Supporting Information of ”The systematic spin of moving pigeons”

1. Proof of Eqs. (4) of the main body

Next, we find that the boundary condition is that θ always located in the interval $[0, 2\pi]$, thus the integral of the solution to the fokker-planck equation, $P(x, t|y, 0)$, can express the the probability that the process is still inside the interval at time t, which can be called $P_i(t, y)$

$$Prob\{T > t\} = P_i(t, y) = \int_0^{\pi} P(x, t|y, 0)\mathrm{d}x \tag{1}$$

We all see that since the fokker-planck equation is time homogenous(since $F(\theta)$ and D_θ is time independent),adding the time origin will not affect the results.

$$Prob\{T > t\} = P_i(t, y) = \int_0^{\pi} P(x, 0|y, -t)\mathrm{d}x \tag{2}$$

2. Proof of Eqs. (7/8) of the main body

Introducing the backwards Fokker-Planck equation(from section 3.4 and 3.6 of Gardiner's Handbook of Stochastic Methods), the corresponding equation is

$$\frac{\partial}{\partial t}R(\theta, t) = D_\theta \frac{\partial^2}{\partial \theta^2}[P(\theta, t)] - F(\theta)\frac{\partial}{\partial \theta}[P(\theta, t)].$$

Since $P_i(t, y)$ is merely the integral of the $R(y, t)$ over x, then the $P_i(t, y)$ still satisfy the same equation of $R(y, t)$. To solve the equation, we need to know the boundary condition.Since the process is inside the interval at t=0,we have $P_i(0, y) = 1$.If the process starts at the boundaries,then at t=0 it already reach the exit(since it is immediately be absorbed when reaches the boundaries), thus $P_i(t, a)$=$P_i(t, b)$=0 for all t≥0.The differential equation for $P_i(t, y)$ is thus

$$\frac{\partial}{\partial t}P_i(t, y)(\theta, t) = D_\theta \frac{\partial^2}{\partial \theta^2}[P_i(t, y)] - F(\theta)\frac{\partial}{\partial \theta}[P_i(t, y)].$$

with the boundary condition

$$P_i(0, y) = 1, P_i(t, a) = P_i(t, b) = 0 \tag{3}$$

Since $P_i(0, y)$ is the probability that the time-to-exit, T, is greater than t, then the probability distribution of T $D_T(t)$ is

$$D_T(t) = Prob(0 \le T \le t) = 1 - P_i(t, y) \tag{4}$$

and thus the probability density for T is

$$P_T(t) = \frac{\partial}{\partial t} D_T(t) = -\frac{\partial}{\partial t} P_i(t, y) \tag{5}$$

Then the solution to the backwards Fokker-Planck equation therefore gives us all the information about the first-exit time from a given interval.
By using the results above we can obtain a closed-form expression for the average time it takes a process to exit an interval. The average first-passage time is

$$\begin{aligned} \langle T \rangle &= \int_0^\infty t P_T(t) dt = -\int_0^\infty t \frac{\partial}{\partial t} P_i(t, y) \mathrm{d}t & (6) \\ &= -\{t P_i(t, y)|_0^\infty - \int_0^\infty P_i(t, y) \mathrm{d}t\} & (7) \end{aligned}$$

Here the last step is given by integrating by parts, and using the relation $\lim_{t \to \infty} t P_i(t, y) = 0$.Thus the equation turns to be

$$= \int_0^\infty P_i(t, y) \mathrm{d}t \tag{8}$$

Next, we consider that $\langle T \rangle$ is a function of the initial position of the process,y, as we write it as $\langle T \rangle$.Then we integrate both sides of the Fokker-Planck equation we can obtain

$$1 = D_\theta \frac{\partial^2}{\partial \theta^2} [\langle T \rangle] - F(\theta) \frac{\partial}{\partial \theta} [\langle T \rangle]. \tag{9}$$

Solving the differential equation, we can obtain the form of T, thus we can further explain the inherent characteristics of the dynamics system.

