1.4 Exponential functions

1. exponent 指數 a^b

2. exponential function 指數函數 $f(x) = a^x$

3. number e 歐拉數

0.1　Exponent

Define: $\boxed{a^b}$ a 的 b 次方 ("a to the (power of) b", "a to the b-th power", or "the b-th power of a"), a 稱爲底數 (***base***), b 稱爲指數 (***exponent***).　　$2^{\heartsuit}$

(b 個 a 相加 (+) 等於 a 乘以 b ($a \times b$); b 個 a 相乘 (×) 等於 a 的 b 次方 (a^b).)

	$2 \times n$	$\overbrace{2 + \ldots + 2}^{n}$
$\mathbb{N}$	2×3	$2 + 2 + 2 = 6$
$\mathbb{Z}$	2×0	0(有加跟沒加一樣)
	$2 \times (-3)$	$0 - 2 - 2 - 2 = -6$
$\mathbb{Q}$	$2 \times \frac{1}{3}$	$\frac{2}{3} = 2/3$
$\mathbb{R}$	$2 \times \sqrt{3}$	$\approx 2 \times 1.732 = 3.464$

	2^n	$\overbrace{2 \times \ldots \times 2}^{n}$
$\mathbb{N}$	2^3	$2 \times 2 \times 2 = 8$
$\mathbb{Z}$	2^0	1(有乘跟沒乘一樣)
	2^{-3}	$1 \div 2 \div 2 \div 2 = \frac{1}{8} = \frac{1}{2^3}$
$\mathbb{Q}$	$2^{\frac{1}{3}}$	$\sqrt[3]{2}$
$\mathbb{R}$	$2^{\sqrt{3}}$	$\approx 2^{1.732} \approx 3.322$

$$3 < \pi < 4 \qquad\qquad 2^3 < 2^\pi < 2^4$$
$$3.1 < \pi < 3.2 \qquad\qquad 2^{3.1} < 2^\pi < 2^{3.2}$$
$$3.14 < \pi < 3.15 \qquad\qquad 2^{3.14} < 2^\pi < 2^{3.15}$$
$$3.141 < \pi < 3.142 \qquad\qquad 2^{3.141} < 2^\pi < 2^{3.142}$$
$$3.1415 < \pi < 3.1416 \qquad\qquad 2^{3.1415} < 2^\pi < 2^{3.1416}$$
$$3.14159 < \pi < 3.14160 \qquad\qquad 2^{3.14159} < 2^\pi < 2^{3.14160}$$
$$3.141592 < \pi < 3.141593 \qquad 2^{3.141592} < 2^\pi < 2^{3.141593}$$
$$3.1415926 < \pi < 3.1415927 \quad 2^{3.1415926} < 2^\pi < 2^{3.1415927}$$

(存在無窮多的有理數逼近π $\implies$ 無窮多的$2^{\text{有理數}}$逼近2^π.)

♦: **Story of** π: 劉徽: $\pi \approx 3.14$ 稱爲徽率; 祖沖之: $3.1415926 < \pi < 3.141927$; 以 $\dfrac{22}{7}$ 爲約率, $\dfrac{335}{113} \approx 3.1415929$ 爲密率, 稱爲祖率.

Note: $\sqrt{2} \times \sqrt{3} = \sqrt{6} \approx 2.449$, $\sqrt{2}^{\sqrt{3}} \approx 1.414^{1.732} \approx 1.823$.

(Check by yourself: which is large: 2^3 v.s. 3^2? $\frac{1}{2}^{\frac{1}{3}}$ v.s. $\frac{1}{3}^{\frac{1}{2}}$? $\sqrt{2}^{\sqrt{3}}$ v.s. $\sqrt{3}^{\sqrt{2}}$?)

Law of exponents 指數律**:** $(a > 0,\ b, c \in \mathbb{R})$

1 加: $a^{b+c} = a^b \times a^c$; $\quad \overbrace{(a \times \cdots \times a)}^{b} \times \overbrace{(a \times \cdots \times a)}^{c} = \overbrace{a \times \cdots \times a}^{b+c} = a^{b+c}$

2 減: $a^{b-c} = a^b \div a^c$; $\qquad\qquad \overbrace{(a \times \cdots \times a)}^{b} \div \overbrace{(a \times \cdots \times a)}^{c} = a^{b-c}$

3 乘: $a^{bc} = (a^b)^c$; $\qquad\qquad \overbrace{\underbrace{(a \times \cdots \times a)}_{b} \times \cdots \times \underbrace{(a \times \cdots \times a)}_{b}}^{c} = a^{bc}$

4 除: $a^{b/c} = \sqrt[c]{a^b} = (\sqrt[c]{a})^b$; $\qquad\qquad \overbrace{\sqrt[c]{a^b} \times \cdots \times \sqrt[c]{a^b}}^{c} = a^b$

5 分配: $(ab)^c = a^c b^c$; $\overbrace{ab \times \cdots \times ab}^{c} = \overbrace{(a \times \cdots \times a)}^{c} \times \overbrace{(b \times \cdots \times b)}^{c} = a^c b^c$

6 零: $\boxed{a^0 := 1}$, $\boxed{0^b := 0}$, $\boxed{0^0}$: undetermined(未定).

