Riemann Surfaces and Algebraic Curves   Lecture 1

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Riemann surfaces are abstract spaces that "locally look like" the complex plane $\mathbb{C}$.
To say precisely what this means, we need some foundational background material from general topology.

<u>Definition</u>: A <u>metric space</u> is a set X endowed with a function $d : X \times X \to \mathbb{R}$ satisfying for all x, y, z:
(i) $d(x,y) = d(y,x)$ (symmetry)
(ii) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$. (positivity)
(iii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

We think of $d(x,y)$ as the distance between x and y.

<u>Examples</u>: $X = \mathbb{R}$, $d(x,y) = |x-y|$
$X = \mathbb{C}$, $d(x,y) = |x-y| = \sqrt{(x-y)(\bar{x}-\bar{y})}$

$X = \mathbb{R}^n$, $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\| = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}$

If (X,d) is a metric space, and $Y \subset X$ is a subset, then $(Y, d|_{Y\times Y})$ is a metric space.

If (X,d) is a metric space, the <u>open ball</u> with center x_0 and radius $r>0$ is $B_r(x_0) := \{x \in X \mid d(x,x_0) < r\}$
A general subset $Y \subset X$ is called <u>open</u> (w.r.t. d) if for all $x_0 \in Y$, $\exists\, r > 0$ such that $B_r(x_0) \subset Y$.

The complement of an open set is a <u>closed</u> set.
If $X = \mathbb{R}^n$, these notions agree with the usual ones from real analysis, and the theory of open/closed sets in a general metric space is rather similar to that for $\mathbb{R}^n$.

Often, we do not actually care about the metric d, but only what the class of open set is.
As preview of things to come consider the following: Any open subset $U \subset \mathbb{C}$ is a metric space, and is in fact a Riemann surface. The natural notion of isomorphism for metric spaces is distance-preserving bijection (isometry) But complex analysis teaches us that two open sets $U_1, U_2 \subset \mathbb{C}$ are to be considered equivalent if there is a "conformal mapping" $\varphi : U_1 \to U_2$ (φ is a bijection such that φ and φ^{-1} are holomorphic) Such conformal mappings are <u>not</u> usually isometries.

The concept of a "topology" axiomatizes the properties of open sets.

<u>Definition</u>: Let X be a set. A <u>topology</u> on X is a set $\mathcal{T}$ of subsets of X, $\mathcal{T} \subset \mathcal{P}(X)$ such that

(i) $\emptyset \in \mathcal{T}$ $X \in \mathcal{T}$
(ii) if $U_1, \ldots, U_k \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_k \in \mathcal{T}$
(iii) if $\{U_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ is any subcollection (possibly infinite)

then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

A set X equipped with a topology is called a <u>topological space</u>.

A metric induces a topology: If (X,d) is a metric space, define $\mathcal{T} = \{ U \subset X \mid U \text{ is open w.r.t. } d \}$

About the converse direction, if $(X, \mathcal{T})$ is a topological space, it is not necessarily possible to find a metric d that induces $\mathcal{T}$. Even if such a d does exist, it will not be unique.

E.g. $X = \mathbb{R}^n$ $\quad d_p(\vec{x}, \vec{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^p \right)^{1/p}$

induces same topology for all $1 \leq p < \infty$.

If a metric d inducing a given topology $\mathcal{T}$ does exist, we say that $\mathcal{T}$ is <u>metrizable</u>.

All topological spaces in this course are metrizable.

This means we can justify statements about open sets as if we are working in a metric space. A very important consequence of being metrizable is:

<u>Hausdorff property</u>: $\forall x, y \in X$, if $x \neq y$ then there are open sets $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

<u>Theorem</u> Any metrizable topological space has the Hausdorff property.

<u>Proof</u>: Pick a metric d inducing $\mathcal{T}$. If $x \neq y$, then $r = \frac{d(x,y)}{2} > 0$. Take $U = B_r(x)$ and $V = B_r(y)$. ∎

For some technical reasons, we also want to impose the condition that our topological spaces are "separable"

A set $Y \subset X$ is called dense if the only closed set containing Y is X itself. X is called separable if it has a countable dense subset.

Scholium: (separable + metrizable) $\Longleftrightarrow$ (regular + Hausdorff + 2nd countable)

Last recall: $f : X \to Y$ is continuous if $f^{-1}(U) \subset Y$ is open whenever $U \subset X$ is open. f is a homeomorphism if f is a bijection and f and f^{-1} are continuous.

To get a Riemann surface, we start with a separable metrizable topological space $(X, \mathcal{T})$ and then add another layer of structure that encodes a notion of "local complex coordinates" on X. Complex coordinates are "compatible" if they are related by a holomorphic mapping. To wit, recall the notions of analytic / holomorphic functions:

① An analytic function $f(z)$ has a convergent power series expansion centered at any point z_0 in its domain: For each z_0, there exists $R>0$ s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ is valid for } |z - z_0| < R$$

When z represents a complex variable, f is called complex analytic.

(2) A function $f(z)$ of a complex variable z is <u>complex differentiable</u> if for any z in the domain,

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h} \text{ exists.}$$

(3) A function $f(z)$ is a <u>solution to the Cauchy-Riemann eqn</u> if, by regarding $f(z) = f(x+iy)$ as a function of two real variables $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous, and

$$\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0 \qquad \text{(CR equ)}$$

The foundational theorems of complex analysis say that (1), (2), (3) are all equivalent. We use the word "<u>holomorphic</u>" to denote this condition. The point is that is easy to check (3), but (1) is a very strong property.

<u>Definition</u> Suppose $\mathcal{U}_1$ and $\mathcal{U}_2$ are open sets in $\mathbb{C}$. A function $\varphi : \mathcal{U}_1 \to \mathcal{U}_2$ is a <u>holomorphic diffeomorphism</u> if φ is a bijection, φ is holomorphic, and φ^{-1} is holomorphic.

Note that holomorphic functions are continuous, so a holomorphic diffeomorphism is in particular a homeomorphism. Holomorphic diffeomorphisms are also called "conformal maps" since they preserve angles.

Definition Let X be a topological space.
A (Riemann surface) **chart** on X is a pair (U, φ) where $U \subset X$ is an open set, and $\varphi : U \to \mathbb{C}$ is a function such that φ is a homeomorphism $U \to \varphi(U)$.

Definition X as above. Let (U_0, φ_0) and (U_1, φ_1) be two charts. They are **compatible** if either

- $U_0 \cap U_1 = \emptyset$ or
- $U_0 \cap U_1 \neq \emptyset$ and the function

$$\left(\varphi_1|_{U_0 \cap U_1}\right) \circ \left(\varphi_0|_{U_0 \cap U_1}\right)^{-1} : \varphi_0(U_0 \cap U_1) \to \varphi_1(U_0 \cap U_1)$$

is a holomorphic diffeomorphism.

The picture: