FMCW Radar & DOA Techniques

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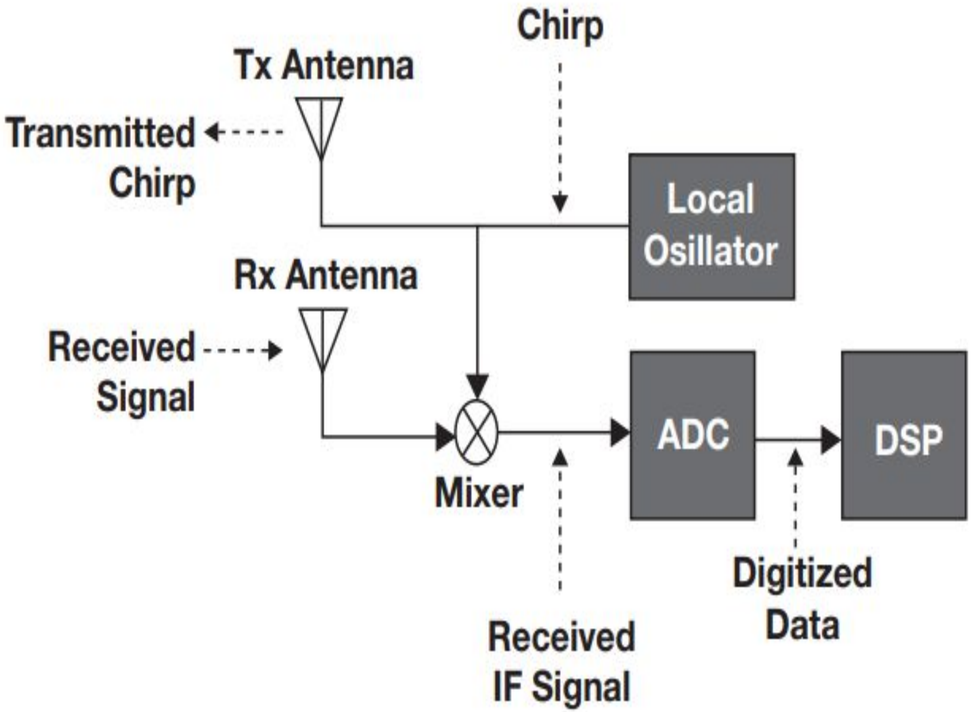
Outline

- FMCW Radar
  - TX antenna and waveform
  - RX antenna and DSP
- Algorithm
  - FFT, MUSIC
  - Coherent integration
- Beam-Forming
  - Uniformed Array vs. Sparse Array
  - Sparse Array:
    - Nested Array
    - CoPrime Array
    - Weight Optimized Array

