Dynamic Programming

Quantitative Macroeconomics [Econ 5725]

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The Finite Horizon Case

- Time is **discrete** and indexed by $t = 0, 1, ..., T < \infty$
- Environment is **stochastic**
    - Uncertainty is introduced via z_t, an exogenous r.v. (or shock)
    - z_t follows a Markov process with **transition function**

      $$Q(z', z) = \Pr(z_{t+1} \leq z' | z_t = z)$$

      with z_0 given.
    - We assume z_t is known at time t, but not z_{t+1}.

- The **instantaneous return function** is $u(x_t, c_t)$
- $u(x_t, c_t)$ is continuous and bounded in x_t and c_t
- The **state variable** $x_t \in X \subset \Re^m, \quad \forall\ t$
- The **control variable** $c_t \in C(x_t, z_t) \subset \Re^n, \quad \forall\ t$

- The **objective function** is

$$E_0 \sum_{t=0}^{T} \beta^t \, u(x_t, c_t)$$

  where

  - $\beta < 1$ is the discount factor
  - E_0 denotes the expectation conditional on $t = 0$

- Subject to a law of motion for x_t, the stochastic process z_t, and given x_0 and z_0.

- The **law of motion** of state x is:

$$x_{t+1} = f(x_t, z_t, c_t)$$

with x_0 given.

1 The state vector (x_t, z_t) completely describes the 'state' of the economy at every t.

2 The additive separability of the objective function implies that the action c_t depends solely on the currents states through a (possibly) time-varying function g_t,

$$g_t : X \times Z \to C, \forall t$$

that is, $c_t = g_t(x_t, z_t)$.

- The function g_t that maps the state vector into choices is the **decision rule**.

- The sequence $\pi_T = \{g_0, g_1, ..., g_T\}$ is the **policy**.

- If each $g_t(x_t, z_t) \in C(x_t, z_t)$, then $\pi_T \in \Pi$, i.e, the policy is **feasible**.

- If each $g_t(x_t, z_t) = g(x_t, z_t) \in C(x_t, z_t) \ \forall \ t$, the policy is **stationary**.

- The expected discounted value of a given feasible policy $\pi_T \in \Pi$ is

$$W_T(x_0, z_0, \pi_T) = E_0 \sum_{t=0}^{T} \beta^t \, u(x_t, g_t(x_t, z_t))$$

where

  - x_{t+1} follows $f(x_t, z_t, g_t(x_t, z_t))$
  - x_0, z_0 is given
  - The expectation is taken with respect to $Q(z', z)$

The Dynamic Programming Problem

An individual maximizes,

$$\max_{g_t(x_t,z_t)\in C(x_t,z_t)} W_T(x_0, z_0, \pi_T)$$

subject to

$$x_{t+1} = f(x_t, z_t, g_t(x_t, z_t))$$

given

$$x_0,\ z_0 \text{ and } Q(z', z)$$

Theorem of the Maximum

If

- the constraint set $C(x_t, z_t)$ is non-empty, compact and continuous,
- $u(\cdot)$ is continuous and bounded,
- $f(\cdot)$ is continuous, and
- Q satisfies the Feller property,

then

- there exists a solution (optimal policy) to the problem above, $\pi_T^* = \{g_0^*, g_1^*, ..., g_T^*\}$, and
- the **value function** $V_T(x_0, z_0) = W_T(x_0, z_0, \pi_T^*)$ is also continuous.

Proof See SLP p.62

Precisely, **value function** $V_T(x_0, z_0)$ is the expected discounted present value of the optimal policy π^*_T,

$$V_T(x_0, z_0) = E_0 \sum_{t=0}^{T} \beta^t \, u(x_t, g^*_t(x_t, z_t))$$

Corollary: If $C(x_t, z_t)$ is convex and $u(\cdot)$ and $f(\cdot)$ are strictly concave in c_t, then $g_t(x_t, z_t)$ is also continuous.

