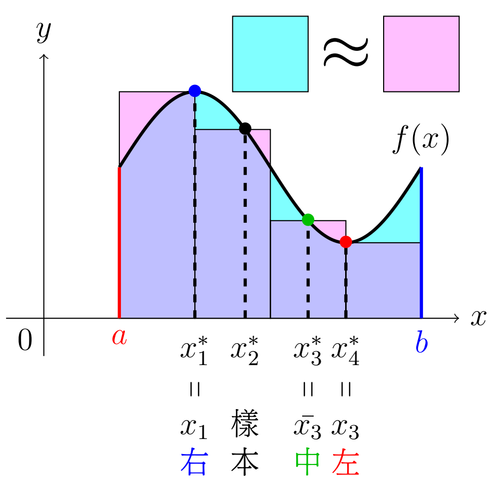
7.7 Approximate integration

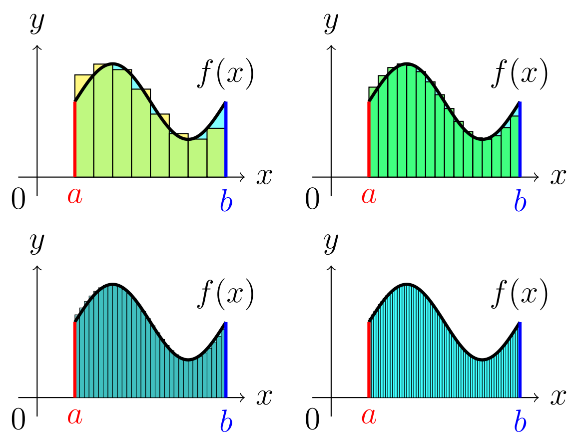
1. Right endpoint rule 右端法 R_n
2. Left endpoint rule 左端法 L_n
3. Trapezoidal rule 梯形法 T_n
4. Midpoint rule 中點法 M_n
5. Simpson's rule 辛普森法 S_{2n}
6. Error bounds 誤差

Ex: $\displaystyle\int_0^1 e^{x^2}\,dx,\ \int_{-1}^1 \sqrt{x^3+1}\,dx$: 求不出來.
Ex: 有時候只是測量所得, 不見得是個函數.

Idea: 用黎曼和 (Riemann sum) 求近似值.
Recall: $f(x)$ is integrable on $[a,b]$,
sample points $x_i^* \in [x_{i-1}, x_i]$, $x_i = a + i\Delta x$, $i = 1, \ldots, n$, $\Delta x = \dfrac{b-a}{n}$.

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x \;\approx\; \sum_{i=1}^n f(x_i^*)\Delta x.$$





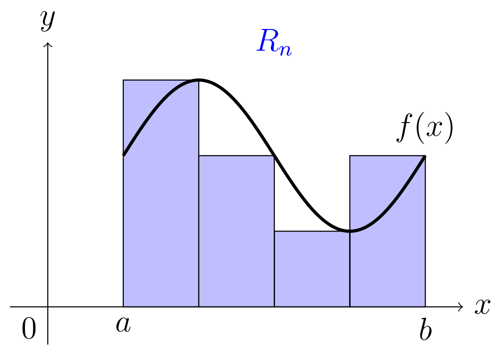
0.1 Right/Left endpoint rule

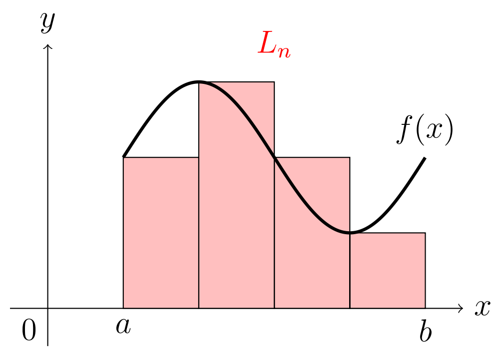
$$\int_a^b f(x)\,dx \approx \boxed{\color{blue}{\boldsymbol{R_n}}} \quad (\text{\color{blue}{右}端點})$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\int_a^b f(x)\,dx \approx \boxed{\color{red}{\boldsymbol{L_n}}} \quad (\text{\color{red}{左}端點})$$

