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Discrete Fourier Transform

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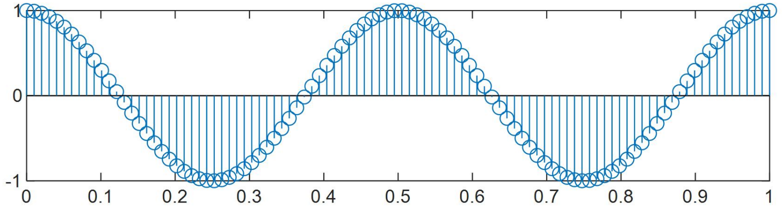
What does DFT do?

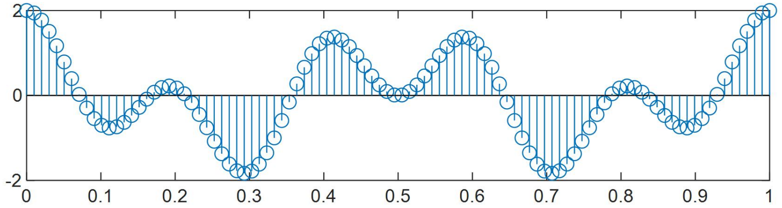
Fourier analysis told us:

Any periodical signal can be expressed as linear combinations of sine and cosine functions

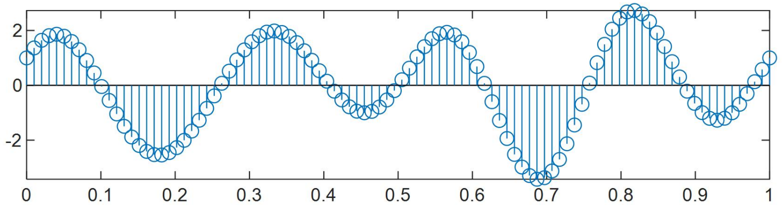
$x_1 = \cos(2\pi * 2t)$



$x_2 = \cos(2\pi * 2t) + \cos(2\pi * 5t)$



$x_3 = \cos\left(2\pi * 2t + \frac{\pi}{3}\right) + 2\sin(2\pi * 4t) + 0.5\cos(2\pi * 5t)$



DFT

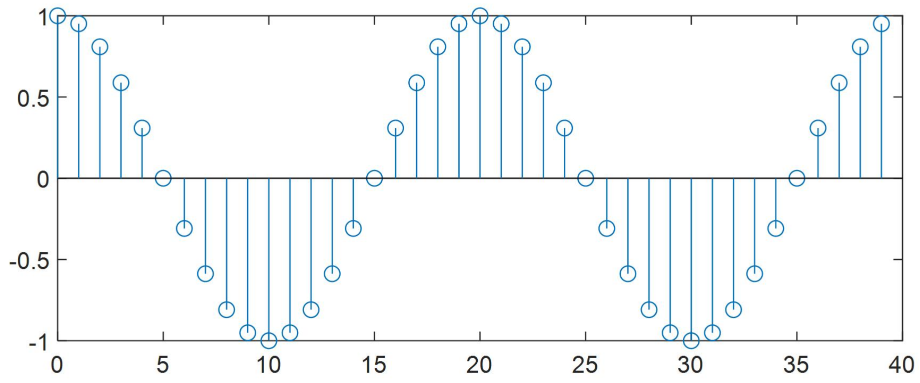
Component1
Frequency: $2HZ$
Amplitude: 1
Phase: 0

Component1	Component2
Frequency: $2HZ$	Frequency: $5HZ$
Amplitude: 1	Amplitude: 1
Phase: 0	Phase: 0

Component1	Component2	Component3
Frequency: $2HZ$	Frequency: $4HZ$	Frequency: $5HZ$
Amplitude: 1	Amplitude: 2	Amplitude: 0.5
Phase: $\pi/3$	Phase: 0	Phase: 0

How does DFT calculate those frequency partials?

Sample a cosine wave 40 times within 2 periods



$$x[n] = \cos\left(2\pi \frac{2n}{40}\right), \quad n = 0,1,\ldots,39$$

Task: Determine how many times does the cosine wave oscillates in 40 samples?

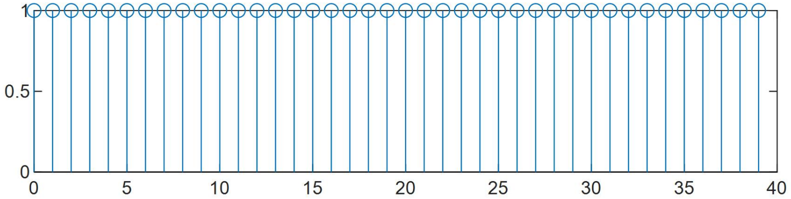
Human: Counting

Computer: Brute force

How does DFT calculate those frequency partials?

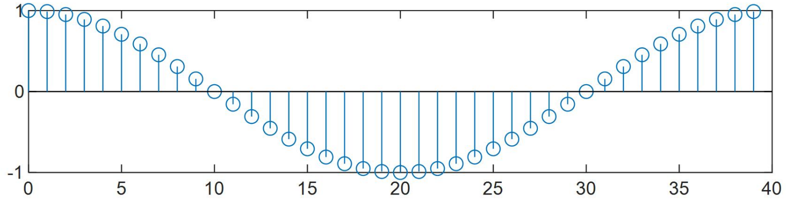
Choose 40 basis signals:

