1. (2) 用极球坐标系，方程化为

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+k^2u=0$$

$$\Rightarrow \frac{2}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial r^2}+k^2u=0$$

$$\Rightarrow \underbrace{2\frac{\partial u}{\partial r}+r\frac{\partial^2 u}{\partial r^2}}+k^2ru=0$$

$$\Rightarrow (ru)''+k^2ru=0$$

（这里''对r求导）

$$ru=C_1e^{ikr}+C_2e^{-ikr}$$

$$\Rightarrow u(r)=C_1\frac{e^{ikr}}{r}+C_2\frac{e^{-ikr}}{r}$$

6.(2) $$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+u\right)=0$$

$$\Rightarrow \exists f,\ \frac{\partial u}{\partial x}+u=f(x)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(e^x \frac{\partial u}{\partial x}\right) = f(x)e^x := F(x)$$

$$\Rightarrow e^x \frac{\partial u}{\partial x} = \underbrace{\int F\,dx}_{:=h(x)} + g(y)$$

$$\Rightarrow \frac{\partial}{\partial x}(e^x u) = f(x)e^x := F(x)$$

$$\Rightarrow e^x u = \underbrace{\int F\,dx}_{:=h(x)} + g(y)$$

$$e^x u = h(x) + g(y)$$

$$u = \underbrace{e^{-x} h(x)}_{:=\mu(x)} + g(y)e^{-x}$$

故通解 $u(x,y) = \mu(x) + g(y)e^{-x}$

6.(3).

首先求一个特解，形如 $v(x)$

$$0 = a^2 \frac{\partial^2 v}{\partial x^2} + 3x^2 \Rightarrow a^2 \frac{d^2 v}{dx^2} = -3x^2$$

$$\Rightarrow v(x) = -\frac{1}{4a^2}x^4$$

再求齐次方程 $\tilde{u}_{tt}=a^2\tilde{u}_{xx}$ 的通解

由达朗贝尔公式，$\tilde{u}(x,t)=f(x+at)+g(x-at)$

故原方程的通解为 $u(x,t)=f(x+at)+g(x-at)-\frac{1}{4a^2}x^4$

9.(2).

和1(2)的解题方法相似

原方程可以转化为

$$(ru)_{tt}=a^2(ru)_{rr}$$

$$\xRightarrow{\text{达朗贝尔}} ru=f(r+at)+g(r-at)$$

$$\Rightarrow u=\frac{1}{r}(f(r+at)+g(r-at))$$

$$u_t=\frac{a}{r}(f'(r+at)-g'(r-at))$$

$$u(r,0)=\frac{1}{r}f(r)+\frac{1}{r}g(r)=\varphi(r)$$

$$u_t(r,0)=\frac{a}{r}f'(r)-\frac{a}{r}g'(r)=\psi(r)$$

故 $f(r)+g(r)=r\varphi(r)$

$$f(r)-g(r)=\frac{1}{a}\int_0^r\tilde{r}\psi(\tilde{r})d\tilde{r}+f(0)-g(0)$$

$$f(r)=\frac{1}{2}r\varphi(r)+\frac{1}{2a}\int_0^r \tilde{r}\psi(\tilde{r})\,d\tilde{r}+\frac{1}{2}(f(0)-g(0$$

$$g(r)=\frac{1}{2}r\varphi(r)-\frac{1}{2a}\int_0^r \tilde{r}\psi(\tilde{r})\,d\tilde{r}-\frac{1}{2}(f(0)-g(0$$

故

$$u=\frac{1}{r}\left(f(r+at)+g(r-at)\right)$$

$$=\frac{1}{r}\left(\frac{1}{2}(r+at)\varphi(r+at)+\frac{1}{2}(r-at)\varphi(r-at)\right)$$

$$+\frac{1}{2ar}\int_{r-at}^{r+at}\tilde{r}\psi(\tilde{r})\,d\tilde{r}$$

9 (4)

$u_{tt}=u_{xx}$ 的通解为

$u=f(x+t)+g(x-t)$

代入定解条件.

$$u\big|_{x+t=0}=g(2x)+f(0)=\varphi(x)$$

$$u\big|_{x-t=0}=f(2x)+g(0)=\psi(x)$$

$$\Rightarrow u(x,t)=\psi\left(\frac{x+t}{2}\right)+\varphi\left(\frac{x-t}{2}\right)-g(0)-f(0)$$

$$=\psi\left(\frac{x+t}{2}\right)+\varphi\left(\frac{x-t}{2}\right)-\varphi(0)$$

10. 令 $\xi = x - at$, $\eta = t$. 即 $\begin{cases} x = \xi + a\eta \\ t = \eta \end{cases}$

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = f(t, x)$$

$$\Downarrow$$

$$\frac{\partial u}{\partial \eta} = f(\eta, \xi + a\eta)$$

$$u(\eta, \xi) = \int_0^{\eta} f(\tau, \xi + a\tau)\, d\tau + \underbrace{u(0, \xi)}_{} = \varphi(\xi)$$

$$u(x, t) = \int_0^{t} f(\tau, x - at + a\tau)\, d\tau + \varphi(x - at)$$

12. 用 u.(1). 做奇延拓.

通解为

$$u = f(x+at) + g(x-at)$$

$$u(0,x) = f(x) + g(x) = \sin x$$

$$u_t(0,x) = a\left(f'(x) - g'(x)\right) = kx$$

$$\begin{cases} f(x) - g(x) = \dfrac{k}{2a}x^2 + C \\ f(x) + g(x) = \sin x \end{cases}$$

$$\Rightarrow u = \frac{1}{2}\left(\frac{k}{2a}(x+at)^2 - \frac{k}{2a}(x-at)\right)$$

$$+ \frac{1}{2}\sin(x+at) + \frac{1}{2}\sin(x-at)$$

$$= kxt + \cancel{\cos x \sin at} \ \sin x \cos at$$