

4.1 Maximum and minimum values

微分應用之一：找極值。

1. Extreme Value Theorem 極值定理 (存在性)
2. Fermat's Theorem 費馬定理 (找極值)

0.1 Extreme Value Theorem

Define: A function $f : D \rightarrow R$ and $c \in D$. $f(c)$ is the

- **absolute maximum** value 絕對最大值 of f on D
if $f(c) \geq f(x)$ for all $x \in D$. (maximum a., n., pl.: -ima)
- **absolute minimum** value 絕對最小值 of f on D
if $f(c) \leq f(x)$ for all $x \in D$. (minimum a., n., pl.: -ima)
(absolute max/min is sometimes called **global** max/min, and both called **extreme values** 極值 of f .)
- **local maximum** value 相對 (局部) 極大值 of f
if $f(c) \geq f(x)$ when x is near c . (有些書用 maximal a.)
- **local minimum** value 相對 (局部) 極小值 of f
if $f(c) \leq f(x)$ when x is near c . (有些書用 minimal a.)

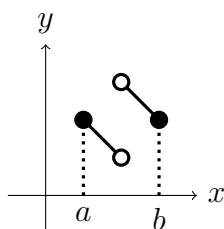
Attention: 絕對最大/小值發生在端點時不是相對極大/小值。

Theorem 1 (Extreme Value Theorem) 閉連續有極值

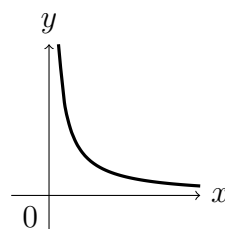
If f is **continuous** on a **closed** interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$. (f has **extreme** values.)

Proof. (省略: 閉區間 = 頭尾固定; 連續 = 中間沒斷; \implies 有高有低.) ■

Note:



not continuous, no max.



not closed, no extreme values.

0.2 Fermat's Theorem

Theorem 2 (Fermat's Theorem) 極處可微導數為零

If f has a **local** maximum or minimum at c , and if **$f'(c)$ exists**, then **$f'(c) = 0$** .

Proof. [反證法]

If $f'(c) > 0$, then $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} > 0$,

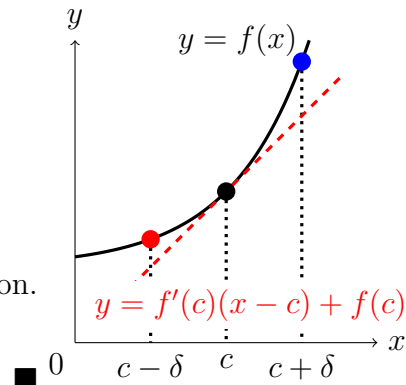
$$\implies f(c+\delta) > f(c) > f(c-\delta)$$

for some $\delta > 0$ small enough.

(The case of $f'(c) < 0$ is similar.)

$\implies f(c)$ is not a local max/min, a contradiction.

Therefore, $f'(c) = 0$.



[直證法] Suppose $f(c)$ is a local max, $f(c) \geq f(c+h)$.

When $h > 0$, $\frac{f(c+h) - f(c)}{h} \leq 0$. $\left(\frac{(-)}{(+)}\right)$

$$\because f'(c) \text{ exists, } f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h} \leq 0.$$

When $h < 0$, $\frac{f(c+h) - f(c)}{h} \geq 0$. $\left(\frac{(-)}{(-)}\right)$

$$\because f'(c) \text{ exists, } f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h} \geq 0.$$

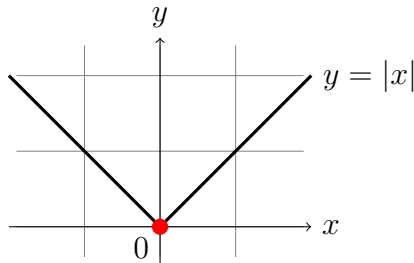
Therefore, $f'(c) = 0$. ■

Remark: f has local max/min: $\begin{cases} f'(c) \text{ exists} \implies f'(c) = 0; \\ f'(c) \text{ does not exist} \implies ? \end{cases}$

Ex: $f(x) = |x|$ has a (absolute)

min(imum value) $f(0) = 0$ at 0,

but $f'(x)$ does not exist at 0 ($\nexists f'(0)$).