0 periods



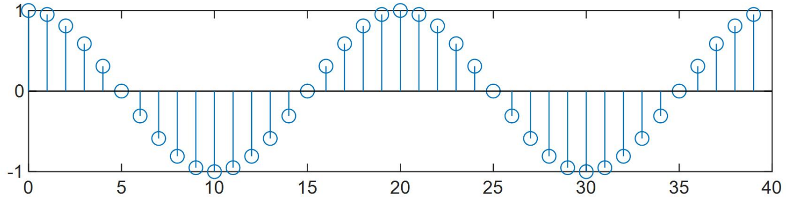
$$x[n] = \cos\left(2\pi \frac{0n}{40}\right), \quad n = 0,1,\ldots,39$$

1 period



$$x[n] = \cos\left(2\pi \frac{1n}{40}\right), \quad n = 0,1,\ldots,39$$

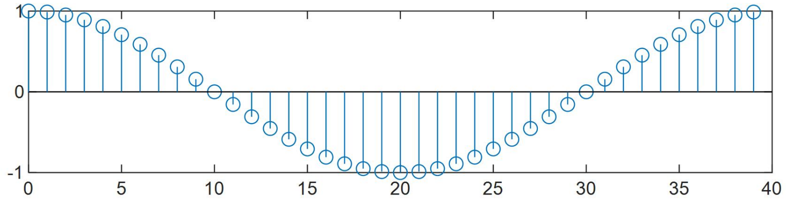
2 periods



$$x[n] = \cos\left(2\pi \frac{2n}{40}\right), \quad n = 0,1,\ldots,39$$

… … …

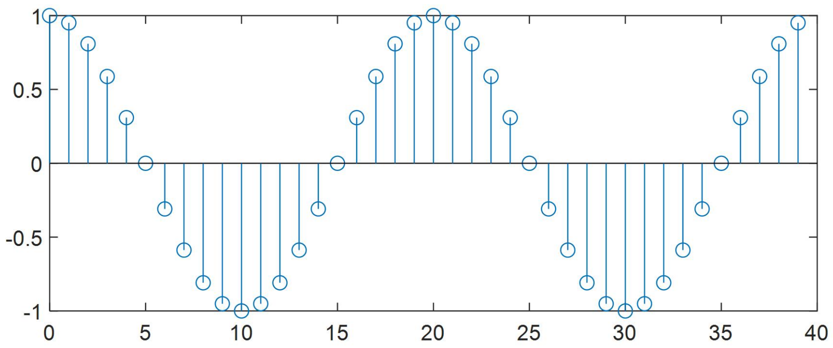
39 period



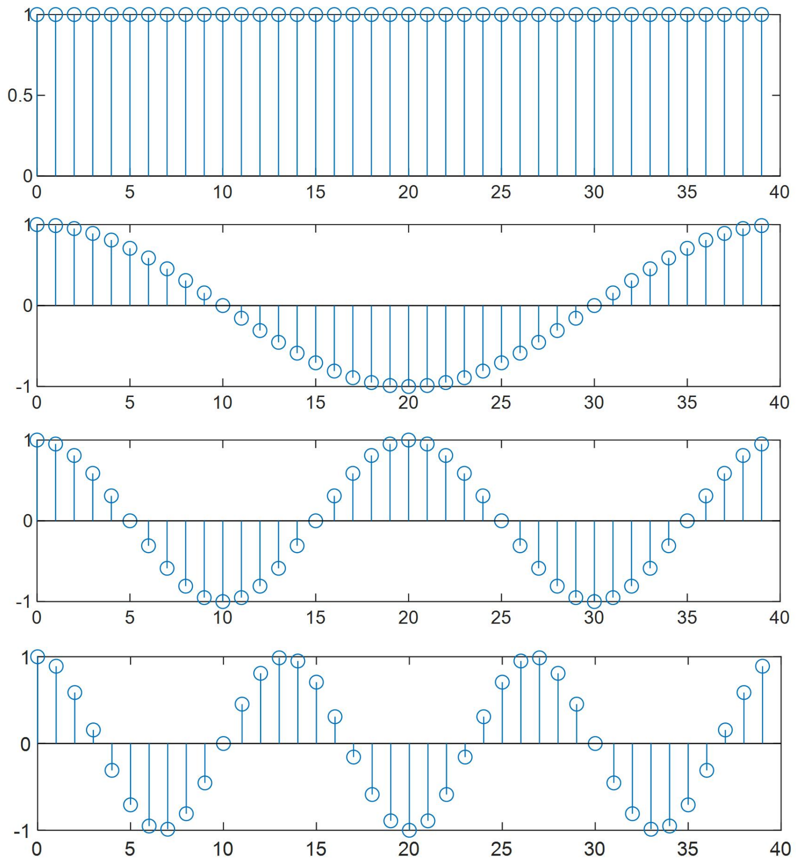
$$x[n] = \cos\left(2\pi \frac{39n}{40}\right), \quad n = 0,1,\ldots,39$$

How does DFT calculate those frequency partials?

Original signal $\cos\left(2\pi \frac{2n}{40}\right)$



Basic signals



$$x[n] = \cos\left(2\pi \frac{0n}{40}\right)$$

$$x[n] = \cos\left(2\pi \frac{1n}{40}\right)$$

$$x[n] = \cos\left(2\pi \frac{2n}{40}\right)$$

$$x[n] = \cos\left(2\pi \frac{3n}{40}\right)$$

Idea: Compare how similar is the original signal to each basis signal

Correlation function: $corr(x, y) = \sum_i x[i]y[i]$

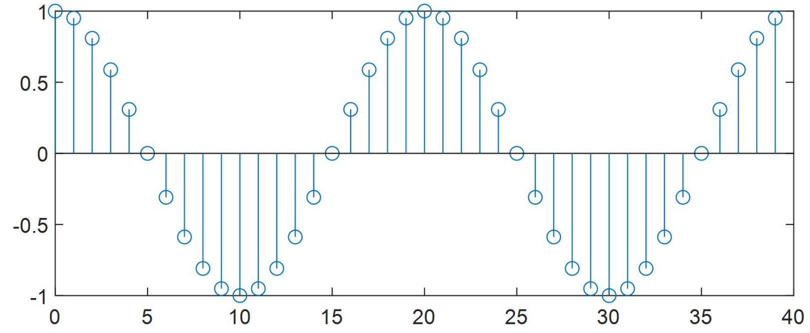
How does DFT calculate those frequency partials?