Toward Bellman's Equation

- Given the existence of a solution, we can write the value function:

$$V_T(x_0, z_0) = \max_{\pi_T} \; E_0 \left\{ u(x_0, c_0) + \sum_{t=1}^{T} \beta^t u(x_t, c_t) \right\}$$

- By the law of iterated expectations, $E_0(x_1) = E_0(E_1(x_1))$, hence

$$V_T(x_0, z_0) = \max_{\pi_T} \; E_0 \left\{ u(x_0, c_0) + E_1 \sum_{t=1}^{T} \beta^t u(x_t, c_t) \right\}$$

- Then, we can cascade the max operator,

$$V_T(x_0, z_0) = \max_{c_0} \; E_0 \left\{ u(x_0, c_0) + \max_{\pi_{T-1}} E_1 \sum_{t=1}^{T} \beta^t u(x_t, c_t) \right\}$$

where $\pi_{T-1} = \{c_1, c_2, ..., c_T\}$

- Rearranging the discount factor,

$$\begin{aligned} V_T(x_0, z_0) &= \max_{c_0} \; E_0 \left\{ u(x_0, c_0) + \beta \max_{\pi_{T-1}} E_1 \sum_{t=1}^{T} \beta^{t-1} u(x_t, c_t) \right\} \\ &= \max_{c_0} \; E_0 \left\{ u(x_0, c_0) + \beta \max_{\pi_{T-1}} W_{T-1}(x_1, z_1, \pi_{T-1}) \right\} \end{aligned}$$

- If we, analogously to above, define the value function

$$V_{T-1}(x_1, z_1) = W_{T-1}(x_1, z_1, \pi^*_{T-1})$$

as the expected present value of the optimal policy with $T-1$ periods left,

- then, we can write

$$V_T(x_0, z_0) \;\; = \;\; \max_{c_0} \; u(x_0, c_0) + \beta \; E_0 \; V_{T-1}(x_1, z_1)$$

- Omitting time subscripts on x, z and c we can rewrite the previous expression for any number of periods $s \in \{1, 2, ..., T\}$ left to go

$$V_s(x, z) = \max_{c \in C(x,z)} \; u(x, c) + \beta \; E \; V_{s-1}(x', z')$$

where primes denote next period.

Bellman's Equation with Finite Horizon

- Inserting the law of motion (for x) and using the definition of conditional expectation that is captured by the transition function for z, we arrive at

$$V_s(x,z) = \max_{c \in C(x,z)} \; u(x,c) + \beta \int_Z V_{s-1}(f(x,z,c), z')dQ(z',z) \quad (1)$$

where, $x = x_{T-s}$, $z = z_{T-s}$, and $z' = z_{T-s+1}$; and s are periods left to go, T is the total number of periods, and $T - s$ are periods that passed from $t = 0$.

Few important remarks:

- Bellman's equation is useful because reduces the choice of a sequence of decision rules to a sequence of choices for the control variable
- Hence a dynamic problem is reduced to a sequence of static problems,
- This way, it is sufficient to solve the DP problem sequentially $T + 1$ times, as shown in the next section.

Bellman's Principle of Optimality

A consequence of this result is the **Bellman's principle of optimality**:

- If the sequence of functions $\pi_T^* = \{g_0^*, g_1^*, ..., g_T^*\}$ is the optimal policy that maximizes $W_T(x_0, z_0, \pi_T)$,
- then, after j periods with $j + s = T$, $\pi_s^* = \{g_{T-s}^*, g_{T-s+1}^*, ..., g_T^*\}$ is the optimal policy (with elements identical to the original one) that maximizes $W_s(x_j, z_j, \pi_s)$.

Policies with this property are called **time-consistent**.

Backward Induction Algorithm

This leads to the following algorithm to solve the DP problem:

- First, start from the last period, with 0 periods to go (s=0). Then the problem is static and reads:

$$V_0(x_T, z_T) = \max_{c_T \in C(x_T, z_T)} u(x_T, c_T)$$

From here we obtain $g_T^*(x_T, z_T)$.

Then, given a specific form for $u(\cdot)$, we also have a specific functional form for $V_0(x_T, z_T)$.

- Second, go back one period. With 1 period to go (s=1) and using the law of motion $x_T = f(x_{T-1}, z_{T-1}, c_{T-1})$ and transition function Q we can write

$$V_1(x_{T-1}, z_{T-1}) = \max_{c_{T-1} \in C(x_{T-1}, c_{T-1})} u(x_{T-1}, z_{T-1}) + \beta \int_Z V_0(f(x_{T-1}, z_{T-1}, c_{T-1}), z_T) dQ(z_T, z_{T-1})$$

- Third, we keep going back until we reach time 0. This way, we collect the sequence of decision rules into the optimal policy vector.

