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Chapter 13

Relevant Logics

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Once upon a time, modal logic was castigated because it 'had no semantics.' Kripke, Hintikka, Kanger, and others changed all that. In a similar way, when Relevant Logic was introduced by Anderson and Belnap, it too was castigated for 'having no semantics.' Then Routley and Meyer (1982a [1973]) changed all that, along with Urquhart (1992a [1972]), Fine (1992b [1974]), and others. The present overview marks a culmination of that effort. The semantic approach described here brings together a number of hitherto disparate efforts to set out formal systems for logics of relevant implication and entailment. It also makes clear (despite some of our hopes and utterances) that the One True Logic *does not exist*. This is as true for relevant logics as Kripke et al., showed it to be for modal logics. In both cases, subtle (and not so subtle) variations on semantical postulates produce different logics in the same family. The question of which semantical postulates are correct makes no sense without further context, i.e., the questioner needs to answer the question: Correct for what? The question that does remain is: What motivates the relevant *family* of logics? And this is the question that is the main job for this chapter to investigate.

13.1. A Little History

Entailment, one would think, is a *relation*. It is the relation that holds between the *premises* of a valid argument and its *conclusion*. Yet modern symbolic logic, which at least since DeMorgan and Peirce has prided itself on taking relations seriously, failed to do so with respect to the central notion of *logical consequence* that is its business to analyze. Here and later we shall insist on the essentially relational character of a good implication.

The modern history of relevant logics begins at the same point as the history of modal logics – namely, with the disquiet over the thought that the material $\supset$ is a decent implication.[1] With the ink scarcely dry on the first edition of Whitehead and Russell's *Principia Mathematica* (1910–13), C. I. Lewis (1918) was already in print, decrying the paradoxes of 'implication.' The chief ones say (in English), P– A false proposition implies anything.

P+ Anything implies a true proposition.

P– and P+ reflect the well-known truth table for $\supset$, which looks like

$\supset$	F	T
F	T	T
T	F	T

Now what, honestly, would induce a sane human being to suppose that this table captures 'implies'? (Or even, as has sometimes been urged instead, 'if . . . then'?) Until they have been brainwashed with sophistries, elementary logic students grasp the point at once. This table is silly. No, Bertie, 'France is in Australia' does not imply 'The sea is sweet'. And no, Van, it is equally false to say 'If France is in Australia then the sea is sweet'.

To be sure, Logic is the *science* of argument, and like any other science, Logic has a right to simplifying assumptions and a formalism of its own. But it also has the obligation to enrich that formalism, the better to separate good arguments from bad.

Lewis saw it that way, introducing several systems of strict implication to overcome the deficiencies of P– and P+, and, as shall be seen a little later, Lewis' original rejection of material implication is based on ideas very close to those of relevant logicians. Beginning with negation $\sim$ and a binary consistency operator $\circ$, Lewis defined strict implication (our $\rightarrow$), via the rubric

D$\rightarrow$ $\quad A \rightarrow B =_{df} \sim(A \circ \sim B)$

That is, A (strictly) implies B just in case A is inconsistent with the negation of B. The task of formalizing a good theory of implication then becomes one of finding the right postulates for the *binary* possibility operator.

In a certain sense, of course, Lewis believed that P– and P+ are true. He agreed that

CP– $\quad \sim A \supset (A \supset B)$

CP+ $\quad A \supset (B \supset A)$

are logical truths. But he also held that "[t]he relation $A \supset B$ in this calculus has not quite the usual meaning of 'A implies B,' due to the fact that relations of the system are those of extension" (Lewis and Langford, 1959, p. 85). Lewis had no objection to CP– and CP+ as material logical truths. He had no qualm with accepting the material hook (or horseshoe) as a legitimate connective. But he objected to the identification of the hook with *implication*. Pleasantly (as Lewis and his later co-author Langford saw it), CP– and CP+ are not theorem schemes of any of the systems of theirs (1959), when formulated with strict implication replacing material $\supset$.

Less welcome to many later logicians were the paradoxes of strict implication, which Lewis and Langford (1959) considered ineluctable. These say, again in English,

SP− An impossible proposition implies anything.

SP+ Anything implies a necessary truth.

Paradigmatic formal counterparts for Lewis of SP− and SP+ were the following:

XP− $(A \wedge \sim A) \rightarrow B$

XP+ $A \rightarrow (B \vee \sim B)$

So important were XP− and XP+ to Lewis that he and Langford gave 'independent arguments' for them. Here is a version for XP− based on that of Lewis and Langford (1959, p. 250):

1	$A \wedge \sim A$	Hypothesis
2	A	1, ∧Elim
3	$\sim A$	1, ∧Elim
4	$A \vee B$	2, ∨Intro
5	B	3, 4, Disjunctive Syllogism
6	$(A \wedge \sim A) \rightarrow B$	1–5, →Intro

The argument is simple, perhaps even familiar, but is it any good? Is each line really entailed by its premises?[2] It seems that in his 1917 article "The Issues Concerning Material Implication" Lewis had already seen why it is fallacious. There he sets out a dialogue between two characters: *X* and himself (*L*). Here is a relevant part of that dialogue (Lewis, 1917, p. 355):

L. But tell me: do you admit that "Socrates was a solar myth" materially implies $2 + 2 = 5$?

X. Yes; but only because Socrates was *not* a solar myth.

L. Quite so. But if Socrates were a solar myth, would it be true that $2 + 2 = 5$? If you granted some paradoxer his assumption that Socrates was a solar myth, would you feel constrained to go on and grant that $2 + 2 = 5$?