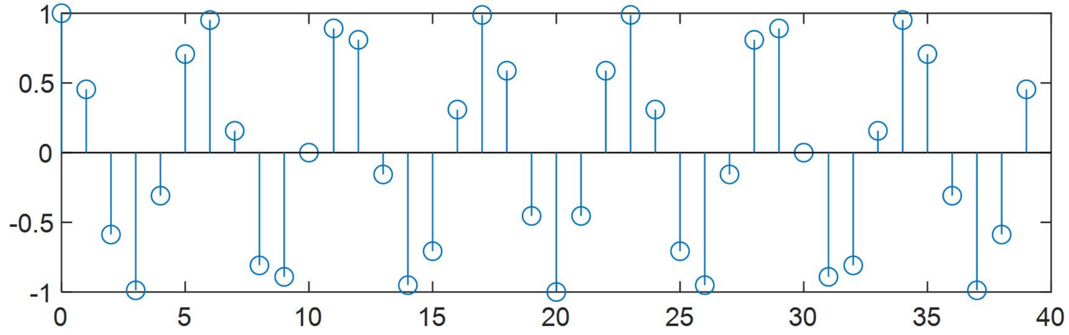
Correlation between the basis signal containing 7 periods in N samples and the original signal

Original signal

8^{th} basis signal

$$\boxed{X[7]} = \sum_{n=0}^{39} \boxed{\cos\left(2\pi \frac{2n}{40}\right)} \boxed{\cos\left(2\pi \frac{7n}{40}\right)}$$





Calculate $X[0], X[1], \ldots, X[39]$

How does DFT calculate those frequency partials?

In this specific example, we found …

For the 2nd and the 38th basis signal, we have

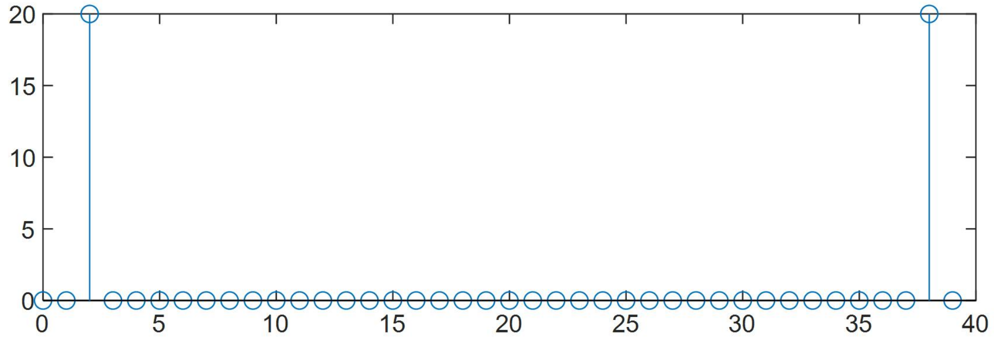
$$\sum_{n=0}^{39} \cos\left(2\pi \frac{2n}{40}\right)\cos\left(2\pi \frac{2n}{40}\right) = \sum_{n=0}^{39} \cos\left(2\pi \frac{2n}{40}\right)\cos\left(2\pi \frac{38n}{40}\right) = 20$$

$$X[2] = X[38] = 20$$

For the other basis signals, we have

$$\sum_{n=0}^{39} \cos\left(2\pi \frac{2n}{40}\right)\cos\left(2\pi \frac{kn}{40}\right) = 0, \;\; \text{where } k = 0, \ldots, 39 \text{ and } k \neq 2{,}38$$

$$X[0] = X[1] = X[3] = \ldots = X[37] = X[39] = 0$$



How does DFT calculate those frequency partials?

$$X[k] = \sum_{n=0}^{39} \cos\left(2\pi \frac{2n}{40}\right) \cos\left(2\pi \frac{kn}{40}\right)$$

General notation

$$X[k] = \sum_{n=0}^{N-1} x[n] \cos\left(2\pi \frac{kn}{N}\right)$$

40 – signal length – N

Original function - $x[n]$

DFT Formula:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi j \frac{kn}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] \cos\left(2\pi \frac{kn}{N}\right) - j \sum_{n=0}^{N-1} x[n] \sin\left(2\pi \frac{kn}{N}\right)$$

Why using complex numbers?

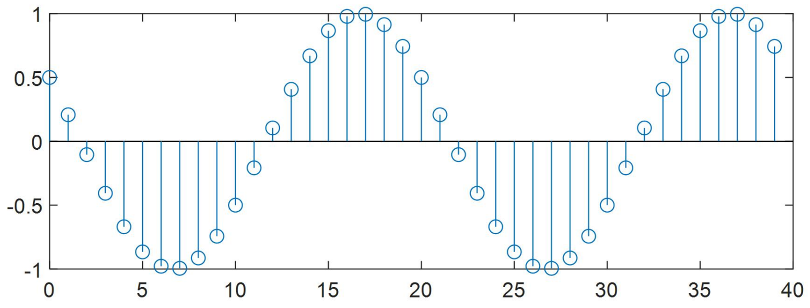
Why complex numbers?

