510 lecture 2

Last time: X a topological space

A chart on X is a pair $(\mathcal{U}, \varphi)$ where $\mathcal{U} \subset X$ is open and $\varphi : \mathcal{U} \to \mathbb{C}$ is a homeomorphism $\mathcal{U} \to \varphi(\mathcal{U})$.

Two charts $(\mathcal{U}_0, \varphi_0)$ and $(\mathcal{U}_1, \varphi_1)$ are compatible if $\mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset$ or else

$$\varphi_1 \circ \varphi_0^{-1} : \varphi_0(\mathcal{U}_0 \cap \mathcal{U}_1) \to \varphi_1(\mathcal{U}_0 \cap \mathcal{U}_1)$$

is a holomorphic diffeomorphism.

Definition: An atlas is a collection $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts such that:

(i) $\forall \alpha, \beta$ $(\mathcal{U}_\alpha, \varphi_\alpha)$ and $(\mathcal{U}_\beta, \varphi_\beta)$ are compatible,

(ii) $\bigcup_\alpha \mathcal{U}_\alpha = X$, i.e. the domains of the charts cover X.

A separable metrizable space X together with an atlas on X determines a Riemann surface.

The function $\varphi_1 \circ \varphi_0^{-1}$ in the compatibility condition is called the transition function between the charts.

<u>lemma</u> let T be the transition function between two compatible charts. then the derivative of T is never zero on its domain.

<u>Pf</u> let $T = \varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_1) \to \varphi_1(U_0 \cap U_1)$
set $S = \varphi_0 \circ \varphi_1^{-1} : \varphi_1(U_0 \cap U_1) \to \varphi_0(U_0 \cap U_1)$

then $S \circ T = \mathrm{Id} : \varphi_0(U_0 \cap U_1) \to \varphi_0(U_0 \cap U_1)$
i.e. $S(T(w)) = w$ for all $w \in \varphi_0(U_0 \cap U_1) =$ domain
by chain rule
$$S'(T(w))\,T'(w) = 1 \quad \text{for all } w \in \text{domain}$$
So $T'(w) \neq 0$.

Suppose $p \in U_0 \cap U_1$ let z denote coordinate in $\varphi_0(U_0)$
let w denote coordinate in $\varphi_1(U_1)$

set $z_0 = \varphi_0(p)$ $w_0 = \varphi_1(p)$. then $T(w_0) = z_0$
The change of coordinates is $z = T(w)$, and we can expand
$$z = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n \quad \text{with } a_1 \neq 0 \qquad (*)$$

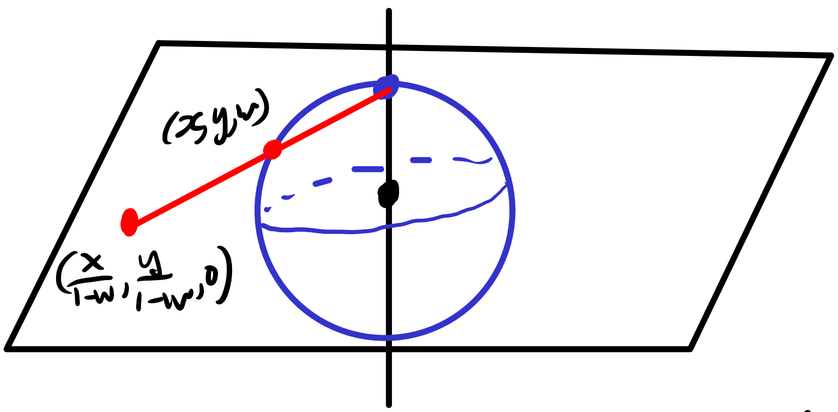
The "classical" way to think of a Riemann surface is as a space X where, near any point p, there are complex coordinate systems (z, w, others) such that any two coordinates whose domains of validity overlap are connected by analytic changes of coordinates like $(*)$.

<u>Example</u> 2-sphere. Let $X = S^2 = \{(x,y,w) \in \mathbb{R}^3 \mid x^2+y^2+w^2=1\}$ be the unit sphere in $\mathbb{R}^3$.

Consider $w=0$ plane as a copy of $\mathbb{C}$

$(x,y,0) \longleftrightarrow z = x+iy$.

Let $\varphi_1 : X \setminus \{(0,0,1)\} \to \mathbb{C}$ be the projection from $(0,0,1)$ onto the $w=0$ plane followed by the identification with $\mathbb{C}$.



$$\varphi_1(x,y,w) = \frac{x}{1-w} + i\frac{y}{1-w}$$

The inverse is $\varphi_1^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$

This shows that φ_1 is a homeomorphism $X \setminus \{(0,0,1)\} \to \mathbb{C}$

Let φ_2 be projection from $(0,0,-1)$ follow by complex conjugation.

