About numerical integration

Refer to Kenneth Judd "Numerical Method in Economics" P260

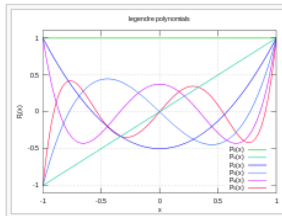
Guass-Legendre Quadrature

From Wiki

Gauss–Legendre quadrature [edit]

For the simplest integration problem stated above, i.e. with $\omega(x) = 1$, the associated polynomials are Legendre polynomials, $P_n(x)$, and the method is usually known as Gauss–Legendre quadrature. With the nth polynomial normalized to give $P_n(1) = 1$, the i^{th} Gauss node, x_i, is the i^{th} root of P_n; its weight is given by (Abramowitz & Stegun 1972, p. 887)

$$w_i = \frac{2}{(1 - x_i^2)\,[P_n'(x_i)]^2}.$$



Graphs of Legendre polynomials (up to $n = 5$)

Some low-order rules for solving the integration problem are listed below.

Number of points, n	Points, x_i	Weights, w_i
1	0	2
2	$\pm\sqrt{1/3}$	1
3	0	$8/9$
	$\pm\sqrt{3/5}$	$5/9$
4	$\pm\sqrt{\left(3-2\sqrt{6/5}\right)/7}$	$\frac{18+\sqrt{30}}{36}$
	$\pm\sqrt{\left(3+2\sqrt{6/5}\right)/7}$	$\frac{18-\sqrt{30}}{36}$
5	0	$128/225$
	$\pm\frac{1}{3}\sqrt{5-2\sqrt{10/7}}$	$\frac{322+13\sqrt{70}}{900}$
	$\pm\frac{1}{3}\sqrt{5+2\sqrt{10/7}}$	$\frac{322-13\sqrt{70}}{900}$

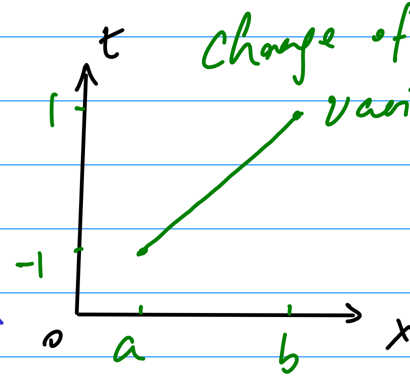
Change of interval [edit]

An integral over $[a, b]$ must be changed into an integral over $[-1, 1]$ before applying the Gaussian quadrature rule. This change of interval can be done in the following way:

Hence, $\int_{-1}^{1} f(x)\,dx \doteq \sum_{i=1}^{n} \omega_i f(x_i)$

In general, we need to compute $\int_a^b f(x)\,dx$

Change of variable



Define $t = -1 + \frac{2}{b-a}(x-a) \Rightarrow x = \frac{(t+1)(b-a)}{2} + a$

Then $\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{(t+1)(b-a)}{2} + a\right) d\,\frac{b-a}{2}t$

$$\doteq \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{(t+1)(b-a)}{2} + a\right) dt$$

Then $\int_a^b f(x)\,dx \doteq \frac{b-a}{2}\sum_{i=1}^{n}\omega_i\, f\left(\frac{(t_i+1)(b-a)}{2}+a\right)$

- Gauss - Laguerre Quadrature

This method deals with the integration of the following kind

$$\int_0^{\infty} f(x)\,e^{-x}\,dx \simeq \sum_{i=1}^{n}\omega_i f(x_i)$$

For more general integral

$t = a - x$

(1) $\boxed{\int_{-\infty}^{a} g(t)\,dt}$

Define $x = -(t-a) \Rightarrow t = a - x$ and $dt = -dx$

Then $\int_{-\infty}^{a} g(t)\,dt = \int_{+\infty}^{0} g(a-x)\,(-dx) = \int_0^{+\infty} g(a-x)\,dx$

$$= \int_0^{+\infty} g(a-x)\,e^{x}\cdot e^{-x}\,dx \xrightarrow[\text{Gauss-Laguerre}]{\text{Now use}}$$

$$= \sum_{i=1}^{n}\omega_i\, g(a-x_i)\,e^{x_i}$$

(2) $\boxed{\int_a^{+\infty} g(t)\,dt}$

Define $x = t - a \Rightarrow t = x + a$ and $dt = dx$

Therefore $\int_a^{+\infty} g(t)\,dt = \int_0^{+\infty} g(x+a)\,dx \xrightarrow[\text{Gauss-Laguerre}]{\text{use}}$

Then $\int_a^{+\infty} g(t)\,dt = \int_0^{+\infty} g(x+a)e^{x}\cdot e^{-x}\,dx$

$$= \sum_{i=1}^{n} \omega_i\, g(x_i+a)e^{x_i}$$