Natural Intrinsic Period of collective motion

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Phase transition of collective behavior ranging from microscopic to macroscopic level is typically studied by observing the designed order parameter, which ignores catching the critical subtle mutagenesis process of symmetry breaking. In this Letter, we introduce the conception of *natural intrinsic period* and establish the theoretical framework for dynamics describing general collective behaviors, which helps accurately estimate the transformation speed of phase transition and determine the ultimate phases under different parameter selections. We find that the dynamical evolution of collective behavior can be directly quantified, and the various phases can be directly detected without designing the order parameters and running simulations. Our theory with numerical examples may provide a specific, succinct, and effective analysis framework for investigating collective behaviors.

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The appealing natural collective behaviors have been widely studied in the past few decades. Simple interaction rules propelling individual particles have been proposed, which are sufficiently realistic to reproduce numerous observed phenomena and hence are beneficial for understanding the biological coordination mystery. As a pioneer work, Vicsek et al. proposed the well known self-propelled particle (SPP) model, where phase transitions from disordered swarm behavior to highly coordinating flocking behavior were observed via the designed consensus order parameter [1]. Following the research line, a series of studies [2–4] assumed that the movement direction of each individual is determined by the average direction of the neighbors within a metric space. However, in the absence of the symmetric interaction, other models [5–7] that individuals only use the instantaneous position information in their vision range for navigation have been proposed. Measured by different order parameters, these models can exhibit various patterns of collective behavior by slightly tweaking the system variables.

The concept of order parameter has been successfully and widely used in characterizing different equilibrium or non-equilibrium system states, such as quantitative description of the degree of order, the degree of aggregation, and the various dynamical patterns [8]. However, while studying the evolution of the SPP models, many complicated configurations of the particles, such as the milling-like or even complicated curved-“8”-like configurations, are still difficult to be identified by typical order parameters [9, 18], although the change of configuration patterns can be obviously observed. This is mainly due to the fact that the order parameter is designed for the targeted observed configuration, which lags behind the observation. The limitation of accurately discriminating different configurations or phases by designing order parameters has become the bottleneck problem in the study of collective behavior. This motivates us to investigating the approach to directly quantify the phase transition and determine the ultimate phase without designing the various forms of order parameter.

In this Letter, we aim at developing a theoretical framework for directly quantifying the phase transition and determine the ultimate phases without designing various targeted forms of order parameter. We started from using stochastic differential equation (SDE) to describe general dynamic collective behavior, and adequately considered the effects of individual visual cones and system noise [11–13]. We introduced the theory of mesoscopic systems [14] and further developed the estimation method based on Fokker-Planck equation [15]. We extracted the uniform key property of the dynamical system and proposed the definition of natural intrinsic period (abbr., NIP), which can directly quantify the effects of the driven and stochastic terms on the system evolution, without being limited by the different dynamical forms. Meanwhile, we find the coefficient of variation (CoV), may well describe the time of formation and dispersion required by the particle group.

We study a general second-order SDE describing the collective behavior of a group of particles, and the i th agent follows

$$\begin{aligned} \dot{x_i} &= v_0 V(\theta_i), \\ \dot{\theta_i} &= F(\theta) + 2\sqrt{D_\theta}\xi_i(t), \end{aligned} \tag{1}$$

where x_i, v_0, θ_i mean the position, speed, movement angle, respectively, $F(\theta)$ represents the driving term and $2\sqrt{D_\theta}\xi_i(t)$ denotes a stochastic process with D_θ being the magnitude of noise with $\xi_i(t)$ is a random variable satisfies $N(0,\sigma^2)$. We will later explain why the model focuses on the direction angle θ rather than other parameters. By specific designing the format of $F(\theta)$, the SPP model (1) will exhibit diverse phase transition phenomena.

Inspiringly, when the particle group shapes as a closed chain, such as $Torus$, *Dumbbell*, or $Twist$, the shaping time is almost a fixed periodic value, whereas it goes to infinite when forming a *worm*. Particularly, when the

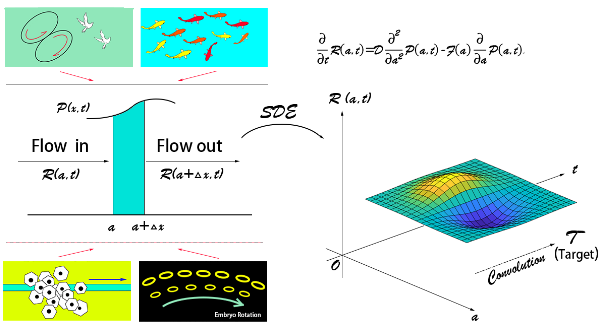

FIG. 1: The calculating mechanism of NIP. Different patterns of collective movements, ranging from microscopic to macroscopic world, may be described by a specific SPP model. By introducing the concept of flowing rate, we can quantify the tiny change of the probability density of the stochastic process. Further calculating the average first-passage time under the boundary conditions yields the final NIP.

particles aggregate simply forming a swarm, the shaping time is unfixed, when observing at different times.

