Approximate AR(1) shock: Ada and Cooper (2003)

- Suppose that ε_t follows an AR(1) process:

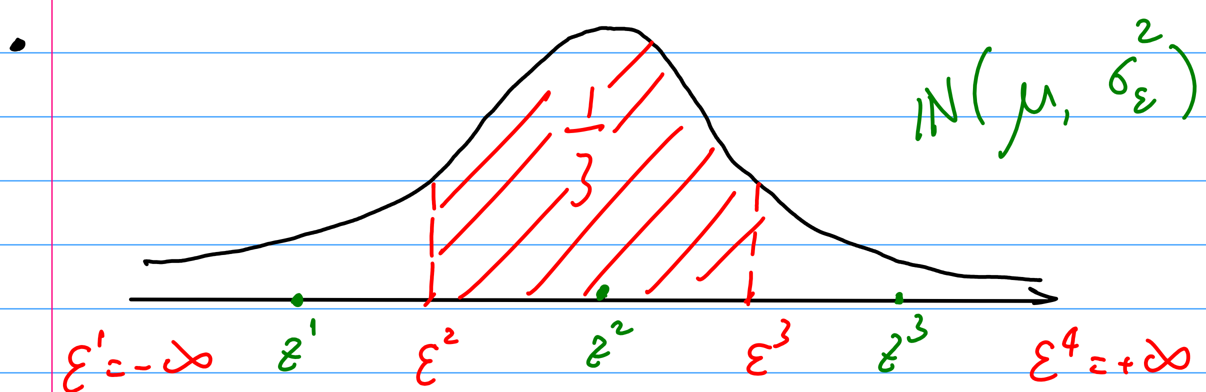
$$\varepsilon_t = \mu(1-\rho) + \rho\,\varepsilon_{t-1} + u_t$$

where $u_t \sim$ i.i.d $N(0, \sigma^2)$

we can note that the stationary distribution of ε_t is that

$$\varepsilon_t \sim N(\mu, \sigma_\varepsilon^2)$$

with $\sigma_\varepsilon = \dfrac{\sigma}{\sqrt{1-\rho^2}}$

- 

(1) Start by discretizing the real line into $N+1$ intervals, defined by the limits

$-\infty = \varepsilon^1, \varepsilon^2, \cdots, \varepsilon^N, \varepsilon^{N+1} = +\infty$

s.t. $\Phi\left(\dfrac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right) - \Phi\left(\dfrac{\varepsilon^{i}-\mu}{\sigma_\varepsilon}\right) = \dfrac{1}{N}$ $(i=1,\cdots,N)$

(2) Now compute the discrete states $\left\{z^i\right\}_{i=1}^{N}$

$$z^i = \mathbb{E}\left(\varepsilon_t \,\middle|\, \varepsilon_t \in [\varepsilon^i, \varepsilon^{i+1})\right)$$

$$= \mu - \sigma_\varepsilon \frac{\phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right) - \phi\left(\frac{\varepsilon^{i}-\mu}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right) - \Phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right)} \qquad \begin{pmatrix} i=2,3,\cdots \\ N-1 \end{pmatrix}$$

$$= \mu - N\cdot\sigma_\varepsilon\cdot\left[\phi\left(\frac{\varepsilon^{i+1}-\mu}{\sigma_\varepsilon}\right) - \phi\left(\frac{\varepsilon^{i}-\mu}{\sigma_\varepsilon}\right)\right]$$

$i=1$ $\quad(\varepsilon^1 = -\infty)$

$$z^1 = \mu - N\cdot\sigma_\varepsilon\, \phi\left(\frac{\varepsilon^{2}-\mu}{\sigma_\varepsilon}\right)$$

$i=N$ $\quad(\varepsilon^{N+1} = +\infty)$

$$z^N = \mu + N\sigma_\varepsilon\, \phi\left(\frac{\varepsilon^{N}-\mu}{\sigma_\varepsilon}\right)$$

(3). Now, one needs to compute the transition prob.

Note that

not σ_ε^2 !

$$\varepsilon_t \mid \varepsilon_{t-1} \sim \mathbb{N}\left(\mu(1-\rho) + \rho\,\varepsilon_{t-1},\ \sigma^2\right)$$

then $\lambda_{i,j} = P\left(\varepsilon_t \in [\varepsilon^j, \varepsilon^{j+1}) \,\middle|\, \varepsilon_{t-1} \in [\varepsilon^i, \varepsilon^{i+1})\right)$

$$= \frac{P(\varepsilon_t \in [\varepsilon^j, \varepsilon^{j+1}), \varepsilon_{t-1} \in [\varepsilon^i, \varepsilon^{i+1}))}{P(\varepsilon_{t-1} \in [\varepsilon^i, \varepsilon^{i+1}))} \quad = \frac{1}{N}$$

$$= N \cdot \int_{\varepsilon^i}^{\varepsilon^{i+1}} P\left(\varepsilon_t \in [\varepsilon^j, \varepsilon^{j+1}) \,\middle|\, \varepsilon_{t-1} = t\right) f(t)\, dt$$

note that $\varepsilon_t \mid \varepsilon_{t-1} = t \sim \mathbb{N}(\mu(1-\rho) + \rho t, \sigma^2)$

then $P\left(\varepsilon_t \in [\varepsilon^j, \varepsilon^{j+1}) \,\middle|\, \varepsilon_{t-1} = t\right)$

$$= \Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) - \Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right)$$

Therefore

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \cdot \int_{\varepsilon^i}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) - \Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 1 $\quad i = 1 \quad (\varepsilon^i = -\infty)$

case 1.1 $\quad j = 1 \quad (\varepsilon^j = -\infty)$ then $\Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right) = 0$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{-\infty}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 1.2 $\quad j = N \quad (\varepsilon^{j+1} = +\infty)$ then $\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) = 1$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{-\infty}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[1 - \Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 1.3 $\quad 2 \le j \le N-1$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{-\infty}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) - \Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 2, $2 \le i \le N-1$

case 2.1 $\quad j = 1 \quad (\varepsilon^j = -\infty)$ then $\Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right) = 0$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \cdot \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 2.2 $\quad j = N \quad (\varepsilon^{j+1} = +\infty)$ then $\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) = 1$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{\varepsilon^{i+1}} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[1 - \Phi\left(\frac{\varepsilon^j - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

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case 3 $\quad i = N \quad (\varepsilon^{i+1} = +\infty)$

case 3.1 $\quad j = 1 \quad (\varepsilon^{j} = -\infty)$

$$\lambda_{i,j} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{+\infty} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 3.2 $\quad j = N \quad (\varepsilon^{j+1} = +\infty)$

$$\lambda_{ij} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{+\infty} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[1 - \Phi\left(\frac{\varepsilon^{j} - \mu(1-\rho) - \rho t}{\sigma}\right)\right] dt$$

case 3.3 $\quad 2 \le j \le N-1$

$$\lambda_{ij} = \frac{N}{\sqrt{2\pi\sigma_\varepsilon^2}} \int_{\varepsilon^i}^{+\infty} \exp\left\{-\frac{(t-\mu)^2}{2\sigma_\varepsilon^2}\right\} \left[\Phi\left(\frac{\varepsilon^{j+1} - \mu(1-\rho) - \rho t}{\sigma}\right) - \Phi\left(\frac{\varepsilon^{j} - \mu(1-\rho) - \rho t}{\sigma}\right)\right) dt$$