If we only compare the original signal with cosine basis signals, the phase information will be lost

$x[n] = \frac{1}{2}\cos\left(2\pi\frac{2n}{40}\right)$ will generate exactly the same

X array

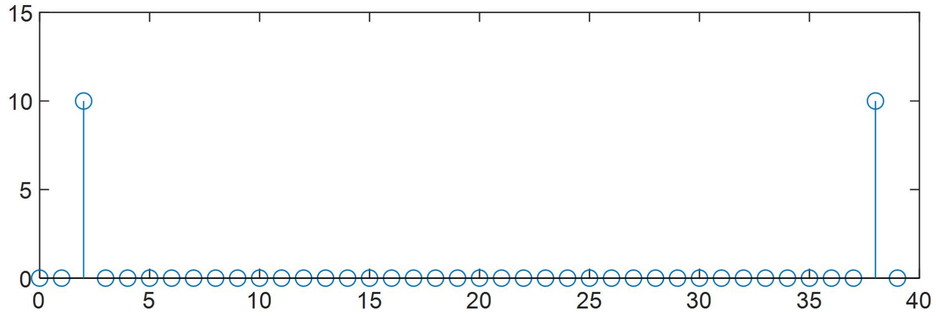
A phase shift example



$$x[n] = \cos\left(2\pi\frac{2n}{40} + \frac{\pi}{3}\right), \quad n = 0,1,\ldots,39$$

$$X[2] = X[38] = \sum_{n=0}^{39} \cos\left(2\pi\frac{2n}{40} + \frac{\pi}{3}\right)\cos\left(2\pi\frac{2n}{40}\right) = 10$$

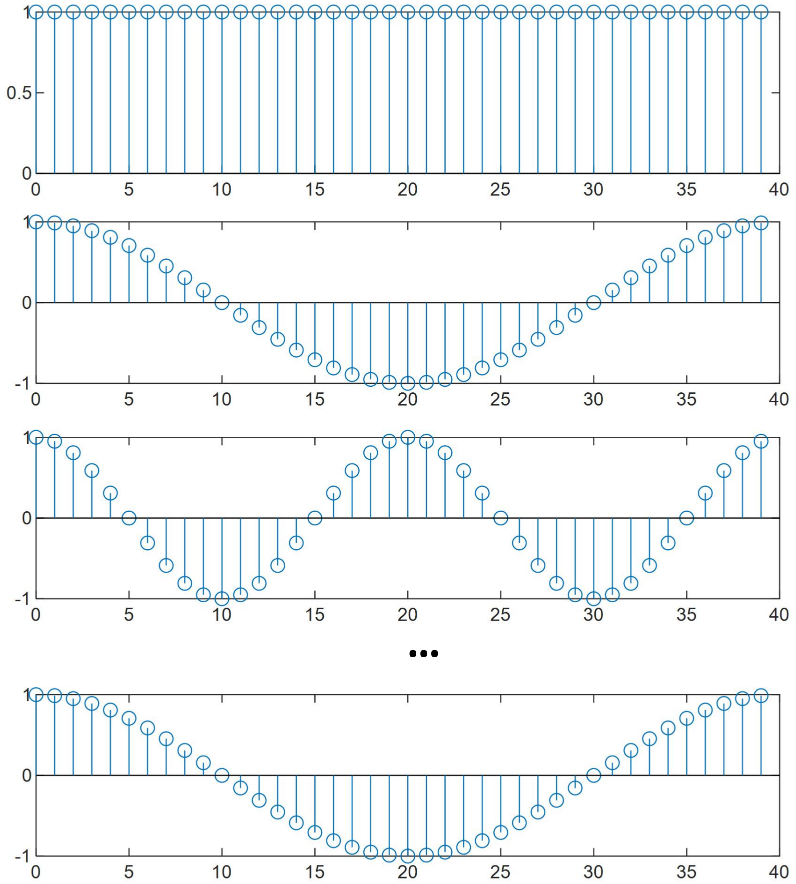
$$X[0] = X[1] = X[3] = \ldots = X[37] = X[39] = 0$$



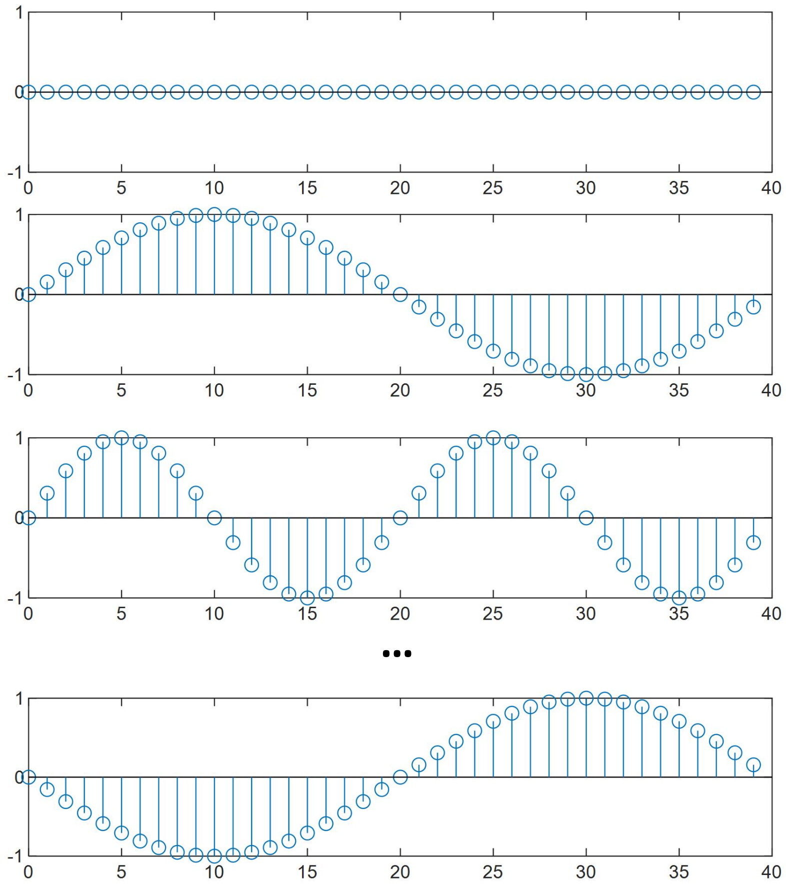
Why complex numbers?