$$= \sum_{i=1}^{n} f(x_{i-1})\Delta x$$





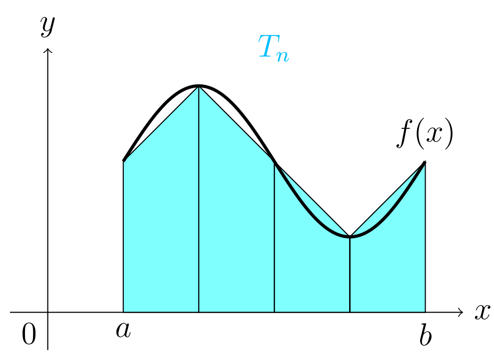
0.2 Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \boxed{\color{cyan}{\boldsymbol{T_n}}} \quad (\text{\color{cyan}{梯}形})$$

$$= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Note: 係數是: 1,2,2,...,2,1.

$$\boxed{\color{cyan}{\boldsymbol{T_n}} = \frac{\color{blue}{\boldsymbol{R_n}} + \color{red}{\boldsymbol{L_n}}}{\mathbf{2}}} (\text{\color{cyan}{梯}形} = \text{\color{red}{左右}端平均})$$

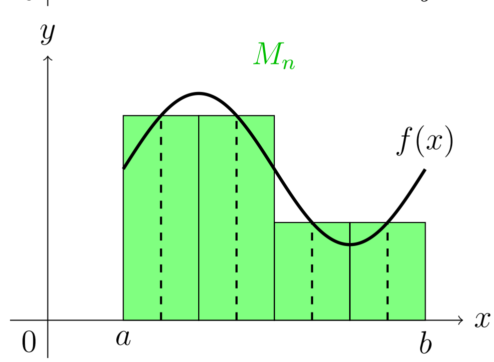


0.3 Midpoint rule

$$\int_a^b f(x)\,dx \approx \boxed{\color{green}{\boldsymbol{M_n}}} \quad (\text{\color{green}{中}點})$$

$$= \sum_{i=1}^{n} f(\bar{x}_i)\Delta x,$$

where $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$.



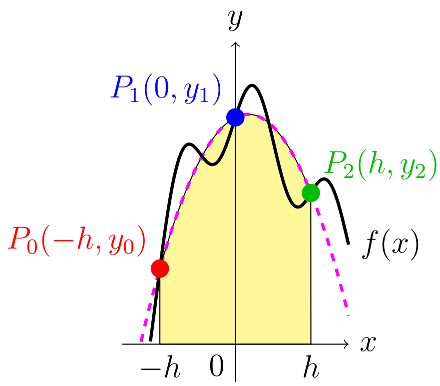
0.4 Simpson's rule

Simpson 考慮偶數 n, 用通過 $(x_{2i-2}, f(x_{2i-2})), (x_{2i-1}, f(x_{2i-1})), (x_{2i}, f(x_{2i}))$ 的抛物線逼近第 $(2i-1)$ 與第 $(2i)$ 段.

(方便計算面積, 把 x_{2i-1} 平移到 0, let $h = \Delta x$.)

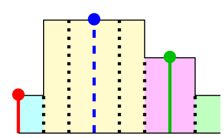
假設抛物線 $y = Ax^2 + Bx + C$ 通過 $P_0(-h, y_0)$, $P_1(0, y_1)$, $P_2(h, y_2)$,

$$\Longrightarrow \begin{cases} y_0 = Ah^2 - Bh + C, \\ y_1 = C, \\ y_2 = Ah^2 + Bh + C. \end{cases}$$



$$\int_{-h}^{h} (Ax^2 + Bx + C)\ dx$$
$$= 2\int_{0}^{h} (Ax^2 + C)\ dx$$
$$= \frac{h}{3}(2Ah^2 + 6C)$$
$$= \frac{h}{3}(y_0 + 4y_1 + y_2),$$

$$\begin{aligned} \int_a^b f(x)\ dx \ &\approx\ \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) \\ &\quad + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &=\ \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \end{aligned}$$

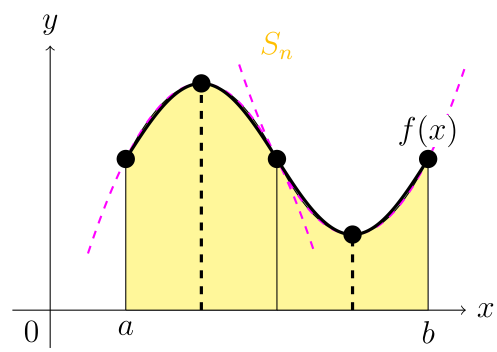


Simpson's Rule

$$\int_a^b f(x)\ dx \ \approx\ \boxed{S_n}$$
$$= \frac{\Delta x}{3}\left[\begin{array}{lll} f(x_0) & +4f(x_1) & +2f(x_2) \\ +4f(x_3) & +2f(x_4) & +\cdots \\ +2f(x_{n-2}) & +4f(x_{n-1}) & +f(x_n) \end{array}\right],$$

where n is even.