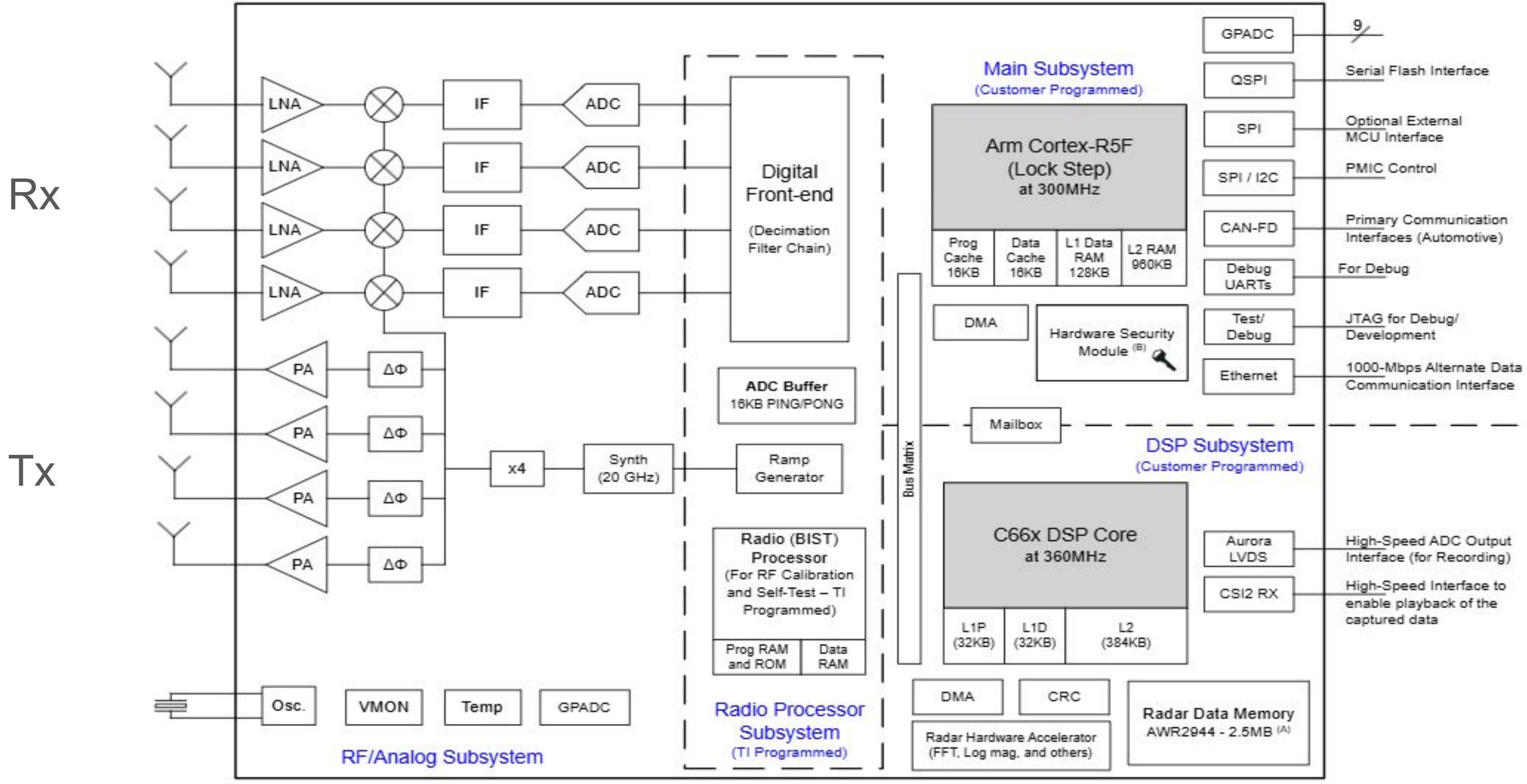
FMCW RADAR

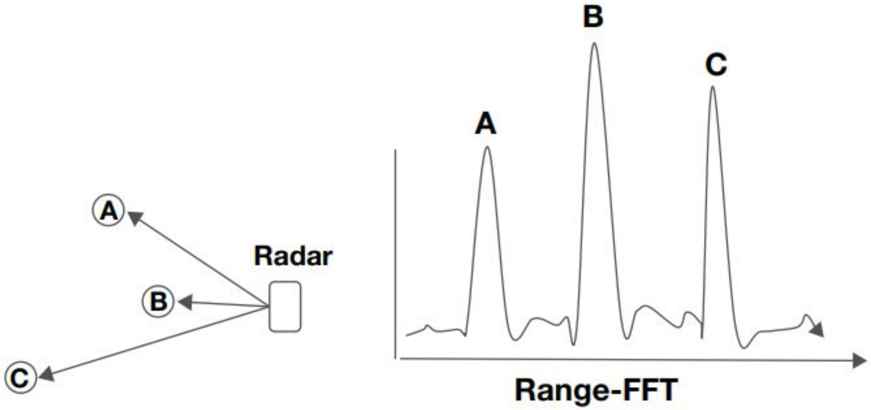


77GHz single chip radar sensor enables automotive body and chassis applications

TEXAS INSTRUMENTS

Block Diagram





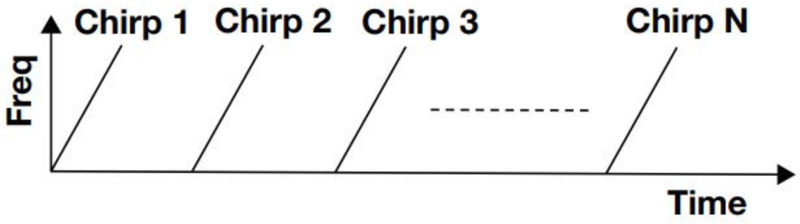
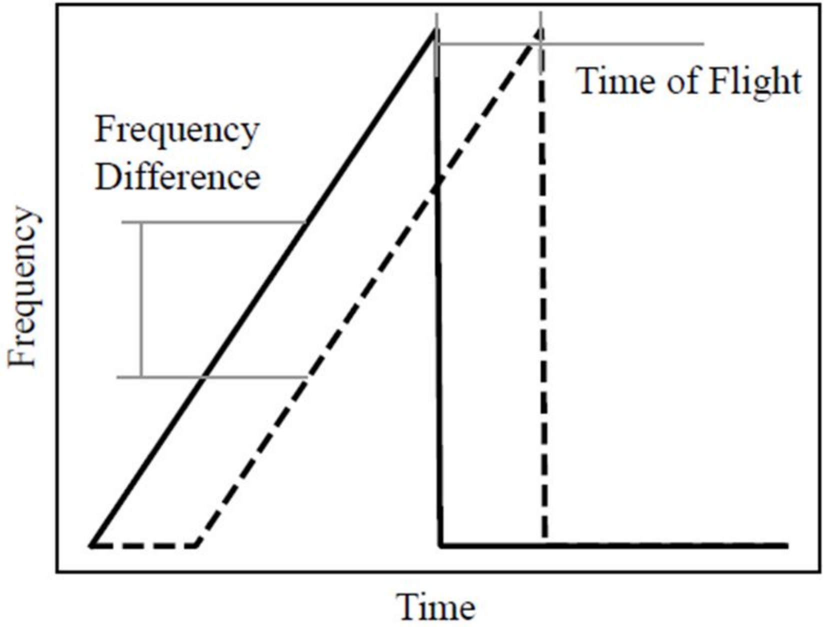


__Figure 2.__ A single frame with N equally spaced chirps.



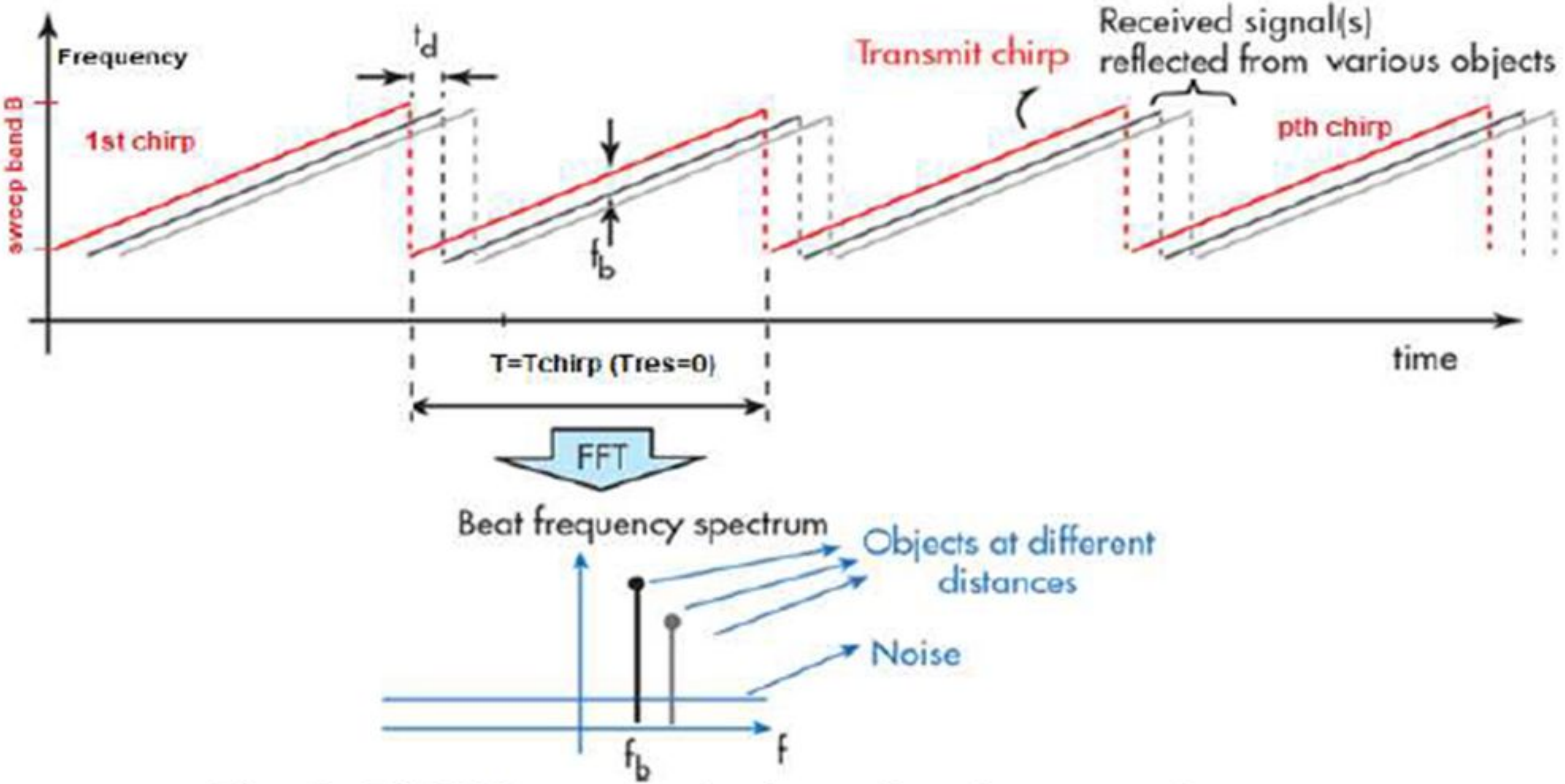
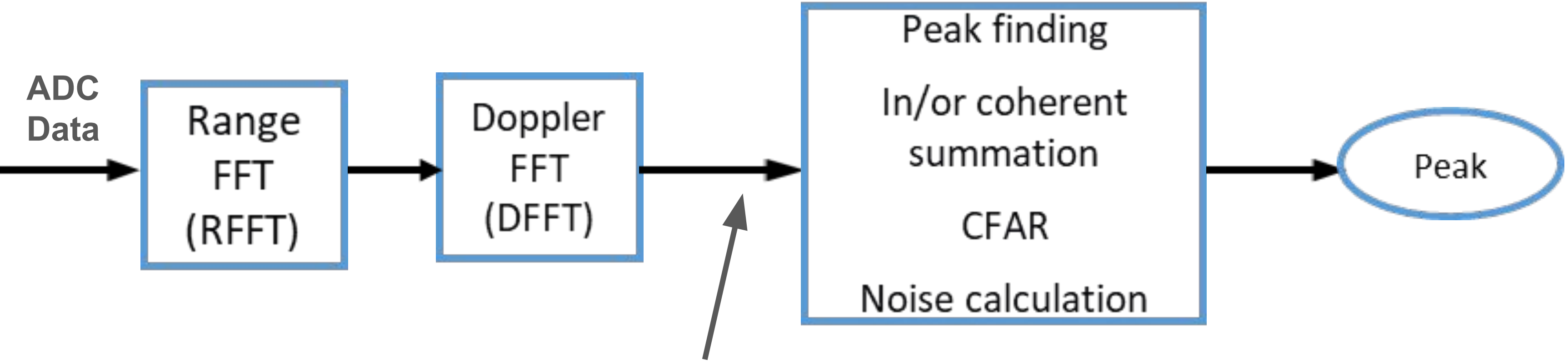
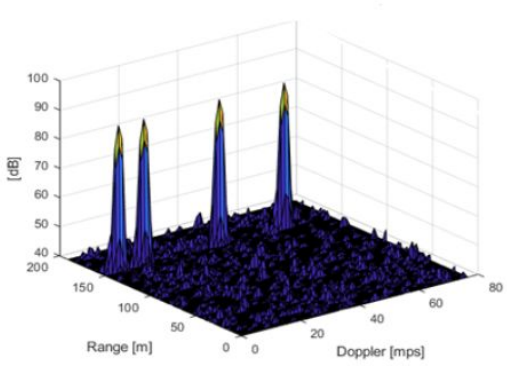


Fig. 3: FMCW range-velocity estimation, waveforms.

