

4.9 Antiderivatives

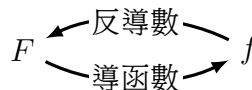
1. anti-derivative [ˌæntaɪ-dəˈrɪvətɪv] 反導函數

Recall: §2.8 導函數:

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ is the derivative of f at where the limit exists.

Define: A function F is called an **antiderivative** 反導 (函) 數 of f on an interval I if $F'(x) = f(x)$ for all x on I .

你 (f') 是我 (f) 的導函數, 我是你的反導數.



Ex: Let $f(x) = x^2$. If $F(x) = \frac{1}{3}x^3$, then $F'(x) = x^2 = f(x)$.

$\frac{1}{3}x^3, \frac{1}{3}x^3 + 1, \frac{1}{3}x^3 + 2, \dots$ are antiderivatives of x^2 . (反導數不止一個)

Theorem 1 If F is an antiderivative of f on I , then **the most general antiderivative** 最一般反導 (函) 數 of f on I is

$$\boxed{F + C}$$

where C is an **arbitrary constant** 任意常數.

凡走過必留下痕跡, (最一般) 反導數必加個大 C 。

Ex: ($\because \frac{d}{dx}C = 0$) $\frac{1}{3}x^3 + C$ is the most general antiderivative of x^2 .

Note: 找 f 的反導數沒指定範圍 (I), 就要看 f 的定義域 (domain).

Example 0.1 Find the most general antiderivative.

(a) $f(x) = \sin x$, (b) $f(x) = \frac{1}{x}$, (c) $f(x) = x^n$, $n \neq -1$.

(a) $(\cos x)' = -\sin x$, $(-\cos x)' = \sin x$. $F(x) = -\cos x + C$.

(b) $(\ln x)' = \frac{1}{x}$ on $(0, \infty)$, (但是 domain 不一樣)

but also $(\ln |x|)' = \frac{1}{x}$ on $(-\infty, 0) \cup (0, \infty)$. $F(x) = \ln |x| + C$.

(c) $(\frac{1}{n+1}x^{n+1})' = \frac{n+1}{n+1}x^n = x^n$ ($n \neq -1$). $F(x) = \frac{1}{n+1}x^{n+1} + C$. ■

Table 1: Table of derivatives and antiderivatives formulas: ($F' = f, G' = g$)

Derivative	Function	Antiderivative
cf'	cf	cF
$f' \pm g'$	$f \pm g$	$F \pm G$
nx^{n-1}	$x^n, n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln x $
$\frac{1}{x}$	$\ln x$	$x \ln x - x$
e^x	e^x	e^x
$\cos x$	$\sin x$	$-\cos x$
$-\sin x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$\ln \sec x $
$\sec x \tan x$	$\sec x$	$\ln \sec x + \tan x $
$2\sec^2 x \tan x$	$\sec^2 x$	$\tan x$
$2\sec^3 x - \sec x$	$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$x \sin^{-1} x + \sqrt{1-x^2}$
$\frac{1}{1+x^2}$	$\tan^{-1} x$	$x \tan^{-1} x - \ln \sqrt{1+x^2}$
$\frac{x}{\sqrt{1-x^2}^3}$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{-2x}{(1+x^2)^2}$	$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$

黑色必背, 藍色多背, 綠色免背, 紅色別背。

Example 0.2 Find g such that $g' = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$.

$$g' = 4 \sin x + 2x^4 - x^{-1/2}; (-\cos x)' = \sin x, (\frac{1}{5}x^5)' = x^4, (2x^{1/2})' = x^{-1/2}.$$

$$g(x) = 4(-\cos x) + 2(\frac{1}{5}x^5) - (2x^{1/2}) + C = -4 \cos x + \frac{2}{5}x^5 - 2\sqrt{x} + C. \quad \blacksquare$$

Example 0.3 Find f such that $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

$$(e^x)' = e^x, (\tan^{-1} x)' = \frac{1}{1+x^2}, f(x) = (e^x) + 20(\tan^{-1} x) + C.$$

$$f(0) = 1 + 20 \cdot 0 + C = -2, C = -3. f(x) = e^x + 20 \tan^{-1} x - 3. \quad \blacksquare$$

Example 0.4 Find f such that $f'' = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$.