X. I suppose you mean to harp on "irrelevant" some more.

In his and Langford's 'independent argument' for XP−, they do not "grant the paradoxer his assumption" that a contradiction holds. What is needed is a way to deal non-trivially with impossible assumptions, like contradictions or Socrates' being a solar myth. Section 13.4 returns to the treatment of impossibilities in relevant logic. The next section 13.2 turns to another aspect of the relevant analysis of the paradoxes.

13.2. Variable Sharing

What *in general* is wrong with the paradoxes of implication? It would seem that, in each, there is an insufficient tie between antecedent and consequent or premise and conclusion. As Lewis says in the dialogue given above, there is a lack of *relevance* here.

Relevant logics ensure that logically true implications do not have antecedents that are completely irrelevant to their consequents. As Ackermann (1956), the father of the theory of relevant entailment, wrote, there should be a connection between the content of the antecedent and the content of the consequent. This connection might seem difficult to enforce. For *content* is a semantic notion. The notion of a logic, on the other hand, is usually taken to be a syntactic concept, specified either in terms of a set of valid proofs or of a set of theorems. The gap between the semantic and the syntactic is bridged in part, however, by the *variable sharing constraint*. A logic, **L**, satisfies the variable sharing constraint iff (if and only if) whenever $A \to B$ is a theorem of **L**, A and B share at least one propositional variable.[3] The variable sharing constraint forces the antecedent and consequent to share some content, for then they are, in part, both about at least one or two or more propositions. Thus, they cannot be *absolutely* semantically irrelevant to one another.

A form of the variable sharing constraint was discovered early in the development of modern logic. Russell's book, *The Principles of Mathematics*, begins in its first chapter with a version of variable sharing (1903, p. 3):

> Pure mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants.[4]

It is not remarkable that Russell demands that all statements of mathematics be implications, since it is well-known that Russell (following Peano) believed that statements of mathematics are formal implications. What is interesting, however, is that Russell demands that the two propositions in an implication of mathematics contain exactly the same variables. The variables discussed here are not usually propositional variables, as they are in the relevant logicians' variable sharing constraint, but Russell does seem to desire that the formal implications of mathematics connect propositions that have content in common. These propositions are supposed to be about the same things (Russell, 1903, section 5).

The variable sharing constraint by itself, however, does not yield an analysis of relevance. Although it is a necessary condition for a logic to be a relevant logic, it is not sufficient. For suppose that one merely accepted all of the theorems of classical logic that satisfied this constraint. One would then still be left with

$$p \to (q \to p)$$

$$\sim(p \to q) \to (q \to r)$$

as well as many other paradoxes.

13.3. The Deduction Theorem

Relevant logics are supposed to capture the relation of entailment or that of implication between propositions. The philosophical notion of entailment was first developed by G. E. Moore in approximately 1920. He defines 'p entails q' to mean 'q is deducible from p' (Moore, 1922, p. 291). Thus, where '$\vdash$' represents the relation of deducibility, the following captures the logics of entailment:

If $A \vdash B$, then it is a theorem of the logic that A *entails* B.

Since implication is logically weaker than entailment, this relationship should also hold for implication. The above condition is known as the ***single premise deduction theorem***.

To understand the importance of the deduction theorem in this context, a little more needs to be said about the notion of relevant deducibility. One standard condition on deducibility relations is that, if a proposition is a premise then it can also be a conclusion. For example, classical logicians and intuitionists take

(PP) $\quad p, q \vdash p$

to be a valid deduction. But relevant logicians do not. For, consider the full deduction theorem:

If $A_1, \ldots, A_n, A \vdash B$, then $A_1, \ldots, A_n \vdash A \rightarrow B$

If one were to accept (PP), then, by two applications of the deduction theorem, one would have to accept

$\vdash p \rightarrow (q \rightarrow p)$

This says that $p \rightarrow (q \rightarrow p)$ is a theorem. But it is a paradox of implication, and it is not wanted. So one cannot accept this standard condition on deducibility.

Instead, relevant logicians have developed a notion of deduction due to Moh Shaw-Kwei (1950) and Church (1951). On this conception of deducibility, $A_1, \ldots, A_n \vdash B$ is ***relevantly valid*** only if $A_1, \ldots, A_n$ may ***all*** be really used in the deduction of B. In (PP), q is ***not*** used in the deduction of p, hence relevant logicians claim that (PP) is not a valid inference.[5]

The requirement that it is ***possible*** to use all premises in a relevant deduction needs itself to be fleshed out. One means, in natural deduction systems, is the method of 'relevance indices.' Such systems are not dealt with here,[6] but the main point can be put briefly. Hypotheses in a proof are tagged, and other steps are indexed by the tags on the hypotheses that are used to produce them. For a tagged hypothesis A to be discharged in a conditional sub-proof, the conclusion C of that sub-proof ***must bear*** (perhaps among others) the tag on the hypothesis A; if it does, $A \rightarrow C$ is inferred by ***discharging*** A, ending the sub-proof. After this application of the $\rightarrow$I rule, the tag on the discharged hypothesis A is removed from the indices on $A \to C$, which inherits its remaining tags, if any, from the conclusion C of the conditional proof. If a tag on a hypothesis does not appear in the steps on which the conditional proof is based, then the hypothesis cannot be discharged in that proof. Thus only hypotheses that are really used can be discharged.