- Finally, given the initial conditions at time 0, we can reconstruct the whole optimal path for the state and control, contingent on any realization of $\{z_t\}_{t=0}^{T}$.

The Infinite Horizon Case

We take the same environment and primitives as before except for:

$$T \to \infty.$$

Important consequences:

- We cannot proceed with backward induction. WHY?
- The DP problem (conditional on the initial state) is the same at each period since we always have the same number of periods left to go (∞), hence the environment is **stationary**.
- Then, the value function will be time invariant as well, $V(x, z)$.

Bellman's Equation

In the infinite horizon case, we can write the Bellman's equation as:

$$V(x,z) = \max_{c \in C(x,z)} \; u(x,c) + \beta \int_Z V(f(x,z,c), z') dQ(z', z) \qquad (2)$$

where we have dropped the time subindexes in the value function.

- The solution to this problem will be a stationary (time-invariant) decision rule $c^* = g^*(x,z)$.

It will be useful to think of (2) as a **functional equation**, i.e. an equation where the unknown is a function $\varphi(x, z)$ belonging to some functional space $\mathcal{C}$.

Then, more generally, the Bellman's equation (2) can be written

$$T(\varphi) = \max_{c \in C(x,z)} \; u(x, c) + \beta \int_Z \varphi(x', z') dQ(z', z)$$

where $T : \mathcal{C} \rightarrow \mathcal{C}$.

We are interested in knowing,

1. if (2) has a solution, that is, if there is a function V belonging to the functional space $\mathcal{C}$ that satisfies the **fixed point** property $V = T(V)$.
2. whether (2) can be thought of as the limit of the DP problem with finite horizon in (1) as $s \to \infty$.

Some Basic Elements for Functional Analysis

A **metric space** $(\mathcal{M}, d)$ is a set $\mathcal{M}$ together with a metric $d : \mathcal{M} \times \mathcal{M} \rightarrow \Re_+$ satisfying the following conditions $\forall\, \varphi, \phi$ and ψ in $\mathcal{M}$:

1. $d(\varphi, \phi) = 0 \Longleftrightarrow \varphi = \phi$
2. $d(\varphi, \phi) = d(\phi, \varphi)$
3. $d(\varphi, \psi) \leq d(\varphi, \phi) + d(\phi, \psi)$

Convergent Sequence. A sequence $\{\varphi_n\}$ in $\mathcal{M}$ is said to converge to $\varphi \in \mathcal{M}$ if: $\forall \delta > 0 \exists N = N(\delta) > 0$ such that $d(\varphi_n, \varphi) < \delta$ if $n > N$.

Cauchy Sequence. A sequence $\{\varphi_n\}$ in $\mathcal{M}$ with the property that: $\forall \delta > 0 \exists N = N(\delta) > 0$ such that $d(\varphi_n, \varphi_m) < \delta$ if $n, m > N$.

Complete Metric Space. A metric space $(\mathcal{M}, d)$ is complete if every Cauchy sequence in $(\mathcal{M}, d)$ converges to a point in the space. This space is also called **Banach space**.

Operator, T: A function T mapping a metric space into itself is called an operator.

Contraction Mapping, T: Let $(\mathcal{M}, d)$ be a complete metric space and $T = (\mathcal{M}, d) \rightarrow (\mathcal{M}, d)$. Then T is said to be a **contraction** with modulus β if there is a number $\beta \in (0, 1)$ such that

$$\forall (\varphi, \phi) \in (\mathcal{M}, d), d\left(T(\varphi), T(\phi)\right) \leq d(\varphi, \phi). \tag{3}$$

That is, a contraction contracts two points so that their images, $T(\varphi)$ and $T(\phi)$, are closer together than φ and ϕ

Blackwell Sufficient Conditions (BSC)

BSC: Let T be an operator on a metric space $(\mathcal{M}, d_\infty)$ where $\mathcal{M}$ is a space function with domain X and d_∞ is the sup metric. Then, T is a contraction mapping with modulus β if it satisfies the following two conditions:

1. (monotonicity) $\varphi \leq \phi \rightarrow T(\varphi) \leq T(\phi), \;\; \forall \varphi, \phi \in \mathcal{M}$

2. (discounting) $T(a + \varphi) \leq a\beta + T(\varphi), \;\; \forall a > 0, \varphi \in \mathcal{M}$

Contraction Mapping Theorem (CMT)

CMT: Let $(\mathcal{M}, d)$ be a complete metric space and let T be a contraction mapping with modulus β. Then, it follows that:

1. T has a unique fixed point φ^* in $\mathcal{M}$
2. For any φ^0 in $\mathcal{M}$, the sequence $\varphi^{n+1} = T(\varphi^n)$ started at φ^0 converges to φ^* in the metric d.

