Distributed entanglement

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Consider three qubits *A*, *B*, and *C* which may be entangled with each other. We show that there is a trade-off between *A*'s entanglement with *B* and its entanglement with *C*. This relation is expressed in terms of a measure of entanglement called the concurrence, which is related to the entanglement of formation. Specifically, we show that the squared concurrence between *A* and *B*, plus the squared concurrence between *A* and *C*, cannot be greater than the squared concurrence between *A* and the pair *BC*. This inequality is as strong as it could be, in the sense that for any values of the concurrences satisfying the corresponding equality, one can find a quantum state consistent with those values. Further exploration of this result leads to a definition of an essential three-way entanglement of the system, which is invariant under permutations of the qubits.

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Quantum entanglement has rightly been the subject of much study in recent years as a potential resource for communication and information processing. As with other resources such as free energy and information, one would like to have a quantitative theory of entanglement giving specific rules about how it can and cannot be manipulated; indeed, such a theory has begun to be developed. The first step in building the theory has been to quantify entanglement itself. In the last few years a number of entanglement measures for bipartite states have been introduced and analyzed [1–7], the one most relevant to the present work being the ''entanglement of formation'' [2], which is intended to quantify the amount of quantum communication required to create a given state. In the present paper we draw on previous work on entanglement of formation [6,7] in order to explore another basic quantitative question: To what extent can an object be simultaneously entangled with two other objects?

Unlike classical correlations, quantum entanglement cannot be freely shared among many objects. For example, given a triple of spin-1/2 particles *A*, *B*, and *C*, if particle *A* is fully entangled with particle *B*, e.g., if they are in the singlet state $(1/\sqrt{2})(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, then particle *A* cannot be simultaneously entangled with particle *C*. (If *A* were entangled with *C*, then the pair *AB* would also be entangled with *C* and would therefore have a mixed-state density matrix, whereas the singlet state is pure.) One expects that a less extreme form of this restriction should also hold: if *A* is *partly* entangled with *B*, then *A* can have only a limited entanglement with *C*. The first goal of this paper is to verify this intuition and express it quantitatively. We will see that the restriction on the sharing of entanglement takes a particularly elegant form in terms of a measure of entanglement called the ''concurrence,'' which is closely related to the entanglement of formation. Further analysis of this result will lead us naturally to a quantity that measures an essential three-way entanglement of the system and is invariant under all permutations of the particles [8].

The present work is related to recent work on the characterization of multiparticle states in terms of invariants under local transformations [9–12]; indeed, both the concurrence and our measure of three-way entanglement are invariants in this sense. Our work is also related to research exploring the connection between entanglement and cloning [13–17]. An example along these lines was studied by Bruß, who asked, in the case of a singlet pair *AB*, to what extent particle *B*'s entanglement with particle *A* can be shared symmetrically and isotropically with a third particle, for a purpose such as teleportation where isotropy is desired [18]. Our investigation is similar in spirit to that of Bruß but has a different focus in that we are looking for a general law that is independent of assumptions about symmetry or isotropy. Some of the results presented here have been mentioned in a recent paper by one of us [19], but the proofs and most of the details and observations have not been previously published. In this paper we confine our attention to binary quantum objects (qubits) such as spin-1/2 particles—we will use the generic basis labels $|0\rangle$ and $|1\rangle$ rather than $|\uparrow\rangle$ and $|\downarrow\rangle$—but the same questions could be raised for larger objects.

We begin by defining concurrence. Let *A* and *B* be a pair of qubits, and let the density matrix of the pair be ρ_{AB}, which may be pure or mixed. We define the ''spin-flipped'' density matrix to be

$$\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y), \tag{1}$$

where the asterisk denotes complex conjugation in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and σ_y expressed in the same basis is the matrix

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

As both ρ_{AB} and $\tilde{\rho}_{AB}$ are positive operators, it follows that the product $\rho_{AB}\tilde{\rho}_{AB}$, though non-Hermitian, also has only real and non-negative eigenvalues. Let the square roots of these eigenvalues, in decreasing order, be λ_1, λ_2, λ_3, and λ_4. Then the concurrence of the density matrix ρ_{AB} is defined as

$$\mathcal{C}_{AB}=\max\{\lambda_1-\lambda_2-\lambda_3-\lambda_4,0\}. \tag{2}$$

For the special case in which the state of AB is pure, one can show that $\mathcal{C}_{AB}=2\sqrt{\det\rho_A}$, where ρ_A is the density matrix of qubit A, that is, the trace of ρ_{AB} over qubit B.

It is by no means obvious from the definition that the concurrence is a measure of entanglement for mixed states. This interpretation comes from earlier work, in which a specific connection is established between the concurrence and the entanglement of formation of a pair of qubits [7]. For the purpose of this paper it is sufficient to note that $\mathcal{C}=0$ corresponds to an unentangled state, $\mathcal{C}=1$ corresponds to a completely entangled state, and the entanglement of formation is a monotonically increasing function of $\mathcal{C}$.[1] At present, the concurrence is defined only for a pair of qubits, not for higher-dimensional systems.

We now turn to the first problem of this paper: given a pure state of three qubits A, B, and C, how is the concurrence between A and B related to the concurrence between A and C? For this special case—a pure state of three qubits—the formula for the concurrence simplifies: each pair of qubits, being entangled with only one other qubit in a joint pure state, is described by a density matrix having at most two nonzero eigenvalues. It follows that the product $\rho_{AB}\tilde{\rho}_{AB}$ also has only two nonzero eigenvalues. We can use this fact and Eq. (2) to write the following inequality for the concurrence $\mathcal{C}_{AB}$ between A and B.[2]

$$\begin{aligned}\mathcal{C}_{AB}^2&=(\lambda_1-\lambda_2)^2=\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2\\&=\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})-2\lambda_1\lambda_2\leqslant\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB}).\end{aligned} \tag{3}$$

Here ρ_{AB} is the density matrix of the pair AB, obtained from the original pure state by tracing over qubit C. Equation (3) and the analogous equation for $\mathcal{C}_{AC}$ allow us to bound the sum $\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2$:

$$\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2\leqslant\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})+\mathrm{Tr}(\rho_{AC}\tilde{\rho}_{AC}). \tag{4}$$

The next paragraph is devoted to evaluating the right-hand side of this inequality.

Let us express the pure state $|\xi\rangle$ of the three-qubit system in the standard basis $\{|ijk\rangle\}$, where each index takes the values 0 and 1:

$$|\xi\rangle=\sum_{ijk} a_{ijk}|ijk\rangle. \tag{5}$$

In terms of the coefficients a_{ijk}, we can write $\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})$ as

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})=\sum a_{ijk}a^*_{mnk}\epsilon_{mm'}\epsilon_{nn'}a^*_{m'n'p}a_{i'j'p}\epsilon_{i'i}\epsilon_{j'j}, \tag{6}$$

where $\epsilon_{01}=-\epsilon_{10}=1$ and $\epsilon_{00}=\epsilon_{11}=0$ and the sum is over all the indices. We now replace the product $\epsilon_{nn'}\epsilon_{j'j}$ with the equivalent expression $\delta_{nj'}\delta_{n'j}-\delta_{nj}\delta_{n'j'}$, and in the first of the two resulting terms (that is, the one associated with $\delta_{nj'}\delta_{n'j}$) we perform a similar substitution for $\epsilon_{mm'}\epsilon_{i'i}$. These substitutions directly give us

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})=2\det\rho_A-\mathrm{Tr}(\rho_B^2)+\mathrm{Tr}(\rho_C^2), \tag{7}$$

where ρ_A, ρ_B, and ρ_C are the 2×2 density matrices of the individual qubits. Because each of these matrices has unit trace, we can rewrite Eq. (7) as

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})=2(\det\rho_A+\det\rho_B-\det\rho_C). \tag{8}$$

By symmetry we must also have

$$\mathrm{Tr}(\rho_{AC}\tilde{\rho}_{AC})=2(\det\rho_A+\det\rho_C-\det\rho_B). \tag{9}$$

Summing these last two equations, we finally get a simple expression for the right-hand side of Eq. (4), namely,

$$\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})+\mathrm{Tr}(\rho_{AC}\tilde{\rho}_{AC})=4\det\rho_A. \tag{10}$$

Equations (4) and (10) give us our first main result:

$$\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2\leqslant4\det\rho_A. \tag{11}$$

We can interpret the right-hand side of Eq. (11) as follows. If we regard the pair BC as a single object, it makes sense to speak of the concurrence $\mathcal{C}_{A(BC)}$ between qubit A and the pair BC, because, even though the state space of BC is four dimensional, only two of those dimensions are necessary to express the state $|\xi\rangle$ of ABC. (The two necessary dimensions are those spanned by the two eigenstates of ρ_{BC} that have nonzero eigenvalues. That there are only two such eigenvalues follows from the fact that A is only a qubit and that the state of the whole system is pure.) We may thus treat A and BC, at least for this purpose, as a pair of qubits in a pure state. As we have mentioned before, the concurrence for this case is simply $2\sqrt{\det\rho_A}$. We can therefore rewrite our result as

$$\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2\leqslant\mathcal{C}_{A(BC)}^2. \tag{12}$$

Informally, Eq. (12) can be expressed as follows. Qubit A has a certain amount of entanglement with the pair BC. This amount bounds A's entanglement with qubits B and C taken individually, and the part of the entanglement that is devoted to qubit B (as measured by the squared concurrence) is not available to qubit C.

We will say more shortly about the case of three qubits in a pure state, but at this point it is worth mentioning a generalization to mixed states. If ABC is in a mixed state ρ, then $\mathcal{C}_{A(BC)}$ is not defined, because all four dimensions of BC might be involved, but we can define a related quantity

[1]The entanglement of formation is given by $E=h(\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathcal{C}^2})$, where h is the binary entropy function $h(x)=-x\log_2x-(1-x)\log_2(1-x)$.

[2]An alternative derivation of this inequality [20] demonstrates that it does not depend on the rank of the density matrix: $\mathcal{C}_{AB}^2\leqslant\lambda_1^2\leqslant\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2=\mathrm{Tr}(\rho_{AB}\tilde{\rho}_{AB})$. We use the more specialized derivation because the intermediate steps will be useful later.

$(\mathcal{C}^2)^{\min}_{A(BC)}$ via the following prescription.[3] Consider all possible pure-state decompositions of the state ρ, that is, all sets $\{(\psi_i, p_i)\}$ such that $\rho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i|$. For each of these decompositions, one can compute the average value of $\mathcal{C}^2_{A(BC)}$:

$$\langle \mathcal{C}^2_{A(BC)} \rangle = \sum_i p_i \mathcal{C}^2_{A(BC)}(\psi_i). \tag{13}$$

The minimum of this average over all decompositions of ρ is what we define to be $(\mathcal{C}^2)^{\min}_{A(BC)}(\rho)$. The following analog of Eq. (12) then holds for mixed states:

$$\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} \leqslant (\mathcal{C}^2)^{\min}_{A(BC)}. \tag{14}$$

To prove this, consider the pure states $|\psi_i\rangle$ belonging to an optimal decomposition of ρ, that is, a decomposition that minimizes $\langle \mathcal{C}^2_{A(BC)} \rangle$. We can write our basic inequality, Eq. (12), for each such pure state and then average both sides of the inequality over the whole decomposition. The right-hand side of the resulting inequality is $(\mathcal{C}^2)^{\min}_{A(BC)}(\rho)$, which is what we want on the right-hand side. On the left-hand side we have two terms: (i) the average of the squared concurrence between A and B over a set of mixed states whose average is ρ_{AB} [i.e., $\mathrm{Tr}_C(\rho)$], and (ii) the average of the squared concurrence between A and C over a set of mixed states whose average is ρ_{AC}. It is a fact that the squared concurrence is a convex function on the set of density matrices.[4] That is, the average of $\mathcal{C}^2$ over the ensemble is greater than or equal to the value of $\mathcal{C}^2$ for the average density matrix. In this case the values of $\mathcal{C}^2$ for the average density matrices are $\mathcal{C}^2_{AB} = \mathcal{C}^2(\rho_{AB})$ and $\mathcal{C}^2_{AC} = \mathcal{C}^2(\rho_{AC})$. The sum of these two quantities must thus be less than or equal to $(\mathcal{C}^2)^{\min}_{A(BC)}(\rho)$, which is what we wanted to prove.