In sequent systems for relevant logics, the real use requirement is enforced by treating premises and their relation to conclusions in a very intensional manner. A sequent (or consecution) is a structure of the form $A_1, \ldots, A_n \vdash B$. And a sequent calculus (also known as a 'consecution calculus' or 'Gentzen system') is a logic for inferring sequents from sequents. Consider the classically and intuitionistically valid inference on sequents from (13.1) to (13.2):

$$B \vdash C \tag{13.1}$$

$$A, B \vdash C \tag{13.2}$$

This inference – the so-called 'weakening' rule – obviously allows one to add arbitrary premises, even those that *cannot* be used in any intuitive sense. True, there is a way to concede that classical or intuitionist logicians who appeal to the weakening rule know what they are talking about. For they interpret the structural ',' as extensional conjunction '$\wedge$'. Following Dunn (1975b [1973]) and Mints (1976), *another* structural connective is introduced, ':', to do this job. So the standard logician does have a good argument to justify weakening. But it applies to ':', and not to ','. So from (13.1), one can justifiably conclude, not (13.2), but

$$A : B \vdash C \tag{13.3}$$

On the Dunn–Mints plan, this leads immediately to

$$A \wedge B \vdash C \tag{13.4}$$

Relevant logicians deny that (13.2) and (13.3) are equivalent. That is, they deny that the premises in a relevantly valid argument are conjoined to one another with standard conjunction. Rather, they think an inference $A_1, \ldots, A_n \vdash B$ is equivalent to $\vdash A_1 \to (\ldots(A_n \to B)\ldots)$. Moreover, the latter is *not* equivalent to $\vdash (A_1 \wedge \cdots \wedge A_n) \to B$ in relevant logics. Instead of taking premises in an inference as bound together by standard extensional conjunction (i.e. whose truth conditions are determined by a truth-table), relevant logicians have introduced another form of conjunction. This is the intensional conjunction that was briefly introduced in section 13.1. It is called *fusion*, written '$\circ$', and goes back (at least) to Church (1951). The central requirement of fusion is that it obey *residuation*:

$$A \circ B \vdash C \text{ iff } A \vdash B \to C$$

In short, fusion needs to satisfy the deduction theorem.[7]

To sum up what has been said so far, relevant logics should satisfy three conditions:

1 They should avoid the paradoxes of implication and, in particular, give a way of dealing with contradictions and other impossibilities non-trivially.
2 They should satisfy the variable sharing constraint.
3 They should contain a deducibility relation that requires all premises in a valid deduction to be capable of being used in that deduction and they should satisfy a deduction theorem.

13.4. The Ubiquity of Inconsistency

Relevant logics avoid XP–. This makes them *paraconsistent* logics. A paraconsistent logic is a logic that somehow tolerates contradictions. This is a very good feature. For contradictions are everywhere. Sad though this fact may be for any pursuit of rationality, candor compels its admission. Here are some of the spots at which inconsistencies break through:

- *Natural science* A theory is said to be 'in difficulties' when it conflicts with the results of observation or with another well-accepted theory. Combined with classical physics, Bohr's early theory of the atom predicts that electrons would radiate energy and fall into nuclei, and that they would not.
- *Foundations of Mathematics* From infinitesimal analysis through the summation of infinite series to the contradictions of set theory, mathematics too has ever been 'in difficulties.'
- *Bad data* A recent census reported one million more married women than married men in the USA. This is unlikely.
- *Metaphysics* Is not Zeno's arrow always both at rest and in motion?
- *Theology* God is three. God is one. Is He off by two?

This is not to suggest that the contradictions in all (or even any) of these cases is ineluctable. Great efforts have been made to resolve (or at worst live with) the associated problems. But no one takes XP– above at face value, to deduce whatever they want (and don't want) from present mistakes (if mistakes they be).

Rather, as Belnap has urged, people reason around any inconsistency in their present beliefs. This section now tries to find some theoretical ground on which to do so. Building on algebraic and semantical work by Białnicki-Birula and Rasiowa (1975), Dunn (1975a [1966]) and others, Routley and Routley (1972) introduced a unary operation on what are called 'worlds' or, more soberly, 'theories', 'set-ups' or 'situations'. Where a is a world, Routley and Routley postulate a companion world a^* such that

T~ $\quad I(\sim A, a) = \mathit{true}$ iff $I(A, a^*) = \mathit{false}$

where I is an interpretation in a model. Two ways to understand the 'Routley star operator' are discussed in section 13.6 below. Given this formalism, however, it is immediately clear how Routley and Routley (renamed Sylvan and Plumwood) provide a semantic refutation of XP–. Here $(p \wedge \sim p) \rightarrow q$ is refuted. Fixing a, set $I(p, a) = true$ and $I(p, a^*) = I(q, a) = false$. Applying T~ this makes the premise (of XP–) true at a but its conclusion false at a. And a purported implication that fails to preserve truth is surely no just candidate to ground a theory of logical consequence.[8]

13.5. Model Theoretic Semantics

Section 13.4 introduced worlds and the Routley star operator. Now it is time to present a semantics for relevant logic in almost all its glory. Some its finer technical details are omitted, and discussion is limited to its philosophically more interesting aspects.

Like Kripke's semantics for modal logics [see chapter 7], the semantics for relevant logic is a world-based semantics. Our frames start with a set of worlds, K. Of these worlds, some are distinguished, and called N ('normal worlds'). A formula is valid in a frame iff it is true on all normal worlds on all interpretations. Like Kripke, there is also an accessibility relation, R, between worlds. His accessibility relation, however, is meant to deal with a unary operator, necessity. Instead, R deals with a binary relation, implication. According to Jónsson and Tarski (1951), it makes formal sense to treat a unary connective by means of a binary relation and to treat a binary connective using a ternary relation.

