5.4 Indefinite integral and net change theorem

1. indefinite integral 不定積分 $\int f(x)\ dx$
2. net change theorem 淨變化定理 $\int v(t)\ dt$ v.s. $\int |v(t)|\ dt$

0.1 Indefinite integral

TFTC (2) 告訴我們積分與反導數的關係: 積 a 到 b = 反導代 b 減 a.
我們叫這個“f的反導數”爲“f的不定積分”.

Define: The ***indefinite integral*** 不定積分 of a continuous f (on its domain)

$$\boxed{\int \boldsymbol{f(x)\ dx = F(x)} \iff \boldsymbol{F'(x) = f(x)}}$$

Note: 定積分 $\displaystyle\int_a^b f(x)\ dx$ 是一個數字 (極限值 $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$);
不定積分 $\displaystyle\int f(x)\ dx$ 是一個函數, 寫成“函數+C”的形式, 也就是 f 最一般的反導數.

定積分與不定積分的關係: (TFTC)

$$\boxed{\int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f(x)\ dx} = \left[\int \boldsymbol{f(x)\ dx}\right]_{\boldsymbol{a}}^{\boldsymbol{b}}}$$

Example 0.1

$$\int x^2\ dx = \frac{x^3}{3} + C, \quad \int \sec^2 x\ dx = \tan x + C, \quad \int \frac{1}{x^2}\ dx = -\frac{1}{x} + C.$$

Note: 雖然用 $F(x)+C$ 來代表 $\displaystyle\int f(x)\ dx$, 但是 C 只在同一個區間有效;
在不同區間要用不同的常數.

$$\int \frac{1}{x^2}\ dx = -\frac{1}{x} + C = \begin{cases} -\dfrac{1}{x} + C_1 & \text{if } x > 0 \\ -\dfrac{1}{x} + C_2 & \text{if } x < 0 \end{cases}$$

Table 1: Table of indefinite integrals:

$\int cf(x)\ dx = c\int f(x)\ dx$	常數倍
$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$	加減
$\int c\ dx = cx + C$	常數
$\int x^n\ dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	(逆 power rule)
$\int \dfrac{1}{x}\ dx = \ln\lvert x\rvert + C$	(注意絕對值)
$\int e^x\ dx = e^x + C$	指數
$\int a^x\ dx = \dfrac{a^x}{\ln a} + C$	
$\int \sin x\ dx = -\cos x + C$	三角
$\int \cos x\ dx = \sin x + C$	
$\int \sec^2 x\ dx = \tan x + C$	
$\int \csc^2 x\ dx = -\cot x + C$	
$\int \sec x \tan x\ dx = \sec x + C$	
$\int \csc x \cot x\ dx = -\csc x + C$	
$\int \dfrac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C$	反三角
$\int \dfrac{1}{1+x^2}\ dx = \tan^{-1} x + C$	
$\int \dfrac{1}{x\sqrt{x^2-1}}\ dx = \sec^{-1} x + C$	
$\int \sinh x\ dx = \cosh x + C$	雙曲三角
$\int \cosh x\ dx = \sinh x + C$	

Example 0.2 $\int (10x^4 - 2\sec^2 x)\ dx = ?$

$$\int (10\,x^4 - 2\sec^2 x)\ dx = 10\int x^4\ dx - 2\int \sec^2 x\ dx$$

$= 10\,\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$. *(共用一個 C 就好)* ■

Example 0.3 $\int \frac{\cos\theta}{\sin^2\theta}\ d\theta = ?$

$$\int \frac{\cos\theta}{\sin^2\theta}\ d\theta = \int \frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}\ d\theta = \int \csc\theta\cot\theta\ d\theta = -\csc\theta + C.$$ ■

(不太容易想到, 也可以用變數變換 §5.5.)

Example 0.4 *(Recall §5.2 Ex 0.2)* $\int_0^3 (x^3 - 6x)\ dx = ?$

$$\int (x^3 - 6x)\ dx = \frac{x^4}{4} - 3x^2 + C, \text{ (找不定積分)}$$

$\int_0^3 (x^3 - 6x)\ dx = \frac{x^4}{4} - 3x^2\Big]_0^3 = \left[\frac{3^4}{4} - 3(3)^2\right] - \left[\frac{0^4}{4} - 3(0)^2\right]$

$= \frac{81}{4} - 27 - 0 + 0 = -6.75.$ *(用 TFTC (2) 比用 $\lim\sum$ 好算多了)* ■

Example 0.5 $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx = ?$

$$\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)\ dx = \frac{x^4}{2} - 3x^2 + 3\tan^{-1} x\Big]_0^2$$

$= \left[\frac{2^4}{2} - 3(2)^2 + 3\tan^{-1} 2\right] - \left[\frac{0^4}{2} - 3(0)^2 + 3\tan^{-1} 0\right]$

$= 8 - 12 + 3\tan^{-1} 2 - 0 + 0 - 0 = -4 + 3\tan^{-1} 2 (\approx -0.67855).$ ■

Example 0.6 $\int_1^9 (2 + t^{1/2} - t^{-2})\ dt = ?$

$$\int_1^9 (2 + t^{1/2} - t^{-2})\ dt = 2t + \frac{2}{3}t^{3/2} + t^{-1}\Big]_1^9$$

$= \left[2(9) + \frac{2}{3}(9)^{3/2} + 9^{-1}\right] - \left[2(1) + \frac{2}{3}(1)^{3/2} + 1^{-1}\right]$

$= 18 + 18 + \frac{1}{9} - 2 - \frac{2}{3} - 1 = 32\frac{4}{9}.$ ■

Skill: 怎麼檢查反導數找得對不對? 微分!

