7.4 Integration of rational functions by partial fractions

變數變換之 — 部份分式法

Type 理解: 有理函數 $\dfrac{P(x)}{Q(x)}$ 的積分.
Idea 分解: 分成會積的分式 (proper fraction) 相加, 使用公式個別積分.
Formula 再構成:

$$\boxed{\int \frac{dx}{x-a} = \ln|x-a| + C}$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C}$$

$$\boxed{\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C}$$

$$\boxed{\int \frac{dx}{x^n} = \frac{-1}{(n-1)x^{n-1}} + C}$$

Additional 1.: 代數基本定理 (**TFTA**): n 次多項式有 n 個根 (in $\mathbb{C}$).
因此可以因式分解(polynomial factorization) 成一次式 $(x-a)$ 或 (irreducible 無法再化簡的)二次式 (x^2+bx+c, $b^2-4c<0$, or $(x-b)^2+c^2$) 的乘積:

$$p(x) = K\prod_{i=1}^{r}(x-a_i)^{d_i}\prod_{j=1}^{s}[(x-b_j)^2+c_j^2]^{e_j},$$

where $K, a_i, b_j, c_j \in \mathbb{R}, d_i, e_j \in \mathbb{N}\cup\{0\}$, $\displaystyle\sum_{i=1}^{r} d_i + \sum_{j=1}^{s} e_j = n$.

Note: $\prod$ 是乘積符號 (product notation), 用法與 $\sum$ 一樣.

Additional 2.: 整係數多項式 ($\mathbb{Z}[x]$) 的因式分解技巧: 一次因式檢驗法.

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{Z}, \quad a_n \neq 0.$$

考慮所有滿足 $k \mid a_n$(最高次係數) 與 $\ell \mid a_0$(常數項) 的 $kx-\ell$.
Ex: $a_n = 2$, $a_0 = 4$, $\implies$ $2x\pm1$, $x\pm1$, $x\pm2$, $x\pm4$.
♦: 牛頓 (有理根) 定理: ℓ/k 是 $p(x)\in\mathbb{Z}[x]$ 的有理根 $\implies$ $k \mid a_n$ & $\ell \mid a_0$.

Partial Fractions Method 部分分式法: $\displaystyle\int \frac{P(x)}{Q(x)}\,dx.$

Step 1. If $\deg(P) < \deg(Q)$: ***proper*** 真分式, let $R(x) = P(x)$ & goto **Step 2**.

If $\deg(P) \geq \deg(Q)$: ***improper*** 假分式, $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$,

用$\boxed{\text{長除法}}$(long division) 求商式 $S(x)$: 用 power rule 的積分公式;

而餘式 $R(Q)$: $\dfrac{R(x)}{Q(x)}$ is proper, goto **Step 2**.

Step 2. $\boxed{\text{因式分解}}$ $Q(x)$ 成一次式與 (irreducible)二次式的乘積: (當首係數是 1.)

$$Q(x) = \prod_{i=1}^{r}(x-a_i)^{d_i}\prod_{j=1}^{s}[(x-b_j)^2+c_j^2]^{e_j}.$$

$\boxed{\text{假設未知數}}$ A_{i_k}'s, B_{j_ℓ}'s, C_{j_ℓ}'s 滿足: (每項都是真分式)

$$\begin{aligned}\frac{R(x)}{Q(x)} &= \sum_{i=1}^{r}\left[\frac{A_{i_1}}{x-a_i}+\frac{A_{i_2}}{(x-a_i)^2}+\cdots+\frac{A_{i_{d_i}}}{(x-a_i)^{d_i}}\right]\\ &\quad+\sum_{j=1}^{s}\left[\frac{B_{j_1}x+C_{j_1}}{(x-b_j)^2+c_j^2}+\frac{B_{j_2}x+C_{j_2}}{[(x-b_j)^2+c_j^2]^2}+\cdots+\frac{B_{j_{e_j}}x+C_{j_{e_j}}}{[(x-b_j)^2+c_j^2]^{e_j}}\right],\end{aligned}$$

$\boxed{\text{通分}}$右式(只看分子), 與 $R(x)$ $\boxed{\text{比較}}$, 得到 A_{i_k}'s, B_{j_ℓ}'s, C_{j_ℓ}'s 的聯立方程組 (假設相同 x 冪次的係數相同, 方程式與未知數的個數一定一樣多), $\boxed{\text{解聯立方程組}}$.