Defining $G = \frac{\partial}{\partial \theta} T(\theta)$, we can transform the second-order differential equation into a first-order differential equation with driving term. The equation can be rewritten as:

$$1 = D_\theta \frac{\partial}{\partial \theta} [G(\theta)] - F(\theta) [G(\theta)]. \tag{10}$$

Then we have

$$\frac{\partial}{\partial \theta} [G(\theta)] = \frac{F(\theta)}{D_\theta} [G(\theta)] + \frac{1}{D_\theta}. \tag{11}$$

We define auxiliary function $H(\theta)$ as:

$$H(\theta) = G(\theta) e^{-\frac{1}{D_\theta} F(\theta) \theta} \tag{12}$$

Then, since $H(\theta)$ is defined preciously thus if $G(\theta)$ is a solution, then we can calculate the equation as

$$\frac{\partial}{\partial \theta} H(\theta) = \frac{\partial H(\theta)}{\partial G(\theta)} \frac{dG(\theta)}{d\theta} + \frac{\partial H \theta}{\partial \theta} \tag{13}$$

Then using the chain rule, it turns to

$$\frac{\partial}{\partial \theta} H(\theta) = \frac{1}{D_\theta} exp[-\frac{1}{D_\theta} F(\theta) \theta] \tag{14}$$

Then

$$H(\theta) = H_0 + \int_0^{\theta} \frac{exp[F(s)s]}{D_\theta}\mathrm{d}s \tag{15}$$

$G(\theta)$ can be calculated as

$$G(\theta) = H_0 exp[\frac{1}{D_\theta}F(\theta)\theta] + exp[\frac{F(\theta)\theta}{D_\theta}]\int_0^{\theta} \frac{exp[F(s)s]}{D_\theta}\mathrm{d}s \tag{16}$$

Then,

$$T(\theta) = \int_0^{\theta} H_0 exp[\frac{F(s)s}{D_s}]\mathrm{d}s + \int_0^{\theta} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{p} \frac{exp[F(p)p]}{D_p}\mathrm{d}p \tag{17}$$

Then using the boundary condition,$T_{\theta=0} = 0$ and $T_{\theta=2\pi} = T_0$,we can obtain the value of H_θ

$$T(\theta) = \int_0^{\theta} H_0 exp[\frac{1}{D_s}F(s)s]\mathrm{d}s + \int_0^{\theta} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{p} \frac{exp[F(p)p]}{D_p}]\mathrm{d}p \tag{18}$$

Since we obtain the boundary equation that $T_{\theta=2\pi} = T_0$, by substituting the equation into the estimation of T, we obtain that

$$T(2\pi) = T_0 \tag{19}$$

$$= \int_0^{2\pi} H_0 exp[\frac{F(s)s}{D_s}]\mathrm{d}s + \int_0^{2\pi} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{s} \frac{exp[F(p)p]}{D_p}\mathrm{d}p \tag{20}$$

Thus,

$$T_0 - \int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s \int_0^{p} \frac{1}{D_p} exp[F(p)p]\mathrm{d}p \tag{21}$$

$$= H_0 \int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s \tag{22}$$

Then, H_0 can be expressed as:

$$H_0 = \frac{T_0 - \int_0^{2\pi} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{p} \frac{exp[F(p)p]}{D_p}\mathrm{d}p}{\int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s} \tag{23}$$

3. simplification of results

H_0 can be expressed as:

$$H_0 = \frac{T_0 - \int_0^{2\pi} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{p} \frac{exp[F(p)p]}{D_p}\mathrm{d}p}{\int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s} \tag{24}$$

Then define $M = \frac{F}{D}$,and $Q = expM\pi$, we have

First we have H_0

$$H_0 = \frac{T_0 - \int_0^{2\pi} exp[\frac{F(s)s}{D_s}]\mathrm{d}s \int_0^{p} \frac{exp[F(p)p]}{D_p}\mathrm{d}p}{\int_0^{2\pi} exp[\frac{1}{D_s}F(s)s]\mathrm{d}s} \tag{25}$$

$$= \frac{M}{exp[2\pi M]-1}[-\frac{1}{D}\int_0^\pi exp[(M+F)s]-exp[Ms]\mathrm{d}s] \tag{26}$$

$$\simeq -\frac{M^3\pi}{D}\frac{1}{Q-\frac{1}{Q}} \tag{27}$$

$$T(\pi) = \int_0^\pi H_0 exp[Ms]\mathrm{d}s + \int_0^\pi exp[Ms]ds\int_0^s \frac{exp[F(p)p]}{D}\mathrm{d}p \tag{28}$$

$$= -\frac{M^3\pi^2}{D}\frac{1}{Q-\frac{1}{Q}} + \frac{M\pi}{D}Q \tag{29}$$

since we exploit such an estimate $\int_0^\pi exp[(M+F)s]-exp[Ms]\mathrm{d}s \simeq \frac{M\pi}{D}exp[Ms]$

4. Theory

Next, we attempt to show the theoretical basis to support the rationality of the article.

For the first, we introduce a lemma:

Lemma 1 *A particle leaves its initial position and moves. When it returns to its initial position and maintains its initial direction, the range of its pointing angle is at least 180 degrees*

Proof 1 *We consider a particle located in the coordinate system, which is starting from the coordinate origin.Initial pointing angle $\theta_0 = 0$ and $\theta \geq 0$ throughout the process.Considering the fact that θ_t is a continuous function.Then if $\theta_{max} \leq \pi$, then the drift along the Y-axis can be expressed as*

$$\int_0^T V_0 sin(\theta_t)dt = 0 \tag{30}$$

Using the assumption we obtain that Integrand function is a Nonnegative function, thus the solution is

$$sin(\theta_t) \equiv 0 \tag{31}$$

However, in this situation, the drift along the X-axis can be expressed as

$$\int_0^T V_0 cos(\theta_t)dt = 0 \tag{32}$$

this equation is unsolvable. Thus the range of θ R_θ must satisfy $R_\theta > \pi$

At the same time,some special figures have the character that the R_θ is just a little bigger than π. Obviously,the R_θ of a circle or a ellipse is exactly 2π. However,the R_θ of a prolate dumbbell may equal to $\frac{3}{2}\pi$ or $\frac{5}{4}\pi$, and the narrower and longer the configuration, the closer R_θ is to π.However,when we are computing the time required for a complete cycle , the exact value of R_θ is required. Thus,considering the abundant phenomenon, we focus on an equivalent problem.

Lemma 2 *The time required for a complete cycle can be used for phenomenon classification :*

Proof 2 *1)When the $\langle T\rangle$ is infinite, then we turn to the analytic expression and have a conclusion that: For the leader, $F_\theta = 0$, then $\dot{\theta}_i = 2\sqrt{D_\theta}\xi_i(t)$, the pointing angle of the leader is approximate to Brownian motion. Thus particles will form a worm.*

2)When the $\langle T\rangle$is stable with respect to the parameter α, we conclude that F_θ is stable which is equivalent to the distribution of particles which located in the i_{th} vision cone.Then it is equivalent to the configuration of particles is stable, forming a closed shape.

3)When the $\langle T\rangle$ is unstable and fluctuate then we get that the system is unordered and chaos.

Lemma 3 *The time required for a complete cycle $\langle T(\theta)\rangle$ is equivalent to $T(\pi)$ when is used for phenomenon classification :*

Proof 3 *1)When the particles form a worm, then $\langle T(\theta)\rangle$ and $T(\pi)$ are all infinite.*

2)When the particles form a ordered configuration, especially a closed shape, then $\langle T(\theta)\rangle$ and $T(\pi)$ are all stable.

