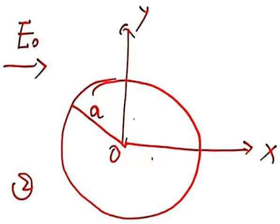
3-15

圆柱的轴与z轴重合，圆柱无限长 $\Rightarrow \frac{\partial \varphi}{\partial z}=0,\ k_z=0$

令 $x=0$ 平面为零电势面，则外电场 E_0 产生的电位

$$\varphi_0 = -E_0 x = -E_0 \rho\cos\phi \quad \cdots\cdots\ ①$$



$g(\phi)=g(\phi\pm 2n\pi) \Rightarrow n^2>0$

$$\varphi = \sum_{n=1}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)(C_n\rho^n + D_n\rho^{-n}) \cdots\ ②$$

$\rho\to\infty$ 时 ①式应与②式统一

$\therefore B_n=0,\quad C_n=0\ (n\neq 1)$

$\therefore \varphi = -E_0\rho\cos\phi + \sum_{n=1}^{\infty} A_n\rho^{-n}\cos n\phi$

边界条件 $\varphi(a)=0 \Rightarrow -E_0 a\cos\phi + \sum_{n=1}^{\infty} A_n a^{-n}\cos n\phi = 0$

$\therefore A_1 = E_0 a^2,\quad A_n=0\ (n\neq 1)$

$$\varphi = -E_0\rho\cos\phi + E_0 a^2\rho^{-1}\cos\phi$$

导体表面感应电荷 $\rho_s = \varepsilon_0 E_\rho = \varepsilon_0(-\nabla\varphi)_\rho = \varepsilon_0\left(-\frac{\partial\varphi}{\partial\rho}\right)\Big|_{\rho=a} = 2\varepsilon_0 E_0\cos\phi$

3-17、3-18.

$$\varphi = \sum_{n=1}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)(C_n\rho^n + D_n\rho^{-n})$$

根据边界条件具有反对称性 $\Rightarrow A_n=0$；$\rho=0$ 处 φ 为有限值 $\Rightarrow D_n=0$

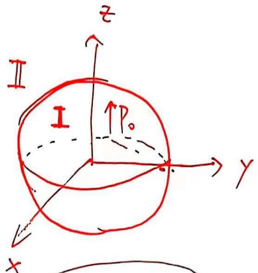
$\therefore \varphi = \sum_{n=1}^{\infty} B_n\rho^n\sin n\phi$

3-14、3-16.

取无穷远处为零电势

对于轴对称的包括极轴在内的球面边值问题

电位 φ 的通解为 $\varphi(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta)$



$r=0$ 处电位应为有限值 $\Rightarrow \varphi_1 = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$

$r\to\infty$ 处电位为 0 $\Rightarrow \varphi_2 = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta)$

代入 $r=a$ 处边界条件
$$\begin{cases}\varphi_1=\varphi_2\\ \varepsilon_0\frac{\partial\varphi_1}{\partial r} - \varepsilon_0\frac{\partial\varphi_2}{\partial r} = \rho_{sb} = \vec{P}\cdot\hat{r} = P_0\cos\theta\end{cases}$$

$$\therefore \begin{cases}\sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) = \sum_{n=0}^{\infty} B_n a^{-(n+1)} P_n(\cos\theta) \Rightarrow B_n = A_n a^{2n+1}\\ \varepsilon_0\left[\sum_{n=1}^{\infty} nA_n a^{n-1}P_n(\cos\theta) + \sum_{n=0}^{\infty}(n+1)B_n a^{-(n+2)}P_n(\cos\theta)\right] = P_0\cos\theta\end{cases}$$

代入

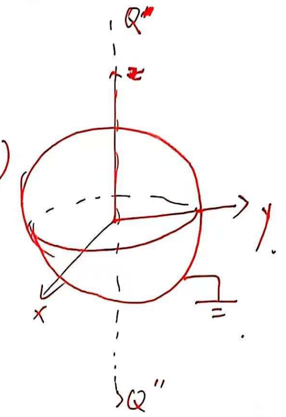
$$\varepsilon_0\left[\frac{A_0}{a} + \sum_{n=1}^{\infty}(2n+1)A_n a^{n-1}P_n(\cos\theta)\right] = P_0\cos\theta.$$

等号两边相应项匹配 $\Rightarrow A_1 = \frac{P_0}{3\varepsilon_0},\ A_n=0\ (n\neq 1) \Rightarrow B_1 = \frac{a^3P_0}{3\varepsilon_0},\ B_n=0\ (n\neq 1)$

$\therefore \varphi_1 = \frac{P_0 r\cos\theta}{3\varepsilon_0},\quad \varphi_2 = \frac{a^3P_0\cos\theta}{3\varepsilon_0 r^2}$

3-20. 设偶极矩 $\vec{p} = ql\hat{z}$

偶极矩产生电位 $\varphi_1 = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$



$\vec{E}_1 = -\nabla\varphi_1 = -\frac{p}{4\pi\varepsilon_0}\nabla\left(\frac{\cos\theta}{r^2}\right) = -\frac{p}{4\pi\varepsilon_0}\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta}\right)\left(\frac{\cos\theta}{r^2}\right)$

$= -\frac{p}{4\pi\varepsilon_0}\left(-\frac{2\cos\theta}{r^3}\hat{r} - \frac{\sin\theta}{r^3}\hat{\theta}\right)$