Step 3. 每項各自積分, 利用$\boxed{\text{變數變換}}$以及

a. (一次式) $\displaystyle\int \frac{dx}{x-a} = \ln|x-a| + C.$ (Let $u = x - a$)

b. (二次式) $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$ (Let $x = a\tan\theta$)

c. (二次式) $\displaystyle\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2) + C.$ (Let $u = x^2 + a^2$)

d. (冪次律) $\displaystyle\int \frac{1}{x^n}\,dx = \frac{-1}{(n-1)x^{n-1}} + C,\ n \geq 1.$

Example 0.1 *(Improper, 一個一次式)* $\displaystyle\int \frac{x^3+x}{x-1}\,dx.$

用長除法:

$$
\begin{array}{rlllll}
 & x^2 & +x & +2 & & \text{(消去最高次項)} \\
\cline{2-5}
x-1) & x^3 & +0 & +x & +0 & \text{(缺項要補 0)} \\
-) & x^3 & -x^2 & & & \\
\cline{2-3}
 & & x^2 & +x & & \\
 & -) & x^2 & -x & & \\
\cline{3-4}
 & & & 2x & +0 & \\
 & & -) & 2x & -2 & \\
\cline{4-5}
 & & & & 2 & \text{(次數不夠就停)}
\end{array}
$$

$$
\begin{aligned}
\int \overset{\frac{P(x)}{Q(x)}}{\frac{x^3+x}{x-1}}\,dx &= \int \Big(\overset{S(x)}{x^2+x+2} + \overset{\frac{R(x)}{Q(x)}}{\frac{2}{\underline{x-1}}}\Big)\,dx \\
&= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\,\underline{\ln|x-1|} + C.
\end{aligned}
$$

$(\displaystyle\int \frac{dx}{x-1} = \ln|x-1| + C)$ ■

Example 0.2 *(多個一次式)* $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

$$
\begin{aligned}
2x^3+3x^2-2x &= x(2x-1)(x+2), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(分母因式分解)} \\
Assume\ \frac{x^2+2x-1}{2x^3+3x^2-2x} &= \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}, \quad \ldots\ldots\ldots\ldots \text{(假設未知數)} \\
x^2+2x-1 &= A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1) \ldots\ldots \text{(通分右式)} \\
&= (2A+B+2C)x^2 + (3A+2B-C)x + (-2A), \ \text{(只看分子部分)}
\end{aligned}
$$

(比較係數, 解聯立方程組)

$$
\left\{\begin{array}{lrcl}
(x^2:) & 2A + B + 2C &=& 1 \\
(x^1:) & 3A + 2B - C &=& 2 \\
(x^0:) & -2A &=& -1
\end{array}\right.
\implies A = \frac{1}{2},\ B = \frac{1}{5},\ C = \frac{-1}{10}.
$$

$$
\begin{aligned}
\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right)dx \\
\text{(注意係數!)} &= \frac{1}{2}\ln|x| + \boxed{\frac{1}{10}}\ln|2x-1| - \frac{1}{10}\ln|x+2| + \underline{K}.
\end{aligned}
$$

(C 用過了改用 K, 每項各自變數變換 $\left\{\begin{array}{ll} u = 2x-1, & du = 2\,dx; \\ v = x+2, & dv = dx. \end{array}\right.$ *)* ■

Example 0.3 $\displaystyle\int \frac{1}{x^2-a^2}\,dx$, *where* $a \neq 0$.