Returning to the case of pure states, one may wonder how tight the inequality (12) is. Could one find, for example, a more stringent bound of the same form, based on a different measure of entanglement? To address this question, consider the following pure state of ABC:

$$|\phi\rangle = \alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle, \tag{15}$$

where the three positions in the kets refer to qubits A, B, and C in that order. For this state, one finds that $\mathcal{C}_{AB} = 2|\alpha\beta|$, $\mathcal{C}_{AC} = 2|\alpha\gamma|$, and $\mathcal{C}_{A(BC)} = 2|\alpha|\sqrt{|\beta|^2 + |\gamma|^2}$. Thus the inequality (12) becomes in this case an equality: $\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} = \mathcal{C}^2_{A(BC)}$. This example shows that for any values of the concurrences satisfying this equality, there is a quantum state that is consistent with those values.

Now let $\Gamma(\mathcal{C})$ be a monotonically increasing function of $\mathcal{C}$ that we might propose as an alternative measure of entanglement. For simplicity let us assume that $\Gamma(0) = 0$ and $\Gamma(1) = 1$. Because of the above example, Γ could satisfy the inequality $\Gamma^2_{AB} + \Gamma^2_{AC} \leqslant \Gamma^2_{A(BC)}$ only if $\Gamma^2(x) + \Gamma^2(y) \leqslant \Gamma^2(\sqrt{x^2+y^2})$ for all non-negative x and y such that $x^2 + y^2 \leqslant 1$. Suppose Γ has this property. Then could there exist some quantum state for which $\Gamma^2_{AB} + \Gamma^2_{AC} = \Gamma^2_{A(BC)}$ but $\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} < \mathcal{C}^2_{A(BC)}$? That is, could Γ yield an equality for some state for which $\mathcal{C}$ gives only an inequality? The answer is no, because if $\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} < \mathcal{C}^2_{A(BC)}$, then $\Gamma^2_{AB} + \Gamma^2_{AC} = \Gamma^2(\mathcal{C}_{AB}) + \Gamma^2(\mathcal{C}_{AC}) \leqslant \Gamma^2(\sqrt{\mathcal{C}^2_{AB} + \mathcal{C}^2_{AC}}) < \Gamma^2(\mathcal{C}_{A(BC)}) = \Gamma^2_{A(BC)}$. Moreover, the only way Γ can match $\mathcal{C}$ in those cases where $\mathcal{C}$ gives an equality is for Γ to be equal to $\mathcal{C}$. In this sense, $\mathcal{C}$ is an optimal measure of entanglement with respect to the inequality given in Eq. (12). Note, however, that the above argument applies only to functions of $\mathcal{C}$. There could in principle be other measures of entanglement that are not functions of $\mathcal{C}$ that could make an equal claim to optimality.

The entanglement of formation is a function of $\mathcal{C}$, but it is a concave function of $\mathcal{C}^2$ and therefore does not satisfy the inequality $E_{AB} + E_{AC} \leqslant E_{A(BC)}$. Consider, for example, the state $(1/\sqrt{2})|100\rangle + (1/2)|010\rangle + (1/2)|001\rangle$. One finds that the relevant entanglements of formation are $E_{AB} = 0.601$, $E_{AC} = 0.601$, and $E_{A(BC)} = 1$. Thus, contrary to what one might expect, the sum of the entanglements of formation between A and the separate qubits B and C is greater than the entanglement capacity of a single qubit. This is not a paradox; it simply shows us that entanglement of formation does not exhibit this particular kind of additivity. (This sense of "additivity" should not be confused with the additivity of entanglement when one combines pairs to make larger systems [7]. It is not known whether entanglement of formation satisfies the latter notion of additivity.)