3)When the particles are aggregate and unordered, $\langle T(\theta)\rangle$ and $T(\pi)$ are all unstable and fluctuate frequently.

Remark 1 *For the Lemma3.1, when particles form a worm, then their is no particle in the vision cone of the Leader, then the differential equation converts to*

$$\dot{\theta}_i = 2\sqrt{D_\theta}\xi_i, \tag{33}$$

in which represents Brownian motion.

Let M_t be the maximum value of standard Brownian motion B_t from 0 to t, as $M_t = \max\limits_{0\le s\le t} B_t$ Then M_t is not less than the probability of any given threshold α is equal to B_t is not less than twice the probability of any given threshold α, as

$$Prob(M_t \ge \alpha) = Prob(B_t \ge \alpha) \tag{34}$$

This result is very easy to find the probability that $Prob(M_t \ge \alpha)$ as

$$Prob(M_t \ge \alpha) = Prob(B_t \ge \alpha) = 2 - 2\phi(\frac{\alpha}{\sqrt{t}}) \tag{35}$$

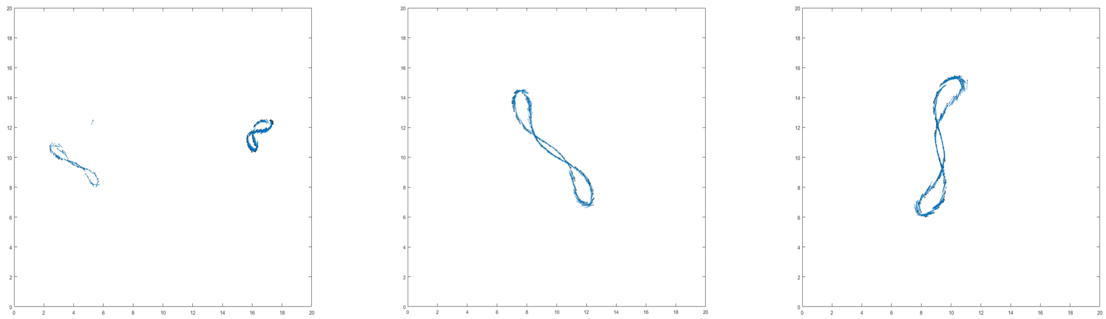
When we focus on a nonstandard Brownian motion M_t', its standard deviation is σ, then

$$Prob(M_t' \ge \alpha) = Prob(B_t \ge \frac{\alpha}{\sqrt{\sigma}}) = 2 - 2\phi(\frac{\alpha}{\sqrt{\sigma t}}) \tag{36}$$

Applying the result into the equation[33], we obtain

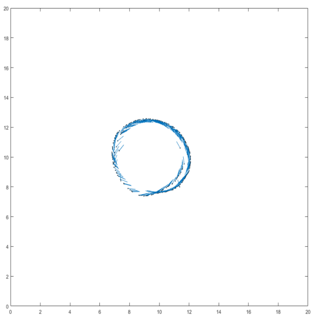
$$Prob(\Delta\theta \ge \theta_0) = Prob(B_t \ge \theta_0) = 2 - 2\phi(\frac{\theta_0}{\sqrt{D_\theta t}}) \tag{37}$$

5. Phenomena

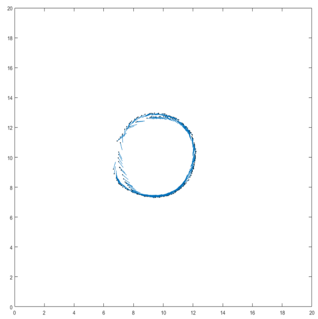


(a) Isolated Torus (b) Milling-like (c) Milling-like

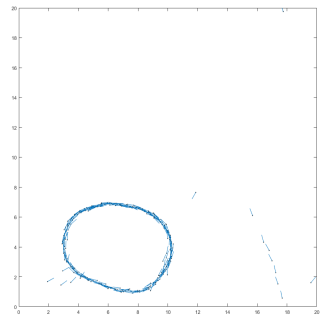
Figure S1: Capital Roman numerals.



