1. Question

In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous

coordinate of the infinity point of this line?

Solution:

In the augmented Euclidean plane, a line and the infinity line cross at the infinity point.

The line l with homogeneous coordinates is $(1, -3, 4)$.

The infinity line l_∞ with homogeneous coordinates is $(0, 0, 1)$.

And the point with homogeneous coordinates is

$$l \times l_\infty = (\begin{bmatrix} -3 & 4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}) = (-3, -1, 0)$$

Therefore, the homogeneous coordinates of the infinity point of the line $x - 3y + 4 = 0$ are

$k(-3, -1, 0)^T$,where $k \neq 0$

2. Question

Compute the Jacobian matrix of $\mathbf{p}_d$ w.t.r$\mathbf{p}_n$, i.e., $\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$.

Solution:

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} \frac{dx_d}{dx} & \frac{dx_d}{dy} \\ \frac{dy_d}{dx} & \frac{dy_d}{dy} \end{bmatrix}$$

We have

$$\begin{cases} x_d = x(1 + k_1 r^2 + k_2 r^4) + 2\rho_1 xy + \rho_2(r^2 + 2x^2) + x k_3 r^6 \\ y_d = y(1 + k_1 r^2 + k_2 r^4) + 2\rho_2 xy + \rho_1(r^2 + 2y^2) + y k_3 r^6 \end{cases}$$

So,

$$\begin{aligned} \frac{dx_d}{dx} &= (1 + k_1 r^2 + k_2 r^4) + x(2k_1 x + 4k_2 r^2 x) + 2\rho_1 y + \rho_2(2x + 4x) + k_3 r^6 + x(6k_3 r^4 x) \\ &= (2k_1 + 4k_2 r^2 + 6k_3 r^4)x^2 + 6\rho_2 x + 2\rho_1 y + 1 + k_1 r^2 + k_2 r^4 + k_3 r^6 \end{aligned}$$

$$\begin{aligned} \frac{dx_d}{dy} &= x(2k_1 y + 4k_2 r^2 y) + 2\rho_1 x + 2\rho_2 y + 6k_3 r^4 xy \\ &= 2\rho_1 x + 2\rho_2 y + (2k_1 + 4k_2 r^2 + 6k_3 r^4)xy \end{aligned}$$

And since symmetry,

$$\frac{dy_d}{dy} = (2k_1 + 4k_2 r^2 + 6k_3 r^4)y^2 + 6\rho_1 y + 2\rho_2 x + 1 + k_1 r^2 + k_2 r^4 + k_3 r^6$$

$$\frac{dy_d}{dx} = 2\rho_1 x + 2\rho_2 y + (2k_1 + 4k_2 r^2 + 6k_3 r^4)xy$$

And the Jacobian matrix is:

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} \frac{dx_d}{dx} & \frac{dx_d}{dy} \\ \frac{dy_d}{dx} & \frac{dy_d}{dy} \end{bmatrix}$$

$$= \begin{bmatrix} 2(k_1+2k_2r^2+3k_3r^4)x_n^2+6\rho_2x_n+2\rho_1y_n+1+k_1r^2+k_2r^4+k_3r^6 & 2\rho_1x_n+2\rho_2y_n+2(k_1+2k_2r^2+3k_3r^4)x_ny_n \\ 2\rho_1x_n+2\rho_2y_n+2(k_1+2k_2r^2+3k_3r^4)x_ny_n & 2(k_1+2k_2r^2+3k_3r^4)y_n^2+6\rho_1y_n+2\rho_2x_n+1+k_1r^2+k_2r^4+k_3r^6 \end{bmatrix}$$

