随机微分方程 (SDE) 总结

名称	SDE 形式	解析解	技巧	分布类型
几何布朗运动 (GBM)	$dX_t = \mu X_t dt + \sigma X_t dW_t$	$X_t = X_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$	Itô 引理 + 指数化	对数正态
Ornstein–Uhlenbeck (OU)	$dX_t = -\theta X_t dt + \sigma dW_t$	$X_t = X_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s)} dW_s$	积分因子法	高斯分布

注

几何布朗运动（*GBM*）：是 *Black–Scholes* 模型的核心......

常见可解析 SDE 一览表（含解法与分布）

名称	SDE 形式	解析解
几何布朗运动 (GBM)	$dX_t = \mu X_t dt + \sigma X_t dW_t$	$X_t = X_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W}$
Ornstein–Uhlenbeck (OU)	$dX_t = -\theta X_t dt + \sigma dW_t$	$X_t = X_0 e^{-\theta t} + \sigma \int_0^t e^{-\theta(t-s}$
Cox–Ingersoll–Ross (CIR)	$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t$	显式分布，不显式路径
Bessel 过程	$dX_t = \frac{n-1}{2X_t}dt + dW_t$	$X_t = \lVert B_t \rVert$ in $\mathbb{R}^n$
对数布朗运动	$dX_t = \mu dt + \sigma dW_t, \quad X_0 = \log S_0$	$S_t = S_0 \exp(X_t)$
线性扩散	$dX_t = (aX_t + b)dt + (cX_t + d)dW_t$	根据参数结构可解

注

几何布朗运动（*GBM*）：是 *Black–Scholes* 模型的核心，最经典的显式解。*OU* 过程：均值回归过程，用于建模利率、速度等，具有高斯平稳性。*CIR* 过程：用于利率与波动率模型，不能为负，解涉及非中心卡方分布，但不能得到显式路径公式。*Bessel* 过程：是多个独立布朗运动的模长，常用于物理中径向扩散建模。*Lamperti* 变换：将非恒定扩散系数的 *SDE* 变换为常系数扩散。

欧拉法

欧拉折线法与阿尔泽拉-阿斯科利 (Arzela-Ascoli) 定理

Conditions for Existence and Uniqueness I

Consider an n-dimensional stochastic differential equation:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \tag{1}$$

where:

- dW_t is an n-dimensional white noise,
- $\mu : \mathbb{R}^n \to \mathbb{R}^n$ is the drift coefficient,
- $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is the diffusion coefficient.

The following conditions ensure the existence and uniqueness of solutions:

- **Global Lipschitz condition**:

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq K_1|x - y|.$$

- **Linear growth condition**:

$$|\mu(x)|^2 + |\sigma(x)|^2 \leq K_2(1 + |x|^2).$$

Under these conditions, the SDE (1) has a unique strong solution.

Numercial approximation

Most SDEs cannot be solved analytically, and numerical methods are used to approximate their solutions. Two main approaches are time-discretization schemes and space-discretization schemes.

空间离散格式

因此，SDE (2.1.2) 的弱解可以通过求解 Fokker-Planck 方程 (2.1.3) 或 Kolmogorov 方程 (2.1.4) 来获得。本文考虑利用这一联系来设计 SDE 的数值解法。

基本思想是使用数值偏微分方程理论，构造 SDE 的无穷小生成元的近似形式。我们通过离散化 SDE 的生成元 $\mathcal{L}$，构造一个离散状态空间上的生成元 Q，其形式如下：

$$Qf(x) = \sum_{i=1}^{K} q(x, y_i(x)) \left[f(y_i(x)) - f(x) \right]$$

在这里，我们引入了一个反应（或跳跃）率函数 $q : \Omega \times \Omega \to [0, \infty)$，以及 K 个反应通道（或跳跃状态）$y_i(x) \in \Omega$，对于每个 $x \in \Omega$，$1 \le i \le K$。这一术语源自化学动力学。