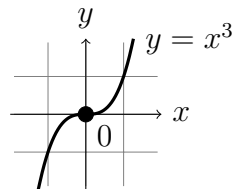


Define: A **critical number** 臨界值 (奇異點) of a function f is a number c in the **domain** of f such that either **$f'(c) = 0$** or **$f'(c)$ does not exist**.

Note: f has local max/min at $c \implies c$ is a critical number of f .
but c is a critical number of f ~~$\not\implies$~~ f has local max/min at c .

兇手 我們之 戀曲 有美好回憶
極值就在臨界值中! 不是每個臨界都是極值。

Ex: $f(x) = x^3$ has a critical number 0, but $f(0)$ is not an extreme value.



Closed interval method: 閉區間法

f is continuous on a closed interval $[a, b]$ 閉連續.

Step 1. 找臨界值: $f(c)$ for critical number $c \in (a, b)$ of f . (沒有不在場證明。)

Step 2. 找邊界點: $f(a), f(b)$.

Step 3. 比大小: The **largest/smallest** value in Steps 1 and 2 is the absolute **maximum/minimum** value of f .

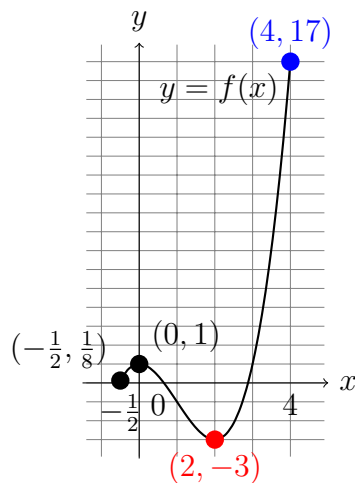
Example 0.1 Find max/min of $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \leq x \leq 4$.

1. $f'(x) = 3x^2 - 6x = 3x(x - 2)$,
 $f'(x) = 0$ when $x = 0$ or $x = 2$, $(\in (-\frac{1}{2}, 4))$
臨界值: $f(0) = 1$, $f(2) = -3$.

2. 邊界點: $f(-\frac{1}{2}) = \frac{1}{8}$, $f(4) = 17$.

3. Choose the largest/smallest in $\{-3, \frac{1}{8}, 1, 17\}$.

Ans: The absolute maximum value is $f(4) = 17$,
the absolute minimum value is $f(2) = -3$. ■



Note: 要寫清楚極值在哪值多少 $f(\dots) = \dots$.

◆ Additional: Proof of Extreme Value Theorem

極值定理的完整證明, 需要學習進階 (advanced) 微積分, 從 實數的完備性公設 (Completeness Axiom of Real Numbers) 開始, 一路證明: 區間套定理 (Nested Intervals Theorem), 波爾札諾-魏爾斯特拉斯定理 (Bolzano-Weierstrass Theorem), 有界定理 (Bounding Theorem); 這裡只給最後一步的證明。

Theorem 3 (Completeness Axiom of Real Numbers (§11))

Every nonempty set of real numbers that has an upper bound has a least upper bound (lub).

⇓

Theorem 4 (Nested Interval Theorem)

Intervals $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots \implies \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

⇓

Theorem 5 (Bolzano-Weierstrass Theorem)

Every bounded sequence has a convergent subsequence.

⇓

Theorem 6 (Bounding Theorem) (閉區間連續函數的值域有界。)

If f is continuous on closed interval $[a, b]$, then the range of f is bounded.

⇓

Proof. (Extreme Value Theorem)

By Bounding Theorem and Completeness Axiom, the range of f has a least upper bound M , claim that “ $\exists c \in [a, b] \ni f(c) = M$ ”.

Suppose not, consider $g(x) = \frac{1}{M - f(x)}$ is continuous on $[a, b]$, by Bounding Theorem, $g(x)$ is bounded above by $L > 0$. Then $\frac{1}{M - f(x)} < L$, $f(x) < M - \frac{1}{L} (< M)$, a lower upper bound than M , a contradiction.

Similarly, the range of f has a greatest lower bound L and $\exists d \in [a, b] \ni f(d) = L$. ■