$x^2-a^2=(x-a)(x+a)$.

Assume $\dfrac{1}{x^2-a^2}=\dfrac{A}{x-a}+\dfrac{B}{x+a}$,

$1=A(x+a)+B(x-a)$ .... (*)

$\quad =(A+B)x+(A-B)a$,

$$\begin{cases} A+B=0 \\ A-B=1/a \end{cases} \implies A=\frac{1}{2a},\ B=-\frac{1}{2a}. \qquad (x\text{ 缺項, 1 當成 } 0x+1.)$$

$$\begin{aligned}\int \frac{1}{x^2-a^2}\,dx &= \int \left(\frac{1}{2a}\frac{1}{x-a}-\frac{1}{2a}\frac{1}{x+a}\right)dx\\ &= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C\\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C.\end{aligned}$$

■

Skill: 解未知數技巧: 不要乘開, x 代入使某項變零的值.

$$\begin{array}{lrcll} \text{Ex: } (*) & 1 &=& A(x+a)+B(x-a) & \\ (\text{代入 } x=a) & 1 &=& A(a+a)+B(a-a) & = 2aA, \\ & \implies A &=& \dfrac{1}{2a}; & \\ (\text{代入 } x=-a) & 1 &=& A(-a+a)+B(-a-a) & = -2aB, \\ & \implies B &=& -\dfrac{1}{2a}. & \end{array}$$

Additional: (不好背, 用部分分式直接推)

$$\boxed{\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C} \overset{\blacklozenge}{=} \left\{\begin{matrix} -\dfrac{1}{a}\tanh^{-1}\dfrac{x}{a}, & \text{for } |x|<a \\ -\dfrac{1}{a}\coth^{-1}\dfrac{x}{a}, & \text{for } |x|>a \end{matrix}\right\}+C.$$

Example 0.4 *(重複的一次式)* $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx.$

$$\begin{array}{rrrrrr} & x & +1 & & & \\ \hline x^3-x^2-x+1) & x^4 & +0 & -2x^2 & +4x & +1 \\ -) & x^4 & -x^3 & -x^2 & +x & \\ \hline & & x^3 & -x^2 & +3x & +1 \\ -) & & x^3 & -x^2 & -x & +1 \\ \hline & & & & 4x & \end{array}$$

$$\boxed{\begin{array}{l}\text{假設分子少分母一次:}\\ \dfrac{Ax+B'}{(x-1)^2}\ \text{(很難積)}\\ =\dfrac{A(x-1)+A+B'}{(x-1)^2}\\ =\dfrac{A}{x-1}+\dfrac{A+B'}{(x-1)^2}\\ =\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}\end{array}}$$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}=x+1+\frac{4x}{x^3-x^2-x+1}.$$

$x^3-x^2-x+1=(x-1)^2(x+1).$

Assume $\dfrac{4x}{x^3-x^2-x+1}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}$, *(Why? ↑)*

$$4x=A(x-1)(x+1)+B(x+1)+C(x-1)^2 \quad\cdots\cdots\cdots\cdots (**)$$
$$=(A+C)x^2+(B-2C)x+(-A+B+C),$$

$$\left\{\begin{array}{rcrcrcl} A & & & + & C & = & 0 \\ & & B & - & 2C & = & 4 \\ -A & + & B & + & C & = & 0 \end{array}\right. \implies A=1,\ B=2,\ C=-1.\ \ (\text{當作 } 0x^2+4x+0.)$$

$$\begin{aligned}\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx &= \int\left(x+1+\frac{1}{x-1}+\boxed{\frac{2}{(x-1)^2}}-\frac{1}{x+1}\right)dx\\ \text{(變數變換冪次律)}\quad &= \frac{x^2}{2}+x+\ln|x-1|\boxed{-\frac{2}{x-1}}-\ln|x+1|+K\\ &= \frac{x^2}{2}+x-\frac{2}{x-1}+\ln\left|\frac{x-1}{x+1}\right|+K.\end{aligned}$$