The truth condition for relevant implication is

T→ $I(A \rightarrow B, a) = true$ iff for all b, c such that $Rabc$, if $I(A, b) = true$, then $I(B, c) = true$.

Notice how this truth condition allows one to avoid, e.g., the paradoxical $A \rightarrow (B \rightarrow B)$ since it does not force $B \rightarrow B$ to hold at all worlds.

To handle negation, there is the Routley star operator, introduced above. The truth conditions for extensional conjunction and disjunction are straightforward:

T∧ $I(A \wedge B, a) = true$ iff $I(A, a) = true$ and $I(B, a) = true$.

T∨ $I(A \vee B, a) = true$ iff $I(A, a) = true$ or $I(B, a) = true$.

Intensional conjunction, or fusion, discussed above, has a more technical condition:

T∘ $I(A \circ B, a) = true$ iff there are some worlds b, c, such that $Rbca$ and $I(A, b) = true$ and $I(B, c) = true$.

This condition looks a bit forbidding, but it can be understood if one follows Lewis in thinking of fusion as a type of binary relative possibility connective. $I(A \circ B, a) = true$ says that, A and B are jointly possible at a in the sense that a recognizes the combination of worlds in which A and B obtain.

The relationship between non-normal worlds and normal worlds in the Routley–Meyer – henceforth *relational* – semantics is also interesting and important. Define a relation on worlds $\leq$ such that

$a \leq b$ iff there is some world n in N such that $Rnab$

and postulate that $\leq$ is reflexive and transitive. Also, place certain constraints on frames and on interpretations so that *hereditariness* holds, namely,

If $I(A, a) = true$ and $a \leq b$, then $I(A, b) = true$.

It is traditional in logic to identify an entailment *relation* on sentences with the truth of $\rightarrow$ statements at one or more points. This tradition is reflected in the relational semantics via the semantical entailment fact below. Given an interpretation I, it is said that

A *entails* B iff, for all worlds a, if $I(A, a) = true$ then $I(B, a) = true$.

Entailment thus being (as usual) truth-preservation over *all* worlds, the matching true $\rightarrow$ statements are those true at all normal worlds n in N. That is, given I,

A *normally implies* B iff, for all n in N, $I(A \rightarrow B, n) = true$.

And now by hereditariness and reflexivity of $\leq$, this important *semantical entailment fact* obtains, for every frame and every interpretation I therein:

Fact (SemEnt): A *entails* B iff A *normally implies* B.

The proof is simple, and so is left to the reader.

This fact simplifies soundness proofs for relevant logics. Suppose that one wants to verify a theorem of the form '$A \rightarrow B$'. Assuming that at an arbitrary world a, $I(A, a) = true$, one then shows $I(B, a) = true$. Applying SemEnt, $A \rightarrow B$ is true on every normal world in the model. Generalizing, $A \rightarrow B$ is true on every normal world in every model structure, hence it is valid. The reader should keep this use of SemEnt in mind when the various semantical postulates that are placed on models in coming sections are discussed. This clarifies greatly the relationship between the postulates and their corresponding axioms.

13.6. Interpreting the Semantics

How is one to understand the various features of the semantics? We do not think that there is a single right answer to this question. Different relevant logics, we think, formalize different notions of entailment or implication. For these, different interpretations of their corresponding semantics are appropriate.

Consider the relation R. One interpretation, due to Barwise (1993) and developed by Restall (1996), takes worlds to be 'sites and channels.' A channel transmits information from site to site. In addition, channels can also be sites and sites can be channels. Where a, b and c are sites, they read $Rabc$ as saying that a is a channel between b and c, and thus $I(B \rightarrow C, a) = \mathit{true}$ as saying that all pairs of sites b, c connected by channel a are such that if B is information available in b, then C is information available in c. For example, for two sites connected by a telephone wire (the channel), what one person says in one site causes a person in the second site to hear certain sounds.

Mares (1996) presents another interpretation that adapts Israel and Perry's (1990) theory of information to the relational semantics. On this, the worlds in frames are situations, in the sense of Barwise and Perry's situation semantics (1983). [See chapter 20.] Situations contain information. A piece of information – an *infon*, to use Devlin's (1991) term – might be about the physical things in the situation, or it might be about connections between other infons. In particular, an infon might be about what information other infons carry. For example, an infon might carry the information that a red light showing on a stove carries the information that the oven is on. These infons that present information about connections between other infons, can be called *informational links*. The accessibility relation R represents the links in situations. If there is a link in a situation a that says that an infon σ carries the information that the infon π also holds, then if $Rabc$ and b contains the infon σ, then c contains the infon π. Links are not only included among the information in a situation, but also impose closure constraints on the set of infons in the situation. For example, if the 'law of nature'

All bodies attract one another

is a link in a and i and j are bodies in a, then i and j attract one another in a. Thus, on the link-interpretation, add the following postulate to the definition of a frame:

$Raaa$

which says that every situation is closed under informational links. And note that the link interpretation demands this closure. On the channel theoretic interpretation, on the other hand, this is an unnatural postulate, for not every channel carries information from itself to itself.