Solution: Compare with sine waves to produce another array

Cosine basis



Sine basis



Basis 0

Basis 1

Basis 2

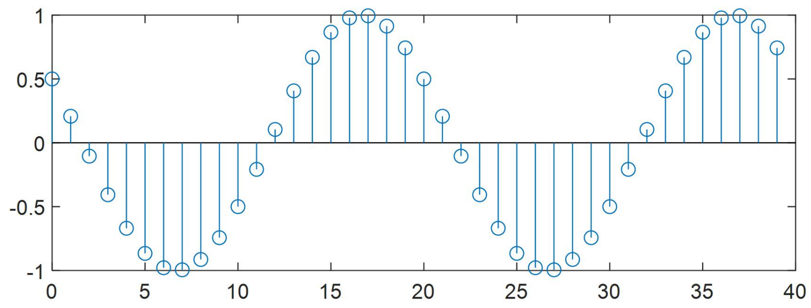
...

Basis 39

Produce $X_{cos}[0], \ldots, X_{cos}[39]$

Produce $X_{sin}[0], \ldots, X_{sin}[39]$

Why complex numbers?



$$x[n] = \cos\left(2\pi\frac{2n}{40} + \frac{\pi}{3}\right), \quad n = 0,1,\ldots,39$$

$$X_{sin}[2] = \sum_{n=0}^{39} \cos\left(2\pi\frac{2n}{40} + \frac{\pi}{3}\right)\sin\left(2\pi\frac{2n}{40}\right) = 10\sqrt{3}$$ and $X_{sin}[38] =- 10\sqrt{3}$

Recall:

$$X_{cos}[2] = X_{cos}[38] = \sum_{n=0}^{39} \cos\left(2\pi\frac{2n}{40} + \frac{\pi}{3}\right)\cos\left(2\pi\frac{2n}{40}\right) = 10$$

$\Rightarrow$

$$X[2] = 10 + 10\sqrt{3}j$$
$$X[38] = 10 - 10\sqrt{3}j$$

The complex number is a tool to store X_{cos} and X_{sin} in a compact form

Why complex numbers?

Compound form

$$X[2] = X_{cos}[2] + jX_{sin}[2] = \sum_{n=0}^{39} x[n]e^{2\pi j\frac{2n}{40}} = 10 + 10\sqrt{3}j$$

$$X[38] = X_{cos}[38] + jX_{sin}[38] = \sum_{n=0}^{39} x[n]e^{2\pi j\frac{38n}{40}} = 10 - 10\sqrt{3}j$$

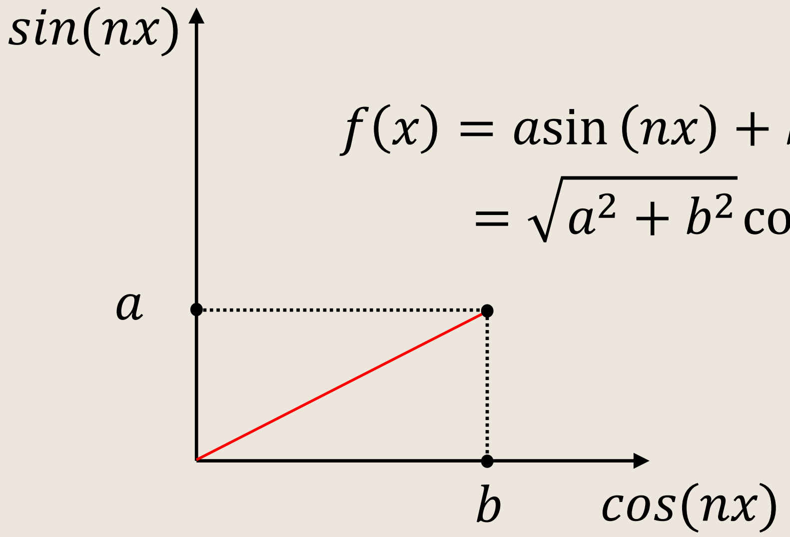
Magnitude: $\sqrt{10^2 + \left(10\sqrt{3}\right)^2} = 20$

Phase: $\arctan\frac{10\sqrt{3}}{10} = \frac{\pi}{3}$

We won’t loose information anymore!

