1. (1) 对 x 作 Fourier 变换

$$\bar{u}(\lambda, y) = \int_{-\infty}^{+\infty} u(x, y) e^{i\lambda x} dx$$

$$F(u_{xx}) = -\lambda^2 \bar{u}, \quad F(u_{yy}) = \frac{d^2 \bar{u}}{dy^2}$$

$$\Delta_2 u = 0 \Rightarrow \frac{d^2 \bar{u}}{dy^2} - \lambda^2 \bar{u} = 0$$

$$\Rightarrow \bar{u} = A(\lambda) e^{|\lambda| y} + B(\lambda) e^{-\lambda y}$$

Since $y \to +\infty$, $u(x,y) \to 0$ $\Rightarrow \cancel{A(\lambda) = 0}$

$$\Rightarrow \bar{u} = C(\lambda) e^{-|\lambda| y} \quad (-\infty < \lambda < +\infty,\ y > 0)$$

由于 $\bar{u}(\lambda, 0) = \bar{F}(u(x,0)) = F(f) = \bar{f}(\lambda)$

$$\bar{u}(\lambda, y) = \bar{f}(\lambda) e^{-|\lambda| y}$$

$$u(x, y) = F^{-1}\left(\bar{f}(\lambda) \cdot e^{-|\lambda| y} \right)$$

$$F^{-1}(\bar{f}(\lambda)) = f(x).$$

$$F^{-1}(e^{-|\lambda| y}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-|\lambda| y - i\lambda x} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} e^{\lambda(y - ix)} + \int_{0}^{+\infty} e^{-\lambda(y + ix)} d$$

$$= \frac{1}{2\pi} \left(\frac{1}{y - ix} + \frac{1}{y + ix} \right)$$

$$= \frac{y}{\pi(x^2 + y^2)}$$

$$\Rightarrow u(x, y) = \int_{-\infty}^{+\infty} f(x) \cdot \frac{y}{\pi\left((x - \xi)^2 + y^2\right)} d\xi$$

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1.(2) 对 x 作 Fourier 变换

$$\bar{u}(t,\lambda)=\int_{-\infty}^{+\infty}u(t,x)e^{i\lambda x}dx$$

$$F(u_{xx})=-\lambda^2\bar{u}(t,\lambda)$$

$$F(u_t)=\frac{d}{dt}\bar{u}$$

$$F(f(t,x))=\bar{f}(t,\lambda)$$

$$\partial_t u=a^2u_{xx}+f(t,x)\Rightarrow\frac{d\bar{u}}{dt}+\lambda^2a^2\bar{u}=\bar{f}(t,\lambda)$$

$$u(0,x)=0\Rightarrow\bar{u}(0,\lambda)=0$$

$$\text{ODE}\Rightarrow\bar{u}(t,\lambda)=\int_0^t\bar{f}(\tau,\lambda)e^{-\lambda^2a^2(t-\tau)}d\tau$$

$$\begin{aligned}\therefore u(t,x)&=F^{-1}\left(\bar{u}(t,\lambda)\right)\\&=\int_0^t F^{-1}\left(\bar{f}(\tau,\lambda)\right)*F^{-1}\left(e^{-\lambda^2a^2(t-\tau)}\right)d\tau\\&=\int_0^t\frac{1}{2a\sqrt{(t-\tau)\pi}}\left(\int_{-\infty}^{+\infty}f(\tau,\xi)e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi\right)d\tau\end{aligned}$$

1.(3) 对 x 作正弦变换 记 $\bar{u}=F(u)$

$$\begin{aligned}F(u_{xx})&=\int_0^{+\infty}u_{xx}(t,x)\sin\lambda x\,dx\\&=\underbrace{u_x\sin(\lambda x)\Big|_0^{+\infty}}_{=0}-\lambda\int_0^{+\infty}u_x(t,x)\cos(\lambda x)dx\\&=-\lambda\left(u(t,x)\cos(\lambda x)\Big|_0^{+\infty}+\lambda\int_0^{+\infty}u(t,x)\sin(\lambda x)dx\right)\\&=\lambda\varphi(t)-\lambda^2\bar{u}(t,\lambda)\end{aligned}$$

$$F(u_t) = \frac{d}{dt}\bar{u}(t,\lambda)$$

$$u_t = a^2 u_{xx} \Longrightarrow \cancel{\frac{d}{dt}\bar{u}(t,\lambda)}$$

$$\frac{d\bar{u}(t,\lambda)}{dt} + \lambda^2 a^2 \bar{u}(t,\lambda) = \lambda a^2 \varphi(t)$$

$$\bar{u}(t,\lambda) = \int_0^t \lambda a^2 \varphi(\tau) e^{-\lambda^2 a^2 (t-\tau)} d\tau$$

$$u(t,x) = \frac{2}{\pi}\int_0^{+\infty} \bar{u}(t,\lambda)\sin(\lambda x)\,d\lambda$$

$$= \frac{2}{\pi}\int_0^{\cancel{+\infty}\, t} a^2\varphi(\tau)\left(\int_0^{+\infty}\lambda e^{-\lambda^2 a^2 (t-\tau)}\sin(\lambda x)\,d\lambda\right)d\tau$$