(Let $u=x-1$, $\displaystyle\int\frac{dx}{(x-1)^2}=\int\frac{du}{u^2}=-\frac{1}{u}+C=-\frac{1}{x-1}+C$;

代入比解聯立快: $(**)$ $4x=A(x-1)(x+1)+B(x+1)+C(x-1)^2$,

$$\left\{\begin{array}{llll} \text{把}(x-1)\text{變零}:x=1 & \implies 4=2B, & B=2; & \\ \text{把}(x+1)\text{變零}:x=-1 & \implies -4=4C, & C=-1; &) \\ \text{代其他好算的}:\ x=0 & \implies 0=-A+B+C, & A=B+C=1. & \end{array}\right.$$ ■

Attention: $Q(x)$ 有 d 重的一次因式 $(x-a)^d$, 就要假設 d 個未知數 $A_1, A_2, \ldots, A_d$:

$$\boxed{Q(x)=(x-a)^d\times\cdots \overset{\text{假設}}{\implies} \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_d}{(x-a)^d}}.$$

Example 0.5 *(二次式)* $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx.$

$x^3+4x = x(x^2+4)$ *irreducible.*

Assume $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}, \ldots$ *(二次式分母的分子要假設一次式)*

$2x^2-x+4 = A(x^2+4)+(Bx+C)x = (A+B)x^2+Cx+4A,$

$$\begin{cases} A+B & = & 2 \\ \qquad C & = & -1 \\ 4A & = & 4 \end{cases} \implies A=1,\ B=1,\ C=-1.$$

$$\begin{aligned}
\int \frac{2x^2-x+4}{x^3+4x}\,dx &= \int \left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)\,dx \qquad \text{(再分開)} \\
&= \int \left(\frac{1}{x}+\frac{x}{x^2+4}-\frac{1}{x^2+4}\right)\,dx \\
&= \ln|x| + \frac{1}{2}\ln(\underline{x^2+4}) - \frac{1}{2}\tan^{-1}\frac{x}{2}+K.
\end{aligned}$$

(Let $u=x^2+4$, $du = 2x\ dx$; $x^2+4>0$, *絕對值可以換掉.)* ■

Example 0.6 *(要配方的二次式.)* $\displaystyle\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx.$

$\dfrac{4x^2-3x+2}{4x^2-4x+3} = 1+\dfrac{x-1}{4x^2-4x+3},$

$4x^2-4x+3$ *is irreducible* $(\because b^2-4ac = [(-4)^2-4\cdot4\cdot3]<0)$.

配方*:* $4x^2-4x+3 = (2x-1)^2+2$, *let* $u=2x-1$, $du = 2\ dx$.

$$\begin{aligned}
\int \frac{4x^2-3x+2}{4x^2-4x+3}\,dx &= \int 1+\frac{x-1}{4x^2-4x+3}\,dx \\
&= x+\int \frac{\frac{u+1}{2}-1}{u^2+2}\cdot\frac{1}{2}\,du \qquad \text{(變數變換)} \\
&= x+\frac{1}{4}\int\left(\frac{u}{u^2+2}-\frac{1}{u^2+2}\right)\,du \\
&= x+\frac{1}{4}\frac{1}{2}\ln(\underline{u^2+2}) - \frac{1}{4}\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}}+C \\
&= x+\frac{1}{8}\ln(\underline{4x^2-4x+3}) - \frac{1}{4\sqrt{2}}\tan^{-1}\frac{2x-1}{\sqrt{2}}+C.
\end{aligned}$$

($4x^2-4x+3>0$, *絕對值可以拿掉; 最後的* u^2+2 *直接換回* $4x^2-4x+3$*.)* ■

Observation: 不能分解的二次式 x^2+bx+c 一定可以配方成 $u^2+a^2>0$, 所以 $\ln|x^2+bx+c|$ 的絕對值都可以換成小括號 $\ln(x^2+bx+c)$; 最後換回 x 的時候若有 u^2+a^2 也可以直接換成 x^2+bx+c (用代的也一樣).

