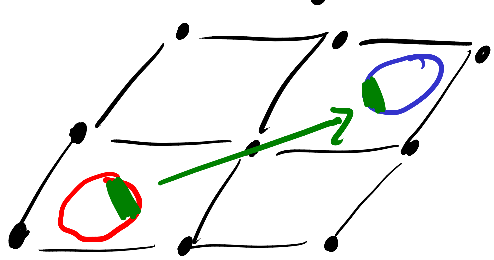
Complex tori: Let $L \subset \mathbb{C}$ be a lattice, i.e. an additive subgroup of the form
$$L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 \mid n, m \in \mathbb{Z}\}$$
such that ω_1 and ω_2 are linearly independent over $\mathbb{R}$.
($\omega_1 \neq 0$, $\omega_2 \neq 0$ and $\omega_1/\omega_2 \notin \mathbb{R}$)

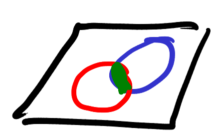
By homework, $L \subset \mathbb{C}$ is a discrete subset.
The quotient group $\mathbb{C}/L$ has a topology:
write $\pi : \mathbb{C} \to \mathbb{C}/L$. Then $U \subset \mathbb{C}/L$ is open iff $\pi^{-1}(U)$ is open.
By homework, this topological space is homeomorphic to $S^1 \times S^1$, a torus.
This doesn't depend on which L we choose.
However, the complex structure (Riemann surface structure)
Does depend on L (it turns out).

Charts: Given $z_0 \in \mathbb{C}/L$, choose a preimage $\tilde{z}_0 \in \pi^{-1}(z_0) \subset \mathbb{C}$.
Choose an open disk $D(\tilde{z}_0, \varepsilon)$ so small that no two elements of $D(\tilde{z}_0, \varepsilon)$ differ by an element of L. Then
$$\pi|_{D(\tilde{z}_0,\varepsilon)} : D(\tilde{z}_0, \varepsilon) \to \pi(D(\tilde{z}_0, \varepsilon)) \text{ is a homeomorphism.}$$

(Since bijective, continuous, open map)
Define a chart around z_0 as $\left(\pi(D(\tilde{z}_0,\varepsilon)), \left(\pi|_{D(\tilde{z}_0,\varepsilon)}\right)^{-1}\right)$

These charts are pairwise compatible: The transition function is a translation by an element of L.





This example has several direct generalizations:

(A) X a Riemann surface, G a group acting by holomorphic automorphisms: $G \times X \to X$.
If action is <u>properly discontinuous</u>:
$(\forall x \in X)(\exists\ U$ open, $x \in U$ such that $gU \cap U \neq \emptyset \Rightarrow g = id.)$
then X/G naturally has the structure of a Riemann surface

E.g. $X = \mathbb{C}$, $G = L$ acting by translation. $X/G = \mathbb{C}/L$

(B) Can consider $\mathbb{C}^N/L$ where $L \cong \mathbb{Z}\vec{\omega}_1 + \cdots + \mathbb{Z}\vec{\omega}_{2N}$ and $\{\vec{\omega}_i\}_{i=1}^{2N}$ are linearly independent over $\mathbb{R}$.
These are higher dimensional complex tori; in fact, there is such a torus attached to any compact Riemann surface, the Jacobian (more on this towards end of course).

A 1d complex torus $\mathbb{C}/L$ is sometimes called a <u>(complex) elliptic curve</u>.

<u>Projective space, projective curves:</u> Last time we constructed $\mathbb{CP}^1$

$\mathbb{CP}^1 = \{\ell \mid \ell \subset \mathbb{C}^2$ is 1-d subspace$\}$
$\ell = [z : w]$ where $(z,w) \in \ell$ is nonzero.