On to the other aspects of the semantics: As has been seen, the relational semantics divides worlds into normal and non-normal worlds. For some logics, the normal worlds (those at which we verify theorems) can be interpreted as possible worlds in the metaphysicians' sense. For these logics, the normal worlds can be taken to be complete (i.e. to satisfy the principle of excluded middle) and to be consistent. But not all relevant logics are characterized by a class of frames that have these properties. Only those logics which have excluded middle as a theorem and for which the rule gamma (γ) is admissible (see section 13.9) have a model theory of this sort.

The star operator has been controversial, but it can be given various reasonable interpretations. First, start with a linguistic interpretation from Meyer and Martin

(1986). The underlying idea here is that there is a distinction between what one actually *asserts* and what one fails to deny (thus what one *weakly asserts*). Here are a couple of sentences of recent interest:

P: In December 1999, NASA listened in vain for signals from the Mars polar lander.

Q: Martians interfered in 1999 with the transmission from their polar region.

All of us, probably, will agree with P. But what of Q? Only supermarket tabloids are likely actually to assert it, perhaps citing P as evidence. But one might, if one pleases, *weakly assert* Q, lacking evidence to support its denial. On this interpretation, a^* comprises the sentences weakly asserted at a.

Another interpretation, due to Dunn (1993), suggests one thinks of two worlds as containing compatible or incompatible information. Suppose that a says that a particular table is round and according to b that table is square. Then a and b can be said to be incompatible with one another. On the other hand, if there are no such conflicts, then the two worlds are compatible. In the language of our formal semantics Cab says that a and b are compatible. The truth condition for negation can be explicated in terms of compatibility alone, namely,

C$\sim$ $I(\sim A, a) = \mathit{true}$ iff for all b such that Cab, $I(A, b) = \mathit{false}$.

In other words, $\sim A$ is true at a if A's being true is somehow incompatible with the other information contained in a. Worlds can be incompatible with themselves; any inconsistent world is. The star operator can now be understood in terms of compatibility, for a^* can be taken to be the maximal world such that a is compatible with it. That is, for any world a, a^* is the world such that

(i) Caa^* and
(ii) for any world b, if Cab, then $b \leq a^*$.

Of course, this definition assumes that, for any world, there always is a maximal world compatible with it.[9]

13.7. Some Main Systems of Relevant Logic

This section presents some of the central systems of relevant logic. It does not present all the systems that people have proposed or on which important work has been done. Rather, it looks at only enough to give the reader the flavor of the systems and an idea of the range of relevant logics.

From the outset here, a relevant $\rightarrow$ has been considered *essentially relational*. Before going into the details of specific systems, one might pause to think about relations. The most famous ones are *binary* (2-place), e.g., *brother*, *sister*, *parent*, *child* – not to mention =, ∈, <. Even modal logics have a philosophically motivated (Kripke) binary relational semantics [see chapter 7]. Yet the ***key to the universe***, discussed below, involves a step up, at least to ternary (3-place) relations. (There are more than a few of these as well, e.g., ***between***, ***sum***, ***product***, ***jealous***.) Consider now relational ***composition***. Use 'B', 'P', 'U' respectively for ***brother***, ***parent***, and ***uncle***. Then x is the uncle of y iff $\exists z(Bxz \wedge Pzy)$. The notation for this will set

$$Uxy =_{df} P(Bx)y$$

which one can abbreviate, on the obvious convention, to

$$Uxy =_{df} PBxy$$

Thus, the result of composing two binary relations is another binary relation. Things become more interesting when composing general n-ary relations. For one thing, composing two 3-place relations yields a 4-place one (and so on, pushing n as high as one likes). For another thing, the order in which 3-place relations are composed definitely matters. For it is important to distinguish $\exists x(Rabx \wedge Rxcd)$ from $\exists x(Raxd \wedge Rbcx)$. This can be done by extending the conventions just introduced in the 2-place case and writing:

$$Rabcd =_{df} R(Rab)cd =_{df} \exists x(Rabx \wedge Rxcd)$$

$$Ra(bc)d =_{df} Ra(Rbc)d =_{df} \exists x(Raxd \wedge Rbcx)$$

(thus, employing again the device of associating the composed relations ***to the left***, inserting explicit parentheses otherwise. The iterated occurrences of R can also be dropped, which

(a) reduces visual clutter and
(b) clarifies connections between candidate logical axioms and matching combinators – ***the key to the universe***, discussed below.)

Now for some systems: Start with the logic **B**. This system (or at least its ***positive*** part **B+**) may be taken as a base system in much the same sense as the logic **K** is taken to be the base normal modal logic. The language used, and that has been assumed throughout this chapter, includes the unary connective ~ (for negation), the binary connectives ∧ (extensional conjunction) and → (implication or entailment). Extensional disjunction, ∨, is another primitive for **B+**; otherwise it is defined, along with ↔, as usual:

$$A \vee B =_{df} \sim(\sim A \wedge \sim B)$$

$$A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$$

The axiom schemes and rules for **B** are as follows:

1 $A \to A$

2 $(A \wedge B) \to A$

3 $(A \wedge B) \to B$

4 $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

5 $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C))$

6 $A \to (A \vee B)$

7 $B \to (A \vee B)$

8 $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

9 $\sim\sim A \to A$

Modus Ponens $\dfrac{\vdash A \to B \qquad \vdash A}{\vdash B}$

Adjunction $\dfrac{\vdash A \qquad \vdash B}{\vdash A \wedge B}$

Affixing $\dfrac{\vdash B \to B' \qquad \vdash A' \to A}{\vdash (A \to B) \to (A' \to B')}$