0.2 Net Change Theorem

Recall: f is continuous on $[a,b]$, $\int_a^b f(x)\ dx = F(b) - F(a)$, $F'(x) = f(x)$.

F 是 f 的反導數, 代入得到: $\int_a^b F'(x)\ dx = F(b) - F(a)$.

$F'(x)$ 代表 rate of change 改變率, $F(b) - F(a)$ 代表 net change 淨改變.

Theorem 1 (Net Change Theorem 淨改變定理)
The integral of a rate of change is the net change:

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Example 0.7 $V(t)$ 是水量, $V'(t)$ 是注水率.

$$\int_{t_1}^{t_2} V'(t)\ dt = V(t_2) - V(t_1).$$

Example 0.8 $n(t)$ 是人口, $n'(t)$ 是人口成長率.

$$\int_{t_1}^{t_2} n'(t)\ dt = n(t_2) - n(t_1).$$

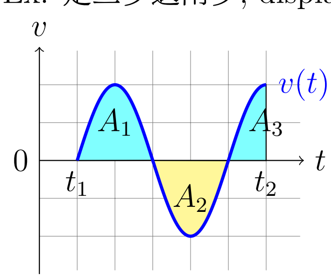
Example 0.9 $s(t)$ 是位置, $s'(t) = v(t)$ 是速率.

$$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1).$$

$\int_{t_1}^{t_2} v(t)\ dt = s(t_2) - s(t_1)$: the net change of position = displacement.

$\int_{t_1}^{t_2} |v(t)|\ dt$: the total distance traveled. (加上絕對值)

Ex: 走三步退兩步, displacement: $+3-2=1$, distance: $+3+2=5$.



$$\int_{t_1}^{t_2} v(t)\ dt = A_1 - A_2 + A_3$$

$$\int_{t_1}^{t_2} |v(t)|\ dt = A_1 + A_2 + A_3$$

Example 0.10 *A particle 粒子 moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).*
(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during this time period.

(a) $\displaystyle\int_1^4 v(t)\,dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4$

$$= \left[\frac{(4)^3}{3} - \frac{(4)^2}{2} - 6(4)\right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} - 6(1)\right] = -\frac{9}{2} = -4.5.$$

(b) (去掉絕對值, 要先知道 $v(t)$ 甚麼時候是正的/負的.)
$t^2 - t - 6 = (t-3)(t+2)$, $v(t) > 0$ *when* $t > 3$ *or* $t < -2$, *and* $v(t) < 0$ *when* $-2 < t < 3$, $[1, 4]$ 分成 $[1, 3] \cup [3, 4]$ 兩段.

分段與去絕對值同時完成

$$\int_1^4 |v(t)|\,dt = \int_1^3 -v(t)\,dt + \int_3^4 v(t)\,dt$$

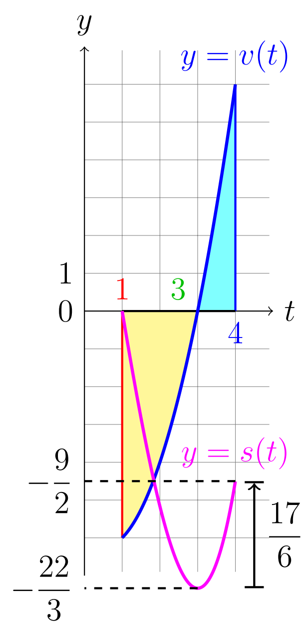
反導數同樣只差一個負號

$$= \left[-\left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4$$

$$= \left\{\left[-\left(\frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3)\right)\right] - \left[-\left(\frac{(1)^3}{3} - \frac{(1)^2}{2} - 6(1)\right)\right]\right\}$$

$$+\left\{\left[\frac{(4)^3}{3} - \frac{(4)^2}{2} - 6(4)\right] - \left[\frac{(3)^3}{3} - \frac{(3)^2}{2} - 6(3)\right]\right\}$$

$$= \frac{22}{3} + \frac{17}{6} = \frac{61}{6} \ (\approx\ 10.17).$$



Ans: Displacement 位移 -4.5 *m; traveled distance* 旅行距離 $\frac{61}{6}$ *m.* ■

胡適: 大膽假設, 小心求證。— 大膽列式, 小心計算。