令 h 为空间步长参数，p 为一个正实数，用以设定该方法的逼近阶数。我们要求无穷小生成元 Q 满足以下两个条件：

(Q1) 局部 p 阶精度：

Asymptotic Analysis of Mean Holding Time in 1D I

Space-discretization scheme(Bou-Rabee and Vanden-Eijnden, 2018). [1]Simulate the evolution of SDEs based on the space-discretization Kolmogorov equation and random walk.
The time lapse between two grid points: $Y(0) = x_i$ and $Y(t^e) = x_{i-1}$ where $x_i > x_{i-1} \gg 0$ satisfies

$$t^e = \int_{x_{i-1}}^{x_i} \frac{dx}{|\mu(x)|}.$$

For simplicity, we assume that the spacing between grid points is uniform: $\delta x_i^+ = \delta x_i^- = \delta x$. By using integration by parts and the mean value theorem, observe that can be written as:

$$t^e = \frac{\delta x}{|\mu(x_i)|} - \int_{x_{i-1}}^{x_i} (x - x_{i-1})\mu(x)^{-2}\mu'(x)dx$$

$$= t^* - \frac{1}{2}\frac{\mu'(\xi)}{\mu(\xi)^2}\delta x^2$$

for some $\xi \in (x_{i-1}, x_i)$, and where we have introduced: $t^* = \delta x / |\mu_i|$.

$$t^u = ((Q_u)_{i,i+1} + (Q_u)_{i,i-1})^{-1} = \frac{\delta x^2}{2 + |\mu_i|\delta x}$$

This expression can be rewritten as

$$t^u = t^* - t^* \frac{2}{2 + |\mu_i|\delta x}.$$

we see that t^u approaches t^* as $|x_i|$ becomes large, as the next Proposition states.

命题

For any $\delta x > 0$, the mean holding time of $\tilde{Q}_u$ satisfies:

$$\frac{|t^u - t^*|}{t^*} \to 0 \quad \text{as } |x_i| \to \infty.$$

Likewise, if the second term decays faster than the first term, then the relative error between t^e and t^* also tends to zero, and thus, the estimate predicted by Q_u for the mean holding time asymptotically agrees with the exact mean holding time.

The second term decays sufficiently fast if, e.g., the leading order term in $\mu(x)$ is of the form $-ax^{2p+1}$ for $p \geq 0$ and $a > 0$.

Repeating these steps for the mean holding time predicted by $\tilde{Q}_c$ yields:

$$t^c = ((Q_c)_{i,i+1} + (Q_c)_{i,i-1})^{-1} = \frac{\delta x^2}{2} \operatorname{sech}\left(\mu_i \frac{\delta x}{2}\right).$$

Asymptotic Analysis of Mean Holding Time in 1D IV

It follows from this expression that even though Q_c is a second-order accurate approximation to L, it does not capture the right asymptotic mean holding time, as the next Proposition states.

命题

For any $\delta x > 0$, the mean holding time of $\tilde{Q}_c$ satisfies:

$$\frac{|t^c - t^*|}{t^*} \to 1 \quad \text{as } |x_i| \to \infty.$$

Simply put, the mean holding time of Q_c converges to zero too fast.

Zu 的工作：引进 $\tilde{Q}^u$ 并且与精确时间 t^e 比较 I

Q_c 方案具有二阶近似精度，但其平均保持时间的渐近行为较差，且不适应小噪声情况。
Q_u 方案可以克服这些困难，但仅具有一阶近似精度。泊松过程可以被近似为确定性过程的漂移加上布朗运动,Q_u 方案可能会因泊松近似 SDEs 漂移项会导致非零方差，而产生人为的扩散效应. 寻找一种更有效的 Q 方法变得迫切
在 [2] 中，提出了一种改进的 Q_u 方案 $\tilde{Q}_u$ 方案，通过减少公式(5.1)中的 $sigma$ 来补偿额外的人为扩散项，
注意在 [2] 中的 SDE 形式为

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW, \quad X_t(0) \in \Omega, \tag{5.1}$$

因此此时的 $M_{ii}(x) = \frac{1}{2}\sigma_{ii}^2(x)$
改进的 $\tilde{Q}_u$ 格式定义如下:

Zu 的工作：引进 $\tilde{Q}^u$ 并且与精确时间 t^e 比较 II

$$\tilde{Q}_u f(x) = \sum_{i=1}^{n} \left[\frac{\mu_i(x) \vee 0}{h_i(x)} + \frac{M_{ii}^+(x)}{h_i(x) h_i^+(x)} \right] \left(f(x + h_i^+(x) e_i) - f(x) \right) + \left[\frac{-\mu_i(x) \wedge 0}{h_i(x)} + \frac{M_{ii}^-(x)}{h_i(x) h_i^-(x)} \right] \left(f(x - h_i^-(x) e_i) - f(x) \right). \quad (2)$$

其中

$$M_{ii}^+(x) = \frac{1}{2} \left(\sigma_{ii}^2(x) - |\mu_i(x)| h_i^+(x) \right) \vee 0, \quad M_{ii}^-(x) = \frac{1}{2} \left(\sigma_{ii}^2(x) - |\mu_i(x)| h_i^-(x$$