What if one can calculate the time required for the particle group to move through a specific shape, since we always observe that one individual starts from the focal position, keep spinning, until turning to the focal position, and it is exactly equal to the shaping time for the particle group. The key finding in this study is just the NIP, which describes the time required for the particle group to move through a specific shape.

Thereby, we can classify the generated phases by NIP. If NIP tends to a large value, and the coefficient of variation, CoV, is relatively stable, then the state of the particles keeps stable, and the inter-particle alignment is the dominant driving force, called *ordered aperiodic state* (OAS). If NIP stabilizes around a moderate value, or it oscillates slightly around a certain value, we identify it as *ordered periodic state* (OPS). If NIP does not keep stable, then the shaping process is chaotic, called *disordered state* (DS).

Next, we show the detailed methodology. To numerical analyze the system, we can transfer Eq. (1) into

$$d\theta_i = F(\theta)dt + 2\sqrt{D_\theta}dW. \tag{2}$$

Through transformation based on Itô SDE [14], and let $P(\theta, t)$ denote the probability density of the stochastic process, which satisfies $lim_{x\to\pm\infty}P(x,t) = 0$ and $\int_{-\infty}^{\infty} P(x,t)\mathrm{d}x = 1$. Then $P(\theta,t)\mathrm{d}\theta$ is the probability of the average angle $\theta(t) \in [\theta, \theta+\mathrm{d}\theta)$. Thus, $P(\theta,t)$ satisfies the following Fokker-Planck equation [15]

$$\frac{\partial}{\partial t}P(\theta,t) = \frac{\partial^2}{\partial\theta^2}[D_\theta P(\theta,t)] - \frac{\partial}{\partial\theta}[(F(\theta))P(\theta,t)]. \tag{3}$$

The derivation process is schematically illustrated in Fig. 1. We denote the flowing rate of the probability function across the point x at time t as $RT(x,t)$. It is worth noting that the rate at which the probability changes is defined as the difference between the probability flowing in from the left side and flowing out from the right side of the interval, such that there will be a change of the sign. Thus, $\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}RT(x,t)$.

Considering there exist fixed final position and time,since we choose the final position of $\theta = \pi$.The necessary condition for the formation of the closed graph is that the time required for the particle flow to slew pi exists and remains stable, and it is obviously valid to measure the time corresponding to a longer slew angle, but since our defined T measures the angular displacement of the continuous oscillation, the calculation of large angular displacements in complex configurations is prone to fallacies. We then use the backwards Fokker-Planck equation [16] to describe the process of evolution. Define $P_i(y,t)$ as

$$P_i(t,y) = \int_0^\pi P(x,0|y,-t)\mathrm{d}x, \tag{4}$$

which represents that the probability belongs to the interval $[0,\pi]$ describing the selected range of angle during all time t. Logically, $P(x,0|y,-t)$ represents the probability with $x \subset [0,\pi]$ during time t. Imitating the idea of flow mentioned above, we denote $R(y,t) = P(x,0|y,-t)$. Explicit benefit is that R still satisfies the linear differential equation that P satisfies without causing any additional change. By introducing the backwards Fokker-Planck equation (from section 3.4 and 3.6 of [16]), we obtain

$$\frac{\partial}{\partial t}R(\theta,t) = D_\theta\frac{\partial^2}{\partial\theta^2}[R(\theta,t)] - F(\theta)\frac{\partial}{\partial\theta}[R(\theta,t)]. \tag{5}$$

Further denote the probability density for T is $P_T(t)$. And $P_i(0,y)$ is the probability that the time to exit the interval. Accordingly, we can obtain the closed-form expression for the average first-passage time, which is the mathematical expectation of T, representing the most frequent value of T under repeated experiments

$$\langle T\rangle = \int_0^\infty tP_T(t)dt == \int_0^\infty P_i(t,y)\mathrm{d}t. \tag{6}$$