Example 0.7 *(重複的二次式.)* $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$

$$\textit{Assume}\ \frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2},\qquad \textit{(設少了會算錯)}$$

$$\begin{aligned}1-x+2x^2-x^3&=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\quad \ldots (***)\\&=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A,\end{aligned}$$

$$\left\{\begin{array}{rrrrcr} A+B & & & & = & 0\\ & C & & & = & -1\\ 2A+B & & +D & & = & 2\\ & +C & & +E & = & -1\\ A & & & & = & 1\end{array}\right. \Longrightarrow \begin{array}{l}A=1\\B=-1\\C=-1\\D=1\\E=0\end{array}$$

直接解 *(easy)*, 或是代入 *(∗∗∗)(hard, but learn it)*

$$\left\{\begin{array}{lll} x=0 & \Longrightarrow & 1=A;\\ x^2=-1 & \Longrightarrow & -1=-D+Ex,\ D=1,\ E=0;\quad \textit{(用比較係數)}\\ x=\pm1 & \Longrightarrow & -2=B+C,\ 0=B-C,\ B=C=-1.\end{array}\right.$$

$$\begin{aligned}&\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx\\ &= \int\left(\frac{1}{x}-\frac{x}{x^2+1}-\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx\\ &= \ln|x|-\frac{1}{2}\ln(x^2+1)-\tan^{-1}x-\frac{1}{2(x^2+1)}+K.\end{aligned}$$

■

Attention: $Q(x)$ 有 e 重的二次因式 $[(x-b)^2+c^2]^e$, 就要假設 $2e$ 個未知數 $B_1,C_1,B_2,C_2,\ldots,B_e,C_e$:

$$\boxed{Q(x)=[(x-b)^2+c^2]^e\times\cdots \overset{\text{假設}}{\Longrightarrow} \frac{B_1x+C_1}{(x-b)^2+c^2}+\cdots+\frac{B_ex+C_e}{[(x-b)^2+c^2]^e}}.$$

Example 0.8 *(不要放棄嘗試變數變換)* $\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,dx.$

Let $u = x(x^2+3) = x^3+3x$, *then* $du = 3(x^2+1)\,dx$, $(x^2+1)\,dx = \dfrac{1}{3}\,du$.

$\displaystyle\int \frac{x^2+1}{x(x^2+3)}\,dx = \int \frac{du}{3u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+3x| + C.$ ■

(Try yourself: $\dfrac{x^2+1}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3} = \dfrac{1}{3}\dfrac{1}{x} + \dfrac{2}{3}\dfrac{x}{x^2+3}$.*)*

Rationalizing substitutions 有理代換: 分式裡有開 n 次根的函數 $\sqrt[n]{g(x)}$, let $u = \sqrt[n]{g(x)}$, 然後換成沒有根式的有理函數再積分.
例如: 積分時看到 $\sqrt{x^2+1}$, 變數變換用 $u = x^2+1$ 或許沒有 $u = \sqrt{x^2+1}$ 來得簡化, 平平是變數變換, 撇步不同, 過程不同, 雖然答案是一樣的.

Example 0.9 *(有理代換)* $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx.$

Let $u = \sqrt{x+4}$, *then* $du = \dfrac{1}{2u}\,dx$, $x = u^2-4$.