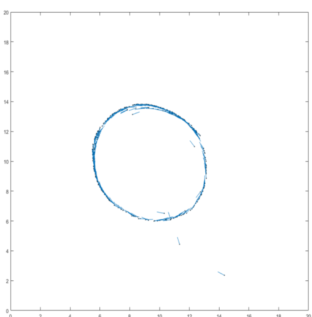
(a) Bidirectional vortex



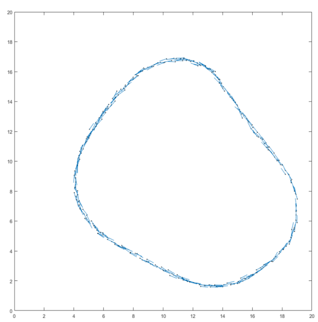
(b) Bidirectional vortex



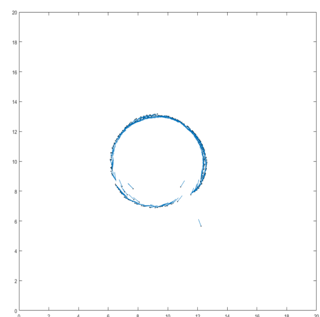
(c) Bidirectional vortex



(d) Bidirectional vortex

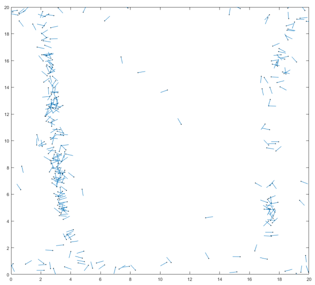


(e) Directional vortex

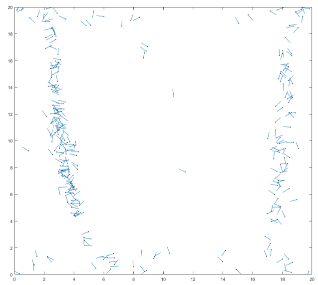


(f) Bidirectional vortex

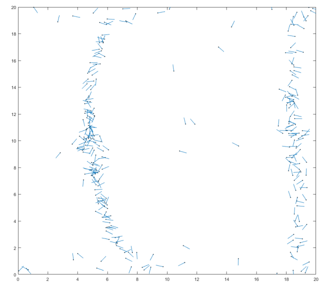
Figure S2: At appropriate values of β,especially when β is near $\frac{\pi}{2}$, various periodic results will be formed according to different noise intensity D_θ.When the NIP is stable near a certain value,the specific performance is that NIP is neither excessively enormous nor too small,simultaneously, CoV expresses small value. It is worth noting that the loop that forms a one-way flow is apparently easy to understand, as long as the latter mimics the actions of the previous particle. Interestingly, sometimes they form bidirectional vortices, but completely opposite flows meet each other without seeming to affect each other. However, the bidirectional particle flow is very fragile, easy to mutate into a twist.