Ex: $\quad 2^{3+5} \quad = 2^3 \times 2^5 = 2^8,$

$\qquad\quad 2^{3-5} \quad = 2^3 \div 2^5 = 2^{-2} = \dfrac{1}{2^2},$

$\qquad\quad 2^{3 \times 5} \quad = (2^3)^5 = 2^{15},$

$\qquad\quad 2^{3/5} \quad = \sqrt[5]{2^3} = \sqrt[5]{2}^3,$

$\qquad (2 \times 3)^5 = 2^5 \times 3^5 = 2^5 \cdot 3^5 = 2^5 3^5.$

Note: (方根唸法) For $0 < a \in \mathbb{R}$,
$\sqrt{a}(= \sqrt[2]{a})$: a 的平方根 ("square root of a");
$\sqrt[3]{a}$: a 的立方根 ("cube root of a");
$\sqrt[n]{a}$: a 的 n 次方根 ("the n-th root of a").

Extend: For $0 < a \in \mathbb{R}$ and $n \in \mathbb{Z}$, $(-a)^n = \begin{cases} a^n & \text{if } n \text{ is even;} \\ -a^n & \text{if } n \text{ is odd.} \end{cases}$
Ex: $(-1)^0 = (-1)^2 = 1$, $(-1)^1 = (-1)^{-1} = -1$.

0.2 Exponential function

$f(x) = x^n$, $n \in \mathbb{N}$ **power function** 冪次函數 ($x^{1/2} =?$ $x^{\sqrt{2}} =?$ $x^x =?$)
Define: The **exponential function of base** $a(>0)$ 指數函數 (以 a 為底)

$$\boxed{f(x) = a^x}$$
←變數在指數
← 底數大於零

Question: Why $a > 0$?
Answer: $(-1)^{1/2} = \sqrt{-1} := i$, 0^{-1} undefined, $\therefore$ 只考慮正底數 $a > 0$.

Question: How to define a^x?

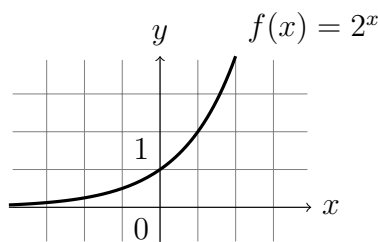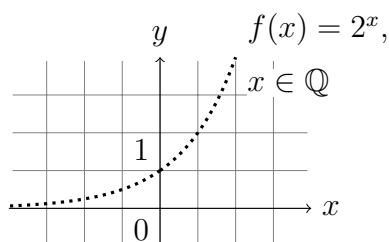1. $x = n \in \mathbb{N}$, $a^n := \overbrace{a \times \ldots \times a}^{n}$.

2. $x = 0$, $a^0 := 1$.

3. $x = -n$, $a^{-n} := \dfrac{1}{a^n}$.

4. $x = \dfrac{p}{q} \in \mathbb{Q}$, $a^{p/q} := \sqrt[q]{a^p} = (\sqrt[q]{a})^p$.

5. $x \in \mathbb{R} \setminus \mathbb{Q}$, $x \approx r \in \mathbb{Q}$, $a^x :\approx a^r$.

6. Domain 定義域: $(-\infty, \infty) = \mathbb{R}$.
   Range 值域: $(0, \infty)$.

(Practice by yourself: find $f(x) = 2^x$ for $x = 3, 0, -3, \dfrac{1}{3}, \sqrt{3}$.)

Note: Intervals 區間表示: ("$($,$)$"不含端點小括號, "$[$,$]$"包含端點中括號.)

$$\left(a, b\right) = \{x : a < x < b\}, \quad \left(a, b\right] = \{x : a < x \leq b\},$$

$$\left[a, b\right) = \{x : a \leq x < b\}, \quad \left[a, b\right] = \{x : a \leq x \leq b\},$$

