7.2 Trigonometric integrals

1. $\int \sin^m x \cos^n x \ dx$
2. $\int \tan^m x \sec^n x \ dx$
3. $\int \sin mx \cos nx \ dx$, $\int \sin mx \sin nx \ dx$, $\int \cos mx \cos nx \ dx$

三角函數的積分攻略: 換換換$\to$變數變換$\to$分部積分.

0.1 $\displaystyle\int \sin^m x \cos^n x \ dx$

Recall: $\int \sin x \ dx = -\cos x + C$, $\int \cos x \ dx = \sin x + C$.

- **Case a.** $n = 2k+1$ is odd.
  (Let $\boldsymbol{u = \sin x}$, $du = \cos x \ dx$, use $\cos^2 x = 1 - \sin^2 x$.)

$$\begin{aligned}&\int \sin^m x \cos^n x \ dx \\ &= \int \sin^m x \cos^{2k+1} x \ dx \ = \int \sin^m x (\cos^2 x)^k \cdot \cos x \ dx \\ &= \int \sin^m x (1-\sin^2 x)^k \ d\sin x \ = \int u^m (1-u^2)^k \ du.\end{aligned}$$

- **Case b.** $m = 2k+1$ is odd.
  (Let $\boldsymbol{u = \cos x}$, $du = -\sin x \ dx$, use $\sin^2 x = 1 - \cos^2 x$.)

$$\begin{aligned}&\int \sin^m x \cos^n x \ dx \\ &= \int \sin^{2k+1} x \cos^n x \ dx \ = \int (\sin^2 x)^k \cos^n x \cdot \sin x \ dx \\ &= \int (1-\cos^2 x)^k \cos^n x (-\ d\cos x) \ = \int -(1-u^2)^k u^n \ du.\end{aligned}$$

- **Case c.** m and n are even.
  使用 $\cos^2 x = 1 - \sin^2 x$ or $\sin^2 x = 1 - \cos^2 x$ 換成只由 $\sin^2 x$ or $\cos^2 x$ 組成的多項式, 再用 **Half/double angle formula** 半/倍角公式:

$$\boxed{\boldsymbol{\sin^2 x = \frac{1-\cos 2x}{2}}}, \ \boxed{\boldsymbol{\cos^2 x = \frac{1+\cos 2x}{2}}}, \ \sin x \cos x = \frac{\sin 2x}{2}.$$

Example 0.1 $\int \cos^3 x \ dx.$

Let $u = \sin x$, $du = \cos x \ dx$.

$$\int \cos^3 x \ dx = \int \cos^2 x \cdot \cos x \ dx = \int (1 - \sin^2 x) \ d\sin x = \int 1 - u^2 \ du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C. \qquad \blacksquare$$

Example 0.2 $\int \sin^5 x \cos^2 x \ dx.$

Let $u = \cos x$, $du = -\sin x \ dx$, $\sin x \ dx = - \ du$.

$$\int \sin^5 x \cos^2 x \ dx = \int \sin^4 x \cos^2 x \cdot \sin x \ dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x (- \ d\cos x) = \int -u^2 + 2u^4 - u^6 \ du$$

$$= -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C. \qquad \blacksquare$$

Example 0.3 $\int_0^{\pi} \sin^2 x \ dx.$

$$\int \sin^2 x \ dx = \int \frac{1}{2}(1 - \cos 2x) \ dx = \frac{1}{2}\int \ dx - \frac{1}{2}\int \cos 2x \ dx$$

$$= \frac{1}{2}\int \ dx - \frac{1}{2 \times 2}\int \cos 2x \ d(2x) = \frac{x}{2} - \frac{1}{4}\sin 2x + C, \ (u = 2x, \ dx = \frac{1}{2} \ du.)$$

$$\int_0^{\pi} \sin^2 x \ dx = \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}. \qquad \blacksquare$$

Example 0.4 $\int \sin^4 x \ dx.$

$$\int \sin^4 x \ dx = \int (\sin^2 x)^2 \ dx = \int \left[\frac{1}{2}(1 - \cos 2x)\right]^2 \ dx$$

$$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x \ dx = \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}(1 + \cos 4x) \ dx$$

$$= \int \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x \ dx$$

$$= \frac{3}{8}\int \ dx - \frac{1}{2 \times 2}\int \cos 2x \ d(2x) + \frac{1}{8 \times 4}\int \cos 4x \ d(4x)$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \qquad \blacksquare$$

0.2 $\displaystyle\int \tan^m x \sec^n x \ dx$

Recall: $\int \sec^2 x \ dx = \tan x + C$, $\int \sec x \tan x \ dx = \sec x + C$.