This is a very useful theorem because it ensures the **existence** and **uniqueness** of a fixed point under very general conditions and provides (through point 2) an algorithm to compute the fixed point, by simple iteration.

The CMT is also known as the Banach Fixed Point Theorem.

V is a Fixed Point

Show that the Value Function in Bellman's Equation (2) is a Fixed Point of a Contraction:

- First, consider the functional equation (3),

$$T(\varphi) = \max_{c \in C(x,z)} u(x,c) + \beta E\left[\varphi(f(x,z,c),z')\right]$$

that defines a mapping T with domain equal to the space of continuous and bounded functions $\mathcal{M}$.

Two results:

  - Given the assumption on the boundedness of u and φ, it is immediate to show that $T\varphi$ is bounded as well.
  - By the Theorem of the Maximum, we obtain that if φ is continuous, then $T\varphi$ is continuous as well (just interpret φ as V_{s-1} and $T\varphi$ as V_s).

- Second, equip the space $\mathcal{M}$ with the sup norm d_∞. Since $(\mathcal{M}, d_\infty)$ is a complete metric space (as you have proved earlier) we can use the Blackwell's sufficient conditions to check if T is a contraction mapping:
  - (monotonicity) Take $\varphi_1 \geq \varphi_2$, then

$$\begin{aligned} T(\varphi_1) &= \max_c \{u(x,c) + \beta\ E\left[\varphi_1(f(x,z,c),z')\right]\} \\ &\geq \max_c \{u(x,c) + \beta\ E\left[\varphi_2(f(x,z,c),z')\right]\} \\ &= T\varphi_2 \end{aligned}$$

  - (discounting) For any function ϕ, positive real numbers $a > 0$ and $\beta \in (0,1)$

$$\begin{aligned} T(\varphi + a) &= \{u(x,c^*) + \beta\ E\left[\varphi(f(x,z,c^*),z') + a\right]\} \\ &= \{u(x,c^*) + \beta\ E\left[\varphi(f(x,z,c^*),z')\right] + \beta a\} \\ &= T(\varphi) + \beta a \end{aligned}$$

Hence,

- It follows that T is a contraction mapping, hence, together with a complete metric space, it satisfies CMT.
- We can therefore use the contraction mapping theorem in characterizing the solution of the DP problem in infinite horizon.
- This is extremely useful because we can interpret the value function of the infinite horizon DP problem as the fixed point of a contraction mapping.

Two important results,

1. We know that the infinite horizon Bellman's equation (2) has a solution V and this solution is unique under general conditions.
2. We can interpret the infinite horizon value function as the limit as $T \to \infty$ of a finite horizon DP problem.
3. The CMT provides a computational algorithm (its point 2 above), a value function iteration (VFI).

Value Function Iteration (VFI) Algorithm

1. To compute our unknown V, which we showed that is fixed point, we can start iterating on (2) from any initial (continuous and bounded) function φ_0, and we are certain to converge to the solution V — Value Function Iteration (VFI) algorithm.

2. Roughly speaking: let $V_0 = \zeta$ be the initial guess of the value function. Iterating on it, $V_1 = T(\zeta)$, $V_2 = T(V_1)$, ..., $V_{n+1} = T(V_n)$ we will find convergence at some iteration N, that is, that $V^* = V_{N+1} = T(V_N)$ is approximately V_N.

Contractions have the feature of guaranteeing that the fixed point V (weakly) preserves the properties of the functions φ on which we iterate, such as **monotonicity**, **continuity** and **concavity**.