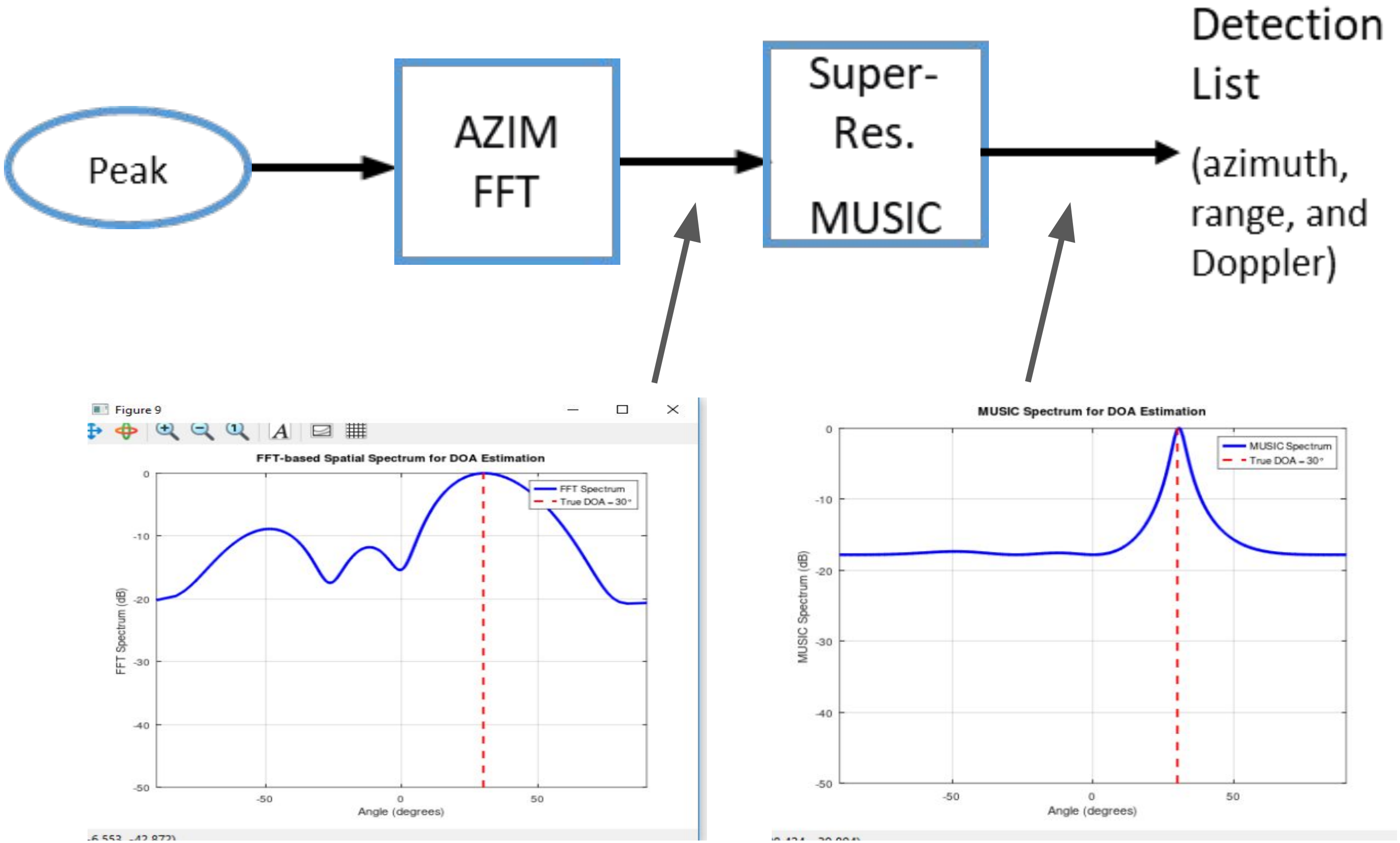
Radar Signal processing flow



Example of Range-Doppler (RD) Map



Radar Signal processing flow



Direction of Arrive (DOA)

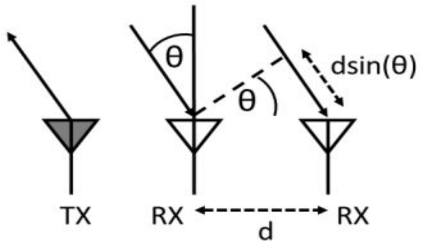


Figure 2.3: Sketch of an antenna array to resolve the AoA.

there are two objects at different angles but with the same velocity and range, there will be two distinct peaks after this last FFT. The angle(s) can then be found by

$$\theta = \arcsin\left(\frac{\lambda\phi}{2\pi d}\right), \tag{2.12}$$

where d here is the RX spacing. If $d = \lambda/2$, the field of view is maximized and the above equation simplifies conveniently.

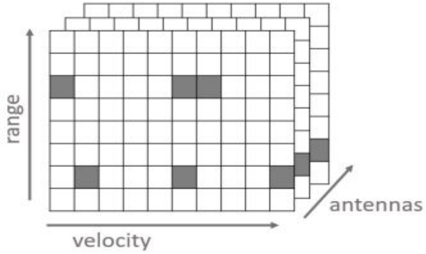


Figure 2.4: The range velocity cube along whose third dimension the angle-FFT is performed.

Algorithms

2D FFT

Range FFT (1st Dimension)

- Input: ADC data $s[n,m,p] = A \cdot e^{j2\pi(f_b n T_s + f_d m T_{\mathrm{PRI}} + \frac{pd\sin\theta}{\lambda})}$
- FFT across samples (n) for each chirp (m) and antenna (p):

$$S[k,m,p] = \sum_{n=0}^{N-1} s[n,m,p] \cdot e^{-j2\pi\frac{kn}{N}}$$

  - k: Range bin, N: Number of samples, Maps f_b to range: $R = \frac{f_b c T_c}{2B}$

Doppler FFT (2nd Dimension)

- FFT across chirps (m) for each range bin (k) and antenna (p):

$$S[k,l,p] = \sum_{m=0}^{M-1} S[k,m,p] \cdot e^{-j2\pi\frac{lm}{M}}$$

  - l: Doppler bin, M: Number of chirps, Maps f_d to velocity: $v = \frac{f_d\lambda}{2}$

Range-Doppler Heatmap

- Compute power spectrum:

$$P[k,l] = \left|\frac{1}{P}\sum_{p=0}^{P-1} S[k,l,p]\right|^2$$

  - Plot: X-axis (l) → Velocity, Y-axis (k) → Range, Color → Power (dB)

Spatial FFT (3rd Dimension)

- Identify peaks in range-Doppler map $P[k,l]$, then FFT across antennas (p) for each peak ($k_{\mathrm{peak}}, l_{\mathrm{peak}}$):

$$S[k_{\mathrm{peak}}, l_{\mathrm{peak}}, q] = \sum_{p=0}^{P-1} S[k_{\mathrm{peak}}, l_{\mathrm{peak}}, p] \cdot e^{-j2\pi\frac{qp}{P}}$$