3. Question

Compute Jacobian matrix of r w.r.t. d, i.e. $\frac{d\mathbf{r}}{d\mathbf{d}^T}$

α=sinθ,β=cosθ,γ=1-cosθ

Solution:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & & \\ & \cos\theta & \\ & & \cos\theta \end{bmatrix} + (1-\cos\theta)\begin{bmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3 \end{bmatrix} + \sin\theta\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} \beta+\gamma n_1^2 & \gamma n_1n_2-\alpha n_3 & \gamma n_1n_3+\alpha n_2 \\ \gamma n_1n_2+\alpha n_3 & \beta+\gamma n_2^2 & \gamma n_2n_3-\alpha n_1 \\ \gamma n_1n_3-\alpha n_2 & \gamma n_2n_3+\alpha n_1 & \beta+\gamma n_3^2 \end{bmatrix}$$

So,

$$\mathbf{r} = \begin{bmatrix} \beta+\gamma n_1^2 \\ \gamma n_1n_2-\alpha n_3 \\ \gamma n_1n_3+\alpha n_2 \\ \gamma n_1n_2+\alpha n_3 \\ \beta+\gamma n_2^2 \\ \gamma n_2n_3-\alpha n_1 \\ \gamma n_1n_3-\alpha n_2 \\ \gamma n_2n_3+\alpha n_1 \\ \beta+\gamma n_3^2 \end{bmatrix}$$

1. For θ,

$$\because \theta = ||d||_2 = \sqrt{d_1^2+d_2^2+d_3^2}$$
$$\therefore \frac{d\theta}{dd_i} = \frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}} = \frac{d_i}{\theta} = n_i$$

2. For n_i,

$$\because n_i = \frac{d_i}{\theta} = \frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}}$$
$$\therefore \frac{dn_i}{dd_i} = \frac{\sqrt{d_1^2+d_2^2+d_3^2} - d_i\frac{d_i}{\sqrt{d_1^2+d_2^2+d_3^2}}}{d_1^2+d_2^2+d_3^2} = \frac{1-n_i^2}{\theta}$$
$$\therefore \frac{dn_i}{dd_j} = \frac{-d_i\frac{d_j}{\sqrt{d_1^2+d_2^2+d_3^2}}}{d_1^2+d_2^2+d_3^2} = -\frac{n_in_j}{\theta}, j \neq i$$

And then for some formula in the same shape,

1. For the formula $\beta+\gamma n_i^2$

$$\frac{d(\beta+\gamma n_i^2)}{dd_i} = -\alpha n_i + \alpha n_i n_i^2 + \gamma 2 n_i \frac{1-n_i^2}{\theta} = \frac{2\gamma n_i(1-n_i^2)}{\theta} + \alpha n_i(n_i^2-1), i=i$$

$$\frac{d(\beta+\gamma n_i^2)}{dd_j} = -\alpha n_j + \alpha n_j n_i^2 - \gamma 2 n_i \frac{n_i n_j}{\theta} = -\frac{2\gamma n_i^2 n_j}{\theta} + \alpha n_j(n_i^2-1), j \neq i$$

2. For the formula $\gamma n_j n_k + \alpha n_i$,

$$\begin{aligned}\frac{d(\gamma n_j n_k \pm \alpha n_i)}{dd_i} &= \alpha n_i n_j n_k - \gamma\frac{n_i n_j}{\theta} n_k - \gamma\frac{n_i n_k}{\theta} n_j \pm \beta n_i^2 \pm \alpha\frac{1-n_i^2}{\theta} \\ &= n_i(\alpha n_j n_k \pm \beta n_i) + \frac{\pm\alpha(1-n_i^2) - 2\gamma n_i n_j n_k}{\theta}, i=i \\ \frac{d(\gamma n_j n_k \pm \alpha n_i)}{dd_j} &= \alpha n_j n_j n_k + \gamma\frac{1-n_j^2}{\theta} n_k + \gamma n_j \frac{-n_j n_k}{\theta} \pm \beta n_i n_j \pm \alpha\frac{-n_i n_j}{\theta} \\ &= n_j(\alpha n_j n_k \pm \beta n_i) + \frac{\gamma n_k(1-2n_j^2) \mp \alpha n_i n_j}{\theta}, j \neq i\end{aligned}$$