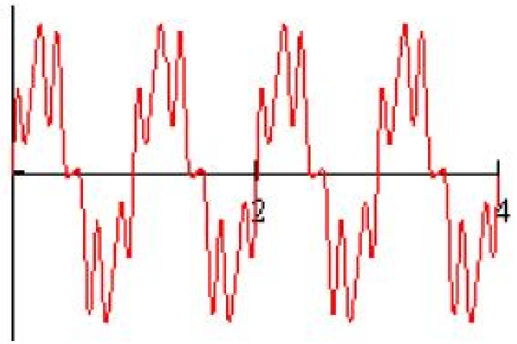
Math behind

$$f(x) = a\sin(nx) + b\cos(nx)$$
$$= \sqrt{a^2 + b^2}\cos\left(nx + \arctan\frac{a}{b}\right)$$

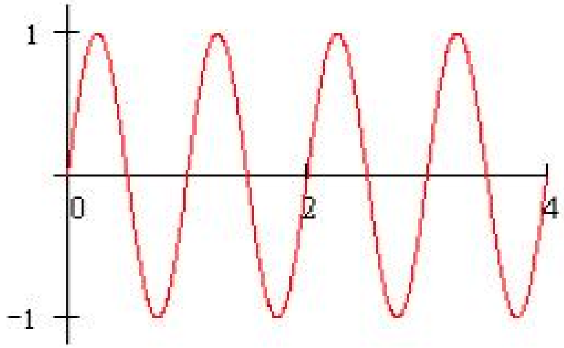


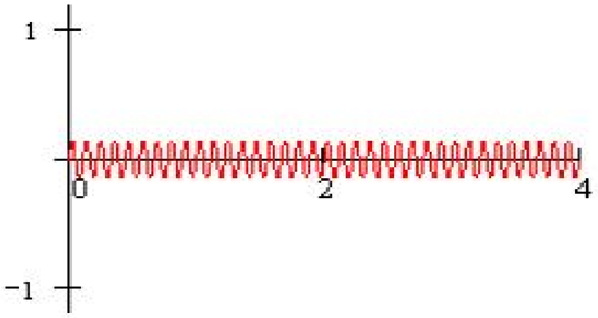
Fourier Series

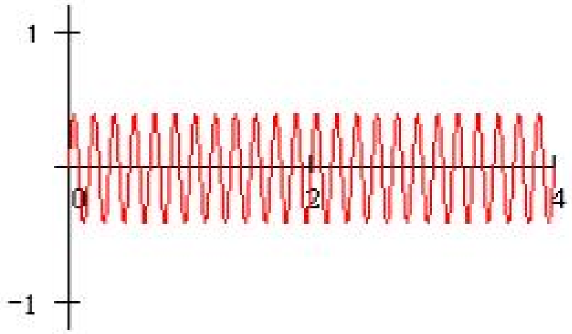
- For any periodic function $f(t)$, how to extract the component of f at a specific frequency?