Contraposition $\dfrac{\vdash A \to \sim B}{\vdash B \to \sim A}$

To form the positive fragment **B+** of **B**, chop axiom 9 and the contraposition rule. To these systems, relevant logicians add various axiom schemes. For example, the logic **R** results from adding to **B** the schemes:

10 $(A \to B) \to ((B \to C) \to (A \to C))$ (*Suffixing*)

11 $(A \to (A \to B)) \to (A \to B)$ (*Contraction*)

12 $(A \to (B \to C)) \to (B \to (A \to C))$ (*Permutation*)

13 $(A \to \sim B) \to (B \to \sim A)$ (*Contraposition*)

14 $(A \to \sim A) \to \sim A$ (*Reductio*)

With the addition of these schemes, there is no need for the affixing and contraposition rules – they can be derived. (Adding these axiom schemes to **B+** yields the positive fragment **R+** of **R**, and similarly for the other logics mentioned below.)

E results from adding to **B** suffixing, contraction, contraposition, reductio and another axiom, e.g., 15 $(((A \to A) \wedge (B \to B)) \to C) \to C$

E was supposed to formalize the notion of entailment – 'the converse of deducibility.' Entailment was motivated as *both* relevant and necessary. **R** was supposed to be the 'demodalized,' but nonetheless *relevant* version of **E**. The thought was that **R** has approximately the same relationship to **E** that classical propositional logic has to **S4**. A natural hope was that by adding an **S4**-like necessity, **R** could be extended to a system **NR** that would prove equivalent to **E**, parsing the entailment of **E** as *strict* relevant implication (Routley and Meyer, 1982b [1972]). Although Kripke and others confirmed this for some fragments of **E**, the project unfortunately collapsed when Maksimova (1973) exhibited a *non-theorem* of **E** which is nonetheless *provable* on **NR** translation.

The system **T** of 'ticket entailment' results from adding to **B** suffixing, contraction, and prefixing, which is

16 $(B \to C) \to ((A \to B) \to (A \to C))$

as well as contraposition (axiom 13) and reductio (axiom 14). Here an arrow formula is taken as an inference ticket, '$A \to B$', saying that the inference from A to B is justified (Anderson and Belnap, 1975, section 6).

Using the abbreviations introduced above, table 13.1 presents some correspondences between propositional theses and semantic theses in the relational semantics. Also listed are the names of *associated combinators*, whose significance becomes clear in section 13.8.

Table 13.1

Combinator	*Thesis name*	*Thesis*	*Semantic postulate*
B	Prefixing	$(B \to C) \to ((A \to B) \to (A \to C))$	$Rabcd \Rightarrow Ra(bc)d$
B$'$(= ***CB***)	Suffixing	$(A \to B) \to ((B \to C) \to (A \to C))$	$Rabcd \Rightarrow Rb(ac)d$
W	Contraction	$(A \to (A \to B)) \to (A \to B)$	$Rabc \Rightarrow Rabbc$
C	Permutation	$(A \to (B \to C)) \to (B \to (A \to C))$	$Rabcd \Rightarrow Racbd$
C*(= ***CI***)	Assertion	$A \to ((A \to B) \to B)$	$Rabc \Rightarrow Rbac$
K	Weakening 1	$A \to (B \to A)$	$Raba$
K*(= ***KI***)	Weakening 2	$A \to (B \to B)$	$Rabb$
	Double negation	$\sim\sim A \to A$ and $A \to \sim\sim A$	$a^{**} = a$
	Contraposition	$(A \to \sim B) \to (B \to \sim A)$	$Rabc \Rightarrow Rac^*b^*$

13.8. Combinators: Connecting Proof Theory to Semantics

This section presents the mathematical motivations behind the various semantical postulates. Clearly, the immediate mathematical motivation in each case is that the postulate works – it does define a class of frames which characterizes the corresponding logical system. But, there is much more to it than this. There is an elegant relationship between the conditions on relational frames and the branches of mathematics called 'combinatory logic' (**CL**) and 'lambda calculus' (**LC**).[10]

Combinatory logic was devised by Curry in approximately 1930 as a very general way to represent and study operators and combinations of operators.[11] For example, from (Hindley and Seldin, 1986, p. 20), consider the arithmetic operation of addition. Addition is commutative, i.e., $x + y = y + x$. Let the addition function be represented by +. Then $+(x, y) = +(y, x)$. Adopting the usual conventions of **CL**, all functions may be treated as 1-place and parentheses and commas dropped while associating to the left: $+xy = +yx$. Now introduce an operator $\boldsymbol{C}$ such that, for any function f, $\boldsymbol{C}fxy = fyx$. Then it can be said of the addition operator that $+ = \boldsymbol{C}+$.

Combinatory logic studies operators, called ***combinators***, that, like $\boldsymbol{C}$, describe the behavior of functions. It begins with a small stock of combinators, and defines other combinators from them. For example, there is another combinator that describes how functions are composed. This is the combinator $\boldsymbol{B}$, and it obeys the equation:

$$\boldsymbol{B}(f, g)x = f(g(x))$$

where f and g are functions. $\boldsymbol{B}$ says that the result of applying the composition of two functions to an object is the same as the result of the application of the first function to the result of applying the second function to the object.