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}\,dx &= \int \frac{u}{u^2-4}\cdot 2u\,du = \int \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right)\,du \\
&= 2u + 2\ln\left|\frac{u-2}{u+2}\right| + C \\
&= 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C.
\end{aligned}$$

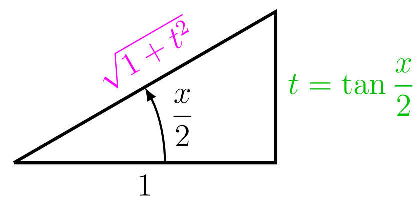
(Let $u = x+4$ 好做嗎? *Try yourself.)* ■

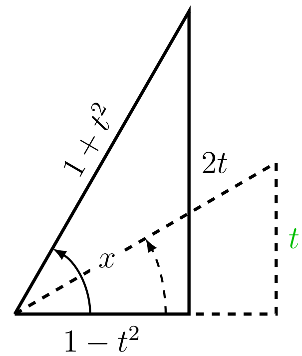
Additional: Weierstrass substitution 魏爾斯特拉斯變換

又稱 Tangent half-angle substitution 正切半角變換, 把三角函數換成有理函數。以德國數學家 Karl Theodor Wilhelm Weierstrass (1815–1897) 命名。

> The world's sneakiest substitution is undoubtedly.
> 世界上最卑鄙的變換是無庸置疑地。 — Michael Spivak

Example 0.10 (Ex 7.4.59.) *Let* $t = \tan\frac{x}{2}$, $x \in (-\pi, \pi)$. *Then*





(a) $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$, $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$.

(b) $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$,

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = (\frac{1}{\sqrt{1+t^2}})^2 - (\frac{t}{\sqrt{1+t^2}})^2 = \frac{1-t^2}{1+t^2}$,

$\tan x = \frac{2t}{1-t^2}$, $\cot x = \frac{1-t^2}{2t}$, $\sec x = \frac{1+t^2}{1-t^2}$, $\csc x = \frac{1+t^2}{2t}$.

(c) $x = 2\tan^{-1} t$, $dx = \frac{2}{1+t^2}\, dt$. ■

(Try yourself: Exercise 7.4.60–63:

$$\int \frac{dx}{1-\cos x},\ \int \frac{dx}{3\sin x - 4\cos x},\ \int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x},\ \int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\, dx.)$$

⧫ 7.5 Strategy for integration (optional)

積分戰略 (5.3 $\sim$ 5.5, 7.1 $\sim$ 7.4).

- 分開: 加減常數倍, 部分分式.

  $\int \sqrt{x}(1+\sqrt{x})\ dx = \int \sqrt{x}\ dx + \int x\ dx.$

  $\int \frac{1}{x^4-1}\ dx = \int \frac{1/4}{x-1}\ dx + \int \frac{1/4}{x+1}\ dx + \int \frac{1/2}{x^2+1}\ dx.$

- 變形: 三角函數 (定義, 恆等式, 半角).

  $\int \frac{\tan\theta}{\sec^2\theta}\ d\theta = \int \frac{\sin\theta}{\cos\theta}\cos^2\theta\ d\theta = \int \sin\theta\cos\theta\ d\theta = \int \frac{1}{2}\sin 2\theta\ d\theta.$

  $\int (\sin x + \cos x)^2\ dx = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\ dx$

  $= \int\ dx + 2\int \sin x\cos x\ dx.$

  $\int \tan^2 x\sec x\ dx = \int \sec^3 x\ dx - \int \sec x\ dx.$

  $\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2(x/2)} = \int \csc^2\frac{x}{2}\ d(\frac{x}{2}).$

- 變換: 有理, 三角.

  $\int \frac{x}{x^2-1}\ dx = \int \frac{1/2}{x^2-1}\ d(x^2-1) = \int \frac{\tan\theta}{\sec^2\theta}\cdot\sec^2\theta\ d\theta.$

  $\int \sin\theta\cos\theta\ d\theta = \int \sin\theta\ d(\cos\theta).$

- 其他: 同乘 (小心 0), 減化.