is composed of the following components

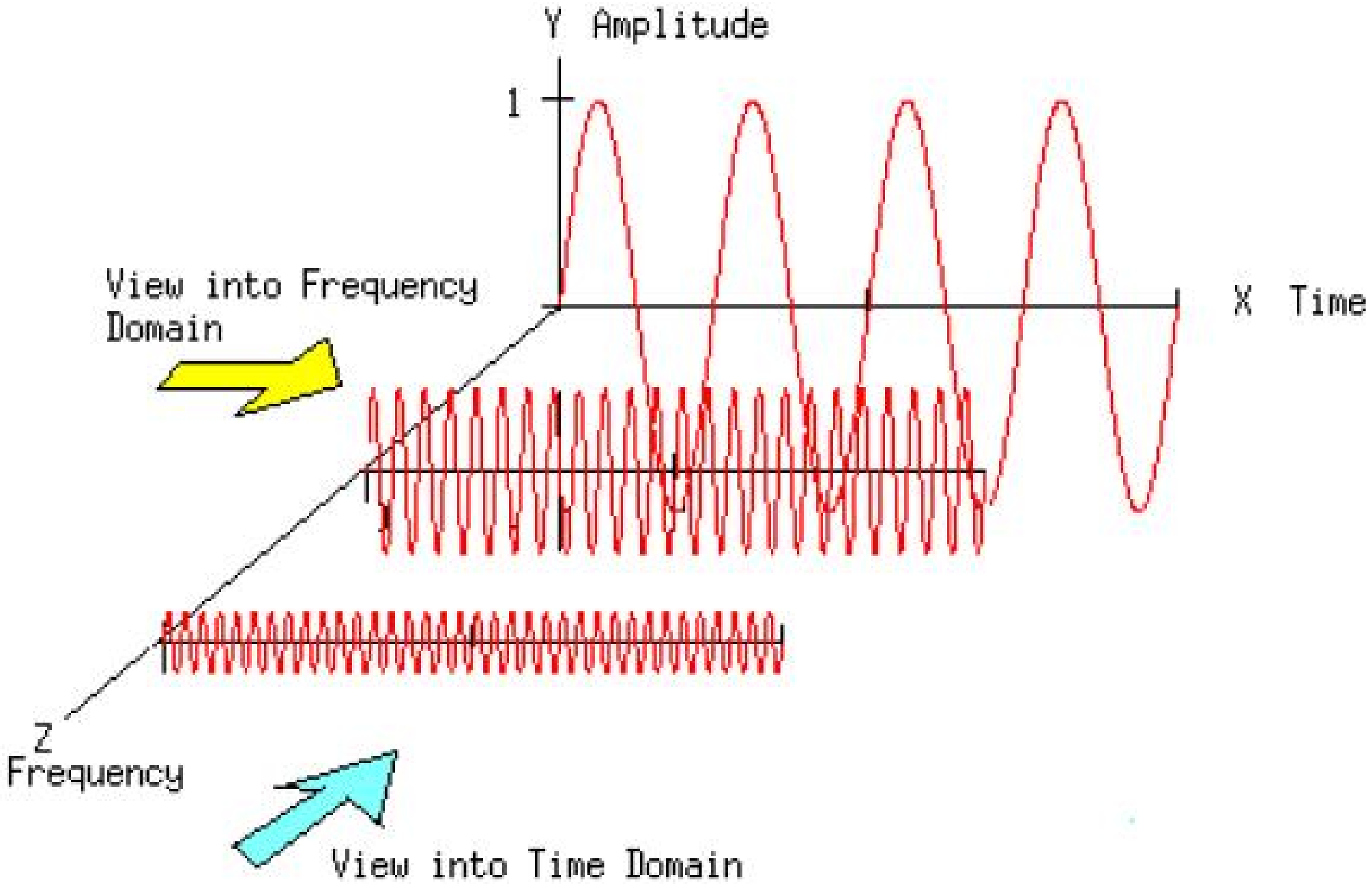






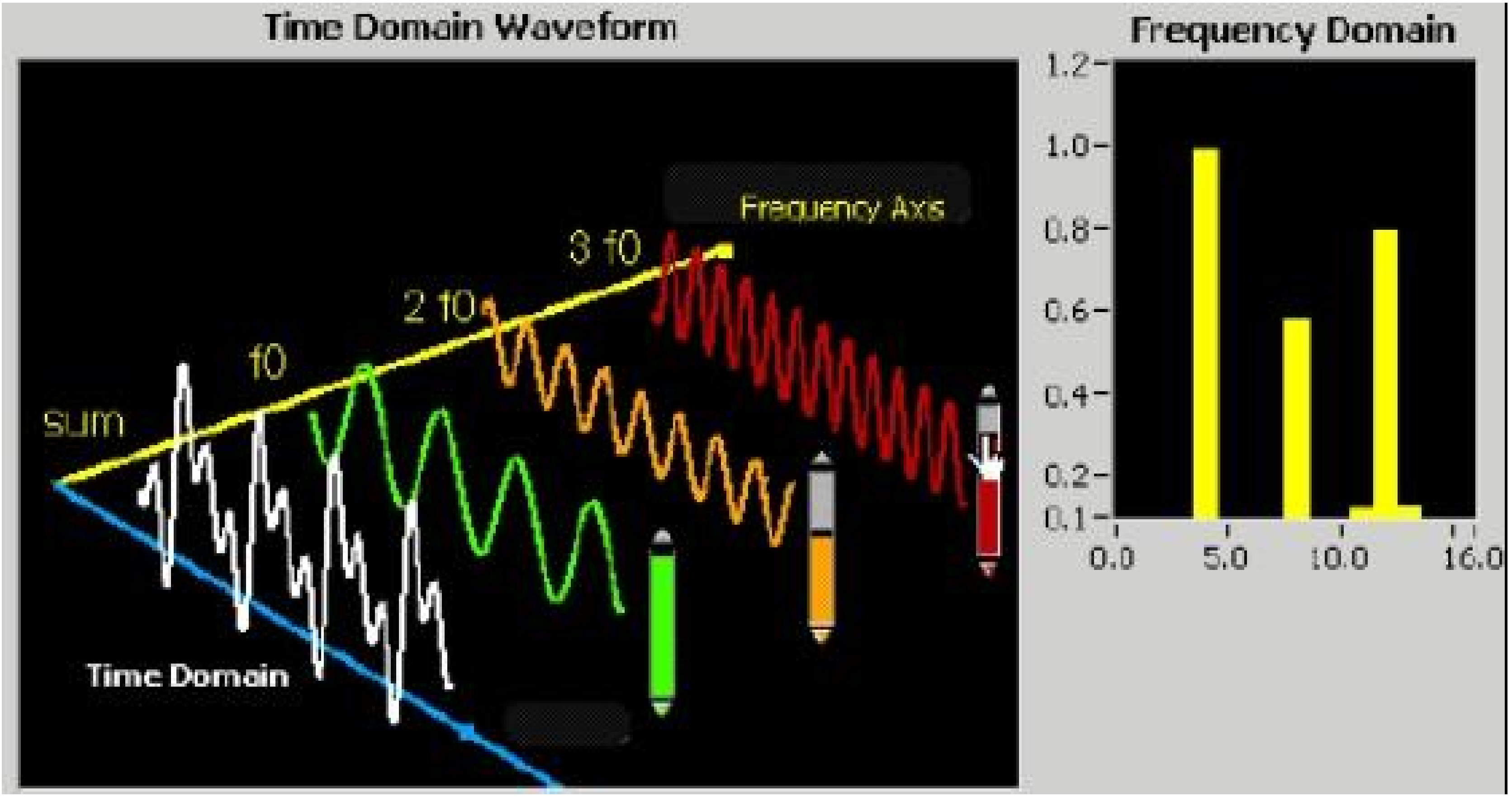
Fourier Series

For any periodic function $f(t)$, how to extract the component of f at a specific frequency?



Fourier Series

- For any periodic function $f(t)$, how to extract the component of f at a specific frequency?



Fourier Transforms

Fourier transform of $f(t)$ (maybe is not periodic) is defined as

$$F(\mu)=\int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}dt$$

Inverse Fourier transform

$$f(t)=\int_{-\infty}^{+\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

How to get these formulas?

Let's start the story from Fourier series to Fourier transform…

Fourier Series

- For any periodic function $f(t)$, how to extract the component of f at a specific frequency?

