Functional Analysis

Wei Qi Yan

AUT, New Zealand

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Metric Space

The concept distance or metric is a measure of how elements are close to each other.

Metric

A distance (or metric) on a metric space $\mathbf{X}$ is a function:
$d : \mathbf{X}^2 \rightarrow \mathbb{R}^+$; $(x, y) \mapsto d(x, y) = \|x - y\|$, $x, y, z \in \mathbf{X}$.

- Triangle inequality: $d(x, y) \leq d(x, z) + d(z, y)$
- Symmetry: $d(x, y) = d(y, x)$
- Equality: $d(x, y) = 0 \Leftrightarrow x = y$

Therefore,

- $d(x, y) \geq |d(x, z) - d(z, y)|$
- $x_1, x_2, \cdots, x_n \in \mathbf{X}$, $d(x_1, x_n) \leq \sum_{i=1}^{n-1} d(x_i, x_{i+1})$

Convergence

A sequence $x_1, \cdots, x_n$ in a metric space $\mathbf{X}$ converges to a limit x, written as $\lim_{n\to\infty} x_n = x$ if $\forall \varepsilon > 0$, $\exists N$, $n \gg N \Rightarrow x_n \in B_\varepsilon(x)$.

Remarks:

- In a metric space, a sequence $x_1, \cdots, x_n$ can only converge to one limit.
- Any neighborhood of x contains all the sequences from some point onwards.
- The sequence points get arbitrarily close to the limit.
- If $x_n \to x$ and $x_n \in \mathbf{A}$, then $x \in \bar{\mathbf{A}}$ (closed set).
- A sequence which does not converge is said to diverge.

Continuity

A function $f : \mathbf{X} \to \mathbf{Y}$ between metric spaces is continuous if it preserves convergence,

$$x_n \to x \in \mathbf{X} \Rightarrow f(x_n) \to f(x) \in \mathbf{Y}.$$

Remarks:

- Continuous $\Leftrightarrow \forall x \in \mathbf{X}, \forall \varepsilon > 0, \exists \delta > 0,\ d_{\mathbf{X}}(x, x') < \delta$, $d_{\mathbf{Y}}(f(x), f(x')) < \varepsilon$.
- Continuous $\Leftrightarrow \forall \mathbf{V} \subseteq \mathbf{Y}$ is open set, $f^{-1}(\mathbf{V}) \subseteq \mathbf{X}$ is open set.
- Continuous $\Leftrightarrow \lim_{x' \to x} f(x') = f(x), \forall x \in \mathbf{X}$.

Completeness

A Cauchy sequence is the one that $d(x_n, x_m) \to 0$ as $n, m \to \infty$, namely,
$\forall \varepsilon > 0$, $\exists N$, if $n, m \geq N$, then $d(x_n, x_m) < \varepsilon$.

Remarks:

- Two sequences $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ are defined to be asymptotic when $d(x_n, y_n) \to 0$ as $n \to \infty$.
- A sequence $x_1, \cdots, x_n$ is Cauchy, if and only if (i.e., iff), every subsequence of $x'_1, \cdots, x'_m$ is asymptotic to $x_1, \cdots, x_n$, $m \leq n$.
- For $x_1, \cdots, x_n$ being asymptotic to $y_1, \cdots, y_n$, if $x_1, \cdots, x_n$ is Cauchy then so is $y_1, \cdots, y_n$; if $x_1, \cdots, x_n$ converges to x, then so does $y_1, \cdots, y_n$. The asymptotic is an equivalence relationship.

Complete Metric Space

A metric space is complete if every Cauchy sequence in it converges.

- The real number space $\mathbb{R}$ is complete.
- Every metric space $\mathbf{X}$ is complete.
- Let $\mathbf{X}$ and $\mathbf{Y}$ be complete metric spaces, then, a subset $\mathbf{F} \subset \mathbf{X}$ is complete $\Leftrightarrow$ $\mathbf{F} \subset \mathbf{X}$ is closed.
- $\mathbf{X} \times \mathbf{Y}$ is complete.
- A uniformly continuous function maps any Cauchy sequence to a Cauchy sequence.
- A function is *uniformly continuous*, where $\delta > 0$ is independent on x.
- A function $f : \mathbf{X} \to \mathbf{Y}$ is a *Lipschitz map* if $\exists c > 0$, $\forall x, x' \in \mathbf{X}$, $d_{\mathbf{Y}}(f(x), f(x')) \leq c \cdot d_{\mathbf{X}}(x, x')$.

Metric Space

Lipschitz Map

A function $f : \mathbf{X} \to \mathbf{Y}$ is a *Lipschitz map* if $\exists c > 0$, $\forall x, x' \in \mathbf{X}$, $d_{\mathbf{Y}}(f(x), f(x')) \leq c \cdot d_{\mathbf{X}}(x, x')$.