We have just seen that there are some states for which the inequality (12) becomes an equality. Of course there are other states for which the inequality is strict. As we will see, it turns out to be very interesting to consider the *difference* between the two sides of Eq. (12). This difference can be thought of as the amount of entanglement between A and BC that cannot be accounted for by the entanglements of A with B and C separately. In the following paragraphs we refer to this quantity as the "residual entanglement."

Let the system ABC be in a pure state $|\xi\rangle$, and, as before, let the components of $|\xi\rangle$ in the standard basis be a_{ijk}:

$$|\xi\rangle = \sum_{ijk} a_{ijk} |ijk\rangle. \tag{16}$$

According to Eqs. (3) and (10) and the discussion following Eq. (11), the residual entanglement is equal to

$$\mathcal{C}^2_{A(BC)} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC} = 2(\lambda_1^{AB}\lambda_2^{AB} + \lambda_1^{AC}\lambda_2^{AC}), \tag{17}$$

where λ_1^{AB} and λ_2^{AB} are the square roots of the two eigenvalues of $\rho_{AB}\tilde{\rho}_{AB}$, and λ_1^{AC} and λ_2^{AC} are defined similarly. We now derive an explicit expression for the residual entanglement in terms of the coefficients a_{ijk}.

[3]According to Eq. (12), the quantity that is traded off between the pair AB and the pair AC is not the concurrence itself but rather the square of the concurrence. For this reason, it is convenient in this discussion of mixed states to take $\mathcal{C}^2$ rather than $\mathcal{C}$ as the quantity to be averaged and minimized.

[4]It follows from Ref. [7] that the concurrence is a convex nonnegative function on the set of density matrices for two qubits. The square of the concurrence is therefore also convex.

We focus first on the product $\lambda_1^{AB}\lambda_2^{AB}$. This product can almost be interpreted as the square root of the determinant of $\rho_{AB}\tilde{\rho}_{AB}$. But $\rho_{AB}\tilde{\rho}_{AB}$ is an operator acting on a four-dimensional space, and two of its eigenvalues are zero; so its determinant is also zero. However, if we consider the action of $\rho_{AB}\tilde{\rho}_{AB}$ only on its range, then $\lambda_1^{AB}\lambda_2^{AB}$ will be the square root of the determinant of this restricted transformation.

The range of $\rho_{AB}\tilde{\rho}_{AB}$ is spanned by the two vectors $|v_0\rangle = \Sigma_{ij}a_{ij0}|ij0\rangle$ and $|v_1\rangle = \Sigma_{ij}a_{ij1}|ij1\rangle$. (These vectors also span the range of ρ_{AB}.) To examine the action of $\rho_{AB}\tilde{\rho}_{AB}$ on this subspace, we consider its effect on vectors of the form

$$x|v_1\rangle + y|v_2\rangle \equiv \begin{pmatrix} x \\ y \end{pmatrix}.$$

This effect is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R\begin{pmatrix} x \\ y \end{pmatrix}, \tag{18}$$

where R is a 2×2 matrix. The product $\lambda_1^{AB}\lambda_2^{AB}$ is the square root of the determinant of R. One finds that

$$R_{ij} = \sum a_{klj}a^*_{mni}\epsilon_{mp}\epsilon_{nq}a^*_{pqr}a_{str}\epsilon_{sk}\epsilon_{tl}, \tag{19}$$

where the sum is over all repeated indices. (We have ordered the factors so as to suggest the expression $\rho_{AB}\tilde{\rho}_{AB}$ from which R is derived.) Taking the determinant of R involves somewhat tedious but straightforward algebra, with the following result:

$$\lambda_1^{AB}\lambda_2^{AB} = \sqrt{\det R} = |d_1 - 2d_2 + 4d_3|, \tag{20}$$

where

$$d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2;$$

$$\begin{aligned} d_2 = {} & a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} \\ & + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} \\ & + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}; \end{aligned}$$

$$d_3 = a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}. \tag{21}$$

We can get a mental picture of this expression by imagining the eight coefficients a_{ijk} attached to the corners of a cube. Then each term appearing in d_1, d_2, or d_3 is a product of four of the coefficients a_{ijk} such that the ''center of mass'' of the four is at the center of the cube. Such configurations fall into three classes: those in which the four coefficients lie on a body diagonal and each one is used twice (d_1), those in which they lie on a diagonal plane (d_2), and those in which they lie on the vertices of a tetrahedron (d_3). Within each category, all the possible configurations are given the same weight.