$$f'(x) = 4x^3 + 3x^2 - 4x + C, f(x) = x^4 + x^3 - 2x^2 + Cx + D,$$

$$f(0) = D = 4, f(1) = C + D = 1, C = -3.$$

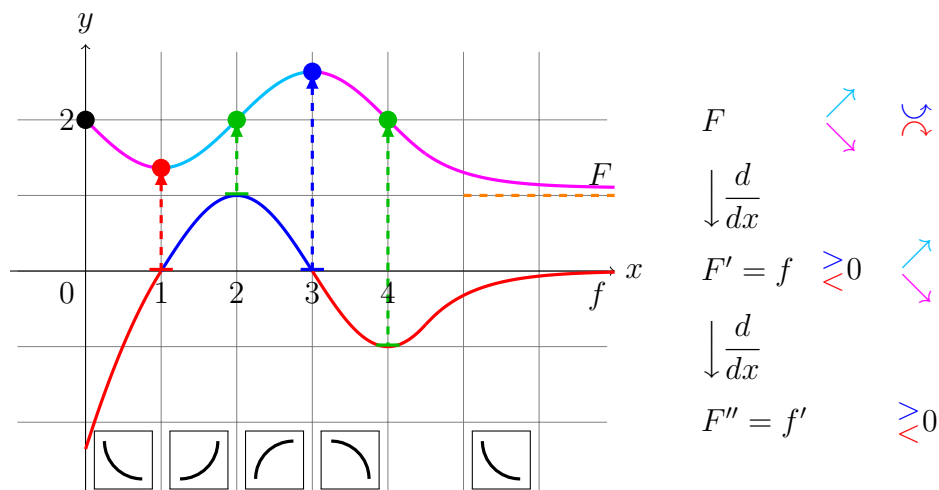
$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4. \quad \blacksquare$$

Example 0.5 Sketch an antiderivative F of a given f with $F(0) = 2$.

$f(1) = f(3) = 0$ and f change sign $- \rightarrow +$ at 1 and $+ \rightarrow -$ at 3
 $\Rightarrow F$ has local min at 1 and local max at 3.

$f'(2) = f'(4) = 0$ and f' change sign $+ \rightarrow -$ at 2 and $- \rightarrow +$ at 4
 $\Rightarrow F$ has IP at 2 and 4.

When $x \rightarrow \infty$, $f \rightarrow 0 \Rightarrow F \rightarrow c$ constant $\Rightarrow F$ has a H.A. $y = c$. \blacksquare



Additional: $f \rightarrow c \Rightarrow F \rightarrow cx + d \Rightarrow F$ has a S.A. $y = cx + d$.

Application: Rectilinear[ˈrɛktɪˈlɪnɪə] **motion** 直線運動

$s(t)$ position function, 位置

$v(t) = s'(t)$ velocity function, 速率

$a(t) = v'(t) = s''(t)$ acceleration[ækˌseləˈreɪʃən] function. 加速度

$s(t)$ is an antiderivative of $v(t)$, $v(t)$ is an antiderivative of $a(t)$.

Example 0.6 $a(t) = 6t + 4$, $v(0) = -6$ cm/s, $s(0) = 9$ cm. Find $s(t)$.

$$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C, v(0) = C = -6. \implies v(t) = 3t^2 + 4t - 6.$$

$$s(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6t + D = t^3 + 2t^2 - 6t + D, s(0) = D = 9.$$

$$\implies s(t) = t^3 + 2t^2 - 6t + 9.$$

Note: Gravity 重力加速度: -9.8 m/s² (往下).

Example 0.7 A ball is thrown upward with a speed of 15 m/s from the edge of cliff 140 m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?
 於 $(s(0) =) 140$ m 崖向上以 $(v(0) =) 15$ m/s 丟球. $s(t) = ?$ max at $t = ?$
 $s(t) = 0$, $t = ?$

$$a(t) = -9.8, v(t) = -9.8t + C,$$

$$v(0) = C = 15,$$

$$\implies v(t) = -9.8t + 15 \text{ m/s.}$$

$$s(t) = -4.9t^2 + 15t + D,$$

$$s(0) = D = 140,$$

$$\implies s(t) = -4.9t^2 + 15t + 140 \text{ m.}$$

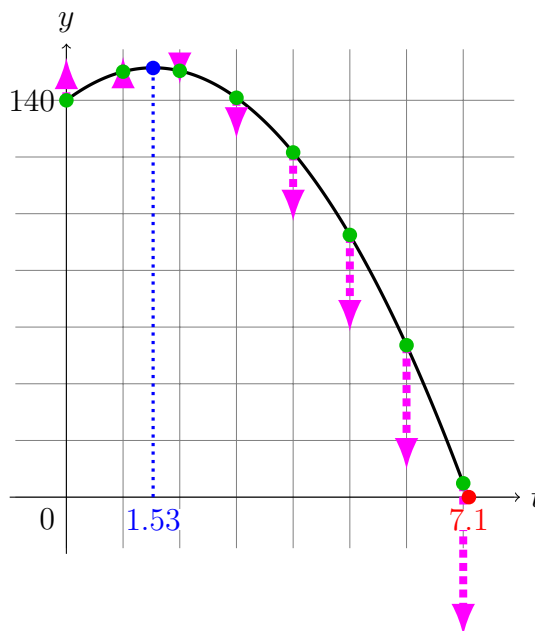
$$s'(t) = v(t) = 0$$

$$\text{when } t = \frac{15}{9.8} \approx 1.53 \text{ s.}$$

$$s(t) = -4.9t^2 + 15t + 140 = 0$$

$$\text{when } t = \frac{15 \pm \sqrt{2969}}{9.8} \approx 7.1 \text{ s.}$$

(負不合, negative time)



Ans: $s(t) = -4.9t^2 + 15t + 140$ m, $\frac{15}{9.8}$ s, $\frac{15 + \sqrt{2969}}{9.8}$ s.