- **Equivalence** (or bi-Lipschitz): If f is bijective, then both f and f^{-1} are Lipschitz.
- **Contraction**: If it is Lipschitz, then constant $0 < c < 1$.
- **Isometry**: If f preserves distances $\forall x, x'$, then $d_{\mathbf{Y}}(f(x), f(x')) = d_{\mathbf{X}}(x, x')$.
- **The fixed point theorem**: Let $\mathbf{X} \neq \varnothing$ be a complete metric space, every contraction map $f : \mathbf{X} \mapsto \mathbf{X}$ has a unique fixed point $x = f(x)$ and the iteration $x_{n+1} := f(x_n)$ converges to it for any x_0.

Metric Space

Separable Set

A metric space is separable if it contains a countable dense subset, then $\exists \mathbf{A} \subseteq \mathbf{X}$, $\mathbf{A}$ is countable and $\bar{\mathbf{A}} = \mathbf{X}$.

- Any subset of a separable metric space is separable.
- The product of two separable spaces is separable.
- The image of a separable space under a continuous map is separable.

Bounded Set

A set $\mathbf{B}$ is bounded if the distance between any two points in the set has an upper bound, i.e., $\exists r > 0$, $\forall x, y \in \mathbf{B}$, $d(x, y) \leq r$. The least upper bound is called the diameter of the set

$$diam\ \mathbf{B} := \sup_{x,y \in \mathbf{B}} d(x, y).$$

Bounded Set

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Remarks:

- Any subset of a bounded set is bounded.
- The union of a finite number of bounded sets is bounded,
- A subset $\mathbf{B} \subseteq \mathbf{X}$ is totally bounded if it can be covered by a finite number of ε-balls, for small radii ε, $\mathbf{B} \subseteq \bigcup_{n=1}^{N} \mathbf{B}_\varepsilon(a_n)$.
- A set $\mathbf{B}$ is totally bounded means every sequence in $\mathbf{B}$ has a Cauchy subsequence.
- A uniformly continuous function maps totally bounded sets to totally bounded sets.

Compact Set

A set **K** is compact if given any cover of balls, there is a finite sub-collection of them that still cover the set (a subcover) $K \subseteq \bigcup_i B_{\varepsilon_i}(a_i) \Rightarrow K \subseteq \bigcup_{n=1}^{N} B_{\varepsilon_{i_n}}(a_{i_n})$.

Remarks:

- A set is compact means any open cover of it has a finite subcover. Compact sets are closed.
- A closed subset of a compact set is compact.
- A finite union of compact sets is compact.
- Continuous functions map compact sets to compact sets.
- Continuous functions preserve compactness.
- Uniformly continuous functions preserve total boundedness.
- Lipschitz continuous functions preserve boundedness.
- A set **K** is compact means **K** is complete and totally bounded.

Questions?

Vector Space

Vector Space

- A vector space $\mathbf{V}$ over a field $\mathbb{F}$ is a set on which are defined an operation of vector addition $+ : \mathbf{V} \to \mathbf{V}$ satisfying associativity, commutativity, zero, and inverse axioms.
- For every $x, y, z \in \mathbf{V}$,
  $x + (y + z) = (x + y) + z$; $x + y = y + x$;
  $0 + x = x$; $x + (-x) = 0$.
- An operation of scalar multiplication $\mathbb{F} \times \mathbf{V} \to \mathbf{V}$ satisfies the distributive laws. Namely, for every $\lambda, \mu \in \mathbb{F}$,
  $(\lambda\mu)x = \lambda(\mu x)$; $1x = x$; $\lambda(x + y) = \lambda x + \lambda y$;
  $(\lambda + \mu)x = \lambda x + \mu x$.

- Every vector space has a basis.

Normed Space

Norm

A norm on a real vector space $\mathbf{X}$ is a function: $\mathbf{X} \to \mathbb{R}$, $\mathbf{u} \mapsto \|\mathbf{u}\|$, satisfies:

- $\forall\ \mathbf{u} \in \mathbf{X}$, $\|\mathbf{u}\| \geq 0$.
- $\forall\ \mathbf{u} \in \mathbf{X}$, $\alpha \in \mathbb{R}$, $\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|$.
- $\forall\ \mathbf{u}, \mathbf{v} \in \mathbf{X}$, $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. (Minkowski's inequality)

Normed Space

A normed space $\mathbf{X}$ is a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with a function called the norm $\|\cdot\| : \mathbf{X} \mapsto \mathbb{R}$, for any $x, y \in \mathbf{X}$, $\lambda \in \mathbb{F}$, $\|x + y\| \leq \|x\| + \|y\|$, $\|\lambda x\| = \lambda\|x\|$, $\|x\| = 0 \Leftrightarrow x = 0$.