This picture immediately yields an interesting fact: the quantity $\lambda_1^{AB}\lambda_2^{AB}$ is invariant under permutations of the qubits. (A permutation of qubits corresponds to a reflection or rotation of the cube, but each d_i is invariant under such actions.) This means in particular that we need not carry out a separate calculation to find $\lambda_1^{AC}\lambda_2^{AC}$, since we know we will get the same result. We can now therefore write down an expression for the residual entanglement:

$$\mathcal{C}^2_{A(BC)} - \mathcal{C}^2_{AB} - \mathcal{C}^2_{AC} = 4|d_1 - 2d_2 + 4d_3|. \tag{22}$$

Note that the residual entanglement does not depend on which qubit one takes as the ''focus'' of the construction. In our calculations we have focused on entanglements with qubit A, but if we had chosen qubit B instead, we would have found that

$$\mathcal{C}^2_{B(CA)} - \mathcal{C}^2_{BC} - \mathcal{C}^2_{BA} = 4|d_1 - 2d_2 + 4d_3|. \tag{23}$$

The residual entanglement thus represents a collective property of the three qubits that is unchanged by permutations; it measures an essential three-qubit entanglement. If we call this quantity τ_{ABC}, we can summarize the main results of this paper in the following equation:

$$\mathcal{C}^2_{A(BC)} = \mathcal{C}^2_{AB} + \mathcal{C}^2_{AC} + \tau_{ABC}. \tag{24}$$

In words, the entanglement of A with BC can be manifested in three forms—entanglement with B, entanglement with C, and an essential three-way entanglement of the triple—and these three forms must share the total entanglement. As an example, consider the Greenberger-Horne-Zeilinger state $(1/\sqrt{2})(|000\rangle + |111\rangle)$ [21]. For this state the concurrence of each qubit with the rest of the system is 1, the quantity τ_{ABC} is also 1, and all the pairwise concurrences are zero (the qubits in each pair are classically correlated but not entangled). Thus Eq. (24) in this case becomes $1 = 0 + 0 + 1$.

Finally, we note that the expression for τ_{ABC} in terms of d_1, d_2, and d_3 [Eq. (22)] can be rewritten, after a little more algebra, in a more standard form:

$$\tau_{ABC} = 2\left|\sum a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'} \times \epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}\right|, \tag{25}$$

where the sum is over all the indices. This form does not immediately reveal the invariance of τ_{ABC} under permutations of the qubits, but the invariance is there nonetheless.

It would be very interesting to know which of the results of this paper generalize to larger objects or to larger collections of objects. At this point it is not clear how one might begin to generalize this approach to qutrits or higher dimensional objects, because the spin-flip operation seems peculiar to qubits [22]. On the other hand, it appears very likely that at least some of these results can be extended to larger collections of qubits. The one solid piece of evidence we can offer is the existence of a generalization of the state $|\phi\rangle$ of Eq. (15) to n qubits:

$$|\phi\rangle = \alpha_1|100\dots0\rangle + \alpha_2|010\dots0\rangle + \alpha_3|001\dots0\rangle + \cdots + \alpha_n|000\dots1\rangle. \tag{26}$$

One can show that for this state, the following equality holds:

$$\mathcal{C}_{12}^2 + \mathcal{C}_{13}^2 + \cdots + \mathcal{C}_{1n}^2 = \mathcal{C}_{1(23\dots n)}^2, \tag{27}$$

where the qubits are now labeled by numbers rather than letters. We are willing to conjecture that the corresponding inequality, analogous to Eq. (12), is valid for all pure states of n qubits.

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