A characterization method to describe the noise immunity of marginal ferromagnetism

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Abstract

Abstract We induce such a family of ferromagnetic models, including the exponential family, the logarithmic family, capable of reproducing one of the most interesting properties of collective behavior in starling flocks, namely the fact that strong collective order coexists with the scale-free correlation of microscopic degrees of freedom, i.e., the speed of the bird. The key idea of this theory is that the single-particle potential required to bound the microscopic degrees of freedom modulus around a finite value is marginal, i.e., it has zero curvature at the critical point. Such a family clearly has endogenous noise immunity, and to characterize this property. We characterize the endogenous noise immunity of the bare potential family by expanding the Gibbs free energy $g(m)$ around the velocity reference value, the divergence of the velocity reference value m_0 from the typical value of the velocity (derived from the assumption of invariance of the adjacency matrix).

1. Introduction

As a consequence of spontaneously broken continuum symmetry, non-zero order parameters and mass-less Goldstone modes is developed from Ferromagnetic models with O(n) rotational symmetry in their low-temperature phase [1,2]. Conversely, throughout the symmetry-breaking phase, Goldstone modes produce infinite susceptibility and correlation lengths [3].In terms of the spatial extent of the correlation function, the volume correlation length is infinite in a system of finite size has a practical consequence, which is the spatial extent of the correlation function expands with the size L of the system. In other words, there is no intrinsic length scale in the system other than L itself, and this physical state is often referred to as a system with scale-free correlation [4]. Since the velocity change of each individual must be consistent with the velocity change of other individuals when the group is under perturbation in order to propagate information in the system, the velocity control mechanism behind scale-free systems is of interest to us because fluctuations in individual velocities cannot occur randomly. Thus, the collective response is inseparable from the velocity control mechanism. Further scale-free correlation of velocity cannot be explained as an effect of spontaneous destruction of continuous symmetry [5]; In fact, in standard statistical physics, the fluctuations of the order parameters are severely suppressed in the ordered phase, so they are very short-range correlated [6]. The velocity control mechanism itself is intrinsically resistant to perturbations, especially in groups of birds and fish, which naturally tend to have a large degree of polarization, and noise itself of yet can induce complex phenomena, such as proper noise does not necessarily break up the group, but produces complex local phenomena. This endogenous noise immunity naturally relies on the setting of speed control mechanisms.

Here we give two seemingly ordinary naked potential energy families. Although they seem to be simple and common in form, we will use an unusual method to judge their endogenous anti-noise ability later, because scale-free When the system is disturbed by a small amount, there is only a slight speed change without any other signs. Unless it is disturbed by more than the critical noise, the commonly used methods such as the maximum entropy method will face failure. This has a new enlightenment for simulating different populations.

2. General theory

The starting point is the pseudo-Hamiltonian with the harmonic potential:

$$H(\{v_i\}) = \frac{J}{2}\sum_{i,j} n_{i,j}(v_i - v_j)^2 + \sum_{i}^{N} V(v_i), \tag{1}$$

H is a cost function (or effective Hamiltonian), whose derivative with respect to vi represents the social force acting on the particle's velocity (the effective friction coefficient in front of v_i in can be set to 1 through an appropriate rescaling of time)[10]. In order to implement an imitation dynamics we can use the following - very general - cost function,where the first term represents the imitation interaction between particle velocities with strength J and the second term is the velocity control term, which affects each particle independently. The adjacency matrix n_{ij} is 1 for interacting neighbors and 0 otherwise, and the self-propelled part of the dynamics , implies that the interaction network depends on time, $n_{ij} = n_{ij}(t)$. The first term in the cost function favors neighboring individuals with similar velocities, and thus it contains the dynamic alignment component first explored in the seminal Vicsek model ; however, in the Vicsek model, the velocity of the particles remains constant, $v_{ij} = v_0$, and thus the velocity simulation affects only the direction. Here, on the other hand, we want to study velocity fluctuations and their correlations, so we relax the Vicsek constraint for fixed velocities: the first term in H favors the mutual imitation of direction and velocity, while the control potential, $V(v_i)$, keeps the velocity of each particle around the natural reference value, v_0. Note also that Vicsek dynamics is discrete, whereas here we want to work in the continuous time limit.