Remarks:

- $\|x - y\| \geq \|x\| - \|y\|$.
- $\|x_1 + x_2 + \cdots + x_n\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_n\|$.

Normed Space

Given

$$\|(a_n)\|_{\mathcal{L}^2} = \sqrt{\sum_{n=0}^{\infty} \|a_n\|^2}$$

and

$$\|(b_n)\|_{\mathcal{L}^2} = \sqrt{\sum_{n=0}^{\infty} \|b_n\|^2}$$

Cauchy's Inequality

$$|\sum_{n=0}^{\infty} a_n b_n| \leq \|(a_n)\|_{\mathcal{L}^2} \|(b_n)\|_{\mathcal{L}^2}$$

$$\sqrt{\sum_{n=0}^{\infty} \|a_n + b_n\|^2} \leq \|(a_n)\|_{\mathcal{L}^2} + \|(b_n)\|_{\mathcal{L}^2}$$

Normed Space

Norm Properties

- Vector addition, scalar multiplication, and norm are continuous.
- If $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ converge, then $x_n + y_n$, λx_n, and $\|x_n\|$ converge, namely,

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}(\lambda x_n) = \lambda \lim_{n\to\infty} x_n$$

$$\lim_{n\to\infty}\|x_n\| = \|\lim_{n\to\infty} x_n\|$$

Questions?

Banach Space

A Banach space is a complete normed space. In a Banach space $\mathbf{X}$, the following statements are equivalent:

$$\sum_{n=0}^{\infty} \mathbf{u}_n \leq \infty$$

and

$$\lim_{j\to\infty} \sum_{n=j+1}^{k} \mathbf{u}_n = 0$$

where $k \geq n$.

Remarks:

- If the induced metric $d(x, y) := \|x - y\|$ is complete, the normed space is called a Banach space.
- Every normed space can be completed as a Banach space.
- A normed space is separable, iff there is a countable subset.

Remarks:

- If the metric $d(x, y) := \|x - y\|$ is complete, the normed space is called a Banach space.
- A linear mapping between Banach spaces is called compact if it maps bounded sets to totally bounded sets.
- Compact linear maps are continuous (originally called completely continuous).
- If T and S are compact operators, then so are $T + S$ and λT.
- The identity map $I : X \to X$ is not compact when the Banach space is infinite dimensional.
- It is enough to show that T maps the unit ball to a totally bounded set for T to be compact.

Remarks:

- A linear mapping between Banach spaces is called compact if it maps bounded sets to totally bounded sets.
- If T is compact and S continuous linear, then ST and TS are compact.
- If T_n are compact and $T_n \to T$, then T is compact.
- Convergence in norm: $T_n \to T \Leftrightarrow \|T_n - T\| \to 0$.
- Strong convergence: $T_n x \to Tx, \forall x \in \mathbf{X} \Leftrightarrow \|T_n x - Tx\|_{\mathbf{Y}} \to 0,\ \forall x \in \mathbf{X}$.
- Weak convergence: $T_n \to T \Leftrightarrow \phi T_n x \to \phi T x,\ \forall x \in \mathbf{X}$, $\phi \in \mathbf{Y}$.
- Uniform Bounded Theorem: A Banach space $\mathbf{X}$ and $T_i \in B(\mathbf{X}, \mathbf{Y})$,$\forall i$, $\|T_i\| \leq C$, $C \geq 0$.
- If $T_n \in B(\mathbf{X}, \mathbf{Y})$, $T_n x \to T(x)$, then T is linear and continuous, $\|T\| \leq \liminf_n \|T_n\|$.

Hahn-Banach Theorem: Analytic Version

Let $\mathbf{V}$ be a normed linear space over $\mathbb{C}$, let $\mathbf{W}$ be a subspace of $\mathbf{V}$, let $g : \mathbf{W} \to \mathbb{C}$ be a continuous linear functional on $\mathbf{W}$, then, there exists a continuous linear extension $f : \mathbf{V} \to \mathbb{C}$ and $\|f\|_{\mathbf{V}^*} = \|g\|_{\mathbf{W}^*}$.

Hahn-Banach Theorem: Geometric Version

Let $\mathbf{A}$ and $\mathbf{B}$ be two nonempty and disjoint convex subsets of a real normed linear space $\mathbf{V}$. If $\mathbf{A}$ is open, there exists a closed hyperplane which separates $\mathbf{A}$ and $\mathbf{B}$, i.e., there exists $f \in \mathbf{V}^*$ and $\alpha \in \mathbb{R}$ such that $f(x) \leq \alpha \leq f(y)$, $\forall x \in \mathbf{X}$, $y \in \mathbf{Y}$.

Questions?