$= \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$

偶极矩关于球面的镜像电荷 $Q' = -\frac{a}{\frac{l}{2}}q,\ D' = \frac{a^2}{\frac{l}{2}}$

$Q'' = \frac{a}{\frac{l}{2}}q,\ D'' = \frac{a^2}{\frac{l}{2}}$

Q', Q'' 在球心处产生的电场为 $\vec{E}_2 = \frac{Q'}{4\pi\varepsilon_0 D'^2}(-\hat{z}) + \frac{Q''}{4\pi\varepsilon_0 D''^2}\hat{z} = 2\frac{\frac{a}{\frac{l}{2}}q}{4\pi\varepsilon_0\left(\frac{a^2}{\frac{l}{2}}\right)^2}\hat{z}$

$= \frac{p}{4\pi\varepsilon_0 a^3}\hat{z} = \frac{p}{4\pi\varepsilon_0 a^3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$

$l \to 0$ 时 Q', Q'' 位于无穷远处，空腔内近似看作均匀电场 $\vec{E}_2$

产生的电位 $\varphi_2 = -\vec{E}_2\cdot\vec{z} = -\frac{pz}{4\pi\varepsilon_0 a^3} = -\frac{pr\cos\theta}{4\pi\varepsilon_0 a^3}$

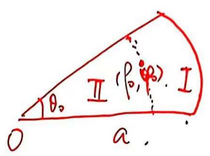
空腔内总电位 $\varphi = \varphi_1 + \varphi_2 = \frac{p\cos\theta}{4\pi\varepsilon_0}\left(\frac{1}{r^2} - \frac{r}{a^3}\right)$

表面处总场 $\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{3p\cos\theta}{4\pi\varepsilon_0 a^3}\hat{r}$

感应电荷 $\rho_s = -\varepsilon_0 E = -\frac{3p\cos\theta}{4\pi\varepsilon_0 a^3}$

3-27. $\begin{cases} \nabla^2 G = -\frac{1}{\rho}\delta(\rho-\rho_0)\delta(\varphi-\varphi_0) \\ G|_{边界} = 0 \end{cases}$

以 $\rho = \rho_0$ 面将扇形区域分为两个区域 I 和 II



$G_1 = \sum(A_{1n}\cos n\varphi + B_{1n}\sin n\varphi)(C_{1n}\rho^n + D_{1n}\rho^{-n})$

$G_2 = \sum(A_{2n}\cos n\varphi + B_{2n}\sin n\varphi)(C_{2n}\rho^n + D_{2n}\rho^{-n})$

$\varphi = 0$ 处 $G_1 = G_2 = 0 \Rightarrow A_{1n} = A_{2n} = 0$；$\rho \to 0$ 处 φ 有限 $\Rightarrow D_{2n} = 0$

$\therefore G_1 = \sum(C_{1n}\rho^n + D_{1n}\rho^{-n})\sin n\varphi$

$G_2 = \sum C_{2n}\rho^n \sin n\varphi$

$\rho = a$ 处 $G_1 = 0 \Rightarrow D_{1n} = -C_{1n}a^{2n}$；$\varphi = \theta_0$ 处 $G_1 = G_2 = 0 \Rightarrow n = \frac{m\pi}{\theta_0}$

$\therefore G_1 = \sum C_{1n}\left[\left(\frac{\rho}{a}\right)^n - \left(\frac{\rho}{a}\right)^{-n}\right]\sin n\varphi$

$G_2 = \sum C_{2n}\left(\frac{\rho}{a}\right)^n \sin n\varphi$

在 $\rho = \rho_0$ 面上 $\begin{cases} G_1 = G_2 \\ \frac{\partial G_2}{\partial\rho} - \frac{\partial G_1}{\partial\rho} = \frac{1}{\rho_0}\delta(\varphi-\varphi_0) \end{cases} \Rightarrow \begin{cases} C_{1n}\left[\left(\frac{\rho_0}{a}\right)^n - \left(\frac{\rho_0}{a}\right)^{-n}\right] = C_{2n}\left(\frac{\rho_0}{a}\right)^n \\ \sum C_{2n}\frac{n}{a}\left(\frac{\rho_0}{a}\right)^{n-1}\sin n\varphi - \sum C_{1n}\left[\frac{n}{a}\left(\frac{\rho_0}{a}\right)^{n-1} + \frac{n}{a}\left(\frac{\rho_0}{a}\right)^{-n-1}\right]\sin n\varphi \\ = \frac{1}{\rho_0}\delta(\varphi-\varphi_0) \end{cases}$

$\begin{cases} \cdots\cdots \\ \frac{\theta_0}{2}\left[C_{2n}\frac{n}{a}\left(\frac{\rho_0}{a}\right)^{n-1} - C_{1n}\frac{n}{a}\left[\left(\frac{\rho_0}{a}\right)^{n-1} + \left(\frac{\rho_0}{a}\right)^{-n-1}\right]\right] = \frac{1}{\rho_0}\sin n\varphi_0 \end{cases} \Rightarrow \begin{cases} C_{1n} = -\frac{\left(\frac{\rho_0}{a}\right)^n}{n\theta_0}\sin(n\varphi_0) \\ C_{2n} = -\frac{\left(\frac{\rho_0}{a}\right)^n - \left(\frac{\rho_0}{a}\right)^{-n}}{n\theta_0}\sin(n\varphi_0) \end{cases}$