  - p: Antenna index (0 to $P-1$), q: Angle bin, P: Number of antennas

Peak Detection & Angle Estimation

- Find angle peak in $|S[k_{\mathrm{peak}}, l_{\mathrm{peak}}, q]|^2$:

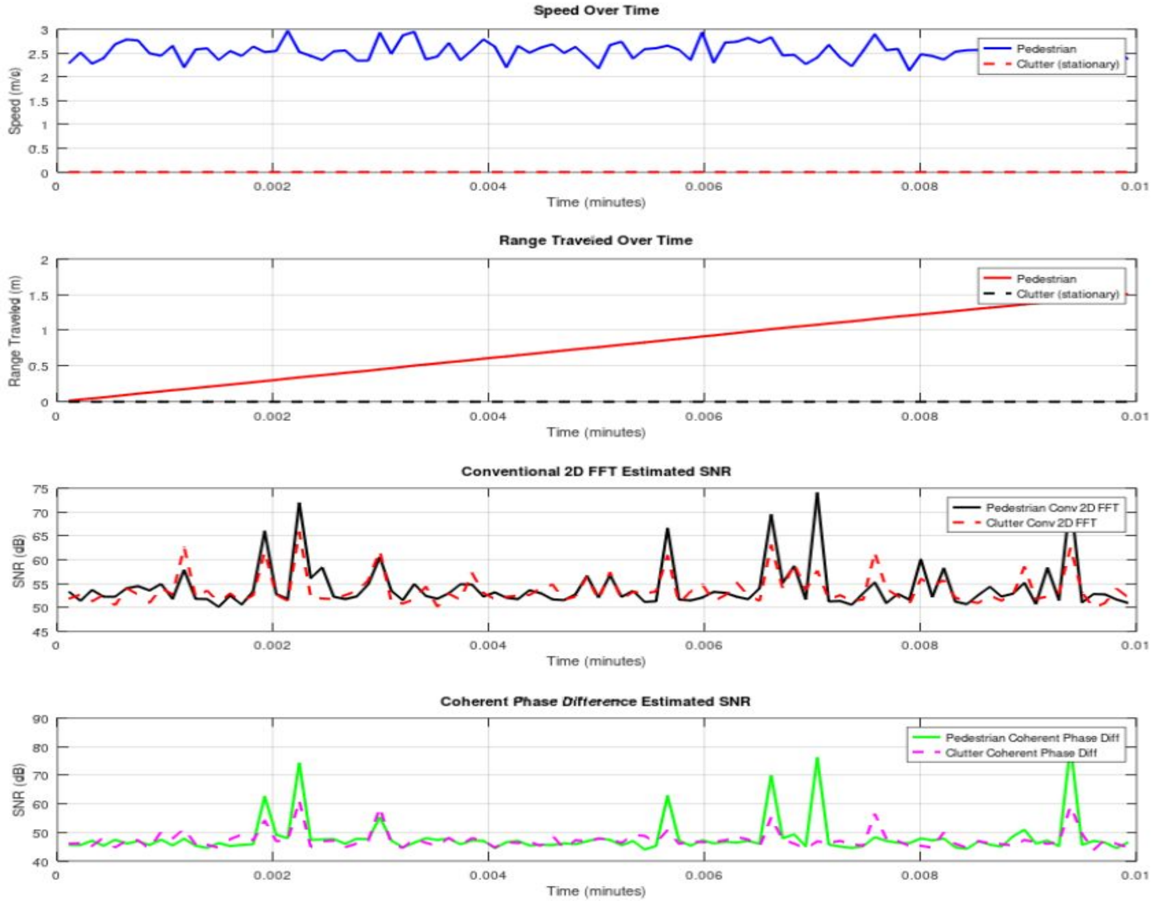
$$q_{\mathrm{peak}} = \arg\max_q |S[k_{\mathrm{peak}}, l_{\mathrm{peak}}, q]|^2$$

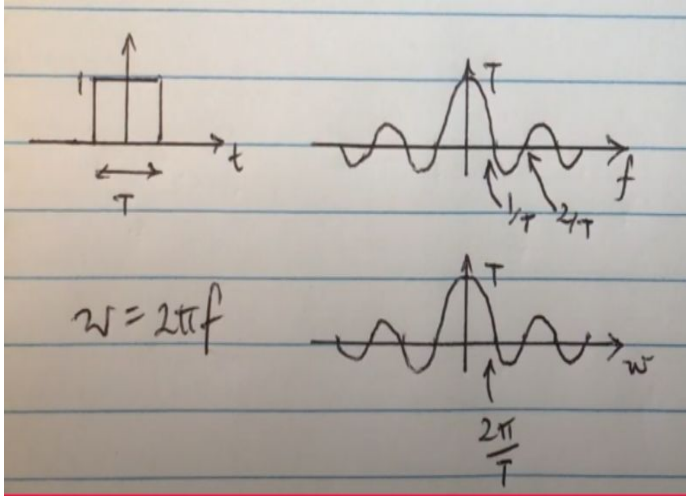
- Estimate angle of arrival (AoA):

$$\theta = \sin^{-1}\left(\frac{q_{\mathrm{peak}}\lambda}{Pd}\right)$$

  - d: Antenna spacing, λ: Wavelength

Identify moving target next to strong clutters





2D FFT

1. Received Beat Signal Model (Equation (1))

The beat signal received from a mixture of stationary clutter and a moving pedestrian is:

$$s(l,k) = A_c \cdot e^{j2\pi f_r \frac{l-1}{L}} \cdot e^{j2\pi \Delta\varphi_c} + A_m \cdot e^{j2\pi f_r \frac{l-1}{L}} \cdot e^{j2\pi f_D T(k-1)}$$

where:

- $l = 1, 2, \ldots, L$ is the fast-time sample index within a ramp,
- $k = 1, 2, \ldots, K$ is the slow-time ramp index,
- A_c and A_m are the amplitudes of clutter and moving target respectively,
- f_r is the range frequency corresponding to the target range,
- $\Delta\varphi_c$ is the initial phase of the clutter (constant over ramps),
- f_D is the Doppler frequency of the moving target,
- T is the pulse repetition interval (PRI).

2. Range FFT (Equation (2))

For each ramp k, the range FFT is computed as:

$$X(m,k) = \sum_{l=1}^{L} s(l,k) \cdot w(l) \cdot e^{-j2\pi \frac{(l-1)(m-1)}{L}}, \quad m = 1, 2, \ldots, L$$

where:

- $w(l)$ is the window function applied to reduce sidelobes (rectangular in the script),
- $X(m,k)$ is the complex range spectrum at range bin m and ramp k.

3. Conventional Doppler FFT

For each range bin m, the Doppler FFT across ramps is:

$$RD(m,n) = \sum_{k=1}^{K} X(m,k) \cdot v(k) \cdot e^{-j2\pi \frac{(k-1)(n-1)}{K}}, \quad n = 1,2,\ldots,K$$

where:

- $v(k)$ is the Doppler window (rectangular in the script),
- $RD(m,n)$ is the 2D range-Doppler map.

4. Coherent Phase Difference Method

4.1 Phase Difference Calculation

Calculate the phase difference between consecutive ramps for each range bin m:

$$PD(m,k) = X(m,k+1) \cdot X^*(m,k), \quad k = 1,2,\ldots,K-1$$

- $X^*(m,k)$ is the complex conjugate of $X(m,k)$,
- This operation suppresses stationary clutter (constant phase), retaining moving targets (changing phase).

4.2 Doppler FFT on Phase Difference

Perform Doppler FFT on the phase difference data:

$$RD_{PD}(m,n) = \sum_{k=1}^{K-1} PD(m,k) \cdot e^{-j2\pi \frac{(k-1)(n-1)}{K-1}}, \quad n = 1,2,\ldots,K-1$$

- $RD_{PD}(m,n)$ is the clutter-suppressed range-Doppler map emphasizing moving targets.

