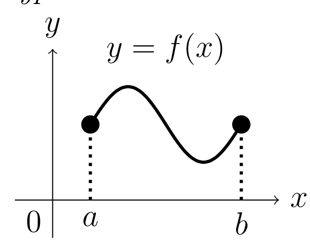
4.2 The Mean Value Theorem

微分應用之二: 瞬間即平均.

1. Rolle's Theorem 羅爾定理
2. Mean Value Theorem 均值定理

0.1 Rolle's Theorem

Theorem 1 (Rolle's Theorem)
Let f be a function that satisfies the following three hypothesis:
1. f is ***continuous*** *on $[a, b]$,* 閉連續
2. f is ***differentiable*** *on (a, b),* 開可微
3. $\boldsymbol{f(a) = f(b)}$. 頭尾同
*Then $\exists\ \boldsymbol{c \in (a, b)} \ni$(*某處水平*)*



$$\boxed{\boldsymbol{f'(c) = 0}}.$$

Proof.
Case 1. $f(x) = k$ is a constant function. $\Longrightarrow f'(c) = 0, \forall\ c \in (a, b)$.
Case 2. $f(x) > f(a)$ for some $x \in (a, b)$. Hypothesis 1 + Extreme Value Theorem $\Longrightarrow$ f has max in (a, b) ($f(a) = f(b)$ are not). Hypothesis 2 + Fermat's Theorem $\Longrightarrow \exists\ c \in (a, b) \ni f'(c) = 0$.
Case 3. $f(x) < f(a)$ for some $x \in (a, b)$. Similarly, f has min in (a, b) and $\exists\ c \in (a, b) \ni f'(c) = 0$. ■

Example 0.1 經過同一點時, 期間會有速率為零.

Proof. *Let $s(t)$ be position function, then $v(t) = s'(t)$ is the velocity function. By Rolle's Theorem, $s(a) = s(b)$, then $\exists\ c \in (a, b) \ni v(c) = 0$.* ■

Example 0.2 *$x^3 + x - 1$ has exactly one real root.*

Proof. *$\because f(x) = x^3 + x - 1$ is continuous and differentiable on $\mathbb{R}$.*
*(*勘根定理證明有根*) $f(0)f(1) = (-1) \cdot 1 < 0$, $\exists\ c \in (0, 1) \ni f(c) = 0$.*
*(*證明只有一根*) Suppose there are two roots a, b, i.e. $f(a) = f(b) = 0$.*
By Rolle's Theorem, $\exists\ c \in (a, b) \ni f'(c) = 0$.
But $f'(x) = 3x^2 + 1 > 0$ for all x, a contradiction.
Therefore, f has exactly one root. ■

0.2 Mean Value Theorem

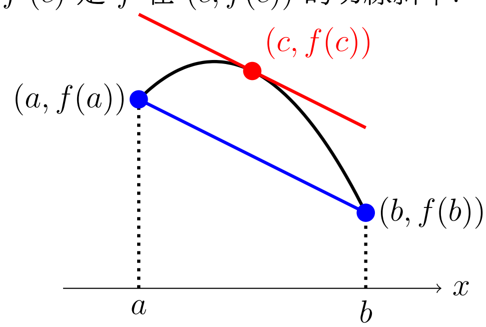
Theorem 2 (Mean Value Theorem) *Let f be a function that satisfies the following two hypothesis.*
1. f is ***continuous*** *on $[a,b]$,* 閉連續
2. f is ***differentiable*** *on (a,b),* 開可微
Then $\exists\ \boldsymbol{c \in (a,b)} \ni$

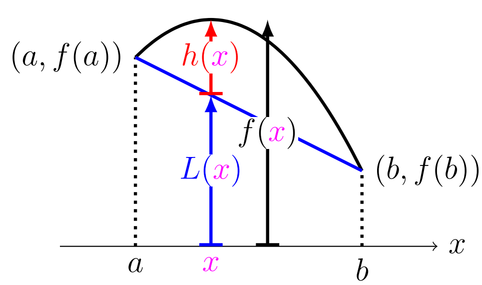
$$\boxed{\boldsymbol{f'(c) = \frac{f(b)-f(a)}{b-a}}}.$$

Equivalently,

$$\boxed{\boldsymbol{f(b)-f(a) = f'(c)(b-a)}}.$$

Note: $\dfrac{f(b)-f(a)}{b-a}$ 是從 $(a,f(a))$ 到 $(b,f(b))$ 的割線斜率.
$f'(c)$ 是 f 在 $(c,f(c))$ 的切線斜率.