Instead of using special notation for functions, combinatory logic attempts to be perfectly general, not distinguishing notationally between functions and other entities. To this end, '$\boldsymbol{B}xyz > x(yz)$' will be written to represent the above reduction rule.[12]

In addition to the combinators themselves, one needs to understand the ***type schemes*** of combinators.[13] A type scheme will be interpreted as a schematic formula. In a 'Curry type,' the only pieces of logical notation are type variables p, etc., and formulas A, etc., built out of these variables by implication $\rightarrow$ (and parentheses). For example, the scheme $A \rightarrow B$ is the type of a combinator which takes entities of type A to entities of type B. An easy combinator to type is the identity combinator, $\boldsymbol{I}$, whose reduction rule is $\boldsymbol{I}x > x$. Clearly, it takes entities of any type A and returns an entity of the same type. So its type scheme is $A \rightarrow A$. The type of $\boldsymbol{K}$ is also easy to understand. Its reduction rule is $\boldsymbol{K}xy > x$. $(\boldsymbol{K}x)$ applied to any entity returns x. So, it is a function from a thing of type A to a function from any entity to a thing of type A. Consider $\boldsymbol{K}3$, where 3 is the third positive integer. This is a ***constant function***, which returns the value 3 for any argument of any type. So $\boldsymbol{K}$ itself is of the type $A \rightarrow (B \rightarrow A)$. Table 13.2 lists some combinators with their reduction rules and principle type schemes.

A logic can be thought of in terms of a set of combinators. The relevant logic **R**, for example, can be thought of as $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{I}$, $\boldsymbol{W}$ logic because it contains as axiom schemes the type schemes of these combinators. In addition, **R** contains as theorems all the type schemes for the combinations of these combinators. For instance, the type scheme of $\boldsymbol{CI}$ is $A \rightarrow ((A \rightarrow B) \rightarrow B)$, which is a theorem of **R**.

Table 13.2

Combinator (Name)	*Reduction rule*	*Type-scheme*
$\boldsymbol{I}$ (Identity)	$Ix > x$	$A \to A$
$\boldsymbol{B}$ (Composition)	$Bxyz > x(yz)$	$(B \to C) \to ((A \to B) \to (A \to C))$
$\boldsymbol{B}'$	$B'xyz > y(xz)$	$(A \to B) \to ((B \to C) \to (A \to C))$
$\boldsymbol{C}$	$Cxyz > xzy$	$(A \to (B \to C)) \to (B \to (A \to C))$
$\boldsymbol{W}$ (Diagonalization)	$Wxy > xyy$	$(A \to (A \to B)) \to (A \to B)$
$\boldsymbol{K}$ (Constant function)	$Kxy > x$	$A \to (B \to A)$
$\boldsymbol{S}$ (Strong composition)	$Sxyz > xz(yz)$	$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

To understand the relationship between combinators and the relational semantics, the notion of a *theory* is needed. The language used is a fragment of propositional language. For now, only formulas containing propositional variables, parentheses, extensional conjunction and relevant implication are considered. Then a theory of a logic **L** is defined to be a set of formulas X such that

(i) (***adjunction***) if A is in X and B is in X, then $A \wedge B$ is in X and
(ii) (***entailment***) if $A \to B$ is in **L** then, if A is in X, then B is also in X.

Then a model may be created out of the set of theories of **L**. The ternary accessibility of relation on this set will be defined later. Now, introduce a ***fusion operator*** on theories, $\circ$. (This is the same symbol as was used for intensional conjunction in the syntax; that will be explained later.) Fusion is defined on theories as:

If X and Y are theories of **L**, then $X \circ Y =_{df} \{B\colon \exists A(A \to B \in X \,\&\, A \in Y)\}$.

It can be shown that for any relevant logic that contains the implication–conjunction fragment of **B**, the fusion of two theories is also a theory.[14] The properties that fusion has in a structure of this sort depend on the choice of relevant logic for **L**, but here is one general property of fusion that will be needed later:

Fusion Fact If $X \subseteq Y$, then $X \circ W \subseteq Y \circ W$.

(The proof of this fact is easy and is left to the reader.)

One way of looking at the variations between the structure of theories of different logics is by investigating the combinators under which they are closed. To explain: take the combinator $\boldsymbol{C}$. For ease of expression omit the fusion operator and merely write 'xy' for the fusion of x and y, and associate to the left as above. Applied to theories, the combinator equation for $\boldsymbol{C}$ says that $Cxyz = xzy$. Closure of a structure of theories under $\boldsymbol{C}$ means that for any theories x, y and z of **L**, $xyz = xzy$.

Now for the fun part: Recall that the principle type scheme for $\boldsymbol{C}$ is the permutation scheme $(A \to (B \to C)) \to (B \to (A \to C))$. One can prove that the theory structure for **L** is closed under *C* iff all instances of that type scheme are theorems of **L**. This might seem to be an amazing coincidence, but it is not merely a coincidence. It is, as promised, ***the key to the universe***. The same correspondence holds for all the combinators listed above.

There's more! *Curry types* (near enough, pure → formulas) were assigned to combinators above. Types become more interesting if ∧ along with → are thrown in. This fact was independently discovered by workers in **LC** in the late 1970s, led by Coppo et al. (1980); but it is already reflected in the behavior of relevant theories under *fusion*. Here's the scoop.

Think again about formulas of the form $(A \to (B \to C)) \to (B \to (A \to C))$, which according to us (and Curry) are mates of the combinator *C*. This is not, however, a theorem scheme of the basic relevant logic **B**; but it does give rise to a ***theory*** – namely, the set of all formulas which are provably entailed in the basic relevant logic by conjunctions of permutation principles. It is this theory – call it *C* also – which so wonderfully interacts with the fusion operation ∘ on theories. Other cases are similar.