The interaction term in the cost function H involves the full velocity vector and therefore regulates the mutual adjustment of velocity and flight direction: individuals that tend to have similar directions also tend to have similar velocities.

Where all sums are from 1 to the number of particles in the system N. We are dealing with an active system, so the matrix $n_{ij} = n_{ij}(t)$ depends on time. However, due to the large polarization of the real flock, previous starling data showed that the relaxation time scale of $n_{ij}(t)$ is significantly larger than the relaxation time scale of the velocity , so a quasi-equilibrium approach to the problem is justified; from now on, we will consider the time-independent n_{ij}.

Considering that in the deeply ordered flocking phase[12] the time scale to reshuffle the interaction network is much larger than the time of local relaxation.To recover the interaction-free case, J is set to 0 in order to simulate the depth-ordered phase, the equations of motion are decoupled into N independent equations, and the amplitude of the Gaussian noise source is written explicitly, then appear,

$$\frac{\mathrm{d}r_i}{\mathrm{d}t} = v_i, \tag{2}$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = -\frac{\partial V}{\partial v_i}\frac{v_i}{|v_i|} + \sqrt{2T}\xi_i, \tag{3}$$

where now $\langle \xi_i(t)\cdot\xi_j(t_0)\rangle = d\delta_{ij}\delta_{tt_0}$. From these, we want to obtain a stochastic differential equation for the speed of the single particle, therefore we use spherical coordinates expressing, in d = 3, $v_i(t) = s_i(sin\theta_i cos\phi_i, sin\theta_i sin\phi_i, cos\theta_i)$ where ϕ_i and θ_i are the two phases of the vector. Within the Ito-scheme of stochastic calculus, we get the equation for the single the speed,

where the noise term depends on the coordinates of the original noise and on the phases,

$$\frac{\mathrm{d}s_i(t)}{\mathrm{d}t} = -\frac{\partial V(s_i)}{\partial s_i} + \frac{2T}{s_i} + \sqrt{2T}\xi_i^s(t), \tag{4}$$

where the noise term depends on the coordinates of the original noise and on the phases,instead of deriving directly downwards from this, we switch to a theory that approximates the spin-wave approach[11], first because the system is highly polarized (due to our assumptions) therefore v can be written as $v_i = v + \epsilon_i$. in which v is average speed and ϵ_i is the deviation of each individual velocity from the average, so plug in it in to eq.1 we have

$$H = NV(v), \tag{5}$$

Here we only leave the potential energy part but omit the interaction term, because we are based on the assumption that for interacting individuals, their velocities are very close so the cross term is considered to be 0; and for those non-interacting cases , we force j=0, meaning that each vector v_i may have a different orientation with respect to any vector v_j. Consider constraints like

$$\sum \epsilon_i = 0 \tag{6}$$

Applying the method of variation to this expression and change the variable from v_i to $\{v_i, \epsilon_i\}$, after computation we have

$$P\{\epsilon_i, v_i\} = \frac{e^{-\frac{NV}{T}}}{\int dv \prod_i d\epsilon_i \delta(\sum_i \epsilon_i) e^{-\frac{NV}{T}}} \tag{7}$$

we change variable from $dv = dv_x dv_y dv_z$ to $dvd\phi d\theta$, gaining the jacobian v^2.

. Under the reasonable approximation then we have

$$P(s) = \frac{s^2 e^{-NV(s)}}{\int dss^2 e^{-NV(s)}} \tag{8}$$

in here, the approximation distribution of s is

$$P(s) = \frac{1}{C} s^{d-1} exp[-\frac{N}{T} V(s)] \tag{9}$$

3. Bare potential class

In statistical physics, the correlation length ξ is connected to the inverse of the quadratic curvature of the (renormalized) potential, calculated at its minimum ; very small curvature implies very large correlation length, so that a divergent ξ is always due to a zero second derivative (or marginal mode) along some direction of the (renormalized) potential.

