信息论第二次小测

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2024 年 3 月 27 日

第 1 题

In this exercise, we provide an information-theoretic proof of the well known number-theoretic result that there are infinitely many prime numbers. For this, consider an arbitrary integer n, and denote the number of primes no greater than n by $\pi(n)$. Take a random variable N uniformly distributed over $\{1, 2, \ldots, n\}$, and write it in its unique prime factorization, $N = p_1^{X_1} p_2^{X_2} \ldots p_{\pi(n)}^{X_{\pi(n)}}$, where $\{p_1, p_2, \cdots, p_{\pi(n)}\}$ are primes no greater than n, and each X_i is the largest power $k \geq 0$ such that p_i^k divides N. By inspecting $H(N)$, prove that $\pi(n) \to \infty$ as $n \to \infty$.

证明: 由于 N 是从 $\{1, 2, ..., n\}$ 中的均匀抽取，从而 $H(N) = \log_2(n)$。此外，由素数分解的唯一性得知，一个 N 的抽样结果可以和一组 $x_1, x_2, ..., x_{\pi(n)}$ 形成一一对应，从而：

$$\log_2(n) = H(N) = H(X_1, X_2, \cdots, X_{\pi(n)}) \leq \sum_{i=1}^{\pi(n)} H(X_i).$$

对每个正整数 $i \leq \pi(n)$，均 $2^{X_i} \leq p_i^{X_i} \leq N \leq n$，从而 $0 \leq X_i \leq \lfloor \log_2(n) \rfloor$. 这样对每个正整数 $i \leq \pi(n)$ 有 $H(X_i) \leq \log_2(\lfloor \log_2(n) \rfloor + 1) \leq \log_2(\log_2(n) + 1)$，亦即

$$\log_2(n) \leq \pi(n) \log_2(\log_2(n) + 1).$$

又因为 $n \to \infty$ 时，不难得到 $\log_2(n) / \log_2(\log_2(n) + 1) \to \infty$, 所以 $\pi(n) \to \infty$. $\square$