Hilbert Spaces

Hilbert space is a Banach space whose norm comes from a scalar (inner or dot) inner product: $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, $< \mathbf{u}, \mathbf{v} >= u_1 v_1 + u_2 v_2 = |\mathbf{u}| \cdot |\mathbf{v}| \cos(\angle(\mathbf{u}, \mathbf{v}))$.

Hilbert spaces

A scalar product on the (real) vector space $\mathbf{X}$ is a function: $(\mathbf{u}, \mathbf{v}) \in \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, $(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u}|\mathbf{v})$.

- $\forall \mathbf{u} \in \mathbf{X}$, $(\mathbf{u}|\mathbf{u}) \geq 0$.
- $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}$, $\alpha, \beta \in \mathbb{R}$, $(\alpha\mathbf{u} + \beta\mathbf{v})|\mathbf{w} = \alpha(\mathbf{u}|\mathbf{w}) + \beta(\mathbf{v}|\mathbf{w})$. $\|\mathbf{u}\| = \sqrt{(\mathbf{u}|\mathbf{u})}$.

A Hilbert space is an inner product space which is complete as a metric space.

Inner Product

An inner product on a vector space X is a positive-definite sesquilinear form $< \cdot, \cdot >: \mathbf{X} \times \mathbf{X} :\mapsto \mathbb{F}$. For $x, y, z \in \mathbf{X}$, $\lambda \in \mathbb{F}$,

- $< x, y + z > = < x, y > + < x, z >$; $< x, \lambda y > = \lambda < x, y >$; $< x, y > = \overline{< y, x >}$; $< x, x > \geq 0$; $< x, x > = 0 \Rightarrow x = 0$.
- Cauchy-Schwarz inequality: $| < x, y > | \leq \|x\| \|y\|$.
- The inner product is continuous: $\lim_{n\to\infty} < x_n, y_n > = < \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n >$.
- A norm is induced from an inner product, iff it satisfies, for all vectors x, y, $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

A Hilbert space is an inner product space which is complete as a metric space.

Orthogonal Spaces

The orthogonal spaces of subsets $\mathbf{A} \subseteq \mathbf{X}$,
$\mathbf{A}^{\perp} := \{\forall x \in X, < x, a >= 0, \forall a \in \mathbf{A}\}$, satisfy, $\mathbf{A} \cap \mathbf{A}^{\perp} \subseteq 0$;
$\mathbf{A} \subseteq \mathbf{B} \Leftrightarrow \mathbf{B}^{\perp} \subseteq \mathbf{A}^{\perp}$; $\mathbf{A} \subseteq \mathbf{A}^{\perp\perp}$; $\mathbf{A}^{\perp}$ is a closed subspace of $\mathbf{X}$.

Remarks:

- If $\mathbf{M}$ is a closed convex subset of a Hilbert space $\mathbf{H}$, then any point in $\mathbf{H}$ has a unique point in $\mathbf{M}$ which is closest to it (Least Squares Approximation).
- An orthonormal basis of a Hilbert space $\mathbf{H}$ is a set of orthonormal vectors $\mathbf{E}$ whose span is dense: $\forall e_i, e_j \in \mathbf{E}, < e_i, e_j >= \delta_{ij}$.
- Parseval's identity (Fourier Series): If $x = \sum \alpha_i e_i$, $y = \sum \beta_i e_i$, $\{e_i\}$ is orthonormal, then $x, y \in \mathbf{H}$, $< x, y >= \sum < x, e_i >< e_i, y >$, $\sum | < x, e_i > |^2 = \|x\|^2$.
- Bessel's Inequality: $x = \sum \alpha_i e_i$, $\sum | < x, e_i > |^2 \leq \|x\|^2$.

Adjoint

The adjoint of an operator $T : \mathbf{X} \to \mathbf{Y}$ between Hilbert spaces, is the operator $T^* : \mathbf{Y} \to \mathbf{X}$ uniquely defined by the relation, i.e., $\forall x \in \mathbf{X}, y \in \mathbf{Y}, < Tx, y >=< y, Tx >$.

Remarks:

- $(S+T)^* = S^* + T^*$, $(ST)^* = T^* S^*$; $TT^* = I$; $T^{**} = T$.
- $(T^* y)^* x = y^* T x$

Dual Relationship

Dual operator $T : \mathbf{X} \to \mathbf{Y}$, $T^\tau : \mathbf{Y}^* \to \mathbf{X}^*$ is uniquely defined by the relation, i.e., $\forall x \in \mathbf{X}, \phi \in \mathbf{Y}^*$, $(T^\tau \phi) x = \phi(Tx)$.

Remarks:

- $(S+T)^\tau = S^\tau + T^\tau$, $(ST)^\tau = T^\tau S^\tau$; $T^{\tau\tau} = T$, $(\lambda T)^\tau = \lambda T^\tau$, $\|T^\tau\| = \|T\|^\tau$.

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Questions?