To find this potential we proceed through general considerations of symmetry and common sense. First, the potential must keep the speed around the reference natural value v0 and it must diverge for large values of the speed; secondly, it must be rotationally symmetric in the whole velocity vector;Third,it should ideally have a simple mathematical form and rotationally symmetric potential that confines the speed around the natural reference value

3.1. Exponential family

The marginal potential energy has the simplest algebraic form necessary to provide zero second derivatives, not only along the transverse direction (Goldstone mode), but also along the longitudinal direction, which is our candidate, yielding moduli in the deeply ordered phase scale-free correlation. The idea is as follows: magnetic susceptibility is given by the reciprocal of the second derivative of the Gibbs free energy at its minimum (i.e., the equilibrium state), and to enter the deeply ordered phase, let the Gibbs free energy be at the second order of the lowest long-range direction A derivative of 0 is easy to imagine. Similar to the previous power function $V = \lambda(\mathbf{s}\cdot\mathbf{s} - v_0^2)^4$, here we naturally set the exponential family

$$V(s) = exp[\lambda(\mathbf{s} \cdot \mathbf{s} - v_0^2)^4] \tag{10}$$

The maximum of this distribution, i.e. the typical mean speed of the flock, is given by $\frac{\partial P}{\partial s} = 0$, which shows,

$$s_{typical} \simeq \sqrt{1 + \sqrt[4]{\frac{T}{2N}(1 + \sqrt{1 + \frac{1}{\lambda}})}} \tag{11}$$

The complete absence of quadratic terms in the expansion of the exponential family seems to indicate that the marginal potential always gives rise to infinite correlation lengths of velocity fluctuations: in fact, this is far from the case for scale-free. The velocity correlation is regulated by the marginal potential (i.e., free energy) but also by fluctuations induced by noise (i.e., entropy): at very low noise, the regulation of free energy dominates, so that velocity fluctuations are indeed scale-free, and by increasing noise, the correlation is increasingly suppressed by entropy fluctuations [7] until sufficiently large entropy takes over everything.

Analyzed in terms of field theory, what happens is that, at finite temperature, entropy provides a non-zero second-order derivative of the renormalization potential, i.e., a non-zero mass of the velocity fluctuations, thus causing a finite correlation length. The reformation curvature only goes to zero at T=0, as well as reverting to zero at the critical point Tc where the entropy takes over the system (T_c is the interesting point, similar to the Curie point in a classical ferromagnetic phase transition), where the velocity correlation length diverges. t=0 signifies that the system possesses scale-free properties, where the transfer of information easily crosses the system scale, and accordingly, the group is very sensitive to external disturbances, and any external disturbance is always accepted and transmitted to each individual prompting a collective response, often manifested as a collective steering and acceleration of the swarm; another critical point of zero curvature is T_c, where the mass of one of the entire vector degrees of freedom m of the swarm is again zero. Mean field analysis [7] shows that the velocity correlation length diverges as.

3.2. Logarithmic family

Similar to the previous idea, we set the logarithmic family corresponding to the exponential family

$$V(s) = ln[1 + \lambda(\mathbf{s} \cdot \mathbf{s} - v_0^2)^4] \tag{12}$$

The maximum of this distribution, i.e. the typical mean speed of the flock, is given by $\frac{\partial P}{\partial s} = 0$, which shows,

$$s_{typical} \simeq \sqrt{1 + \sqrt[4]{\frac{T}{4\lambda(N - T)}}} \tag{13}$$

4. Gibbs free energy

The mean-field Gibbs free energy per particle, g(m), is therefore given by,

$$g(m) = -Jm^2 - \mathbf{m} \cdot \mathbf{x_0(m)} - \frac{1}{\beta} ln \int d\sigma e^{-\beta(S(\sigma) + x_0(m) \cdot \sigma)} \tag{14}$$

with $S(\sigma) = J\sigma^2 + V(\sigma)$ and $\beta = \frac{1}{T}$ is the inverse of T. Notice that g(m) is a O(n)-symmetric function of the modulus of the magnetization. The mean−field Gibbs free energy is linked to the probability of m by the relation, $P(m) = e^{\beta N g(m)}/Z$.