For the deterministic case (see SLP chapter 9 for stochastic environments),

1. If in addition to the assumptions in the contraction mapping theorem we have that *i*) $u(\cdot)$ is strictly concave, and *ii*) X is monotone, then V **is strictly increasing**. (SLP 4.7)

2. If in addition to the assumptions in the contraction mapping theorem we have that *i*) $u(\cdot)$ is strictly concave, and *ii*) X is convex, then V **is strictly concave** and the associated decision rule g **is single-valued and continuous**. (SLP 4.8)

If SLP 4.8 holds (i.e., the decision rule is continuous), we can find the optimal decision rule function using some **continuous method of approximation** — this means we may do not need to solve the decision rule at a lot of points. If SLP 4.8 does not hold, there is not guarantee the decision rule can be approximated with a continuous function, and we will have to use discretization — takes more time.

Few remarks,

It is always nice to have continuity in the optimal decision rule and strict concavity of the value function because then it is easier to search for optimal decisions. However, many interesting models do not have such property:

- Models with convex costs of adjustment (e.g. fixed cost of changing the size of the house).
- Models with non-trivial tax schedule functions.

Examples of trouble,

- There are not nice results for problems with **time-inconsistent preferences** (hyperbolic or quasi-geometric discounting) because agents tomorrow are solving a different problem than agents today. But, you can still solve the problem using backward induction for the finite horizon case.
- Same thing applies in models with lack of commitment — **time-inconsistent policies**. Some tricks that include some state variables as record keeping are proposed in Kydland and Prescott 1977.

Characterization of the Policy Function: The Euler Equation and TVC

The fixed point value function V is associated to an optimal policy function $c^* = g^*(x, z)$.

We can characterize the optimal policy function with the usual methods of calculus, i.e. by differentiation, like in the static case or in the optimal control problem in continuous time.

- Suppose we can invert the function $x' = f(x, z, c)$ with respect to the choice variable, c, we obtain $c = f^{-1}(x, z, x')$.

- Then we substitute out the control variable c in the Bellman's equation and the choice becomes on the state next period x',

$$V(x, z) = \max_{x' \in X} \; u(x', x, z) + \beta \int_Z V(x', z') dQ(z', z) \tag{4}$$

- If we knew that V were differentiable, by taking the FOC with respect to x' we would obtain:

$$u_{x'}(x, x', z) - \beta \int_Z V_{x'}(x', z') dQ(z', z) = 0 \tag{5}$$

- While we are free to make assumptions on u and f, given these assumptions the differentiability of V must be established.

Differentiability of V

The Benveniste and Scheinkman (1979) Theorem:

Let V be a concave function defined on the set X, let $x_0 \in X$, and let $N(x_0)$ be a neighborhood of x_0. If there is a concave differentiable function $\Omega : N(x_0) \to \Re$ such that $\Omega(x) \leq V(x)$, $\forall x \in N(x_0)$ with the equality holding at x_0, then V is differentiable at x_0 and $V_x(x_0) = \Omega_x(x_0)$.

Proof: See SLP p.85.

- Let's apply this result to the Bellman's equation. Define

$$\Omega(x, z) = u(x, g^*(x_0, z)) + \beta \int_Z V(f(x_0, z, g^*(x_0, z)), z')dQ(z', z)$$

- It follows that $\Omega(x_0, z) = V(x_0, z)$ and $\forall x \in N(x_0)$:

$$\Omega(x, z) \leq \max_{c \in C(x,z)} \{u(x, c) + \beta \int_Z V(f(x, z, c), z')dQ(z', z) = V(x, z)$$

  since $g^*(x_0, z)$ is not the optimal policy function for $x \neq x_0$.

- Hence, by the Benveniste and Scheinkman Theorem, V is differentiable.

- Further, assuming that $u()$ is concave and differentiable implies that $\Omega(x, z)$ is concave and differentiable as well, since the integral element is just a constant.

- We can therefore apply the envelope theorem to obtain that:

$$V_{x'}(x', z') = u_{x'}(x', g^*(x', z'))$$

which tells us that the derivative of the value function is the partial derivative of the utility function with respect to the state variable evaluated at the optimal value for the control.