Note: 係數是: 1,4,2,4,2,...,2,4,1.



$$\boxed{S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n}$$ (辛普森 $= \frac{1}{3}$梯形 $+ \frac{2}{3}$中點, 注意下標不同.)

0.5 Error bounds

誤差 (error) 就是: 真正的數值減去逼近的數值.
(>0 低估 (under-estimate), <0 高估 (over-estimate).)

$$E_T = \int_a^b f(x)\,dx - T_n,\quad E_M = \int_a^b f(x)\,dx - M_n,\quad E_S = \int_a^b f(x)\,dx - S_n.$$

Theorem 1 *If $|f''(x)| \le K$ on $[a,b]$, then*

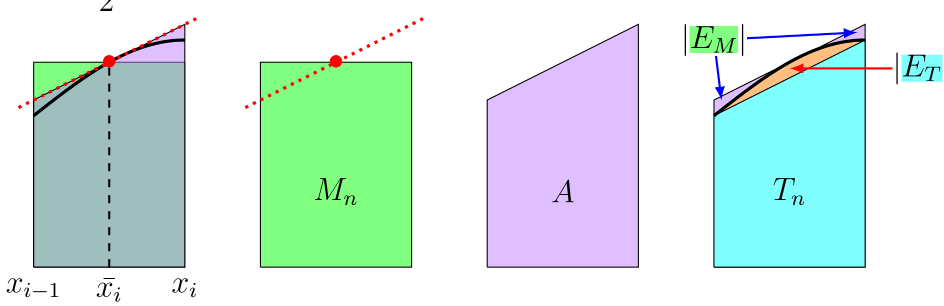
$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \textit{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Theorem 2 *If $|f^{(4)}(x)| \le K$ on $[a,b]$, then*

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

Observation:

1. The larger n, the more accurate approximation. n 越大, 近似值越準.
2. 左右端點法的誤差 $\pm$ 相反 (R_n 多算 $\iff$ L_n 少算, 反之亦然);
當 n 加倍, 誤差剩 $\frac{1}{2}$.
3. T_n & M_n 比 R_n & L_n 精確.
4. T_n & M_n 的誤差 $\pm$ 相反 (T_n 多算 $\iff$ M_n 少算, 反之亦然);
當 n 加倍, 誤差剩 $\frac{1}{4}(=\frac{1}{2^2})$.
5. $|E_M| \approx \frac{1}{2}|E_T|$, 中點比梯形準 (誤差小) 一倍.



Note: $M_n = A$, E_T =最右圖中的橙色>0, $E_M = -$最右圖中的紫色<0;
所以差負號 ($\pm$ 相反), 而且紫色面積約橙色的一半 (數值一半).

6. S_n 比 T_n & M_n 精確 ($\because S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$ and $E_M \approx -\frac{1}{2}E_T$);
當 n 加倍, 誤差剩 $\frac{1}{16}(=\frac{1}{2^4})$.

Example 0.1 (Example 1+2+4+6) $\displaystyle\int_1^2 \frac{dx}{x}$.

(a) & (b) Use Trapezoidal rule and Midpoint rule with $n = 5$ to approximate.

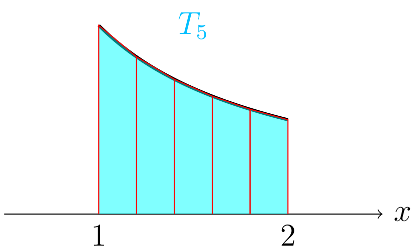
(c) & (d) Their Errors?

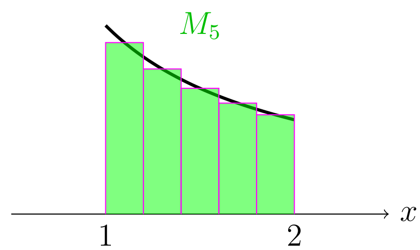
(e) & (f) 多大 n 才會精確到 (accurate to with) 0.0001?