From (14), we can calculate the marginal potential for g(m); we find that g(m) has the property that at T = 0, the free energy has a flat (i.e., marginal) minimum, which, as we will soon see, is the temperature divergence sensitivity of the zero-modulus. This marginal minimum is of course a zero-temperature relic of the marginal potential minimum. Interestingly, by increasing the temperature, a non-zero curvature (i.e., mass) is created at the free energy minimum due to the effect of increasing entropy, which reduces the modulus sensitivity. By further increasing the temperature, g(m) becomes flat again at the finite critical temperature T_c, as would happen with any standard ferromagnetic model. However, at T_c, the order parameter (i.e., the transverse coordinate of the minimum) becomes zero, so that the distinction between modulus and direction is meaningless and there is only a (normal) divergence magnetization. Finally, above T_c, the paramagnetic phase takes over and entropy dominates. Although the free energy is slightly unfamiliar, we believe that the free energy also always tends to lie at the minimum, just as the potential and kinetic energies are bounded, since a larger entropy change is considered sensible.

Working at $T \ll 1$ and expanding g(m) near $m = m_0$ (which is the equilibrium magnetization at T), we calculate the explanation. We want to expand the Gibbs free-energy Eq.9 up to $O(T^2)$ (that is $\frac{1}{\beta^2}$). To accomplish that, we write Eq. 9 expanding the integral in σ, using the saddle point method, which reads,

$$g(m) = -Jm^2 - x_0 m - \frac{1}{\beta} ln[\frac{e^{-\beta}(S_0 + x_0\sigma_0)}{det[S_{\alpha\beta}(\sigma_0)]}(1+\frac{1}{\beta})B_0] \quad (15)$$

$$= -JM^2 + x_0(\sigma_0 - m) + S_0 + \frac{1}{\beta} lndet[S_{\alpha\beta}(\sigma_0)] - \frac{1}{\beta^2}B_0 \quad (16)$$

The saddle point equation for the integrals introduces a new player, the saddle point value $\sigma_0(m)$, and $x_0(m)$ is an auxiliary variable, defined by the saddle point equation for $N \longrightarrow \infty$[7]. And $S_0 = S(\sigma_0)$, $S_{\alpha,\beta}$ is the Hessian matrix of S and $B_0 = B(\sigma_0)$ is the first coefficient of the expansion in $\frac{1}{\beta}$ of the integral in Eq. 15, which will be computed later on. We can write σ_0 and x_0 from saddle point method as expanding around m,

$$\sigma_0 = m + \frac{1}{\beta}C_m + O(\frac{1}{\beta^2}) \quad (17)$$

$$x_0 = -S'(m) - \frac{1}{\beta^2}S''(m) + O(\frac{1}{\beta^2}) \quad (18)$$

where $C_m = C(m)$ is the first coefficient of the expansion in $\frac{1}{\beta}$, that will be computed later; $S'_m = S'(m)$ and $S''_m = S''(m)$ are respectively the first and second derivative of S, from now on this notation will be used for derivatives. If we plug Eq. 17 and Eq. 18 into Eq. 16 and keep all terms up to order $O(\frac{1}{\beta^2})$ we find,

$$g(m) = V(m) + \frac{1}{\beta} lndet[S_{\alpha\beta}(m)] - \frac{1}{\beta^2}[B(m) + \frac{1}{2}S''(m)C^2(m) - \frac{C(m)[det[S_{\alpha\beta}(m)]'}{det[S_{\alpha\beta}]}] \quad (19)$$

To compute the term of order $\frac{1}{\beta^2}$ in Eq. 19 we just need to evaluate the determinant of the Hessian of the function $S(m)$, the Hessian matrix is diagonal and gives,

$$det[S_{\alpha,\beta}] = S''(m)[\frac{S'(m)}{m}]^{n-1} \quad (20)$$

If we take the logarithm and expand near $m^2 \simeq v_0^2$ we have,

$$lndet[S_{\alpha,\beta}] \simeq (m^2 - v_0^2)^2 \quad (21)$$

Thus we get the leading order of the explanation of T .Then Going to next order, we can compute the terms B(m) and C(m) by expanding the integrals of Eq. 16 using the saddle point method. After some calculations we find that the leading order in $(m^2 - v_0^2)$, for the term of order T^2 is given by the term B(m), which reads,