So the answer is as follows:

$$\frac{d\mathbf{r}}{d\mathbf{d}^T} = \begin{bmatrix}
\frac{2\gamma n_1(1-n_1^2)}{\theta} + \alpha n_1(n_1^2-1) & -\frac{2\gamma n_1^2 n_2}{\theta} + \alpha n_2(n_1^2-1) & -\frac{2\gamma n_1^2 n_3}{\theta} + \alpha n_3(n_1^2-1) \\
n_1(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_2(1-2n_1^2)+\alpha n_1 n_3}{\theta} & n_2(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_1(1-2n_2^2)+\alpha n_2 n_3}{\theta} & n_3(\alpha n_1 n_2 - \beta n_3) + \frac{-\alpha(1-n_3^2)-2\gamma n_1 n_2 n_3}{\theta} \\
n_1(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_3(1-2n_1^2)-\alpha n_1 n_2}{\theta} & n_2(\alpha n_1 n_3 + \beta n_2) + \frac{\alpha(1-n_2^2)-2\gamma n_1 n_2 n_3}{\theta} & n_3(\alpha n_1 n_3 + \beta n_2) + \frac{\gamma n_1(1-2n_3^2)-\alpha n_2 n_3}{\theta} \\
n_1(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_2(1-2n_1^2)-\alpha n_1 n_3}{\theta} & n_2(\alpha n_1 n_2 + \beta n_3) + \frac{\gamma n_1(1-2n_2^2)-\alpha n_2 n_3}{\theta} & n_3(\alpha n_1 n_2 + \beta n_3) + \frac{\alpha(1-n_3^2)-2\gamma n_1 n_2 n_3}{\theta} \\
-\frac{2\gamma n_1^2 n_2}{\theta} + \alpha n_1(n_2^2-1) & \frac{2\gamma n_2(1-n_2^2)}{\theta} + \alpha n_2(n_2^2-1) & -\frac{2\gamma n_2^2 n_3}{\theta} + \alpha n_3(n_2^2-1) \\
n_1(\alpha n_2 n_3 - \beta n_1) + \frac{-\alpha(1-n_1^2)-2\gamma n_1 n_2 n_3}{\theta} & n_2(\alpha n_2 n_3 - \beta n_1) + \frac{\gamma n_3(1-2n_2^2)+\alpha n_1 n_2}{\theta} & n_3(\alpha n_2 n_3 - \beta n_1) + \frac{\gamma n_2(1-2n_3^2)+\alpha n_1 n_3}{\theta} \\
n_1(\alpha n_1 n_3 - \beta n_2) + \frac{\gamma n_3(1-2n_1^2)+\alpha n_1 n_2}{\theta} & n_2(\alpha n_1 n_3 - \beta n_2) + \frac{-\alpha(1-n_2^2)-2\gamma n_1 n_2 n_3}{\theta} & n_3(\alpha n_1 n_3 - \beta n_2) + \frac{\gamma n_1(1-2n_3^2)+\alpha n_2 n_3}{\theta} \\
n_1(\alpha n_2 n_3 + \beta n_1) + \frac{\alpha(1-n_1^2)-2\gamma n_1 n_2 n_3}{\theta} & n_2(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_3(1-2n_2^2)-\alpha n_1 n_2}{\theta} & n_3(\alpha n_2 n_3 + \beta n_1) + \frac{\gamma n_2(1-2n_3^2)-\alpha n_1 n_3}{\theta} \\
-\frac{2\gamma n_1 n_3^2}{\theta} + \alpha n_1(n_3^2-1) & -\frac{2\gamma n_2 n_3^2}{\theta} + \alpha n_2(n_3^2-1) & \frac{2\gamma n_3(1-n_3^2)}{\theta} + \alpha n_3(n_3^2-1)
\end{bmatrix}$$