For details of derivation, please refer to the $Supplemental Material$ [17]. Here, we hypothesize that $\langle T\rangle$ has a finite mean, and we must pay attention to this point strictly in the experiments. Applying the boundary conditions, we obtain the T_θ (Average time taken for the pointing angle of the stream to turn to θ)

$$T(\theta) = \int_0^\theta H_0 exp[\frac{F(s)s}{D_s}]\mathrm{d}s + \int_0^\theta exp[\frac{F(s)s}{D_s}]ds\int_0^p \frac{exp[F(p)p]}{D_p}\mathrm{d}p, \tag{7}$$

with

$$H_0 = \frac{T_0 - \int_0^{2\pi} exp[\frac{F(s)s}{D_s}]ds \int_0^p \frac{exp[F(p)p]}{D_p} \mathrm{d}p}{\int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s}. \tag{8}$$

Where H_0 is the parameter derived from the boundary condition, and T_{theta} is the time required for the particle flow to pass theta angle for the first time from an angle of 0 degrees relative to the coordinate system, and whether the system is in a phase transition critical state can be determined by observing whether T_π is stable around a settled value.We set $T_0 = 0$,which is the moment when the particle first reaches $\theta = 0$ with respect to the selected coordinate system ,since we want to know the value of $T_\pi - T_0$ in order to simplify the calculation process. Due to the non-directionality of the cluster motion and the random selection of the coordinate axes, any interval with an interval of π is feasible.

Because only when the collective rotation angle is greater than π, that is, π can be regarded as a $watershed$, it is possible for the particle group to form a periodic shape. Therefore, we can quantify the phases of collective movements via the so-called NIP, $T(\theta)$, with different values and determine whether the particle system forms stable configuration. Note that H_0 is a parameter depending on the initial state, obtained according to the boundary conditions. For details, please refer to the Supplemental Material [17].

To better illustrate the implement process of NIP, we focus on a general minimal cognitive flocking model 2-dimensional space [6] which exhibits various self-organized patterns, by tuning the size of the vision cone and the noise intensity. The numerical experiment was conducted on a two-dimensional square-shaped plane with linear size L and periodic boundary conditions. Initially, N particles are distributed randomly in the square plane. To simplify the model without loss of generality, we regard the particles as point-like items, then the repulsive interaction has not been taken into account.

The differential equation of the ith particle is

$$\begin{aligned} \dot{x_i} =& v_0 V(\theta_i), \\ \dot{\theta}_i =& \frac{\gamma}{n_i} \sum_{j\in\Omega_i} sin(\alpha_{ij} - \theta_i) + 2\sqrt{D_\theta}\xi_i(t), \end{aligned} \tag{9}$$

where x_i represents the position of the ith particle, and v_0 is the speed. Particularly, $V(\theta_i) = (Vcos(\theta_i), Vsin(\theta_i))$ represents the moving direction. γ is the strength of the interaction between different particles. n_i represents the number of particles in the vision cone of the ith particle. α_{ij} denotes the deviation between two particles, and $V(\alpha_{ij}) = \frac{x_i - xj}{||x_i - x_j||}$. The noise term $\xi_i(t)$ satisfies a winner process, $\langle\xi_i(t)\rangle = 0$, and $\langle\xi_i(t_1), \xi_i(t_2)\rangle = \delta_{i,j}\delta(t_1 - t_2)$. The noise intensity is represented as D_θ.

Two key parameters of the model are the vision angle β and the noise intensity D_θ, and the other parameters (v_0, R, γ) are set as (0.5, 5.0, 1.0), respectively. R_0 is the maximum distance that one particle can detect, and the apex angle is defined as $\frac{x_i - x_j}{||x_i - x_j||} \cdot \frac{V_i}{||V_i||} > \cos(\beta)$, in which β is called the cone apex angle. R_0 and β decide the shape of the vision cone. The function $F(\theta)$ in Eq. (9) is time-variant and thus more complicated than ordinary time-invariant ones. Via adjusting the noise intensity and the vision cone, we observe a wealth of phenomena, including worm, curved-“8”-like shape, twine, diffusive gas, aggregation, unstable circle, and so on. The states of the system are grouped into four categories: (1) worm, (2) periodical phase, (3) gas, (4) unstable phase. Typically, one may resort to different forms of order parameters to detect and distinguish them. Here, we can directly calculate NIP from Eq. (9), and obtain the consistent results.