(g) Use Simpson's rule with $n = 10$ to approximate.

(h) 多大 n accurate to with 0.0001?

Proof. $n = 5$, $\Delta x = \dfrac{2-1}{5} = \dfrac{1}{5}$, $x_i = 1 + i\Delta = 1 + \dfrac{i}{5}$.





(a) $T_5 = \dfrac{1}{2}\dfrac{1}{5}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$

$= 0.1\left(1 + \dfrac{2}{1.2} + \dfrac{2}{1.4} + \dfrac{2}{1.6} + \dfrac{2}{1.8} + \dfrac{1}{2}\right) \approx 0.695635.$

(b) $M_5 = \dfrac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$

$= 0.2\left(\dfrac{1}{1.1} + \dfrac{1}{1.3} + \dfrac{1}{1.5} + \dfrac{1}{1.7} + \dfrac{1}{1.9}\right) \approx 0.691905.$

$$\int_1^2 \frac{dx}{x} = \ln x\Big]_1^2 = \ln 2 \approx 0.693147.$$

(c) $E_T = \ln 2 - T_5 \approx -0.002488$.

(d) $E_M = \ln 2 - M_5 \approx 0.001239$.

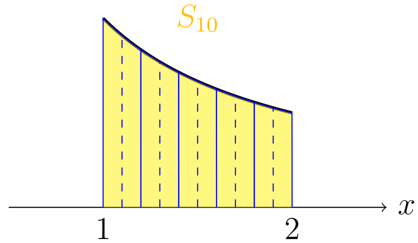
$$|f''(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{1^3} = 2 = K \text{ for } 1 \le x \le 2.$$

(e) $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le \dfrac{2(2-1)^3}{12n^2} < 0.0001$, $n > \dfrac{1}{\sqrt{0.0006}} \approx 40.8$, $n = 41$.

(f) $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le \dfrac{2(2-1)^3}{24n^2} < 0.0001$, $n > \dfrac{1}{\sqrt{0.0012}} \approx 28.9$, $n = 29$.

(Continue)

$n = 10,\ \Delta x = \dfrac{2-1}{10} = \dfrac{1}{10},\ x_i = 1 + i\Delta = 1 + \dfrac{i}{10}.$



(g) $S_{10} = \dfrac{1}{3}\dfrac{1}{10}[f(1)+4f(1.1)+2f(1.2)+4f(1.3)+2f(1.4)+4f(1.5)+2f(1.6)+4f(1.7)+2f(1.8)+4f(1.9)+f(2)] = \ldots = \dfrac{1}{3}T_5 + \dfrac{2}{3}M_5 \approx 0.693150.$

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \le \frac{24}{1^5} = 24 = K.$$

(h) $|E_S| \le \dfrac{K(b-a)^5}{180n^4} \le \dfrac{24(2-1)^5}{180n^4} < 0.0001,\ n > \dfrac{1}{\sqrt[4]{0.00075}} \approx 6.04,\ n = 8.$

(Simpson's rule 要偶數*)* ■

Observation: 相同誤差 0.0001, 梯形 $n \ge 41$, 中點 $n \ge 29$, 辛普森 $n \ge 8$.

Additional: 估計法還有很多, 但是要在計算複雜度與精準度上做選擇.

估計法 approximation	R_n/L_n	T_n	M_n	S_n
複雜度 complexity	small	$<$	$<$	large
精準度 accuracy	rough	$>$	$>$	fine
誤差正比 error $\propto$	$1/n$	$1/n^2$	$1/n^2$	$1/n^4$

(Try yourself)

(Exercise 7.7.49) $\dfrac{1}{2}(T_n + M_n) = T_{2n}$. (Exercise 7.7.50) $\dfrac{1}{3}T_n + \dfrac{2}{3}M_n = S_{2n}$.

Hint:

$$\begin{aligned}
T_n &= \frac{b-a}{2\cdot n}(y_0 + 2y_2 + 2y_4 + \cdots + 2y_{2n-2} + y_{2n}),\\
M_n &= \frac{b-a}{n}(y_1 + y_3 + \cdots + y_{2n-1}),\\
T_{2n} &= \frac{b-a}{2\cdot 2n}(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + \cdots + 2y_{2n-1} + y_{2n}),\\
S_{2n} &= \frac{b-a}{3\cdot 2n}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n}).
\end{aligned}$$