- We can use this result to substitute out the derivative of the value function from (5) to obtain

$$u_{x'}(x, x', z) - \beta \int_Z u_{x'}(x', c') dQ(z', z) = 0 \tag{6}$$

The Euler Equation

- **Euler equation:** If we substitute out next period control through $c' = \varphi(x', x'', z')$, we obtain

$$u_{x'}(x, x', z) - \beta \int_Z u_{x'}(x', x'', z') dQ(z', z) = 0 \tag{7}$$

- To stress that this equation is satisfied by the optimal sequence of states $\{x_t^*\}_{t=0}^{\infty}$ we can re-express (7) it with time subscripts,

$$u_{x_{t+1}}(x_t^*, x_{t+1}^*, z) - \beta \int_Z u_{x_{t+1}}(x_{t+1}^*, x_{t+2}^*, z_{t+1}) dQ(z_{t+1}, z_t) = 0 \tag{8}$$

This is a second order nonlinear difference equation in the state variable.

The Transversality Condition (TVC)

- The **Euler equation** is a second order nonlinear difference equation in the state variable.
- To be able to fully characterize the optimal dynamic path of the state $\{x_t^*\}_{t=0}^{\infty}$, we need two boundary conditions.
- These boundary conditions are
  - Initial conditions x_0 and z_0 given
  - **Transversality Condition (TVC)**: $\lim_{t\to\infty} \beta^t\ u_c(x_t^*, c_t^*)\ x_t^* = 0$

Sufficiency of the Euler Equation and TVC

- It is possible to prove that if a sequence of states $\{x_t^*\}_{t=0}^{\infty}$ satisfies the **Euler equation** and the **TVC**, then it is optimal for the DP problem

- To do so, we will show that the difference D between the objective function evaluated at $\{x_t^*\}_{t=0}^{\infty}$ and at $\{x_t\}_{t=0}^{\infty}$, any alternative feasible sequence of states, is non-negative.

- In this particular proof, we abstract from the stochastic nature of the problem

- First, let's assume concavity and differentiability of the return u.

  Then from concavity it follows that:

$$
\begin{aligned}
D &= \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[u(x_t^*, x_{t+1}^*) - u(x_t, x_{t+1}) \right] \\
&\geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \left[u_{x_{t+1}}(x_t^*, x_{t+1}^*)(x_t^* - x_t) - u_{x_{t+1}}(x_t, x_{t+1})(x_{t+1}^* - x_{t+1}) \right]
\end{aligned}
$$

- Second, since $x_0^* = x_0$, given as initial condition, we can rewrite the sum above as:.

$$D \quad \geq \lim_{T\to\infty}\{\textstyle\sum_{t=0}^{T-1} \beta^t \left[u_{x'}(x_t^*, x_{t+1}^*) - \beta u_{x'}(x_t, x_{t+1})\right](x_{t+1}^* - x_{t+1})$$

$$+\beta^T u_{x'}(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1})\}$$

- Third, since the term in brakets $[\cdot]$ satisfies the Euler Equation, then

$$D \quad \geq \lim_{T\to\infty}\{\beta^T u_{x'}(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1})\}$$

- Fourth, using the Euler equation (8) to substitute the last term, we obtain:

$$
\begin{aligned}
D &\geq \lim_{T\to\infty} \{\beta^T u_{x'}(x_T^*, x_{T+1}^*)(x_{T+1}^* - x_{T+1})\} \\
&= -\lim_{T\to\infty} \{\beta^{T+1} u_{x'}(x_{T+1}^*, x_{T+2}^*)(x_{T+1}^* - x_{T+1})\}
\end{aligned}
$$

- Fifth, since $u_{x'} > 0$ and $x_{T+1} > 0$, then (we can add a last term):

$$
\begin{aligned}
D \quad &\geq -\lim_{T\to\infty}\{\beta^{T+1}u_{x'}(x^*_{T+1}, x^*_{T+2})(x^*_{T+1} - x_{T+1})\} \\
&\qquad - \lim_{T\to\infty}\beta^{T+1}u_{x'}(x^*_{T+1}, x^*_{T+2})x_{T+1}
\end{aligned}
$$

- Using the TVC, we can finally establish that $D \geq 0$.

What's next?

In order to implement solution methods (such as Backward Induction and VFI, among others), we need first to learn some **numerical techniques**:

- First, our object of interest are functions, we deal with functional equations. We will learn how to **approximate functions** in a nice way that a computer can store and handle.
- Second, we will learn how to take **derivatives and integrals** for cases in which analytical solutions are not available.
- Third, we will review how to find roots to **solve (systems of) non-linear equations**.
- Finally, we will go over **numerical optimization** - useful sometimes as a direct approach.

Before going into numerical techniques, we will have a quick review at how we can look at aggregate and survey data in the next class.