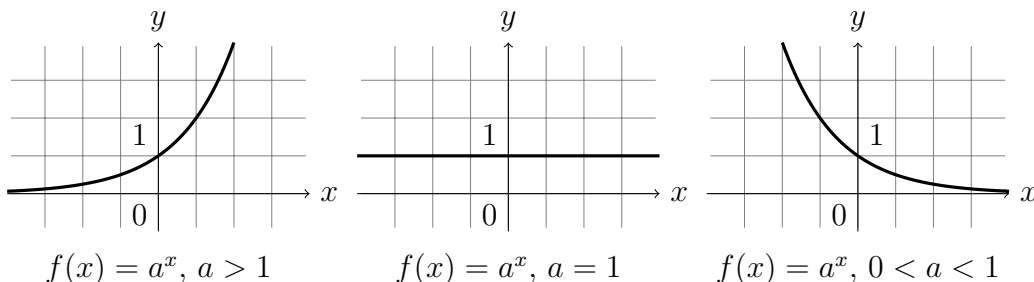$\infty(-\infty)$: 無限大 (小) 不是可確定的數, 必用小括號 "$\ldots, \infty)$" & "$(-\infty, \ldots$".

區間的聯集 (union) $\bigcup$ 與交集 (intersection) $\bigcap$:
$(a, b) \cup (c, d) = \{x : a < x < b \text{ or } c < x < d\}$, $(-\infty, 0) \cup (0, \infty) = \{x \neq 0\}$.
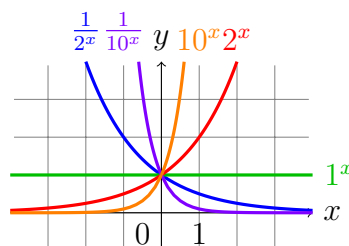$(a, b) \cap (c, d) = \{x : a < x < b \text{ and } c < x < d\}$, $(-\infty, 0) \cap (0, \infty) = \emptyset$.
我找不到, 我到不了, 你所謂的, 無限大或小.

Three types of exponential functions:

$f(x) = a^x$, $a > 1$ $f(x) = a^x$, $a = 1$ $f(x) = a^x$, $0 < a < 1$
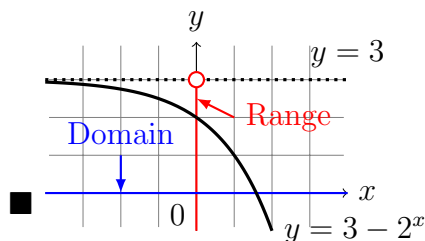
Note: $\boxed{\left(\dfrac{1}{a}\right)^x = \dfrac{1}{a^x} = a^{-x}}$,
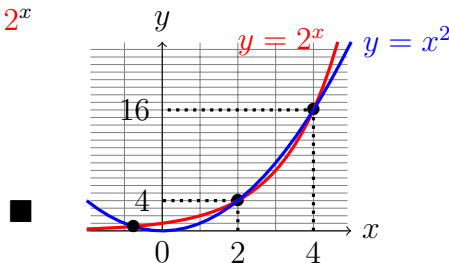
$y = a^{-x}$ 與 $y = a^x$ 以 $x = 0$ (y-軸) 對稱.

Example 0.1 *Range of $f(x) = 3 - 2^x = ?$*

$2^x : \mathbb{R} \to (0, \infty)$ *(所有正數)*
$-2^x : \mathbb{R} \to (-\infty, 0)$ *(所有負數)*
$3 - 2^x : \mathbb{R} \to (-\infty, 3)$ *(所有比 3 小的數)*
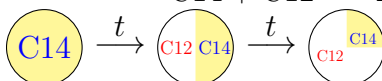*Range: $(-\infty, 3)$ (or $\{x \in \mathbb{R} : x < 3\}$).* ∎

Example 0.2 *How many intersection of 2^x & x^2?*

$2^4 = 16 = 4^2$, $2^2 = 4 = 2^2$.
(只能從圖看出第三點)
Three: $x = 4$, $x = 2$, $x \approx -0.7666$. ∎

Application: Estimate population of human(人口), bacterial(細菌), radiometric dating(放射性定年) C14(C12:5730yr), U235(Pb207) or U238(Pb206).

radiometric dating: (C14) 半衰期 t: $\dfrac{\text{C14}}{\text{C14} + \text{C12}} = (\dfrac{1}{2})^{x/t}$, $x = t \lg \dfrac{\text{C14} + \text{C12}}{\text{C14}}$;
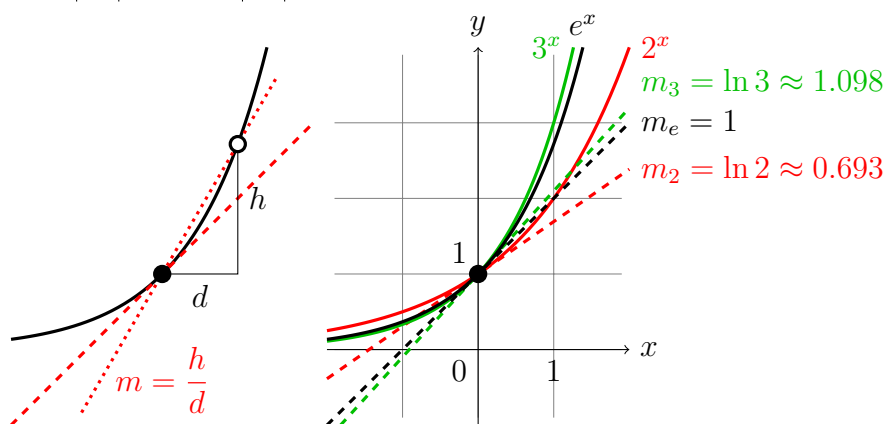