5. SNR Estimation

For a given range bin m (pedestrian or clutter), the SNR is estimated as:

$$\mathrm{SNR} = 10\log_{10}\left(\frac{\max_n |RD(m,n)|^2}{P_{\text{noise}}}\right)$$

and similarly for the coherent phase difference method:

$$\mathrm{SNR}_{\text{coh}} = 10\log_{10}\left(\frac{\max_n |RD_{PD}(m,n)|^2}{P_{\text{noise}}}\right)$$

where:

- P_{noise} is the noise power,
- $\max_n$ denotes the maximum over Doppler bins.

6. Pedestrian Range Update

Assuming constant velocity during each burst, the pedestrian range evolves as:

$$R_{\text{ped}}(t_{burst}) = R_{\text{ped}}(t_{burst-1}) - v_{\text{ped}} \times T_{\text{burst}}$$

where:

- v_{ped} is the pedestrian speed (randomly varying),
- $T_{\text{burst}} = K \times PRI$ is the burst duration.

Summary

Code Block	Mathematical Expression	Description
Signal generation	$s(l, k)$ as above	Beat signal with clutter + moving target
Range FFT	$X(m, k)$	Extract range spectrum per ramp
Doppler FFT (conventional)	$RD(m, n)$	Range-Doppler map
Phase difference	$PD(m, k) = X(m, k+1)X^*(m, k)$	Suppress clutter, keep moving targets
Doppler FFT on PD	$RD_{PD}(m, n)$	Clutter-suppressed range-Doppler map

DOA MUSIC Algorithm

Signal Model

$$\mathbf{x}(t) = \mathbf{A}(\theta)\mathbf{s}(t) + \mathbf{n}(t), \quad \mathbf{A}(\theta) = [\mathbf{a}(\theta_1) \dots \mathbf{a}(\theta_K)]$$

$$\mathbf{a}(\theta) = [1, e^{j\frac{2\pi d \sin\theta}{\lambda}}, \dots, e^{j(M-1)\frac{2\pi d \sin\theta}{\lambda}}]^T$$

Covariance Matrix

$$\hat{\mathbf{R}}_{xx} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}(t_n)\mathbf{x}^H(t_n), \quad E[\mathbf{R}_{xx}] = \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma^2\mathbf{I}$$

Eigenvalue Decomposition

$$\hat{\mathbf{R}}_{xx} = \mathbf{E}_s\mathbf{\Lambda}_s\mathbf{E}_s^H + \mathbf{E}_n\mathbf{\Lambda}_n\mathbf{E}_n^H$$

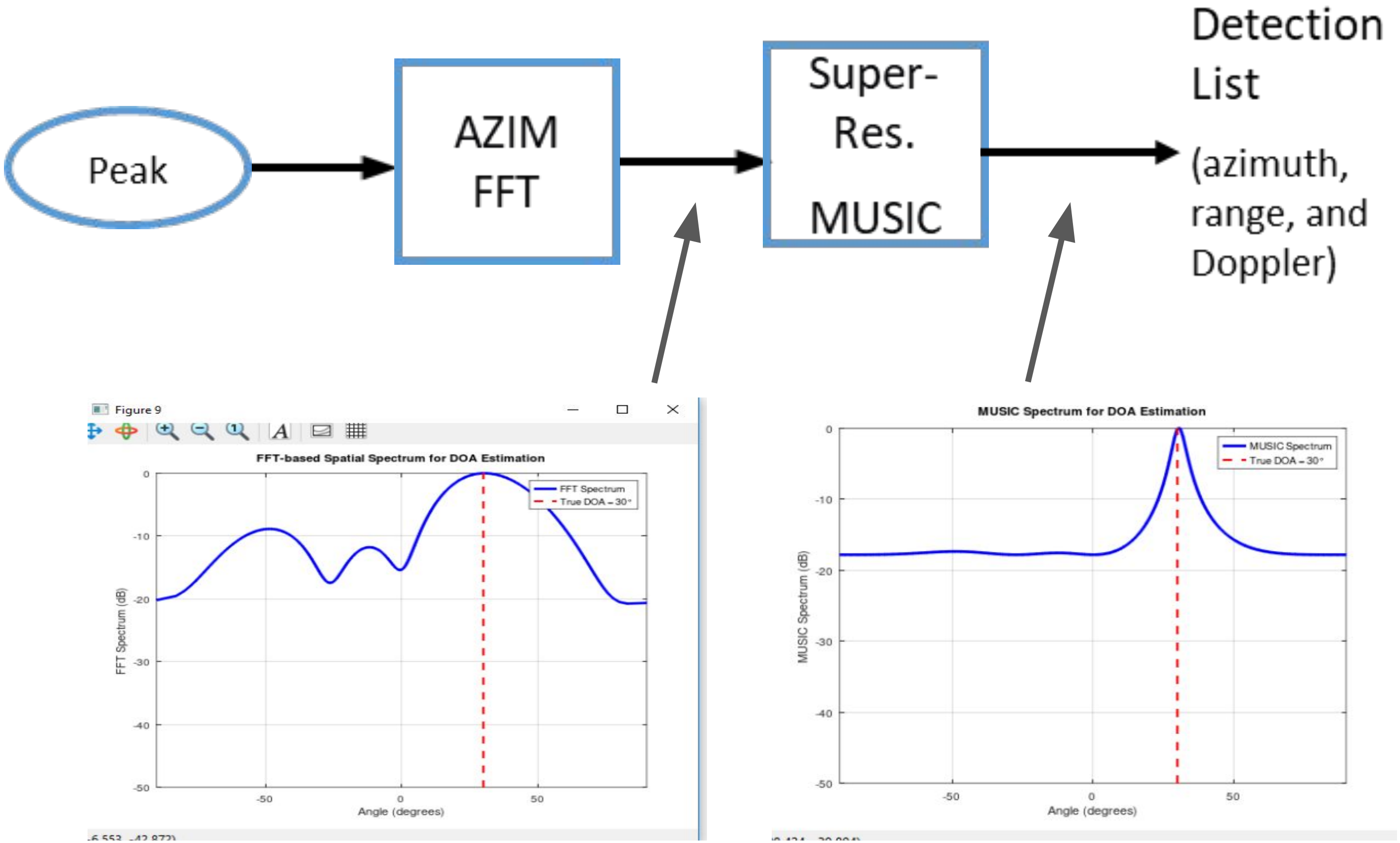
- $\mathbf{E}_s$: Signal subspace, $\mathbf{E}_n$: Noise subspace

MUSIC Spectrum

$$P_{\text{MUSIC}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{E}_n\mathbf{E}_n^H\mathbf{a}(\theta)}$$