Proof. Let $y = L(x) = f(a) + \dfrac{f(b)-f(a)}{b-a}(x-a)$ be the secant line function through $(a,f(a))$ and $(b,f(b))$, and let

$$h(x) = f(x) - L(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

$\because f$ is continuous on $[a,b]$ and differentiable on (a,b), so are L and h. *(1.&2.)*
$\because h(a) = f(a) - f(a) - \dfrac{f(b)-f(a)}{b-a}(a-a) = 0,$
and $h(b) = f(b) - f(a) - \dfrac{f(b)-f(a)}{b-a}(b-a) = 0$, $\therefore h(a) = h(b)$. ...... *(3.)*

By Rolle's Theorem, $\exists\ c \in (a,b) \ni 0 = h'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$,
$\Longrightarrow\ f'(c) = \dfrac{f(b)-f(a)}{b-a}$ or $f(b)-f(a) = f'(c)(b-a)$. ∎

Example 0.3 *A car traveled 180 km in 2 hours, then velocity 90 km/h at least once.*

Example 0.4 $f(0) = -3$, $f'(x) \leq 5$ *for all* x, *how large can* $f(2)$ *be?*

Proof. $\because f$ *is differentiable (and hence continuous) for all* x.
By Mean Value Theorem, $\exists\, c \in (0,2) \ni f(2) - f(0) = f'(c)(2-0)$.
$\Longrightarrow f(2) = 2f'(c) + f(0) \leq 2 \cdot 5 - 3 = 7$. ■

Theorem 3 $\boldsymbol{f'(x) = 0}$ *for all* $x \in (a,b)$ *then* f *is* ***constant*** *on* (a,b).
開可微零導數是常數

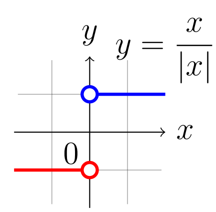
Proof. $\forall\, x_1, x_2 \in (a,b)$, $x_1 < x_2$, f is differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$.

By Mean Value Theorem, $\exists\, c \in (x_1, x_2) \ni f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$, $\Longrightarrow f(x_1) = f(x_2)$.

Therefore, f is constant on (a,b). ■

Corollary 4 $\boldsymbol{f'(x) = g'(x)}$ *for all* $x \in (a,b)$ *then* $f - g$ *is constant on* (a,b)*; that is* $\boldsymbol{f(x) = g(x) + c}$ *where* c *is a constant.* 同導數差常數

Note: 要 (a,b), 不可斷.
Ex: $f(x) = \dfrac{x}{|x|}$ on $D = \{x \neq 0\}$, $f'(x) = 0$ on D, but $f(x)$ is not constant. If choose $D = (0,\infty)$ or $D = (-\infty, 0)$ then f is constant.



Example 0.5 *Prove identity* $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$.

Proof. *Let* $f(x) = \tan^{-1} x + \cot^{-1} x$.
Then $f'(x) = \dfrac{1}{1+x^2} + \dfrac{-1}{1+x^2} = 0$, *so* $f(x)$ *is constant.*

Therefore, $f(x) = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$. ■

Additional: $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $\sec^{-1} x + \csc^{-1} x = \dfrac{\pi}{2}$.
$\left((\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}},\ (\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}.\right)$

Exam 4.2

National Chiao Tung University Campus Run is about 4.5 km. Suppose that you finish it in one hour, and your position function (from the beginning) is continuous (on a closed interval) and differentiable (on an open interval). Prove that your velocity reaches 1.25 m/s at least once during the running.

交大校園路跑約 4.5 km. 假設你一小時跑完, 而且位置函數是閉連續開可微. 證明途中你必定曾經達到速率 1.25 m/s.