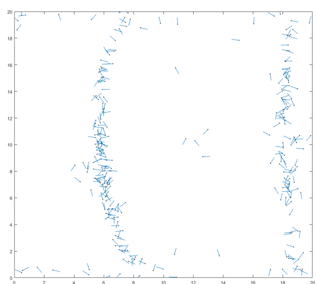
(a) Nematic-Bands



(b) Nematic-Bands

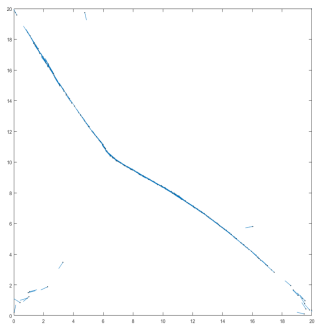


(c) Nematic-Bands

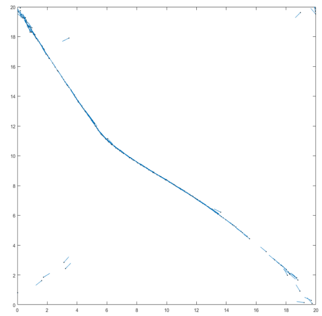


(d) Nematic-Bands

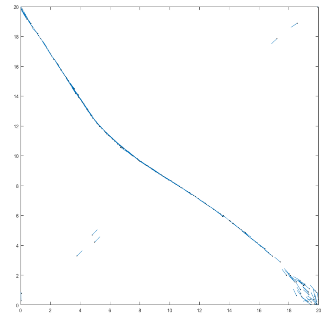
Figure S3: With the increase of noise intensity, particle swarm first behaves as a disordered state and random walk, but when the noise reaches a certain intensity, it will form a more stable liquid crystal state: that is, the position vector of particles is almost stable, but the velocity vector swings back and forth, just like waves.



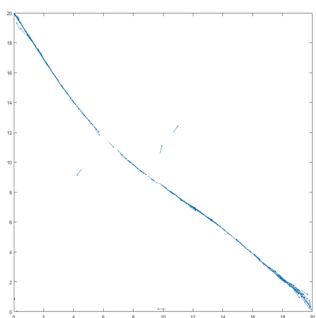
(a) Unitary Worms



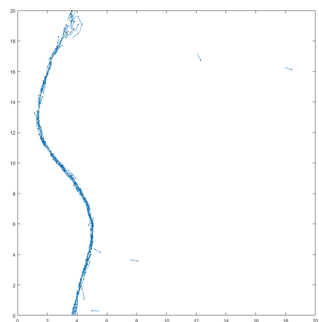
(b) Unitary Worms



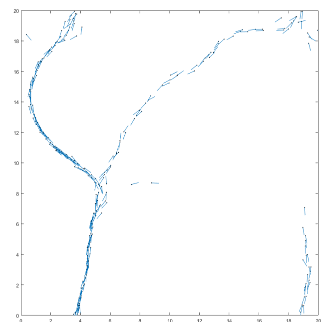
(c) Unitary Worms



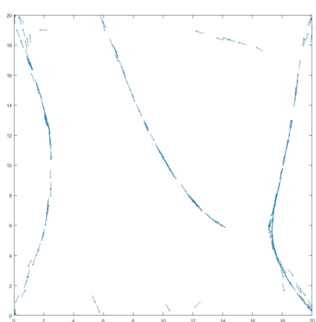
(d) Unitary Worms



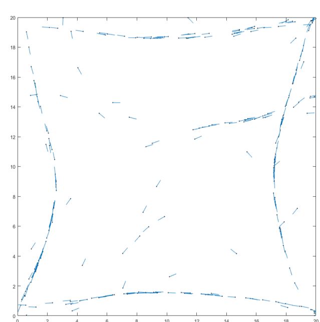
(e) Winding Worms



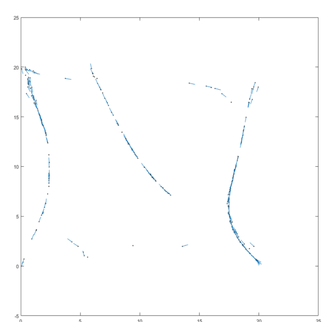
(f) Winding Worms



(g) Isolated Worms



(h) End-to-end Worms



(i) Isolated Worms

Figure S4: When the apex angle of the vision cone β and the noise amplitude D_θ is small, we observe the appearance of a new macrostructure, which we call worms.One agent called "Leader", which can not see anyone in its vision cone and becomes the effective leader of an active particle swarm that spontaneously forms.If keep the vision cone small enough and maintain the evolution time long enough, we can observe the isolated worms phase.