Examples of Riemann surfaces.

To construct examples in a consise manner, it is useful to streamline the process so that we only have to check a minimum of things.

<u>Point-set topology</u>: A space X is <u>regular</u> if any point $x_0 \in X$ any closed set $Y \subset X$ with $x_0 \notin X$ can be separated by open sets ($\exists\, U, V$ open $x_0 \in U$, $Y \subset V$, $U \cap V = \emptyset$)

<u>Lemma</u> Let X be a Hausdorff space, and suppose $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$ is a <u>countable</u> atlas (of charts $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$) Then necessarily X is metrizable and separable.

<u>Proof</u>: The existence of the charts $\varphi : U_\alpha \to V_\alpha$ shows that X is locally metrizable, since each $V_\alpha \subset \mathbb{C}$ is metrizable. This implies that X is regular.
Also, each U_α is 2^{nd} countable. Since there only countably many U_α's, X is 2^{nd} contable. Apply Urysohn metrization theorem: Hausdorff + regular + 2^{nd} countable $\Rightarrow$ metrizable.
Lastly, for a metrizable space the conditions of being separable and being 2^{nd} countable are equivalent.

<u>Remark</u>: Every (separable metrizable) Riemann surface admits a countable atlas: take charts containing each point of a countable dense subset.

So to define a Riemann surface, it suffices to construct a Hausdorff space and a countable atlas on it.

Next, we note that since $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$ is a homeomorphism onto an open set in $\mathbb{C}$, the charts can be used to determine the topology.

Here is the prescription:

(1) Take a set X

(2) Take a <u>countable</u> collection of sets $U_\alpha \subset X$ that cover X $\bigcup_\alpha U_\alpha = X$.

(3) For each α, take an open set $V_\alpha \subset \mathbb{C}$ and a bijection $\varphi_\alpha : U_\alpha \to V_\alpha$.

(4) Define a topology T_α on U_α by declaring that φ_α is homeomorphism.

(5) Define a topology T_X on X by declaring $U \subset X$ is open iff $U \cap U_\alpha \in T_\alpha$ for every α.

(6) It is not necessarily true that T_α is the same as the subspace topology on U_α as a subspace of (X, T_X). However, this is true under the following condition.

$$\forall \alpha, \beta \quad \varphi_\alpha (U_\alpha \cap U_\beta) \text{ is open in } \mathbb{C}.$$

We must check this condition. Then $(U_\alpha, \varphi_\alpha)$ is a chart.

(7) Check that the charts $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ are pairwise compatible.

(8) Check that X is Hausdorff.

In short, we need to supply set theoretic data $X, U_\alpha, \varphi_\alpha$, and then check the things mentioned in (6)(7)(8).

<u>The projective line.</u> $\mathbb{C}^2$ = 2-dimensional complex vector space

Define set $X = \mathbb{CP}^1 = \{ L \subset \mathbb{C}^2 \mid L \text{ is a 1-dimensional } \mathbb{C}\text{-subspace} \}$

if $(z,w) \neq (0,0)$, then $\mathbb{C}\cdot(z,w)$ is a 1-d subspace.
We denote this subspace by $[z:w]$.
Observe
$$[z:w] = [\lambda z : \lambda w] \quad \text{for any } \lambda \in \mathbb{C} \setminus \{0\}.$$
and indeed $[z:w] = [z':w']$ iff $[z':w'] = [\lambda z : \lambda w]$ for some λ.

Take subsets
$$\mathcal{U}_0 = \{ [z:w] \mid z \neq 0 \} \subset \mathbb{CP}^1$$
$$\mathcal{U}_1 = \{ [z:w] \mid w \neq 0 \} \subset \mathbb{CP}^1$$

Define
$$\varphi_0 : \mathcal{U}_0 \to \mathbb{C} \qquad \varphi_0([z:w]) = w/z$$
$$\varphi_1 : \mathcal{U}_1 \to \mathbb{C} \qquad \varphi_1([z:w]) = z/w$$

These are well-defined since rescaling z and w by same λ doesn't change ratio. They are also bijections:
$$\varphi_0^{-1}(x) = [1:x] \qquad \varphi_1^{-1}(x) = [x:1]$$

Need to check:
$$\varphi_0(\mathcal{U}_0 \cap \mathcal{U}_1) = \mathbb{C} \setminus \{0\}$$
$$\varphi_1(\mathcal{U}_0 \cap \mathcal{U}_1) = \mathbb{C} \setminus \{0\}$$
which are open in $\mathbb{C}$. ✓

Transition function $\varphi_1 \circ \varphi_0^{-1}(x) = \varphi_1([1:x]) = \frac{1}{x}$
which is holomorphic as a map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ ✓

Is $\mathbb{CP}^1$ Hausdorff? Let $p, q \in \mathbb{CP}^1$. If both in same chart, we can separate them by disks in that chart. Only other possibility is $p = [1:0]$ and $q = [0:1]$. Then small disk around p in $\mathcal{U}_0$ and around q in $\mathcal{U}_1$ do the trick. ✓

<u>Graph of a holomorphic function.</u> Let $V \subset \mathbb{C}$ be open, and let $g : V \to \mathbb{C}$ be a holomorphic function.
Let $X = \{(z, g(z)) \mid z \in V\} \subset \mathbb{C}^2$.
Define a chart $\mathcal{U} = X$ $\varphi : \mathcal{U} \to V$ $\varphi(z,w) = z$
this is a bijection, as the inverse is $\varphi^{-1}(z) = (z, g(z))$.
Observe that φ is a homeomorphism from the subspace topology on $X \subset \mathbb{C}^2$ to V. Everything we need to check is clear.