$U_0 = \{[z:w] \mid z \neq 0\}$ $\quad \varphi_0 : U_0 \to \mathbb{C}$ $\quad \varphi_0([z:w]) = \frac{w}{z}$

$U_1 = \{[z:w] \mid w \neq 0\}$ $\quad \varphi_1 : U_1 \to \mathbb{C}$ $\quad \varphi_1([z:w]) = \frac{z}{w}$

Transition function $T(\zeta) = \frac{1}{\zeta} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$.

A completely analogous construction starts with $\mathbb{C}^{n+1}$ instead of $\mathbb{C}^2$, and produces an n-dimensional complex manifold $\mathbb{CP}^n$, the complex projective space.

$$\mathbb{CP}^n = \{ \ell \mid \ell \subset \mathbb{C}^{n+1} \text{ is a 1-d subspace} \}$$

Any line ℓ is $\mathbb{C}\cdot(z_0, \dots, z_n)$ for some $(z_0, z_1, \dots, z_n) \neq \vec{0}$ in $\mathbb{C}^{n+1}$. Denote this line $[z_0 : \cdots : z_n]$. Then $[z_0 : z_1 : \cdots : z_n] = [z_0' : z_1' : \cdots : z_n']$ iff $\exists\, \lambda \in \mathbb{C}\setminus\{0\}$ such that $\forall\, i$ $z_i' = \lambda z_i$.

Whether the ith coordinate is zero or not does not depend on choice of representative, so define for $0 \le i \le n$

$$U_i = \{ [z_0 : z_1 : \cdots : z_n] \mid z_i \neq 0 \}$$

$$\varphi_i : U_i \to \mathbb{C}^n$$

$$\varphi_i([z_0 : z_1 : \cdots : z_n]) = \left(\frac{z_0}{z_i}, \frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \frac{z_n}{z_i} \right)$$

ith entry omitted

$$\varphi_i^{-1}(\zeta_0, \zeta_1, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_n) = [\zeta_0 : \zeta_1 : \cdots : \zeta_{i-1} : 1 : \zeta_{i+1} : \cdots : \zeta_n]$$

Transition map: $U_i \cap U_j = \{ [z_0 : z_1 : \cdots : z_n] \mid z_i \neq 0 \text{ and } z_j \neq 0 \}$

$$\varphi_j \circ \varphi_i^{-1}(\zeta_0, \zeta_1, \dots \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_n)$$

$$= \left(\zeta_0/\zeta_j, \zeta_1/\zeta_j, \dots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \dots, \frac{\zeta_{i-1}}{\zeta_j}, \frac{1}{\zeta_j}, \frac{\zeta_{i+1}}{\zeta_j}, \dots, \frac{\zeta_n}{\zeta_j} \right)$$

as map $\mathbb{C}^n \setminus \{\zeta_j = 0\} \to \mathbb{C}^n \setminus \{\zeta_i = 0\}$. Clearly holomorphic.

$n=2$: $\mathbb{CP}^2$ is <u>complex projective plane</u>.

Homogeneous polynomials: let $F(z_0, z_1, \dots, z_n)$ be a polynomial in $n+1$ variables. It is a function on $\mathbb{C}^{n+1}$, but not a function on $\mathbb{CP}^n$, since it may not be constant on lines $\ell \subset \mathbb{C}^{n+1}$!

Suppose F is <u>homogeneous</u> of degree d. This means the sum of the exponents in each term is d: eg

$$z_0^4 + z_1 z_2^3 + z_0 z_1 z_2 z_3 \quad (d=4)$$

Then $F(\lambda z_0, \lambda z_1, \dots, \lambda z_n) = \lambda^d F(z_0, z_1, \dots, z_n)$ any $\lambda \in \mathbb{C} \setminus \{0\}$.
so $F(z_0, z_1, \dots, z_n) = 0 \iff F(\lambda z_0, \lambda z_1, \dots, \lambda z_n)$.

If F vanishes at a point it vanishs on the line through that point. So it make sense to define

$$V(F) = \{ [z_0 : z_1 : \cdots : z_n] \mid F(z_0, z_1, \dots, z_n) = 0 \} \subset \mathbb{CP}^n$$

this is the <u>locus of zeros of F</u>.