Calculation we need to iterate, VT length of the class circular distribution of N particles, the perspective of β, the number of particles in the region of the cone of view can be simulated by the coefficient of variation K multiplied by the density of the line, the distribution of the viewed particles can be roughly randomly distributed among the cone of view, repeatedly brought into the formula for T, and eventually T will be repeatedly passed at a certain level, considered as a stable situation and sampled. The coefficient of variation K is empirically designed as follows: $|\beta - \pi/2|$ for less than $\pi/4$ $k = R - Rsin(\beta)$ when β greater than $\pi/4$ according to $Rsin(\beta)$

In fact, when calculating, we tend to ignore the very small terms and focus on the main terms. Define the $M = \frac{F}{D}$, then we can simply the expression form.

Substituting the entire expression for computation is tedious and unnecessary. Thus the NIP of model (9) can be given as follows:

$$T(\pi) = \frac{\pi}{D}e^{M\pi} - \frac{2\pi}{D}1\frac{1}{Q - \frac{1}{Q}}, \tag{10}$$

and $Q = e^{\pi M}$ is defined.

It is easy to analyze that a relatively rich phenomenon will be produced when the value of D is not too large. F, as the driving term, should be slightly smaller than the value of D, but not smaller than the order of magnitude of D, in order to make the system in a relatively active phase transition region. In this example, F is taken to be of a suitable order of magnitude, probably at the half angle of the apparent cone above and below pi/2, and relatively stable periodic results can be observed. In this case, F is taken to be of the right order of magnitude, around pi/2 of the apparent cone half-angle, to be able to observe relatively stable periodic results. In generalizing the model, F can be both estimated and observed directly in the experiment, which brings good convenience in terms of flexibility of use.

When the values of the apex angle of the vision cone β and the noise intensity D_θ are small, the worm state yields. The foremost particle can not see anyone in its

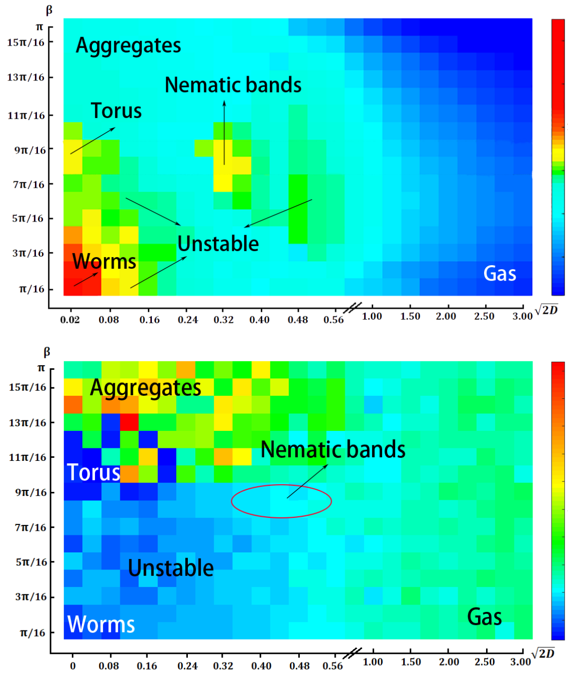

FIG. 2: (a) Heat map of logarithm NIP. The largest value of NIP indicates an OAS state, corresponding to the worm configuration. The moderate values of NIP stands for an OPS state, such as torus, dumbbell, or twist. A small value of NIP indicates a DS state, like irregular molecular motion of gas. (b) Heat map of CoV. To distinguish different periodic results, NIP should be analyzed together with CoV. In OPS state, torus and nematic bands can be further divided according to the value of CoV, since torus has a relatively stable configuration with a small value of CoV. $N = 400$ and $L = 20$. The total experiment time is 2000s and the items are measured after a relaxation time of 500s, and each experiment was repeated 480 times to obtain the results.