这个公式通过调整扩散项 σ，使得 $\tilde{Q}_u$ 格式在漂移项中补偿了泊松近似带来的误差。漂移项的人为扩散效应在小噪声情况下尤为显著，因此 $\tilde{Q}_u$ 格式在小噪声的场景下显著提高了精度

Zu 的工作：引进 $\tilde{Q}^u$ 并且与精确时间 t^e 比较 III

Eric 没有进行 t^u 与精确时间 t^e 之间的比较，Zu 提出改进格式，并且与精确时间 t^e 比较

定理

Assume that $|\mu(x)|$ is large enough and

$$\frac{\mu'(x)}{\mu(x)^2} \sim o\left(\frac{1}{\mu(x)}\right) \quad \textit{as } |x| \to 0.$$

For any $h > 0$, the relative error between t^e and $\tilde{t}^u$ satisfies:

$$\frac{|\tilde{t}^u - t^e|}{t^e} \sim O\left(\frac{\mu'(x)}{\mu(x)}\right) \to 0, \quad \textit{as } |x| \to \infty,$$

where t^e is the time $X(t)$ takes to move a fixed distance h from x to $x - h$.

新工作：t^u 与 t^e 比较，t^u 与 $\tilde{t}^u$ 比较 I

定理

$$\frac{|t^u - t^e|}{|t^e|} \sim O\left(\frac{\mu'(x)}{\mu(x)}\right) \to 0, \quad \textsf{as } |x| \to \infty$$

而且

$$|t^u - t^e| > |\tilde{t}^u - t^e|$$

改进后的 $\tilde{Q}_u$ 格式优于 Q_u 格式

Background on SDE Numerical Methods I

It is well known that under the Lipschitz condition and linear growth condition, an SDE has a unique strong solution and the EM approximation has a strong convergence rate of order $\frac{1}{2}$ (see Kloeden and Platen [3]).

The Euler-Maruyama method performs poorly here due to the locally Lipschitz drift. The Markov chain diverges [4]

Without the linear growth condition, the explicit Euler scheme may not converge to the exact solution of an SDE in the strong mean square sense. Even worse, [5] showed that the moments of the standard EM approximate solution at a finite time may diverge to infinity even if the true solution is finite.

Many implicit methods have therefore been proposed to study the numerical solutions of SDEs with nonlinear coefficients. For example, Higham, Mao and Stuart [6] proved that the implicit EM numerical solutions converge strongly to the exact solutions of SDEs with the globally one-side Lipschitz continuous drift term and globally Lipschitz continuous diffusion term.

Background on SDE Numerical Methods II

On the other hand, some explicit methods have also been proposed for nonlinear SDEs.
For example, [7] proposed the tamed EM schemes to approximate SDEs with the global Lipschitz diffusion coefficient and one-sided Lipschitz drift coefficient, whose numerical solutions converge strongly to the exact solution with order $\frac{1}{2}$.
Moreover, the tamed Milstein [8] and the stopped EM method [9] as well as their variants have also been proposed to deal with the strong converge problem for nonlinear SDEs.
Recently, Mao [10], [11] proposed a new explicit method called the truncated EM method for nonlinear SDEs and established the strong convergence and obtained the convergence rate under the local Lipschitz condition and the Khasminskii-type condition. The authors of [12] showed that the truncated EM method may enable to use a larger stepsize than the tamed Euler method in [13] to achieve the same error.

The **Euler-Maruyama method** uses a time step size Δ, with $t_k = t_0 + k\Delta$. The approximation is given by:

$$\hat{X}^{\Delta}_{t_{k+1}} = \hat{X}^{\Delta}_{t_k} + \mu(\hat{X}^{\Delta}_{t_k})\Delta + \sigma(\hat{X}^{\Delta}_{t_k})(W_{t_{k+1}} - W_{t_k}),$$

where $W_{t_{k+1}} - W_{t_k} \sim \mathcal{N}(0, \Delta)$ are independent Gaussian random vectors. Consider the stochastic differential equation:

$$dX = -X^3 dt + \sigma dW, \quad X(0) \in \mathbb{R}.$$

The solution is geometrically ergodic with a stationary distribution density:

$$\nu(x) = Z^{-1} \exp\left(-\frac{x^4}{2\sigma^2}\right), \quad \text{where} \quad Z = \int_{\mathbb{R}} \exp\left(-\frac{x^4}{2\sigma^2}\right) dx.$$