  $\int \frac{dx}{1-\cos x} = \int \Big(\frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}\Big)\ dx = \int \frac{1+\cos x}{1-\cos^2 x}\ dx$

  $= \int \frac{1+\cos x}{\sin^2 x}\ dx = \int \csc^2 x + \cot x\csc x\ dx.$

- 分部: $\int u\ dv = uv - \int v\ du.$

Example 0.11 $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\ dx$ .... $\displaystyle\left(\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C\right)$

$$\int \frac{\tan^3 x}{\cos^3 x}\ dx = \int \tan^3 x \sec^3 x\ dx \boxed{= \int (\sec^2 x - 1)\sec^2 x\ d(\sec x)}$$

$$= \int v^4 - v^2\ dv;$$

$$\int \frac{\tan^3 x}{\cos^3 x}\ dx = \int \frac{\sin^3 x}{\cos^6 x}\ dx = \int \frac{\cos^2 x - 1}{\cos^6 x}(-\sin x)\ dx$$

$$\boxed{= \int \frac{\cos^2 x - 1}{\cos^6 x}\ d(\cos x)} = \int \frac{u^2 - 1}{u^6}\ du = \int (u^{-4} - u^{-6})\ du. \quad \blacksquare$$

Example 0.12 $\displaystyle\int e^{\sqrt{x}}\ dx$ .... $\left(e^{\sqrt{x}}(2\sqrt{x} - 2) + C\right)$

$$\int e^{\sqrt{x}}\ dx = \int e^u(2u)\ du \boxed{= 2\int u\ d(e^u)} = 2ue^u - \int e^u\ du. \quad \blacksquare$$

Example 0.13 $\displaystyle\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\ dx$

.... $\displaystyle\left(\frac{x^3}{3} + \frac{3x^2}{2} + 19x + \frac{3126}{35}\ln|x - 5| - \frac{1}{10}\ln|x| - \frac{31}{14}\ln|x + 2| + C\right)$

$$\int \frac{x^5 + 1}{x^3 - 3x^2 - 10x}\ dx = \int \left(x^2 + 3x + 19 + \frac{87x^2 + 190x + 1}{x(x - 5)(x + 2)}\right)\ dx. \quad \blacksquare$$

Example 0.14 $\displaystyle\int \frac{dx}{x\sqrt{\ln x}}$ .... $\left(2\sqrt{\ln x} + C\right)$

$$\int \frac{dx}{x\sqrt{\ln x}} \boxed{= \int \frac{1}{\sqrt{\ln x}} d(\ln x)} = \int \frac{du}{\sqrt{u}}. \quad \blacksquare$$

Example 0.15 $\displaystyle\int \sqrt{\frac{1 - x}{1 + x}}\ dx$ .... $\left(\sin^{-1} x + \sqrt{1 - x^2} + C\right)$

$$\int \sqrt{\frac{1 - x}{1 + x}}\ dx = \int \frac{1 - x}{\sqrt{1 - x^2}}\ dx = \int \frac{1}{\sqrt{1 - x^2}}\ dx - \int \frac{x}{\sqrt{1 - x^2}}\ dx$$

$$\boxed{= \int \frac{1}{\sqrt{1 - \sin^2\theta}}\ d(\sin\theta) + \int \frac{1/2}{\sqrt{1 - x^2}}\ d(1 - x^2)} = \int\ d\theta + \int \frac{du}{2u}. \quad \blacksquare$$

但是, 還是有些積不出來. (或許有其他方法.) 例如:

$$\int e^{x^2}\ dx,\ \int e^{-x^2}\ dx,$$

$$\int \frac{e^x}{x}\ dx = \int \frac{1}{xe^x}\ dx,\ \int \frac{e^x}{x^2}\ dx = -\frac{e^x}{x} + \int \frac{e^x}{x}\ dx,$$

$$\int \sin x^2\ dx,\ \int \cos e^x\ dx,$$

$$\int \sqrt{x^3+1}\ dx,\ \int x\sqrt{x^3+1}\ dx,$$

$$\int \frac{1}{\ln x}\ dx,\ \int \frac{\sin x}{x}\ dx,\ \cdots$$

推薦做一做這節的習題作爲綜合練習.