- **Case a-1.** $n = 2k \geq 2$ is even.
  (Let $\boldsymbol{u = \tan x}$, $du = \sec^2 x \ dx$, use $\sec^2 x = 1 + \tan^2 x$.)

$$\begin{aligned}
&\int \tan^m x \sec^n x \ dx \\
&= \int \tan^m x \sec^{2k} x \ dx \ = \ \int \tan^m x (\sec^2 x)^{k-1} \cdot \sec^2 x \ dx \\
&= \int \tan^m x (1 + \tan^2 x)^{k-1} \ d\tan x \ = \ \int u^m (1+u^2)^{k-1} \ du.
\end{aligned}$$

- **Case a-2.** $n = 0$, $m = 1$. $\boxed{\displaystyle\int \boldsymbol{\tan x \ dx = \ln|\sec x| + C}}$.

- **Case a-3.** $n = 0$, $m \geq 2$. (Exercise 7.1.53)

$$\begin{aligned}
\int \tan^m x \ dx &= \int \tan^{m-2} x (\sec^2 x - 1) \ dx \\
&= \int \tan^{m-2} x \cdot \sec^2 x \ dx - \int \tan^{m-2} x \ dx \\
&= \int \tan^{m-2} x \ d\tan x - \int \tan^{m-2} x \ dx \\
&= \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \ dx. \quad \text{(降兩次)}
\end{aligned}$$

  Reduction formula, 最後是 $\displaystyle\int \tan x \ dx = \ln|\sec x| + C$ (if m is odd) or $\displaystyle\int \ dx = x + C$ (if m is even).

- **Case b.** $m = 2k+1$ is odd and $n \geq 1$.
  (Let $\boldsymbol{u = \sec x}$, $du = \sec x \tan x \ dx$, use $\tan^2 x = \sec^2 x - 1$.)

$$\begin{aligned}
&\int \tan^m x \sec^n x \ dx \\
&= \int \tan^{2k+1} x \sec^n x \ dx = \int (\tan^2 x)^k \sec^{n-1} x \cdot \sec x \tan x \ dx \\
&= \int (\sec^2 x - 1)^k \sec^{n-1} x \ d\sec x = \int (u^2-1)^k u^{n-1} \ du.
\end{aligned}$$

- **Case c-1.** $m = 2k \geq 2$ is even and n is odd.

$$\begin{aligned}
&\int \tan^m x \sec^n x \; dx \\
&= \int \tan^{2k} x \sec^n x \; dx = \int (\tan^2 x)^k \sec^n x \; dx \\
&= \int (\sec^2 x - 1)^k \sec^n x \; dx = \sum_{i=0}^{\ell} C_i \int \sec^{2i+1} x \; dx \ldots (\text{續})
\end{aligned}$$

(使用 $\tan^2 x = \sec^2 x - 1$ 變成 $\sec^{\text{奇數次}} x$ 的積分.)

- **Case c-2.** $m = 0$ and $n = 1$. $\boxed{\int \sec x \; dx = \ln |\sec x + \tan x| + C}$.

- **Case c-3.** $m = 0$ and $n \geq 3$ is odd.

用分部積分法: $u = \sec^{n-2} x$, $dv = \sec^2 x \; dx$.

$$\begin{aligned}
\int \sec^n x \; dx &= \int \sec^{n-2} x \cdot \sec^2 x \; dx \\
&= \boxed{\int \sec^{n-2} x \; d\tan x = \sec^{n-2} x \tan x - \int \tan x \; d\sec^{n-2} x} \\
&= \sec^{n-2} x \tan x - \int \tan^2 x \,(n-2) \sec^{n-2} x \; dx \\
&= \sec^{n-2} x \tan x - (n-2) \int (\sec^2 x - 1) \sec^{n-2} x \; dx \\
&= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \; dx + (n-2) \int \sec^{n-2} x \; dx,
\end{aligned}$$

$$\begin{aligned}
(n-1) \int \sec^n x \; dx &= \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \; dx, \\
\int \sec^n x \; dx &= \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \; dx.
\end{aligned}$$

Reduction formula, 最後是 $\int \sec x \; dx = \ln |\sec x + \tan x| + C$.

(降兩次: $n \to (n-2) \to \cdots \to 5 \to 3 \to 1$, 用公式.)

Note: $\int \cot^m x \csc^n x \; dx$ 方法類似.

Example 0.5 $\displaystyle\int \tan^6 x \sec^4 x \; dx$.

Let $u = \tan x$, $du = \sec^2 x \; dx$.

$$\int \tan^6 x \sec^4 x \; dx = \int \tan^6 x \sec^2 x \cdot \sec^2 x \; dx$$
$$= \int \tan^6 x (1 + \tan^2 x) \; d\tan x = \int u^6 + u^8 \; du$$
$$= \frac{u^7}{7} + \frac{u^9}{9} + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C. \qquad \blacksquare$$

Example 0.6 $\displaystyle\int \tan^5 x \sec^7 x \; dx$.