The Euler-Maruyama method performs poorly here due to the locally Lipschitz drift. The Markov chain $\{\hat{X}_{n\Delta}\}$ diverges [4]:

$$\mathbb{E}\left[\left(\hat{X}_{\lfloor t/\Delta \rfloor \Delta}\right)^2\right] \to \infty \quad \text{as} \quad t \to \infty.$$

我们的问题

[6] indicated that many important SDE models satisfy only a local Lipschitz property and, since Brownian paths can make arbitrarily large excursions, the global Lipschitz-based theory is not directly relevant. [6] (实践中，许多重要的随机微分方程（SDE）模型仅满足局部 Lipschitz 条件，并且由于布朗运动路径可能会进行任意大的偏移，基于全局 Lipschitz 条件的理论并不直接适用)

当 $x \to \infty$ 时, 空间离散化的 Mean holding time 和时间离散化方法中，固定空间距离，使用的时间两者之间的比较;

传统的欧拉方法在漂移场较大时通常表现较差.考虑当时间离散的 Euler 格式, 由于布朗运动的性质，可能会出现 $(W_{t_{k+1}} - W_{t_k}) \to \infty$ 的情况，注意到此时由于我们使用的离散格式，

$$\hat{X}^{\Delta}_{t_{k+1}} = \hat{X}^{\Delta}_{t_k} + \mu(\hat{X}^{\Delta}_{t_k})\Delta + \sigma(\hat{X}^{\Delta}_{t_k})(W_{t_{k+1}} - W_{t_k}),$$

, 这样会使得 $\hat{X}^{\Delta}_{t_{k+1}} \to \infty$, 造成数值解进入极端不稳定区域（导致数值解在正负无穷之间振荡）, 当然在连续情况不会发生这种情况, 考虑(**??**), 当 $dW_t \to \infty$, 方程会迅速将轨线拉回到 0 附近

Tamed EM method I

Huzenthaler 在 [7] 提出 Tamed 方法

$$X_{t_{k+1}}^{\Delta} = X_{t_k}^{\Delta} + \frac{\Delta \cdot \mu(X_{t_k}^{\Delta})}{1 + \Delta \|\mu(X_{t_k}^{\Delta})\|} + \sqrt{2} \cdot \sqrt{\Delta} \cdot N(0,1) \tag{3}$$

当 $|X_{t_k}^{\Delta}| < M$，此时对于 $\Delta < \varepsilon$，

$$\frac{\Delta \cdot \mu(X_{t_k}^{\Delta})}{1 + \Delta \|\mu(X_{t_k}^{\Delta})\|} \to 0$$

若 $\forall M > 0, X_{t_k}^{\Delta} > M$，

$$\frac{\Delta \cdot \|\mu(X_{t_k}^{\Delta})\|}{1 + \Delta \|\mu(X_{t_k}^{\Delta})\|} \to 1$$

Tamed EM method II

一旦出现小概率事件，不会出现振荡. 对于我们的方程，此时有

$$X^{\Delta}_{t_{k+1}} - X^{\Delta}_{t_k} = \frac{\Delta \cdot |-X^{\Delta}_{t_k}|^3}{1 + \Delta \cdot |-X^{\Delta}_{t_k}|^3} + \sqrt{2} \cdot \sqrt{\Delta} \cdot N(0,1)$$

于是我们有，对于 $h > 0$

$$t^{\delta} = \frac{h}{\mathbb{E}[X^{\Delta}_{t_{k+1}} - X^{\Delta}_{t_k}]} \sim O(h)$$

同样我们可以得到，如果 M 足够大，我们有 $t^e \to 0, t^{\delta}$ fixed
于是我们有如下结果，

定理

$$\frac{|t^{\delta} - t^e|}{t^e} \to \infty$$

Mao 在 [10] 新的显式方法——截断 EM 方法，其核心思想是通过截断技术控制系数增长，从而保证数值解的收敛性
我们的方程满足 Mao 的方法的使用条件，
Let $u(x) = x^3$, then $u^{-1}(x) = x^{\frac{1}{3}}$.
Let $h(x) = x^{-\frac{1}{4}}$, then $u^{-1}(h(x)) = (x^{-\frac{1}{4}})^{\frac{1}{3}} = x^{-\frac{1}{12}}$.
If $|x| \to \infty$, $|x| \wedge u^{-1}(h(\Delta)) = \Delta^{-\frac{1}{12}}$.
As $|x|$ becomes smaller, it approaches $u^{-1}(h(\Delta))$.
At this point,

$$X^{\Delta}_{t_{k+1}} - X^{\Delta}_{t_k} = -(\Delta^{-\frac{1}{12}}) \cdot \Delta + \sqrt{2} \cdot \sqrt{\Delta} \cdot N(0,1)$$

$$= -\Delta^{-\frac{3}{4}} + \sqrt{2} \cdot \sqrt{\Delta} \cdot N(0,1).$$

Taking expectation:

$$\mathbb{E}[X^{\Delta}_{t_{k+1}} - X^{\Delta}_{t_k}] = -\Delta^{-\frac{3}{4}} + \sqrt{2} \cdot \sqrt{\Delta}.$$