$$B(m) \sim \frac{S_{\alpha\beta\gamma\delta}}{24} \langle y_\alpha y_\beta y_\gamma y_\delta \rangle \quad (22)$$

where $S_{\alpha\beta\gamma\delta}$ is the fourth order derivatives tensor of S and the y_α are Gaussian distributed variables with,

$$\langle y_\alpha \rangle = 0 \quad (23)$$

$$\langle y_\alpha y_\beta \rangle = [\delta_{\alpha\beta}\frac{S'(m)}{m} + \frac{m_\alpha m_\beta}{m^2}\left(S''(m) - \frac{S'(m)}{m}\right)^{-1}]. \quad (24)$$

Thus we get $B(m) \sim Const + O(m^(2) - 1)$, then we get the explanation of g(m),near the lowest point m_0 corresponding to T, focusing on the exponential family and logarithmic family as,

$$g(m) \simeq V(m, m_0) + T(\alpha_2(m_0^2 - m^2)^2 + \alpha_3(m_0^2 - m^2)^3 + ...) + T^2(\alpha_1(m_0^2 - m^2) + \alpha_4(m_0^2 - m^2)^2 + ...) \quad (25)$$

Thus we get interesting and useful discovery, if the potential is set as before, $g_{exp}(m)$ and $g_{ln}(m)$,then the expansion of T in the second half of the expression for g until order=4 is consistent,considering the fact[9],

$$\frac{1}{g''(m)} \sim \xi \sim T^{-\frac{1}{2}} \quad (26)$$

Substitute the previous $s_{typical}$ into it to observe the size of this divergence,and omitting higher order terms, we get

$$\xi_{exp} \sim [(T\alpha_2 + T^2\alpha_4)(2 + 3\sqrt[4]{\frac{T}{2N}(1 + \sqrt{1 + \frac{1}{\lambda}}))]^{-1} \tag{27}$$

$$\xi_{ln} \sim [[(T\alpha_2 + T^2\alpha_4)(2 + 3\sqrt[4]{\frac{T}{4\lambda(N-T)}})]^{-1} \tag{28}$$

The typical velocity value $s_{ypical}t$ from the probability distribution in the vicinity of 0 noise value apparently increases with noise, which diverges from the average velocity m0 deduced from g first increasing to some extreme value and then gradually decreasing to 0. We believe that the root of this divergence is some kind of representation of the endogenous noise immunity of the system, for example, a simple analysis shows that the exponential family is less sensitive to the coefficient λ than the logarithmic family, which laterally indicates that the exponential family itself should have better natural noise immunity; while the original power function family lies in between and is mutually better or worse under different parameter settings.

5. Conclusion and comments

Different groups obviously have heterogeneity; for example, the movement characteristics of a flock of pigeons and a flock of starlings need to be distinguished. Consider models in which both deviations in reference speed and deviations in speed variability are indeed tamed by other individuals as long as the strength of the mimicking interaction J is strong enough, which is never a problem for marginal models, since stronger interactions imply stronger correlations. Thus, in the marginal model, not only scale-free correlations and moderate group velocities can be achieved by simply staying in the ordered phase of strong interactions, but also robustness to heterogeneity. Further, the noise resistance of the endogenous nature of the marginal model cannot be ignored, although from the point of view of mean-field theory, the increase of noise in fact leads directly to a decrease of the correlation length, but even to the extent that the decrease is smaller than the scale of the group, it may not cause the splitting of the group; on the other hand, the correlation length is obviously difficult to be verified by experimental facts when it is very large, but the noise resistance as an always present endogenous system nature, which can be easily extracted from experimental data once the correlation length drops to a level smaller than the system scale. Of course, there are other types of heterogeneity that cannot be studied in our simple model: considering differences in structural size and body weight among individuals within the same species requires a more complex model. The fact that simple heterogeneity is easily tamed in the marginal model makes us optimistic about the robustness of marginal controls in the more general case.

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