$\varphi_2 : X \setminus \{(0,0,-1)\} \to \mathbb{C}$ $\quad \varphi_2(x,y,w) = \frac{x}{1+w} - i\frac{y}{1+w}$

$$\varphi_2^{-1}(z) = \left(\frac{2\,\mathrm{Re}(z)}{|z|^2+1}, \frac{-2\,\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right)$$

The overlap of the domains is $X \setminus \{(0,0,\pm1)\}$
and $\varphi_1(X \setminus \{(0,0,\pm1)\}) = \varphi_2(X \setminus \{(0,0,\pm1)\}) = \mathbb{C} \setminus \{0\}$
the transition function $T : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is $T(z) = \frac{1}{z}$

Since this function is a holomorphic diffeomorphism, these charts are compatible. Since their domains cover X, this is an atlas. Thus we have found a way of making $X = S^2$ into a Riemann surface.

Technical point: Equivalence classes of atlases, maximal atlases.

Giving a space X with an atlas $\mathcal{A}$ is one way of presenting a Riemann surface, but what is a Riemann surface?

The issue is, if we have an atlas, and we add a new chart that is compatible with all the charts in the atlas, this should really be the "same" Riemann surface.

Definition: Atlases $\mathcal{A}$ and $\mathcal{A}'$ on X are equivalent if every chart of $\mathcal{A}$ is compatible with every chart of $\mathcal{A}'$,

Definition An atlas $\mathcal{A}$ is maximal if, whenever $\mathcal{A}'$ is equivalent to $\mathcal{A}$, then $\mathcal{A}' \subset \mathcal{A}$. I.e., every chart compatible with $\mathcal{A}$ is already in $\mathcal{A}$.

Lemma Every atlas is equivalent to a unique maximal atlas.

Definition α: A Riemann surface is a separable metrizable topological space together with an equivalence class of atlases.

Definition β: A Riemann surface is a separable metrizable topological space together with a maximal atlas.

Every α-Riemann surface corresponds to a unique β-Riemann surface and vice versa. You choose which is the "official" definition.

Each atlas $\mathcal{A}$ on X determines a unique Riemann surface (with either definition).

- Other variations on the definition: requiring connectedness, weakening the 2nd-countability condition, equivalent pointset-topological assumptions, etc.

<u>Other kinds of manifolds:</u> A Riemann surface is a "one-dimensional complex manifold."
Other kinds of manifolds can be defined in a completely parallel way by changing the notion of chart and the notion of compatibility. The business about altases is then formally identical.

Type of manifold	Chart φ maps $U \subset X$ to ___	Transition $T = \varphi_1 \circ \varphi_0^{-1}$ is ___
topological	$\mathbb{R}^n$	homeomorphism
smooth C^∞	$\mathbb{R}^n$	C^∞ diffeomorphism
real analytic	$\mathbb{R}^n$	real analytic diffeo.
complex	$\mathbb{C}^n$	complex analytic diffeo
oriented	$\mathbb{R}^n$	Jacobian > 0
PL	$\mathbb{R}^n$	Piecewise linear homeo.
Affine	$\mathbb{R}^n$	Affine linear transformation
with volume measure	$\mathbb{R}^n$	Jacobian $= \pm 1$ (Moser)
symplectic	$\mathbb{R}^{2n}$	symplectic diffeo (Darboux)

It is now interesting to note that if we identify

$$\mathbb{C} \leftrightarrow \mathbb{R}^2$$
$$z = x+iy \leftrightarrow (x,y)$$

then a holomorphic diffeomorphism between open sets in $\mathbb{C}$ is also a C^∞-diffeomorphism between open sets in $\mathbb{R}^2$ (but the converse is not true)

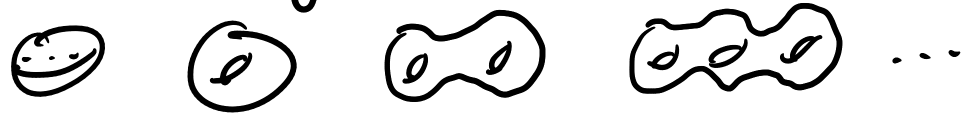
Thus any Riemann surface $(X, \mathcal{A})$ determines a smooth 2-manifold, but not vice versa. Indeed, several Riemann surfaces will determine the same smooth 2-manifold.

Also the Jacobian of a holomorphic diffeomorphism is always positive: $z = T(w)$
using $(\mathrm{Re}(z), \mathrm{Im}(z))$ and $(\mathrm{Re}(w), \mathrm{Im}(w))$ as real coords, we have

$$\text{Jacobian} = |T'(w)|^2 > 0$$

This means that a Riemann surface also has a prefered <u>orientation</u> and is orientable in particular.

Suppose now X is compact. Then the associated real 2-manifold is a compact orientable surface (2-manifold) compact orientable surfaces can be classified.
For each $g \geq 0$, there is a surface Σ_g



g = genus.

Every compact orientable surface is homeomorphic to Σ_g for some g.