population: 起初 N 個, 每 t 時間變成 a 倍 (增加 $a - 1$ 倍): $P(x) = Na^{x/t}$.

0.3 Number e

Define: *Euler's number* 歐拉數 $\boxed{e} \approx 2.718281828$,
是 $y = a^x$ 在 $x = 0$ 切線斜率 (slope) 等於 1 的底數 a 值.
In 1727 Leonhard Eular named "e" for "exponential".

Let m_a be the slope of the tangent line of a^x at $x = 0$.

a	1	2	e	3
m_a	0	0.693...	1	1.098...

Note: 找到 e 有什麼好處? $y = e^{rx}$ 在 $(0, 1)$ 的切線斜率就是 r.

Define: The ***natural exponential function* 自然指數函數**
("e to the (power of) x", or "natural exponential of x")
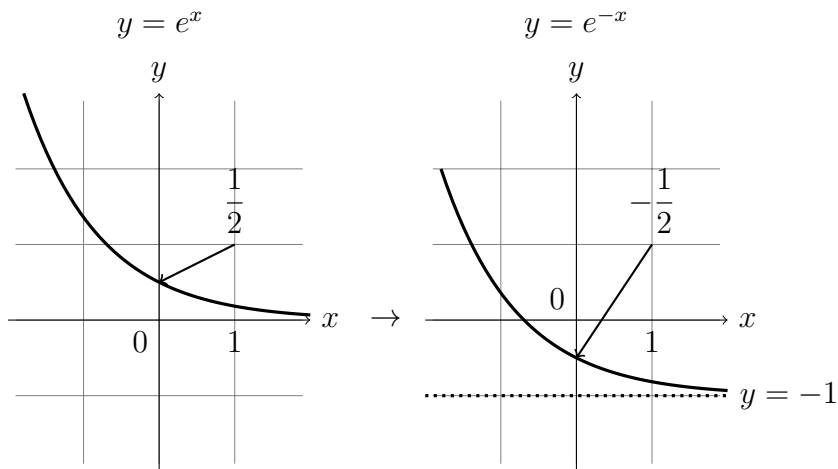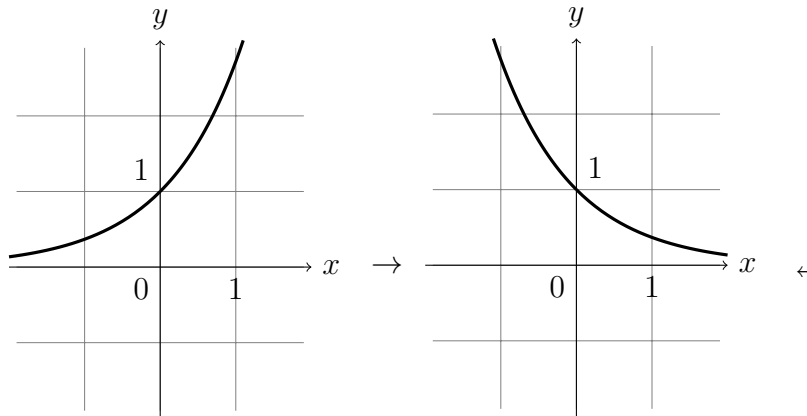
$$\boxed{f(x) = e^x}$$

(Find it out by yourself: What is the difference between e^x, ex and x^e?)

Additional: (See §11)

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}, \ e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

♦: Euler's formula: $e^{ix} = \cos x + i \sin x$, Euler's identity: $e^{i\pi} + 1 = 0$.

Example 0.3 *Draw* $y = \frac{1}{2}e^{-x} - 1$. *(Domain? Range?)*

$$y = e^x$$

$$y = e^{-x}$$

$$y = \frac{1}{2}e^{-x}$$

$$y = \frac{1}{2}e^{-x} - 1$$

(Domain: $(-\infty, \infty) = \mathbb{R}$*; range:* $(-1, \infty)$*.)*

■