A subset of $\mathbb{CP}^n$ obtained as $V(F_1) \cap V(F_2) \cap \cdots \cap V(F_r)$ for $F_1, F_2, \dots, F_r$ homogeneous polynomials (of various degrees) is called a <u>complex projective variety</u>.

If this set is a manifold, we call it <u>smooth</u>.
If $n=2$, $V(F) \subset \mathbb{CP}^2$ is a <u>projective plane curve</u>.
when smooth, it is a Riemann surface.

Affine charts, dehomogenization.

Let $F(z_0, z_1, \dots, z_n)$ be homogeneous of degree d.
$V(F) \subset \mathbb{CP}^n$.
Choose i $0 \le i \le n$, and consider the chart $U_i = \{[z_0 : \cdots : z_n] \mid z_i \neq 0\}$
$\varphi_i : U_i \to \mathbb{C}^n$ is an <u>affine chart</u>.

if $(\zeta_0, \zeta_1, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_n)$ denote coords on $\mathbb{C}^n$,
then the equation $F(z_0, z_1, \dots, z_n) = 0$ goes over to

$$f_i(\zeta_0, \dots, \zeta_n) = F(\zeta_0, \zeta_1, \dots, \zeta_{i-1}, 1, \zeta_{i+1}, \dots, \zeta_n) = 0$$

where $f_i : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.
The intersection $V(F) \cap U_i$ is therefore essentially the <u>affine variety</u>
$V(f_i) = \{ f_i = 0 \} \subset \mathbb{C}^n$.

→ To get affine equation (in ith chart), replace z_i with 1. ←
(dehomogenization)

The projective variety $V(F)$ is smooth iff all of the affine varieties
$V(f_i) \subset \mathbb{C}^n$ $(0 \le i \le n)$ defined by the dehomogenizations are smooth

$$V(F) \text{ smooth} \iff \forall\, 0 \le i \le n \quad V(f_i) \text{ smooth.}$$

By implicit function theorem,

$V(f_i)$ smooth $\iff$ at every point of $V(f_i)$, some partial $\dfrac{\partial f_i}{\partial \zeta_k} \neq 0$.

In principle, we could check this for a given equation F, but there is an easier criterion:

<u>Lemma</u> (Cf Lemma I.3.5) $V(F)$ is smooth iff the only solution to the system of $n+2$ equations in $n+1$ variables

$$\left\{ F=0, \ \frac{\partial F}{\partial z_i}=0 \ (0 \leq i \leq n) \right\}$$

is $(z_0, z_1, \dots, z_n) = (0, 0, \dots, 0)$.

Put another way, $V(F) \subset \mathbb{CP}^n$ is smooth as long as the <u>affine variety</u> $\{F=0\} \subset \mathbb{C}^{n+1}$ is smooth except at the origin.

<u>example</u>: $F = x^2+y^2+z^2$ $\quad \frac{\partial F}{\partial x} = 2x \quad \frac{\partial F}{\partial y} = 2y \quad \frac{\partial F}{\partial z} = 2z$

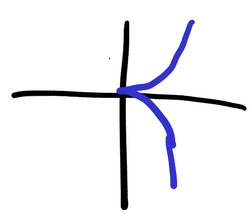
All vanish only at $(x,y,z)=0 \Rightarrow \{[x:y:z] \mid x^2+y^2+z^2=0\} \subset \mathbb{CP}^2$ is smooth.

So this is a Riemann surface.

$F = y^2 z - x^3$. $\quad \frac{\partial F}{\partial x} = -3x^2, \ \frac{\partial F}{\partial y} = 2yz, \ \frac{\partial F}{\partial z} = y^2$

All vanish if $x=y=0$, so $V(F) \subset \mathbb{CP}^2$ is singular at $[0:0:1] \in \mathbb{CP}^2$. In affine chart where $z=1$,

the affine equation is $y^2 = x^3$



This curve has a cusp.