Homework Set Six
ECE 175
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1. Consider the quadratic function

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}.$$

a) show that the function can be written as

$$f(\mathbf{y}) = \sum_i \alpha_i y_i^2$$

where $\mathbf{y}$ is a rotation of $\mathbf{x}$. What is the transformation that maps $\mathbf{x}$ into $\mathbf{y}$, and what are the coefficients α_i?

b) consider the case studied in problem set 2, namely

$$\mathbf{A} = \begin{bmatrix} 5 & b \\ b & 5 \end{bmatrix}$$

with $\mathbf{x}$ is in the range $-100 \leq x_1 \leq 100$, $-100 \leq x_2 \leq 100$, and the three following values of b

$$b \in \{1, 5, 10\}.$$

What are the vectors that point in the direction of the major axes of the ellipses corresponding to the iso-contours of $f(\mathbf{x})$? Using MATLAB, make a contour plot of the function and superimpose a plot of the two vectors, for the three values of b. Hand in the three plots.

c) for the same matrix, plot a slice through the function $f(\mathbf{y})$ for $y_1 = 0$, i.e. the 1D function $f(0, y_2)$ and, $y_2 = 0$. Hand in the two plots obtained for each of the three values of b.

d) explain how the eigenvalues of $\mathbf{A}$ affect the curvature of $f(\mathbf{x})$ and, consequently, in which cases they make $f(\mathbf{x})$ a "bowl", a "saddle", or "one-dimensional".

2. Consider a random variable X distributed according to a Gaussian mixture

$$P_{\mathbf{X}}(\mathbf{x}) = \sum_{i=1}^{C} \pi_i \mathcal{G}(\mathbf{x}, \mu_i, \mathbf{\Lambda}_i)$$

where the covariance matrices $\mathbf{\Lambda}_i$ are diagonal.

a) consider the case where $\mathbf{\Lambda}_i = \sigma_i^2 \mathbf{I}$. Determine the principal components (both orientation and length) of $\mathbf{X}$ as a function of the principal components of a dataset whose covariance is the scatter matrix of the means of $\mathbf{X}$

$$\mathbf{S}_x = \sum_{i=1}^{C} \pi_i [(\mu_i - \mu_x)(\mu_i - \mu_x)^T],$$

and any other necessary parameters of the mixture.

b) suppose that $\mathbf{S}_x = \mathbf{I}$, $\mathbf{X}$ is two-dimensional, and

$$\mathbf{\Lambda}_i = \mathbf{\Lambda} = \begin{bmatrix} c & 0 \\ 0 & 5 \end{bmatrix}$$

for all i. If $\mathbf{\Sigma}$ is the covariance matrix of $\mathbf{X}$, plot the contours

$$\mathbf{x}^T \mathbf{S}_x^{-1} \mathbf{x} = 1$$

and

$$\mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x} = 1$$

for $c = 1$, $c = 5$, and $c = 10$. How do the individual component covariances $\mathbf{\Lambda}_i$ affect the covariance of $\mathbf{X}$?

3. (Computer) While working with the Gaussian classifier in problem set 3, we were unable to utilize the information given in the covariance matrix and assumed the covariance matrix to be identity. This time we will apply PCA on the training data to reduce the dimensionality of our feature space.

1. Implement the PCA algorithm and find out the principal components for the entire dataset `imageTrain`. Plot the top 10 principal components as 28×28 images. Repeat the above, but this time instead of the entire dataset use images from only one class - 'digit5'. Also, for the entire dataset, plot the eigenvalues in a decreasing order.
2. Classification using the PCA subspace on the `imageTest` dataset.
   (a) From the plot of the eigenvalues above, what subspace dimension would be best for classification?
   (b) Calculate the total error rate using subspaces of following dimensions: [5, 10, 20, 30, 40, 60, 90, 130, 180, 250, 350]. Plot these error rates.
   (c) Compare your results with the final error rate obtained in problem set 3.
3. Using the principal components calculated for the class of digit 5 in part 1, find the image from the dataset `imageTest` that is least like a 5. The least 5-like image, would be the one that has maximum energy in the direction orthogonal to the subspace spanned by the principal components of class 5. Assume that the top 40 eigenvectors are the principal components.