Fourier Series

Any periodic function can be expressed as a sum of sines and cosines of different frequencies each multiplied by a different coefficient

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right)$$

more details

Fourier Series

For a periodic function $f(t)$, with period T

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right)$$

where

$$\omega = \frac{2\pi}{T}$$

Redundant!

$$a_0 = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt \,, \qquad a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega t\,dt$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin n\omega t\,dt$$

From Fourier Series to Fourier Transforms

According to Euler formula $e^{j\theta} = \cos\theta + j\sin\theta$

Easy to have

$$\cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}, \sin n\omega t = -j\frac{e^{jn\omega t} - e^{-jn\omega t}}{2}$$

Then, Fourier series become

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

From Fourier Series to Fourier Transforms

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Easy to have

$$\cos n\omega t = \frac{e^{jn\omega t} + e^{-jn\omega t}}{2}, \sin n\omega t = -j\frac{e^{jn\omega t} - e^{-jn\omega t}}{2}$$

Then, Fourier series become

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\frac{e^{jn\omega t} + e^{-jn\omega t}}{2} - jb_n\frac{e^{jn\omega t} - e^{-jn\omega t}}{2}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}e^{jn\omega t} + \frac{a_n + jb_n}{2}e^{-jn\omega t}\right)$$

Then, let

$$c_0 = \frac{a_0}{2}, \frac{a_n - jb_n}{2} = c_n, \frac{a_n + jb_n}{2} = d_n$$

Then,

$$f(t) = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{jn\omega t} + d_n e^{-jn\omega t}\right) \quad (1)$$

From Fourier Series to Fourier Transforms

$$c_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt,$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\left(\cos n\omega t - j\sin n\omega t\right)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega t}dt \quad (2)$$

$$d_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\left(\cos n\omega t + j\sin n\omega t\right)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jn\omega t}dt$$

We can see that

$$d_n = c_{-n}$$

Thus,

$$\sum_{n=1}^{\infty} d_n e^{-jn\omega t} = \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t} = \sum_{n=-\infty}^{-1} c_n e^{jn\omega t} \quad (3)$$

From Fourier Series to Fourier Transforms

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{jn\omega t} + d_n e^{-jn\omega t} \right) \text{ (according to (1))}$$

$$= c_0 e^{j0\omega t} + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} d_n e^{-jn\omega t}$$

$$= c_0 e^{j0\omega t} + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega t} \text{ (according to (3))}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega t}$$,where C_n is defined by (2)

$\boxed{\text{This is the Fourier series in complex form}}$

How about a non-periodic function?

From Fourier Series to Fourier Transforms

$f(t)$ is a non-periodic function

We make a new function $f_T(t)$ which is periodic and the period is T

$$f_T(t) = f(t), if\ \ t \in [-T/2, T/2]$$

If $T \to +\infty$, $f_T(t)$ becomes $f(t)$

According to Fourier series

$$f_T(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega t}, c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-jn\omega t} dt$$

Let $s_n = n\omega$

$$f_T(t) = \sum_{n=-\infty}^{+\infty} \left(\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t} = \frac{1}{T}\sum_{n=-\infty}^{+\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t}$$

From Fourier Series to Fourier Transforms

$$f_T(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t}$$

when $T \to +\infty$

$$f(t) = \lim_{T \to +\infty} f_T(t) = \lim_{T \to +\infty} \frac{1}{T} \sum_{n=-\infty}^{+\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t}$$

$$\Delta s = s_n - s_{n-1} = \omega = \frac{2\pi}{T} \quad \Rightarrow \quad T = \frac{2\pi}{\Delta s}$$

$$f(t) = \lim_{\Delta s \to 0} \frac{\Delta s}{2\pi} \sum_{n=-\infty}^{+\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t}$$

$$= \frac{1}{2\pi} \lim_{\Delta s \to 0} \sum_{n=-\infty}^{+\infty} \left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-js_n t} dt \right) e^{js_n t} \Delta s$$

From Fourier Series to Fourier Transforms

$$f(t)=\frac{1}{2\pi}\lim_{\Delta s\to 0}\sum_{n=-\infty}^{+\infty}\left(\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)e^{-js_n t}dt\right)e^{js_n t}\Delta s$$

when $T\to+\infty(\Delta s\to 0)$

$s_n \rightarrow s$, $\Delta s \rightarrow ds$, $\sum \rightarrow \int$

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(t)e^{-jst}dt\right)e^{jst}ds$$

Denote by $F(s)$

$$\begin{cases} F(s)=\int_{-\infty}^{+\infty} f(t)e^{-jst}dt & \text{Fourier transform} \\ f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(s)e^{jst}ds & \text{Inverse Fourier transform} \end{cases}$$

$$F(\mu)=\int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}dt$$

$$f(t)=\int_{-\infty}^{+\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

From Fourier Series to Fourier Transforms

$$\begin{cases} F(s) = \int_{-\infty}^{+\infty} f(t)e^{-jst}dt \\ f(t) = \dfrac{1}{2\pi}\int_{-\infty}^{+\infty} F(s)e^{jst}ds \end{cases}$$

s here actually is the angular frequency

In the signal processing domain, we usually use another form by substituting s by $s = 2\pi\mu$, where μ is the frequency (measured by Herz)

$$\begin{cases} F(\mu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\mu t}dt \\ f(t) = \int_{-\infty}^{+\infty} F(\mu)e^{j2\pi\mu t}d\mu \end{cases}$$

Discrete Fourier Transform (DFT) in 1D Case

Given a discrete sequence with M points

$$f = [f_0 \ f_1, \ldots., f_{M-1}]$$

Regard it as a periodic signal, thus its basis frequency is $\frac{1}{M}$

For its frequency components, the frequencies are,

$$\frac{1}{M}, \frac{2}{M}, \ldots \frac{M}{M} \equiv \frac{1}{M}(1, 2, \ldots, M)$$

Its DFT is computed as

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi \frac{1}{M} ux}, u = 1, 2, \ldots, M$$

Usually, we write it as,

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi \frac{1}{M} ux}, u = 0, 1, 2, \ldots, M-1$$

Discrete Fourier Transform (DFT) in 1D Case

Thus, f's DFT also has M points

$$F = [F_0 \ F_1, \ldots, F_{M-1}]$$

and

$$F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}, u = 0, 1, 2, \ldots, M-1$$

IDFT $$f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}, x = 0, 1, 2, \ldots, M-1$$

For DFT, there is a fast algorithm for computation, FFT (Fast Fourier Transform)