When $|x| \to \infty$, i.e., $|x| > M$ in numerical terms, the expected value of the time driven by h is:

$$t^{\Delta} = \frac{h}{\mathbb{E}[X^{\Delta}_{t_{k+1}} - X^{\Delta}_{t_k}]} = \frac{h}{-\Delta^{-\frac{3}{4}} + \sqrt{2} \cdot \sqrt{\Delta}} \cdot \Delta \sim O(h \cdot \Delta^{\frac{1}{4}})$$

irst choose Δ and h, then choose M (from the equality condition outside the truncation).
对于空间离散格式，我们有

$$t^{e} = t^{*} - \frac{1}{2}\frac{\mu'(\xi)}{\mu(\xi)^2}\delta x^2$$

如果 M 足够大，我们有 $t^e \to 0, t^{\Delta}$ fixed
于是我们有如下结果，

Truncated EM Method III

定理

$$\frac{|t^{\Delta} - t^{e}|}{t^{e}} \to \infty$$

符号说明

$\mu(X_t)$ 漂移项
$\sigma(X_t)$ 扩散项
$\delta x = h$ 空间离散步长
Δ 时间离散步长
Q_u Eric 提出的有限差分格式
Q_c Eric 提出的有限体积格式
$\tilde{Q}_u$ Zu 提出的改进的有限差分格式
t^e 当 $|x| \to \infty$ 时，精确的 mean holding time
t^* t^e 的主要部分
t^u Q_u 格式的 mean holding time
t^c Q_c 格式的 mean holding time
$\tilde{t}^u$ $\tilde{Q}_u$ 格式的 mean holding time
t^Δ 截断方法运动 h 距离所需要的平均时间
t^δ tamed 方法运动 h 距离所需要的平均时间

References I

- N. Bou-Rabee and E. Vanden-Eijnden, *Continuous-time random walks for the numerical solution of stochastic differential equations*. American Mathematical Society, 2018, vol. 256, no. 1228.
- J. Zu, "Random walk numerical scheme for the steady-state of stochastic differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 121, p. 107200, 2023.
- P. E. Kloeden, E. Platen, P. E. Kloeden, and E. Platen, *Stochastic differential equations*. Springer, 1992.
- J. C. Mattingly, A. M. Stuart, and D. J. Higham, "Ergodicity for sdes and approximations: locally lipschitz vector fields and degenerate noise," *Stochastic processes and their applications*, vol. 101, no. 2, pp. 185–232, 2002.

References II

M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, "Strong and weak divergence in finite time of euler's method for stochastic differential equations with non-globally lipschitz continuous coefficients," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 467, no. 2130, pp. 1563–1576, 2011.

D. J. Higham, X. Mao, and A. M. Stuart, "Strong convergence of euler-type methods for nonlinear stochastic differential equations," *SIAM journal on numerical analysis*, vol. 40, no. 3, pp. 1041–1063, 2002.

M. Hutzenthaler, A. Jentzen, and P. E. Kloeden, "Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients," 2012.

References III

X. Wang and S. Gan, "The tamed milstein method for commutative stochastic differential equations with non-globally lipschitz continuous coefficients," *Journal of Difference Equations and Applications*, vol. 19, no. 3, pp. 466–490, 2013.

W. Liu and X. Mao, "Strong convergence of the stopped euler–maruyama method for nonlinear stochastic differential equations," *Applied Mathematics and Computation*, vol. 223, pp. 389–400, 2013.

X. Mao, "The truncated euler–maruyama method for stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 290, pp. 370–384, 2015.

——, "Convergence rates of the truncated euler–maruyama method for stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 296, pp. 362–375, 2016.

References IV

- L. Hu, X. Li, and X. Mao, "Convergence rate and stability of the truncated euler–maruyama method for stochastic differential equations," *Journal of Computational and Applied Mathematics*, vol. 337, pp. 274–289, 2018.
- S. Sabanis, "Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients," 2016.