<u>Riemann surface of a multivalued function:</u>
Suppose g is a "multivalued" holomorphic function e.g. $g(z) = \log z$ or $g(z) = z^\alpha$. We can define a subset of $\mathbb{C}^2$:

$$X = \left\{ (z,w) \;\middle|\; \begin{array}{l} w \text{ is a value of some single-} \\ \text{-valued branch of } g(z) \end{array} \right\} \subset \mathbb{C}^2$$

We can define charts on X by restricting to $z \in V$ where g has a single-valued branch $\tilde{g}$ on V. Set $\mathcal{U} = \{(z, \tilde{g}(z)) \mid z \in V\}$, and $\varphi : \mathcal{U} \to V$ projection to first coordinate.
The topology this process defines on X does not necessarily agree with the subspace topology of $X \subset \mathbb{C}^2$.

If we apply this to $g(z) = \log z$, then we get
$X = \{(z,w) \mid z = \exp(w)\}$, which is a graph "the other way".

If we apply this to $g(z) = \sqrt{z}$ we get
$X = \{(z,w) \mid z = w^2, z \neq 0\}$ The point $z = 0$ $w = 0$ is missing because there is no single valued branch of $\sqrt{z}$ around $z = 0$. $z = 0$ is called a <u>branch point</u>.

Historically, this was a motivation for the theory of Riemann Surfaces.

Affine plane curves Let $f(z,w) \in \mathbb{C}[z,w]$ be a polynomial in two variables with complex coefficients.

The zero locus of f is the set $X = \{(z,w) \mid f(z,w) = 0\} \subset \mathbb{C}^2$.

Implicit function theorem: Suppose $(z_0, w_0) \in X$, and $\frac{\partial f}{\partial w}(z_0, w_0) \neq 0$. Then there is a unique holomorphic function $g(z)$ defined in an open set $V \ni z_0$ such that $g(z_0) = w_0$ and $f(z, g(z)) = 0$ for all $z \in V$.

Near (z_0, w_0), X coincides with the graph of $g(z)$.

How to remember the theorem:

Differentiate $f(z,w) = 0$

$$\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial w} dw = 0$$

Try to solve for dw in terms of dz: $dw = -\left(\frac{\partial f}{\partial w}\right)^{-1} \frac{\partial f}{\partial z} dz$

Need $\frac{\partial f}{\partial w} \neq 0$ to do this.

IFT says that, if you can solve for dw in terms of dz, you can solve for w in terms of z locally.

Similarly: $\frac{\partial f}{\partial z} \neq 0 \Rightarrow$ solve for dz in terms of dw

$\Rightarrow$ solve for z in terms of w locally.

The polynomial $f(z,w)$ is nonsingular if, for every $(z_0, w_0) \in X = \{f = 0\}$ at least one of $\frac{\partial f}{\partial w}(z_0, w_0)$ and $\frac{\partial f}{\partial z}(z_0, w_0)$ is non zero.

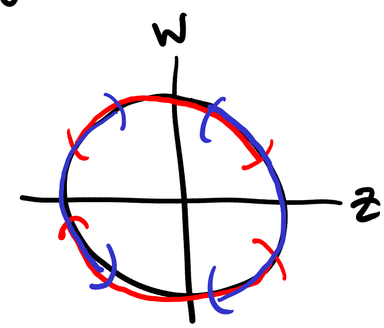
I.e., $f, \frac{\partial f}{\partial w}, \frac{\partial f}{\partial z}$ do not all vanish at same point.

Proposition Let $f(z,w)$ be a nonsingular polynomial. Then $X = \{(z,w) \mid f(z,w)=0\} \subset \mathbb{C}^2$ has a Riemann surface structure.

The atlas is constructed by systematically applying the implicit function theorem. At points where $\frac{\partial f}{\partial w} \neq 0$, we find $V \subset \mathbb{C}$ and $g : V \to \mathbb{C}$ such that locally
X is the graph $\{(z, g(z)) \mid z \in V\}$ (φ = projection to first coord)
At points where $\frac{\partial f}{\partial z} \neq 0$, we find $V \subset \mathbb{C}$ and $h : W \to \mathbb{C}$ so that locally
X is the "other way" graph $\{(h(w), w) \mid w \in W\}$
(φ = projection to second coordinate)

It is easy to see these charts are compatible, see Miranda.

Picture



$$f(z,w) = z^2 + w^2 - 1$$

$$\frac{\partial f}{\partial z} = 2z \quad \frac{\partial f}{\partial w} = 2w$$

so nonsingular.

"$w = \pm\sqrt{1-z^2}$" let V =

branch cuts

Then $\sqrt{1-z^2}$ has two single valued branches in this domain.
By moving the branch cuts, can cover all points except $w=0, z=\pm 1$.

Swap roles of w, z, can use similar charts to cover all points except $z=0, w=\pm 1$. Ultimately we cover everything.