Let $u = \sec x$, $du = \sec x \tan x \; dx$.

$$\int \tan^5 x \sec^7 x \; dx = \int \tan^4 x \sec^6 x \cdot \sec x \tan x \; dx$$
$$= \int (\sec^2 x - 1)^2 \sec^6 x \; d\sec x = \int u^{10} - 2u^8 + u^6 \; du$$
$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C = \frac{1}{11}\sec^{11} x - \frac{2}{9}\sec^9 x + \frac{1}{7}\sec^7 x + C. \qquad \blacksquare$$

Example 0.7 $\displaystyle\int \tan^3 x \; dx$.

$$\begin{aligned}
\int \tan^3 x \; dx &= \int \tan x \tan^2 x \; dx \\
&= \int \tan x (\sec^2 x - 1) \; dx \\
&= \int \tan x \cdot \sec^2 x \; dx - \int \tan x \; dx \\
&= \int \tan x \; d\tan x - \int \tan x \; dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.
\end{aligned} \qquad \blacksquare$$

Note: $\displaystyle\int \tan^3 x \; dx = \frac{1}{2}\sec^2 x - \ln|\sec x| + C$ 也對.

$$\because \int \tan x \sec^2 x \; dx = \int \sec x \cdot \tan x \sec x \; dx = \int \sec x \; d\sec x$$
$= \dfrac{1}{2}\sec^2 x + C = \dfrac{1}{2}\tan^2 x + \dfrac{1}{2} + C$. (常數通通被 C(任意常數) 吸收.)

Example 0.8 (♡考) $\int \sec^3 x\ dx$.

Let $u = \sec x$ *and* $dv = \sec^2 x\ dx$*, then* $du = \sec x \tan x$ *and* $v = \tan x$.

$$\begin{aligned}
\int \sec^3 x\ dx &= \int \sec x \cdot \sec^2 x\ dx \\
&= \boxed{\int \sec x\ d\tan x \ = \ \sec x \tan x - \int \tan x\ d\sec x} \\
&= \sec x \tan x - \int \tan x \sec x \tan x\ dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\ dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x\ dx \\
&= \sec x \tan x - \int \sec^3 x\ dx + \int \sec x\ dx \\
&= \sec x \tan x - \int \sec^3 x\ dx + \ln|\sec x + \tan x|,
\end{aligned}$$

$$\begin{aligned}
2\int \sec^3 x\ dx &= \sec x \tan x + \ln|\sec x + \tan x|, \\
\int \sec^3 x\ dx &= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.
\end{aligned}$$

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加入你的不定積分表:

$$\boxed{\int \sec^3 x\ dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C}.$$

(背身體健康, 背萬事如意. 有背有保庇, 沒背要會積.)

Question: 記不住策略怎麼辦?

Answer:

1. 三角恆等式換換換, $\sin^{偶數次} x$ 或 $\cos^{偶數次} x$ 要用倍角.
2. 變數變換變變變: $\begin{cases} \sin^m x \cos^n x, & 猜\ u = \sin x\ 或\ \cos x; \\ \tan^m x \sec^n x, & 猜\ u = \tan x\ 或\ \sec x. \end{cases}$
3. 分部積分分分分: $\sec^{奇數次} x$; $\tan x$, $\sec x$, $\sec^3 x$ 的最好背起來.

0.3 $\int \sin mx \cos nx\ dx$, $\int \sin mx \sin nx\ dx$, $\int \cos mx \cos nx\ dx$

Recall: Sum/difference formula 和/差角公式:

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B\\
\sin(A-B) &= \sin A\cos B - \cos A\sin B\\
\cos(A+B) &= \cos A\cos B - \sin A\sin B\\
\cos(A-B) &= \cos A\cos B + \sin A\sin B
\end{aligned}$$

Product to sum formula 積化和差:

$$\begin{aligned}
\sin A\cos B &= \frac{1}{2}[\sin(A-B)+\sin(A+B)]\\
\sin A\sin B &= \frac{1}{2}[\cos(A-B)-\cos(A+B)]\\
\cos A\cos B &= \frac{1}{2}[\cos(A-B)+\cos(A+B)]
\end{aligned}$$

Example 0.9 $\int \sin 4x \cos 5x\ dx.$

$$\begin{aligned}
\int \sin 4x\cos 5x\ dx &= \int \frac{1}{2}[\sin(4x-5x)+\sin(4x+5x)]\ dx\\
&= \int \frac{1}{2}[\sin(-x)+\sin 9x]\ dx\\
&= \frac{1}{2}\int(-\sin x+\sin 9x)\ dx\\
&= \frac{1}{2}\int -\sin x\ dx + \frac{1}{2}\int \frac{1}{9}\sin 9x\ d(9x)\\
&= \frac{1}{2}\cos x - \frac{1}{18}\cos 9x + C.
\end{aligned}$$

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♦: ***Fourier series*** 傅立葉級數:

$$f(x)=\sum_{i=1}^{\infty} a_n \sin nx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\ dx.$$