- Peak in $P_{\text{MUSIC}}(\theta)$ gives θ

Radar Signal processing flow

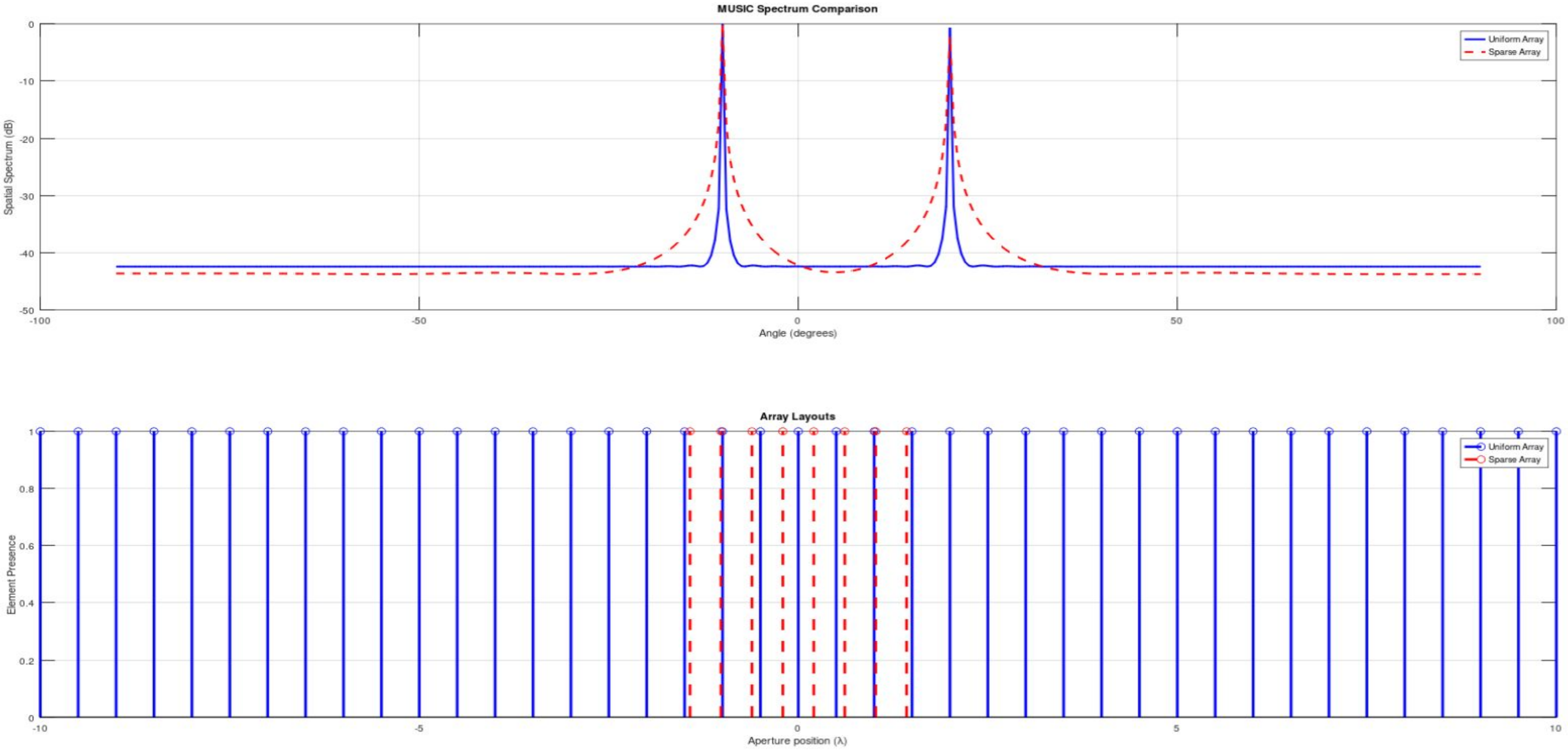


Passive Array Beamforming

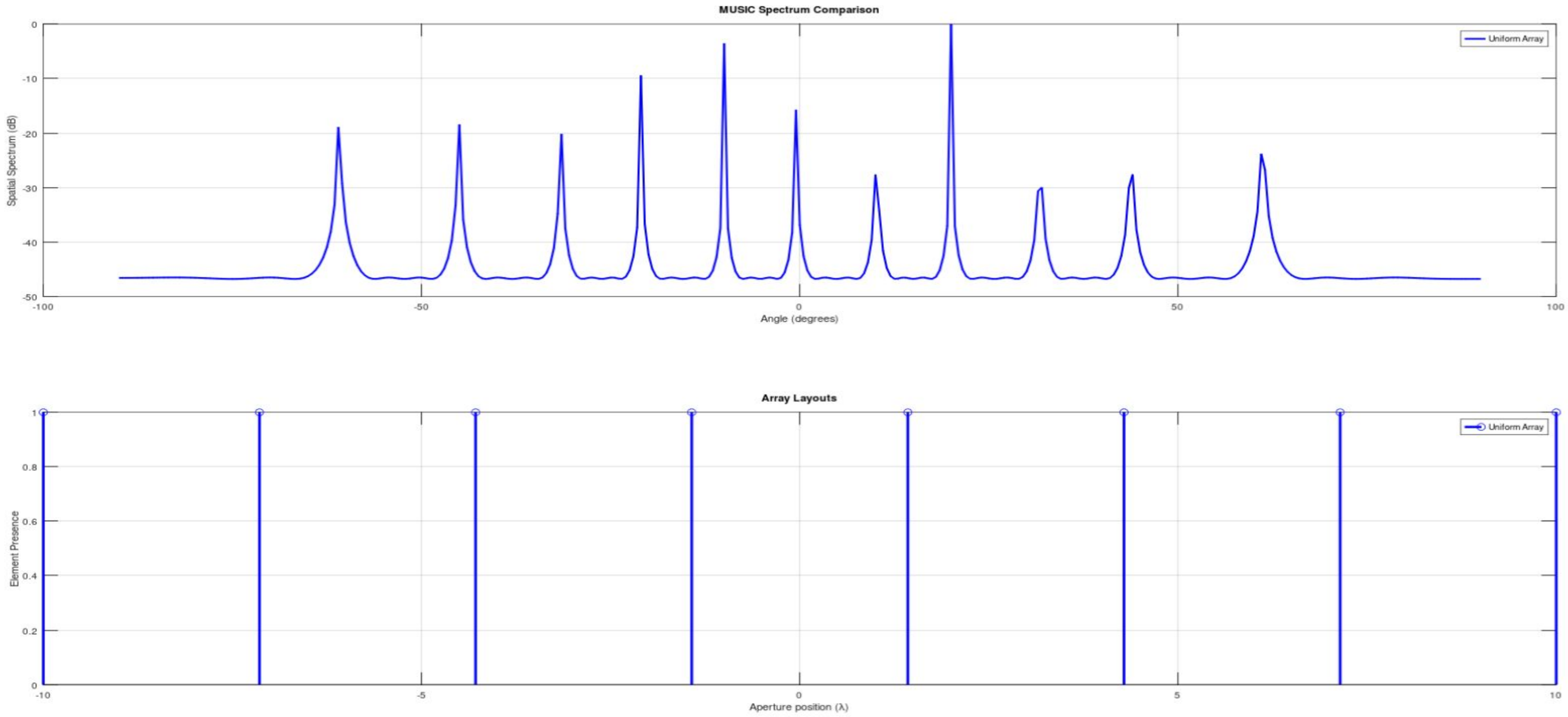
Beamforming Techniques

1. Uniform Array
2. Sparse + MRA
3. Sparse + Coprime
4. Sparse + Weight Optimization

Sparse (Red) Array vs. Uniformed (Blue) Array Comparison



Uniformed Array d = $2.5\,\lambda$, array aperture = $20\,\lambda$, many grating lobes



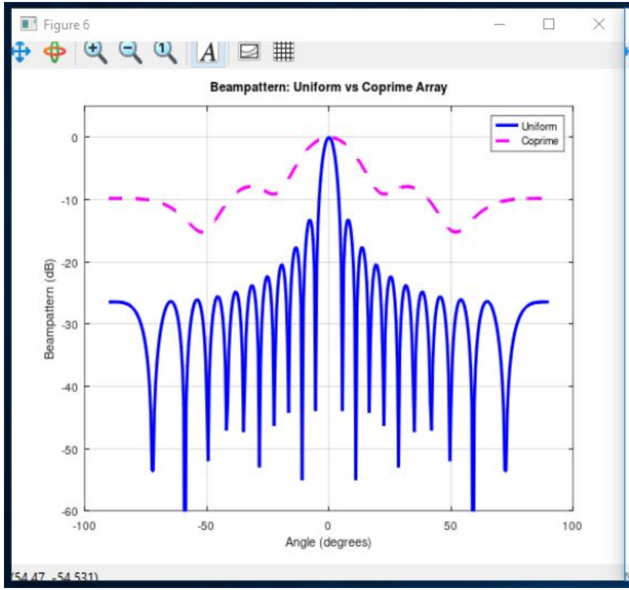


Figure 6
Beampattern: Uniform vs Coprime Array
Uniform
Coprime
Beampattern (dB)
Angle (degrees)

