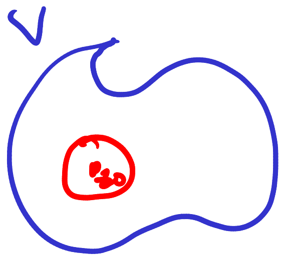
Holomorphic and meromorphic functions

<u>Theory on $\mathbb{C}$:</u> Let $V \subset \mathbb{C}$ be open, $f: V \to \mathbb{C}$ functn. Take $z_0 \in V$. Recall that f is <u>holomorphic</u> at z_0 if it has a convergent power series expansion



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{valid in a disk around } z_0$$

f is holomorphic on V if it is holomorphic at every point of V.

Suppose now f is only defined on $V \setminus \{z_0\}$, but it is holomorphic throughout $V \setminus \{z_0\}$. Then there are three possibilities for the behavior of f at z_0:

① f has <u>a removable singularity</u> at z_0: there is a holomorphic function $\tilde{f}: V \to \mathbb{C}$ such that $f = \tilde{f}|_{V \setminus \{z_0\}}$

② f has a <u>pole at z_0</u>: There is $n \in \mathbb{N}$ such that $(z - z_0)^n f(z)$ has a removable singularity at z_0.

③ f has an <u>essential singularity</u> at z_0:

Example: $\exp\left(\frac{1}{z - z_0}\right)$.

The function $f: V \setminus \{z_0\} \to \mathbb{C}$ has a <u>Laurent series</u> at z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

defined by $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$ C a circle enclosing z_0.

The Laurent series converges within an annulus $\{r < |z-z_0| < R\}$, which may be empty.

① f has removable singularity at $z_0 \iff$ no negative degree terms.
② f has pole at $z_0 \iff$ only finitely many negative degree terms.
③ f has essential singularity at $z_0 \iff$ infinitely many nonzero negative degree terms.

Pole case, write $f(z) = \sum_{n=N}^{\infty} a_n (z-z_0)^n$.

A function is <u>meromorphic</u> if its only singularities are poles (or removable). Usually if a function has a removable singularity we will tacitly identify it with its holomorphic extension.

Now let X be a Riemann surface, $W \subset X$ an open set, $p \in W$, $f: W \to \mathbb{C}$ a function. The idea is that the charts on X allow us to represent f locally as a function from an open set in $\mathbb{C}$.

<u>Definition</u> <u>f is holomorphic at p</u> if there is a chart $\varphi: U \to V \subseteq \mathbb{C}$ such that $p \in U$ and the function

$$f \circ \varphi^{-1} : \varphi(U \cap W) \to \mathbb{C} \quad \text{is holomorphic at } \varphi(p)$$

f is holomorphic in W if it is holomorphic at each point of W.

<u>Lemma</u> (a) f is holomorphic at p iff for <u>every</u> chart $\varphi: U \to V \subseteq \mathbb{C}$ with $p \in U$, the function $f \circ \varphi^{-1}: \varphi(U \cap W) \to \mathbb{C}$ is holomorphic at p

(b) f is holomorphic in W iff there is a set of charts $\{\varphi_i : U_i \to V_i\}$ with $W \subseteq \bigcup U_i$ such that for all i $f \circ \varphi_i^{-1}: \varphi(U_i \cap W) \to \mathbb{C}$ is holomorphic on its domain.

(c) if f is holomorphic at p, f is holomorphic in a neighborhood of p.

<u>Proof</u>: (a) suppose $\varphi_1 : U_1 \to V_1$ and $\varphi_2 : U_2 \to V_2$ are two charts with $p \in U_1$, $p \in U_2$ then

$$(f \circ \varphi_1^{-1}) = (f \circ \varphi_2^{-1})(\varphi_2 \circ \varphi_1^{-1})$$

since the transition function $T = \varphi_2 \circ \varphi_1^{-1}$ is assumed to be holomorphic (compatibility of charts), $(f \circ \varphi_1^{-1})$ will be holomorphic if $f \circ \varphi_2^{-1}$ is (composition of holomorphic is holomorphic).

(b) clear from def. (c) follows from analogous property in $\mathbb{C}$.