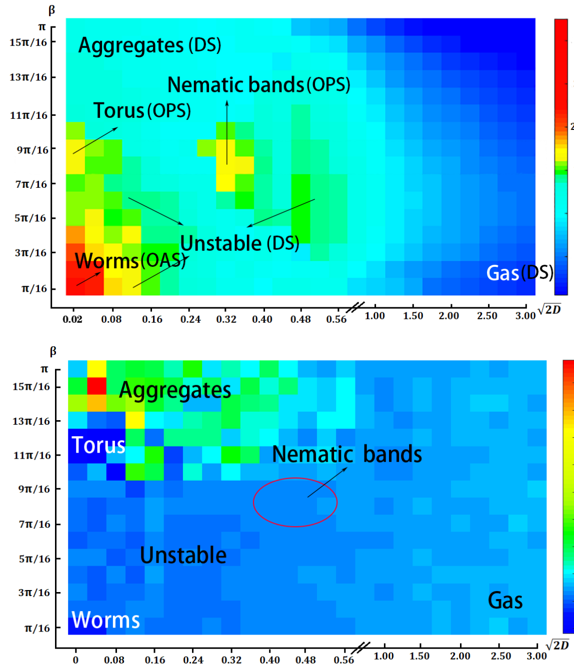

FIG. 3: Theoretical estimation of NIP and Cov. It is highly consistent with the experimental results, which means that we can directly classify the configurations via calculating NIP and CoV. Note that there exist difference between theoretical results and experimental results, mainly because calculating NIP needs an initial value and numerical experiments may have uncertainty due to the stochastic noise.

vision cone, driving the others to form a worm like state. The particles under this configuration move ahead in a coherent direction and it is almost impossible to return to the original position. Thus the value of NIP is giant and CoV is relatively small.

Under appropriate values of β, especially when β is around $\frac{\pi}{2}$, various periodic shapes will be formed according to different noise intensity D_θ. Therein, NIP is stable near a certain value, neither excessively enormous nor too small, and CoV has a small value, simultaneously. In this case, the agent will form a stable periodical configuration, such as torus (a phase of bidirectional milling, in which part of the particles rotate clockwise, while others anticlockwise), or nematic bands (with one or several bands, where only two direction prevails). Sometimes there exist shapes like curved-"8" or circles, which depends on the particular initial distribution of particles.

For larger values of D_θ, particles aggregate with no obvious generated shapes. If β is small, the structure will be looser, whereas β is larger, particles become compactly together. That is, a chaotic and disordered phase yields.

Unstable phases can be observed from the heat map in Fig. 2. Under this circumstance, Cov fluctuates obviously and constantly, and the exhibited pattern is that the particles repeatedly switch between stable shapes and unstable states.

Moreover, in the experiment, we found that the periodic boundary played a crucial role in obtaining stable experimental results, and the number and density of individuals also affect the stable configuration. However, once the parameters are determined, NIP shows explicit calculation results, which is independent of the initial experimental conditions and the specific evolution process. A high degree of consistence can be observed from the comparison between theoretical results in Fig. 3 and experimental results in Fig. 2, which indicates the effectiveness of NIP in quantifying the configurations of the SPP model (9).

For a general 3D model, it is difficult to find a definite rotation axis or rotation plane as a reference to determine the accumulation path of angular displacement. In a general model, it is difficult to determine the direction of accumulation of angular displacements, and then it is difficult to infer the pattern of particle flow. In short, even simple interaction laws can produce unpredictable phenomena in a three-dimensional model. The only statistical angular displacements in 2D models are positive and negative angles, but when considering 3D, the z-axis pointing often has to be determined before the definition of the statistics can be given. This means that it is difficult to quantify the characteristics of the model through NIP metrics; conversely, for some relatively less cluttered models, especially for phenomena that are prevalent in nature, for example, the motion of cells often interacts in a plane, the motion of fish in shallow water often has some imaginary plane of rotation, and schools of sardines in the ocean often rotate around an axis of rotation to confuse predators, such Such a problem can be reduced to a two-dimensional problem by transforming the axes, for example, by compressing along the axes, eliminating displacements in a certain direction, or by decomposing the NIP in each coordinate direction and then compounding it. In other words, if a compound motion in a three-dimensional space can be decomposed into a compound of a motion in a two-dimensional plane and a regular motion in a coordinate axis, then the problem does not pose any additional challenges. However, for general systems determined by coupled differential equations, further study is needed.

The flow method and the SDE method can be combined concisely and effectively with other statistical methods, such as test estimation based on large samples, and extended to high-dimensional vector spaces to study one-to-one variable relationships. It is also very flexible to use. It can be estimated based on samples or directly observe the relationship between variables for quantitative analysis.

To sum up, we have proposed a new framework, NIP, for quantifying and detecting the phases of collective behaviors of SPPs describing by SDE. NIP also shows the measurable influence of tuning parameters on phase transitions, which may help determine the indistinguishable order of phase transition. We theoretically prove the physical significance of NIP general SPP models, and numerically verify the effectiveness in both the 2-dimensional and 3-dimensional cases. The form of NIP encrypts the natural characteristics of the system, whose computational complexity is determined by the dynamics. By means of coordinate transformation and matrix computation, we suggest that NIP may be extended to even higher dimensional dynamical systems. Finally, it would be interesting to further investigate the simplification or equivalent form of NIP and the effective tool for systems describing by partial differential equations.

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