

3.2 The product and quotient rules

1. derivative of product 函數乘積的導函數 $(fg)' = f'g + fg'$.
2. derivative of quotient 函數商數的導函數 $(f/g)' = (f'g - fg')/g^2$.

0.1 Derivative of product

The Product rule: $(fg)' = f'g + fg'$

前面的微分乘以後面加前面乘以後面的微分

If f and g are differentiable, then $f(x)g(x)$ is differentiable and

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} & \quad (\text{一加一減 } f(x)g(x+h)) \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x+h) + f(x)[g(x+h) - g(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \lim_{h \rightarrow 0} g(x+h) + \lim_{h \rightarrow 0} f(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x)g(x) + f(x)g'(x). \quad (\text{與課本順序前後不同}) \end{aligned}$$

($\because g$ 可微分 \implies 連續 $\iff \lim_{h \rightarrow 0} g(x+h) = g(x)$.)

◆: 課本上是用 $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$, 得到 $(fg)' = fg' + gf'$. 符號順序不太一樣, 原理是一樣的.

Example 0.1 $f(x) = xe^x$, find f' and $f^{(n)}$.

$$f' = (x \cdot e^x)' = (x)'e^x + x(e^x)' = e^x + xe^x = (x+1)e^x.$$

$$f'' = ((x+1) \cdot e^x)' = (x+1)'e^x + (x+1)(e^x)' = e^x + (x+1)e^x = (x+2)e^x.$$

Use induction on n we have $f^{(n)} = (x+n)e^x$. ■

Example 0.2 $f(t) = \sqrt{t}(a+bt)$, find f' .

$$[\text{Sol 1}]: f' = (\sqrt{t})'(a+bt) + \sqrt{t}(a+bt)' = \frac{1}{2\sqrt{t}}(a+bt) + \sqrt{t}b = \frac{a+3bt}{2\sqrt{t}}.$$

$$[\text{Sol 2}]: f' = (at^{1/2} + bt^{3/2})' = \frac{1}{2}at^{-1/2} + \frac{3}{2}bt^{1/2}. \quad \blacksquare$$

Note: $f' = f'(t) = \frac{d}{dt}f(t) \neq \frac{d}{d\mathbf{x}}f(t)(=?).$

0.2 Derivative of quotient

The Quotient rule: $\boxed{(f/g)' = (f'g - fg')/g^2}$

上面的微分乘以下面減上面乘以下面的微分再除以下面兩次

If f and g are differentiable, then

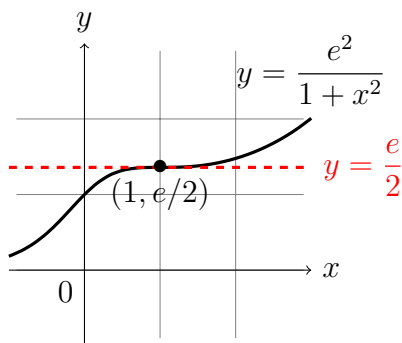
$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2}.$$

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h)/g(x+h) - f(x)/g(x)}{h} \quad (\text{通分}) \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)} \quad (\text{一加一減 } f(x)g(x)) \\ &= \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]g(x) - f(x)[g(x+h) - g(x)]}{hg(x+h)g(x)} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \lim_{h \rightarrow 0} \frac{g(x)}{g(x+h)g(x)} \\ & \quad - \lim_{h \rightarrow 0} \frac{f(x)}{g(x+h)g(x)} \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \end{aligned}$$

Example 0.3 Find $y = e^x/(1+x^2)$ 在 $x = 1$ 的切線.

$$\begin{aligned} y|_{x=1} &= \frac{e}{2}, \implies \text{切點 } (1, \frac{e}{2}). \\ y' &= \frac{dy}{dx} = \frac{(e^x)'(1+x^2) - e^x(1+x^2)'}{(1+x^2)^2} \\ &= \frac{e^x(1+x^2) - e^x(2x)}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}, \\ y'|_{x=1} &= \frac{e^1(1-1)^2}{(1+1^2)^2} = 0. \\ \text{切線: } y &= y'|_{x=1}(x-1) + y|_{x=1} \\ &= 0(x-1) + \frac{e}{2} = \frac{e}{2}. \end{aligned}$$

(法線?)



Skill: 公式怎麼背? 寫題目, 先抄公式, 再把每項換上去.

◆ Additional: Stories in Exercises

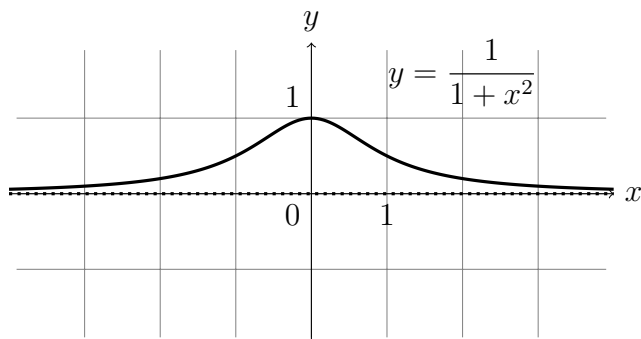
(Exercise 3.2.35) 曲線 $y = \frac{1}{1+x^2}$ 稱為 *witch of (Maria) Agnesi* 阿涅西的女巫/箕舌線. — Maria Gaetana Agnesi, 義大利女數學家.

1630 Fermat 費馬首發.

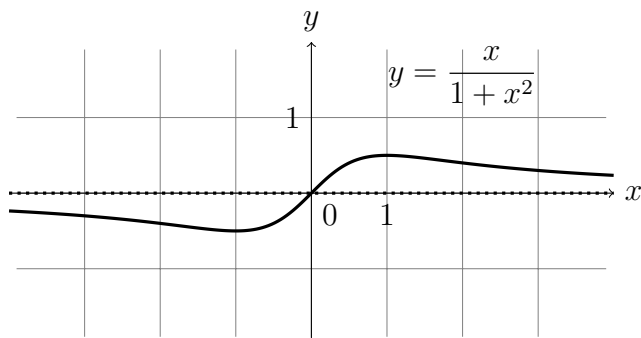
1718 Grandi 葛蘭迪命名“versoria”, 意為帆繩.

1748 Agnesi 阿涅西書中譯為“versiera”[義大利文], 與女巫同義.

1801 Colson 柯爾森誤譯為“witch”.



(Exercise 3.2.36) 曲線 $y = \frac{x}{1+x^2}$ 稱為 *serpentine* 蛇狀線 (/蛇紋石).



(Exercise 3.2.64) **Reciprocal Rule** 倒數律

$$\frac{d}{dx} \left[\frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2}$$