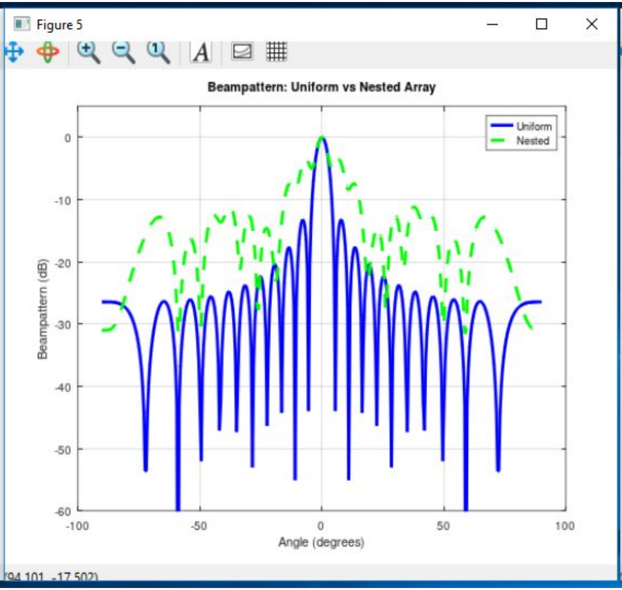


Figure 5
Beampattern: Uniform vs Nested Array
Uniform
Nested
Beampattern (dB)
Angle (degrees)

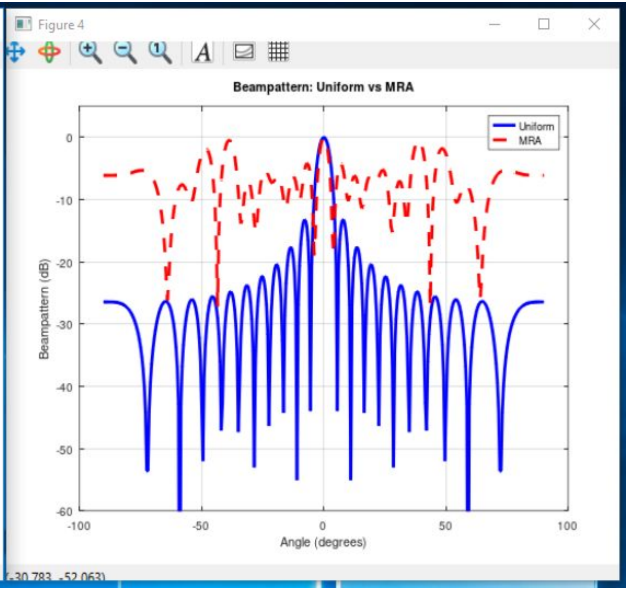


Figure 4
Beampattern: Uniform vs MRA
Uniform
MRA
Beampattern (dB)
Angle (degrees)

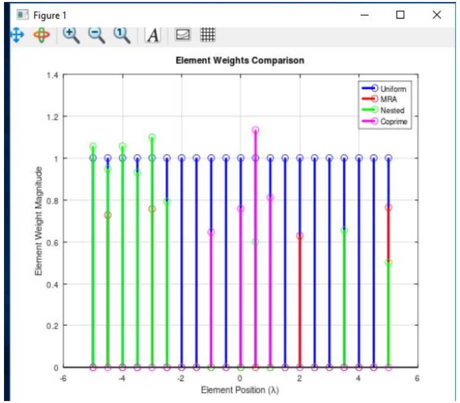
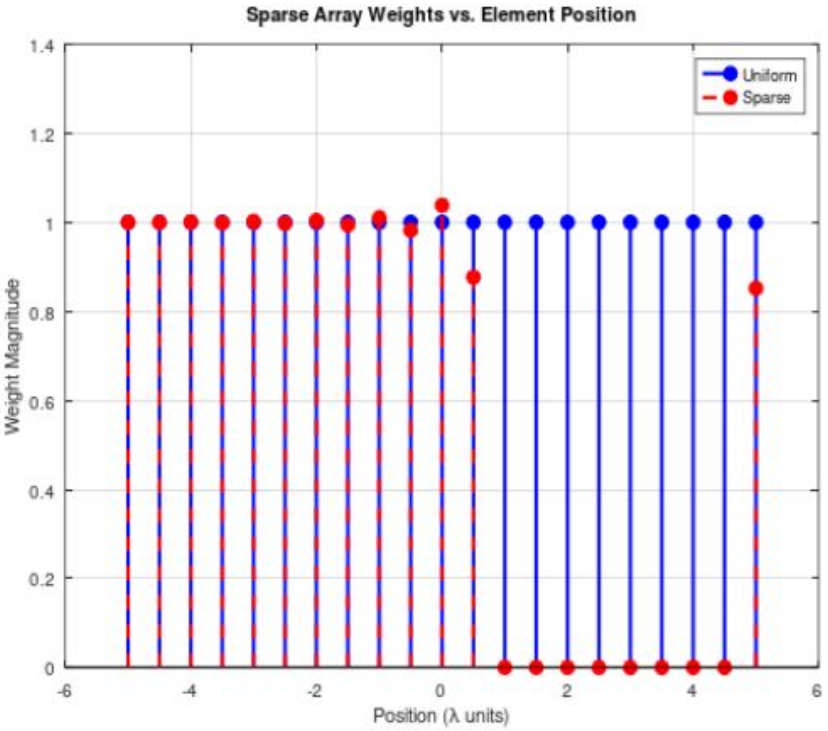
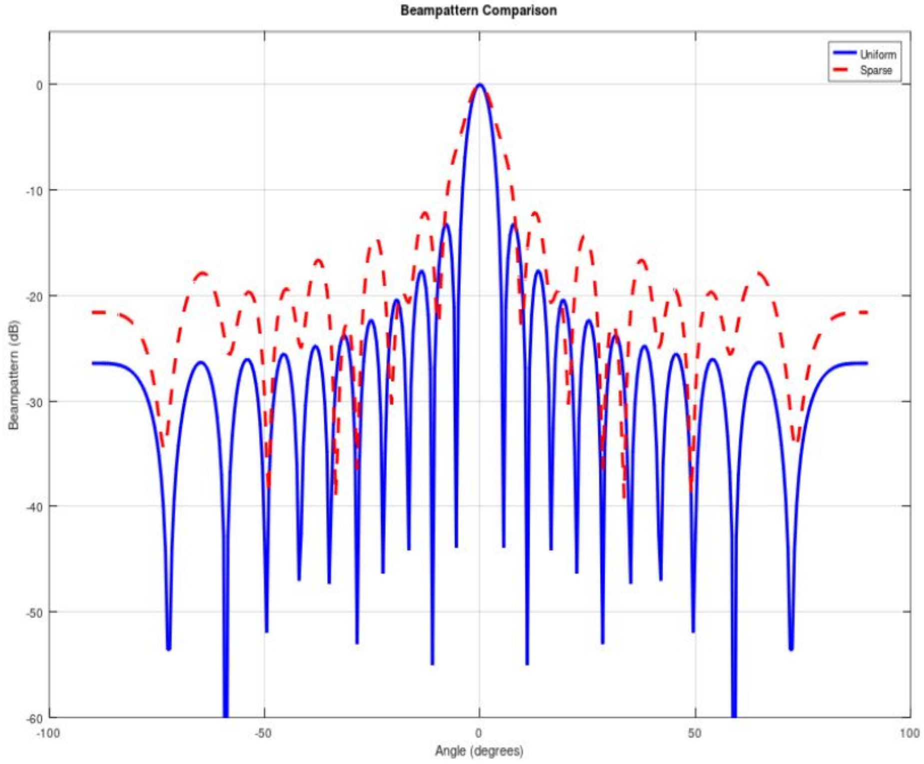


