第三次作业.

10.(3)

首先求一特解 $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - a^2 \dfrac{\partial u}{\partial t} + Ae^{-2x} = 0 & ① \\ u(t,0) = u(t,l) = 0 & ② \end{cases}$

首先, $w_1 = -\frac{A}{4} e^{-2x}$ 满足 ①

注意到 $w_1(0) = -\frac{A}{4}$ $\quad w_2(0) = -\frac{A}{4} e^{-2l}$.

故考虑 $w_2 = -\dfrac{\frac{A}{4}(e^{-2l}-1)}{l} x - \frac{A}{4}$

则 $w = w_1 - w_2 = -\frac{A}{4}(e^{-2x}-1) + \frac{A}{4}(e^{-2l}-1)\frac{x}{l}$ 同时满足 ①②

原方程转化为

$$\begin{cases} \dfrac{\partial^2 v}{\partial x^2} - a^2 \dfrac{\partial v}{\partial t} = 0 \\ v(t,0) = v(t,l) = 0 \\ v(0,x) = T_0 + \frac{A}{4}(e^{-2x}-1) - \frac{A}{4}(e^{-2l}-1)\frac{x}{l}. \end{cases}$$

其 $v = u - w$

由分离变量容易得到 $v = \sum_{n=1}^{\infty} a_n e^{-\frac{n^2\pi^2}{a^2 l^2} t} \sin \frac{n\pi x}{l}$

$t=0$ 时 $\quad v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$

故 $a_n = \frac{2}{l}\left(\int_0^l T_0 \sin \frac{n\pi x}{l} dx + \frac{A}{4}\int_0^l e^{-2x} \sin \frac{n\pi x}{l} dx - \frac{A}{4}\frac{(e^{-2l}-1)}{l}\int_0^l x \sin \frac{n\pi x}{l} dx\right)$

$= \frac{2T_0}{n\pi}(1-(-1)^n) + \frac{2Al^2(1-(-1)^n e^{-2l})}{n\pi(4l^2+n^2\pi^2)}$

中间用到的一些积分

$$\int_0^l \sin\frac{n\pi x}{l}\,dx = \frac{l}{n\pi}\left(1-(-1)^n\right)$$

$$\int_0^l e^{-2x}\sin\frac{n\pi x}{l}\,dx$$

$$= \mathrm{Im}\int_0^l e^{-2x}e^{\frac{in\pi}{l}x}\,dx$$

$$= \mathrm{Im}\,\frac{1}{\frac{in\pi}{l}-2}\,e^{\left(\frac{in\pi}{l}-2\right)x}\Big|_0^l = \frac{n\pi l}{4l^2+n^2\pi^2}\left(1-(-1)^n e^{-2l}\right)$$

$$\int_0^l x\sin\frac{n\pi x}{l}\,dx$$

$$= \mathrm{Im}\int_0^l x\,e^{\frac{in\pi}{l}x}\,dx$$

$$= \mathrm{Im}\,\frac{l}{in\pi}\int_0^l x\,d\,e^{\frac{in\pi}{l}x}$$

$$= \mathrm{Im}\,\frac{l}{in\pi}\left((-1)^n l-\int_0^l e^{\frac{in\pi x}{l}}\,dx\right)$$

$$= \frac{(-1)^{n-1}l^2}{n\pi}.$$

故. $$u = -\frac{A}{4}\left(e^{-2x}-1\right)+\frac{A}{4}\left(e^{-2l}-1\right)\frac{x}{l}+\sum_{n=1}^{\infty}a_n e^{-\frac{n^2\pi^2}{a^2l^2}t}\sin\frac{n\pi x}{l}$$

$$a_n = \frac{2T_0}{n\pi}\left(1-(-1)^n\right)-\frac{2Al^2\left(1-(-1)^n e^{-2l}\right)}{n\pi\left(4l^2+n^2\pi^2\right)}$$

最后一次作业

8. 即求

$$\begin{cases} u_t = a^2 u_{xx} + bu \\ u|_{t=0} = \delta(x) \end{cases}$$

作傅立叶变换

$$\begin{cases} \frac{d}{dt}\bar{u} = a^2(-i\lambda)^2\bar{u} + b\bar{u} \\ \bar{u}|_{t=0} = 1 \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}\bar{u} = (b - a^2\lambda^2)\bar{u} \\ \bar{u}|_{t=0} = 1 \end{cases}$$

故 $\vec{u}(\lambda) = e^{(b-a^2\lambda^2)t}$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(b-a^2\lambda^2)t} e^{-i\lambda t}\, dt = \frac{1}{2a\sqrt{\pi t}} e^{bt - \frac{x^2}{4a^2 t}}$$

9. (1) 首先求基本解

$$\begin{cases} u_t + a u_x = 0 \\ u|_{t=0} = \delta(x) \end{cases}$$

作傅立叶变换

$$\begin{cases} \frac{d}{dt}\bar{u} - ia\lambda\bar{u} = 0 \\ \bar{u}|_{t=0} = 1 \end{cases} \Rightarrow \bar{u}(\lambda) = e^{-ia\lambda t}$$

故 $u(x,t) = \delta(x - at)$

原方程的解为 $u(x,t) = \int_{-\infty}^{\infty} \delta(y - at)\varphi(x-y)\, dy$

$$+ \int_0^t dt \int_{-\infty}^{\infty} \delta(y - a(t-\tau)) f(\tau, x-y)\, dy = \varphi(x-at) + \int_0^t f(\tau, x-a(t-\tau))$$