分部积分

$$= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{\tau - t}\left(\int_0^{+\infty}\sin(\lambda x)\,d e^{-\lambda^2 a^2 (t-\tau)}\right)d\tau$$

$$= \frac{1}{\pi}\int_0^t \frac{\varphi(\tau)}{t-\tau}\left(\int_0^{+\infty} e^{-\lambda^2 a^2 (t-\tau)}\,d(\sin\lambda x)\right)d\tau$$

$$= \frac{1}{\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\left(\int_0^{+\infty} e^{-\lambda^2 a^2 (t-\tau)}\cos\lambda x\,d\lambda\right)d\tau$$

$$= \frac{1}{2\pi}\int_0^t \frac{x\varphi(\tau)}{t-\tau}\sqrt{\frac{\pi}{a^2(t-\tau)}}\,e^{-\frac{x^2}{4a^2(t-\tau)}}\,d\tau$$

$$= \frac{x}{2a\sqrt{\pi}}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{\frac{3}{2}}}\,e^{-\frac{x^2}{4a^2(t-\tau)}}\,d\tau$$

2.(2) 对t作Laplace变换 $L(u(t,x)) = U(p,x)$

$$L(u_t) = pU(p,x) - u(0,x)$$

$$= pU(p,x) - u_1$$

$$L(u_{xx}) = \frac{\partial^2 U}{\partial x^2} = \frac{d^2 U}{dx^2}$$

$$u_t = a^2 u_{xx} \Rightarrow \frac{d^2 U(p,x)}{dx^2} - \frac{p}{a^2} U(p,x) = -\frac{u_1}{a^2}$$

$$U_x(p,0) = \int_0^{+\infty} u_x(t,0) e^{-pt} dt = 0$$

$$U(p,l) = \int_0^{+\infty} u(t,l) e^{-pt} dt = \frac{u_0}{p}$$

$$\Rightarrow U(p,x) = \frac{u_0 - u_1}{p} \cdot \frac{ch(\frac{\sqrt{p}}{a}x)}{ch(\frac{\sqrt{p}}{a}l)} + \frac{u_1}{p}$$

$$\Rightarrow u(t,x) = L^{-1}\left(\frac{u_1}{p}\right) + L^{-1}\left(\frac{u_0-u_1}{p} \frac{ch(\frac{\sqrt{p}}{a}x)}{ch(\frac{\sqrt{p}}{a}l)}\right)$$

$\frac{u_0-u_1}{p} \frac{ch(\frac{\sqrt{p}}{a}x)}{ch(\frac{\sqrt{p}}{a}l)}$ 的极点都是1阶.

$$p_0 = 0 \qquad p_k = \frac{\pm(2k-1)a\pi i}{2l}$$

$$\therefore L^{-1}\left(\frac{u_0-u_1}{p} \cdot \frac{ch(\frac{\sqrt{p}}{a}x)}{ch(\frac{\sqrt{p}}{a}l)}\right) = \sum_{k=0}^{\infty} Res\left(\frac{u_0-u_1}{p} \frac{ch(\frac{\sqrt{p}}{a}x)}{ch(\frac{\sqrt{p}}{a}l)} e^{pt}, p_k\right)$$

$$= u_0 - u_1 + \sum_{k=0}^{\infty} \frac{(u_0-u_1)4l^2}{(2k+1)^2 a^2 \pi^2} \cdot \cos\left(\frac{2k+1}{2l}\pi x\right)(-1)^{k+1} \frac{(2k+1)^2 a^2 \pi^2}{l^2}$$

$$\cdot \exp\left(-\frac{(2k+1)^2 a^2 \pi^2}{4l^2} t\right)$$

$$\Rightarrow u(t,x) = u_0 + \sum_{k=0}^{\infty} (-1)^{k+1} \frac{4(u_0-u_1)}{(2k+1)\pi} \cos\left(\frac{(2k+1)\pi}{2l}x\right) e^{-\frac{(2k+1)^2 a^2 \pi^2}{4l^2} t}$$

2.(4). 对 t 作 Laplace 变换 $\mathcal{L}(u)=U$

$$\mathcal{L}(u_{tt}) = p^2U - pu(0,x) - u_t(0,x) = p^2U - b$$

$$\mathcal{L}(u_{xx}) = \frac{d^2U}{dx^2}$$

$$u_{tt} = a^2u_{xx} \Rightarrow \frac{d^2U}{dx^2} - \frac{p^2}{a^2}U = -\frac{b}{a^2}$$

$$U = Ae^{\frac{p}{a}x} + Be^{-\frac{p}{a}x} + \frac{b}{p^2}$$

$$U_x = \frac{p}{a}Ae^{\frac{p}{a}x} - \frac{p}{a}Be^{-\frac{p}{a}x}$$

由 $\lim\limits_{x\to+\infty} u_x = 0$ 知 $A=0$

由 $U(p,0)=0$ 知 $B=-\frac{b}{p^2}$

$$U(p,x) = \frac{b}{p^2}\left(1-e^{-\frac{p}{a}x}\right)$$

$$\mathcal{L}^{-1}\left(\frac{b}{p^2}\right) = bt$$

$$\mathcal{L}^{-1}\left(\frac{b}{p^2}\cdot e^{-\frac{p}{a}x}\right) = b\left(t-\frac{x}{a}\right)h\left(t-\frac{x}{a}\right)$$

$$\Rightarrow u = \mathcal{L}^{-1}(U) = bt - b\cdot\left(t-\frac{x}{a}\right)\cdot h\left(t-\frac{x}{a}\right)$$