<u>Notation</u> X a R.S., $W \subset X$ open.

$$\mathcal{O}_X(W) = \mathcal{O}(W) = \{ f : W \to \mathbb{C} \mid f \text{ is holomorphic} \}$$

$\mathcal{O}_X(W)$ is a ring [The function $W \mapsto \mathcal{O}_X(W)$ is a sheaf of rings on X]

Meromorphic functions on X are defined similarly:

Let $W \subset X$, $p \in W$, and suppose $f : W \setminus \{p\} \to \mathbb{C}$ is holomorphic. We say f has a removable/pole/essential singularity at p if there is a chart $\varphi: U \to V$ with $p \in U$ such that $f \circ \varphi^{-1}$ has that type of singularity.

We need to see that the type of singularity does not depend on which chart we use to check it. For this, it is convenient to use an alternative characterization in terms of growth rates:

For $f: V \setminus \{z_0\} \to \mathbb{C}$ ($V \subseteq \mathbb{C}$ open)

z_0 is removable singularity $\iff \lim_{z \to z_0} |f(z)| = c < \infty$.

z_0 is pole $\iff \lim_{z \to z_0} |f(z)| = +\infty$

z_0 is essential singularity $\iff \lim_{z \to z_0} |f(z)|$ does not exist.

Considering $(f \circ \varphi_1^{-1}) = (f \circ \varphi_2^{-1})(\varphi_2 \circ \varphi_1^{-1})$ $\qquad f: W \setminus \{p\} \to \mathbb{C}$,

because $\varphi_2 \circ \varphi_1^{-1}$ is a homeomorphism, the behaviour of $|f \circ \varphi_1^{-1}|$ and $|f \circ \varphi_2^{-1}|$ must be the same at corresponding points. Thus the type of singularity does not depend on the chart chosen.

A function $f: W \to \mathbb{C}$, $W \subseteq X$ open, X R.S. is called meromorphic if its only singularities are poles (or removable).

Set $\mathcal{M}_X(W) = \mathcal{M}(W) = \{ f: W \to \mathbb{C} \mid f \text{ meromorphic on } W \}$

$\mathcal{M}_X(W)$ is a <u>field</u>. $\quad \mathcal{M}_X(X) = \{\text{global meromorphic functions}\}$ is called the <u>function field</u> of X.

Order of a meromorphic function at a point.

Let $f(z) = \sum a_n (z - z_0)^n$ be meromorphic at z_0.

Define $\mathrm{ord}_{z_0}(f) = \min\{n \mid a_n \neq 0\} \in \mathbb{Z}$.

If f is a meromorphic function on a Riemann surface X, $p \in X$, define

$$\mathrm{ord}_p(f) = \mathrm{ord}_{z_0}(f \circ \varphi^{-1})$$

where $\varphi : U \to V$ $p \in U$ is some chart. $z_0 = \varphi(p)$

We need to check this does not depend on the chart.
Suppose $\psi : U' \to V'$ is another chart near p. $w_0 = \psi(p)$

$$\left.\begin{aligned} f \circ \varphi^{-1}(z) &= \sum c_n (z - z_0)^n \\ f \circ \psi^{-1}(w) &= \sum d_n (w - w_0)^n \end{aligned}\right\} \text{ two local coordinate representations of } f.$$

The change of variables has the form

$$z - z_0 = \sum_{n \geq 1} a_n (w - w_0)^n \qquad a_1 \neq 0.$$

Suppose $\sum c_n (z - z_0)^n = c_{n_0}(z - z_0)^{n_0} + (\text{higher order terms})$ $(c_{n_0} \neq 0)$

$$= c_{n_0} a_1^{n_0} (w - w_0)^{n_0} + (\text{higher order terms}) = \sum d_n (w - w_0)^n$$

since $c_{n_0} \neq 0$ and $a_1 \neq 0$, we find $d_{n_0} = c_{n_0} a_1^{n_0} \neq 0$

So order is the same in both Laurent series. ∎

<u>Lemma</u> Suppose f is meromorphic at p

f is holomorphic at $p \iff \mathrm{ord}_p(f) \geq 0$

f is holomorphic at p and $f(p)=0 \iff \mathrm{ord}_p(f) > 0$

f has pole at $p \iff \mathrm{ord}_p(f) < 0$

$\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g) \qquad (f \neq 0, g \neq 0)$

$\mathrm{ord}_p(f/g) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$

$\mathrm{ord}_p(1/f) = -\mathrm{ord}_p(f)$

$\mathrm{ord}_p(f \pm g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$

equality holds as long as $\mathrm{ord}_p(f)$ and $\mathrm{ord}_p(g)$ differ.