Figure 1
Element Weights Comparison
Uniform
MRA
Nested
Coprime
Element Weight Magnitude
Element Position (λ)


Sparse Array Weights vs. Element Position
Uniform
Sparse
Weight Magnitude
Position (λ units)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-6
-4
-2
0
2
4
6


Beampattern Comparison
Uniform
Sparse
Beampattern (dB)
Angle (degrees)
0
-10
-20
-30
-40
-50
-60
-100
-50
0
50
100

1. Wavelength and Aperture

- Wavelength: $\lambda = \frac{c}{f_c}$
- Aperture size: $A = 10\lambda$
- Element spacing: $d = \frac{\lambda}{2}$

2. Array Grid Setup

- Number of elements: $N = \left\lfloor \frac{A}{d} \right\rfloor + 1$
- Element positions: $x_n = \text{linspace}\left(-\frac{A}{2}, \frac{A}{2}, N\right)$

3. Steering Matrix (Array Manifold)

For each angle θ in the grid:

$$A_{k,n} = \exp\left(j\frac{2\pi}{\lambda}x_n \sin(\theta_k)\right)$$

where

- $A_{k,n}$ is the response of the n-th element at angle θ_k,
- x_n is the position of the n-th element,
- λ is the wavelength,
- θ_k is the angle in degrees (converted to radians for sin).

4. Uniform Weights and Desired Response

- Uniform weights: $w_{\text{uniform}} = \mathbf{1}_N$ (vector of ones)
- Desired beampattern: $d_{\text{desired}} = Aw_{\text{uniform}}$

5. Sparse Array Selection (Greedy Algorithm)

For each iteration k (up to K elements):

- Select the element j that minimizes the cost function:

$$\text{error} = \|d_{\text{desired}}(\text{mainlobe}) - \text{recon}(\text{mainlobe})\|_2 + 3\|d_{\text{desired}}(\text{sidelobe}) - \text{recon}(\text{sidelobe})\|_2$$

where

- $\text{recon} = A_{\text{temp}}w_{\text{temp}}$,
- $w_{\text{temp}} = \text{pinv}(A_{\text{temp}})d_{\text{desired}}$,
- A_{temp} is the steering matrix for the currently selected elements.

6. Final Sparse Weights

- For the selected elements:

$$w_{\text{sel}} = \text{pinv}(A_{\text{sel}})d_{\text{desired}}$$

- Full sparse weight vector:

$$w_{\text{sparse}}(n) = \begin{cases} w_{\text{sel}}(i), & \text{if } n = \text{selected}(i) \\ 0, & \text{otherwise} \end{cases}$$

7. Beampattern Calculation

- Uniform array power pattern:

$$P_{\text{uniform}}(\theta) = |A(\theta) w_{\text{uniform}}|^2$$

- Sparse array power pattern:

$$P_{\text{sparse}}(\theta) = |A(\theta) w_{\text{sparse}}|^2$$

- Normalize and convert to dB:

$$P_{\text{dB}} = 10 \log_{10} \left(\frac{P}{\max(P)} \right)$$

8. Visualization

- Plot $P_{\text{uniform,dB}}$ and $P_{\text{sparse,dB}}$ versus θ
- Plot $|w_{\text{uniform}}|$ and $|w_{\text{sparse}}|$ versus normalized position x_n/λ

Summary Table

Summary Table

Step	Equation / Concept
Wavelength	$\lambda = c/f_c$
Element positions	$x_n = \text{linspace}(-A/2, A/2, N)$
Steering matrix	$A_{k,n} = \exp(j2\pi x_n \sin(\theta_k)/\lambda)$
Uniform weights	$w_{\text{uniform}} = \mathbf{1}_N$
Desired response	$d_{\text{desired}} = A w_{\text{uniform}}$
Sparse weights	$w_{\text{sel}} = \text{pinv}(A_{\text{sel}}) d_{\text{desired}}$
Sparse vector	$w_{\text{sparse}}(n) = w_{\text{sel}}(i)$ if $n = \text{selected}(i)$
Beampattern	$$P(\theta) =

- The response of this beamformer is given by

$$P(\omega, \theta) = \sum_{m=0}^{M-1} e^{-j\omega\tau_m} w_m = \mathbf{w}^H \mathbf{a}(\omega, \theta), \tag{2}$$

where $\mathbf{a}(\omega, \theta)$ is the steering vector and $\mathbf{w}$ is the coefficient vector

$$\begin{aligned} \mathbf{a}(\omega, \theta) &= \left[1 \; e^{-j\omega\tau_1} \; \ldots \; e^{-j\omega\tau_{M-1}}\right]^T \\ \mathbf{w} &= \left[w_0^* \, w_1^* \; \ldots \; w_{M-1}^*\right]^T . \end{aligned} \tag{3}$$

- For a particular beamforming task, the optimum coefficient vector $\mathbf{w}$ is dependent on both the frequency and direction of the impinging signals (certainly also the array layout).

Sparse Array Design (for BW < 20MHz)
Sparse + Weight Optimization

Theorem:

2. Design of Narrowband Sparse Arrays

- In practice, l_1 norm is used as an approximation to the l_0 norm:

$$\min ||\mathbf{w}||_1 \quad \text{subject to} \quad ||\mathbf{p}_r - \mathbf{w}^H \mathbf{A}||_2 \leq \alpha \, . \tag{13}$$

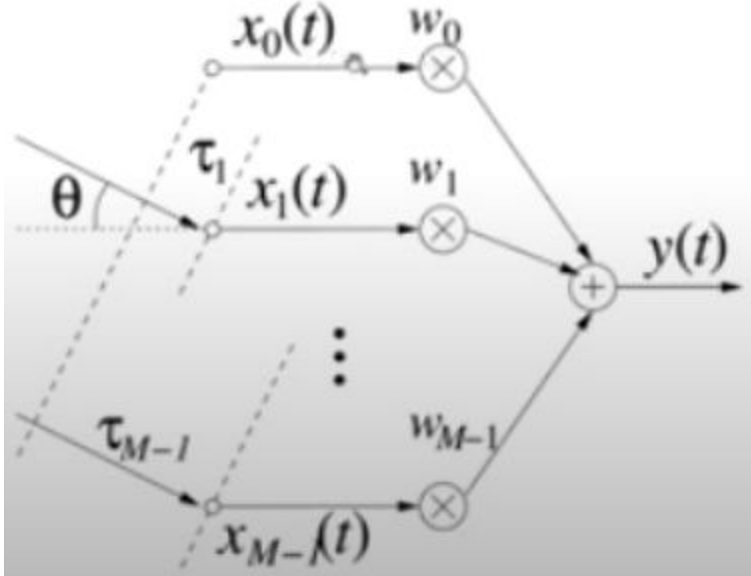
- Sparsity of the solution can be enhanced by minimizing a series of weighted l_1 minimizations [9]. At iteration i, we have

$$\min \sum_{m=0}^{M-1} a_m^i |w_m^i| \quad \text{subject to} \quad ||\mathbf{p}_r - \mathbf{w}^H \mathbf{A}||_2 \leq \alpha \, , \tag{14}$$

where $a_m^i = (|w_m^{i-1}| + \eta)^{-1}$ and the very small constant η is required to ensure stability.

A small coefficient w_m^{i-1} leads to a large weighting term a_m^i, and therefore heavily penalized at the next iteration, while a large coefficient give a small weighting term, more likely to be replicated at the next iteration.

Narrowband beamforming is achieved by changing the phase and amplitude of the received array signals, and the beamformer output $y(t)$ is given by an instantaneous linear combination of the spatial samples $x_m(t)$, $m = 0, 1, \ldots, M-1$.



$$y(t) = \sum_{m=0}^{M-1} x_m(t) w_m \quad (1)$$

$\tau_m = d_m \sin\theta / c$ with d_m being the distance from sensor 0 to sensor m, and w_m is the beamformer coefficient.

The End