There are profound semantical and combinatorial facts underlying these correspondences. Check again the postulates in table 13.1 on the relevant accessibility relation R induced by various candidate axiom schemes. Note that, in the abbreviated relational notation, the postulates ***look like*** the matching reduction rules for corresponding combinators, as listed in table 13.2. Think yet again of *C*. Its combinatorial reduction rule sets $Cxyz = xzy$. In section 13.7, in table 13.1, the permutation axiom scheme was matched with the semantic postulate (often called 'Pasch') $Rxyzw \Rightarrow Rxzyw$. Note that the second and the third arguments are ***reversed***, just as they are in the **CL** equality governing *C*. Combinator fans will note similar linkages with the other suggested postulates and axioms, such as *B′*, *W*, *CI*.

Meyer and Routley (1972) were already in print with a ***key to the universe*** remark, induced by the shape of relevant semantics and the corresponding algebras. They knew even then of the formulas-as-(Curry) types connections between combinators and theorems of pure → intuitionist logic. But there were other candidate relevant axioms that appeared as though they should fit into the scheme; but which did not do so. Table 13.3 extends the two preceding tables (13.1 and 13.2) by incorporating columns from both.

Both of these candidate axiom schemes contain ∧ along with →. W^* is also known as *WI*, *SII*, or $\lambda x \cdot xx$. It has no Curry type. Yet the principles with which it has been mated – conjunctive ***modus ponens*** deductively and total reflexivity of the 3-place

Table 13.3

Combinator (name)	*Reduction rule*	*Semantic postulate*	*Type-scheme*
W^* (Duplication)	$W^*x > xx$	$Raaa$	$((A \to B) \wedge A) \to B$
WB	$WBxy > x(xy)$	$Rabc \Rightarrow Ra(ab)c$	$((B \to C) \wedge (A \to B)) \to (A \to C)$

relation R semantically – are natural (and famous). ***WB does*** have a Curry type – but it is the dull $(A \to A) \to (A \to A)$, not the more exciting conjunctive syllogism here.

These correspondences point to a deep relationship between theories, fusion and combinators. Return to our minimal positive logic **B+**; more particularly, to its implication–conjunction part **B**$\wedge$, pronounced 'Band,' which is determined by the $\to\wedge$ axioms 1–4 of section 13.7, with the *modus ponens*, adjunction and affixing rules governing these particles. For technical reasons (largely having to do with modeling the *bad* combinator ***K***), it is useful to extend **B**$\wedge$ to include also the *Church constant* T, subject to the axiom schemes

$$(T1) \qquad A \to T$$

$$(T2) \qquad T \to (T \to T)$$

The truth-condition on T is that it shall be marked *true* at every world, which is a dull (and mainly silly) thing to do.[15] The resulting system is called **B** $\wedge$ **T** (say 'Bat'). Then, applying Barendregt et al. (1983) it can be seen that there is a model of **LC** (and hence of **CL**) in the theories of **B** $\wedge$ **T**.[16] For the *filters* of the **LC** algebraists are nothing but the *theories* of the relevant logicians. And the non-empty theories of **B** $\wedge$ **T** have all the right properties, along the lines explored above, to make true the provable equalities of **CL**. Identify each combinator (***B***, ***C***, ***W***, ***K***, ***S***, ***W****, ***I***, etc.) with the set of all formulas of the corresponding scheme, closed in the appropriate way to make it a theory of **B** $\wedge$ **T**.

To be specific: Consider the combinator ***I***, whose type scheme is $A \to A$. Since a theory is closed under the adjunction condition (i) on page 295 everything of the form $(A \to A) \wedge (B \to B)$ will also belong to the theory ***I***. And since theories are closed under the condition of provable entailment (ii) on page 295 ***I*** will contain yet further members. Trivially, the top truth T is one such member; more interestingly, formulas entailed by members of ***I***, like any $(A \wedge B) \to A$, also belong to ***I***. In a nutshell, as readers may verify, ***I*** will consist *exactly* of the theorems of our minimal relevant logic. It follows, as night follows day, that, for every theory x, $\boldsymbol{I}x = x$.

Other combinators are similar – except that, since their corresponding schemes are only *sometimes* available, only those logics for which the schemes are valid will be closed under them. For illustrative purposes, consider the 'wicked' combinator ***K***. Its reduction rule is $\boldsymbol{K}xy = x$. While hopefully one has by now excluded its mate $A \to (B \to A)$ from one's own preferred logic, the displayed equality holds already for all non-empty theories x, y at the **B** $\wedge$ **T** level. For let $A \in x$. By definition of ***K***, $B \to A \in \boldsymbol{K}x$ for *all* B. Something B belongs to the non-empty theory y – at least T, if nothing more salutary shows up. (And *that* is why T was added to **B**$\wedge$.) Detaching, $A \in \boldsymbol{K}xy$, which establishes the inclusion from right to left. The converse inclusion is also demonstrable. Using the ideas of Dezani, Motohama and their colleagues – especially Proposition 9.6 of Dezani-Ciancaglini et al. (1998, p. 70) – all the other demonstrable equalities of the λβ-calculus (and hence of its definable **CL** subsystem) are likewise modeled in the theories of **B** $\wedge$ **T**.

That **CL** is the key to the (relevant semantical) universe means, so far, that

(a) there is a minimal relevant logic **B**$\wedge$ based on $\rightarrow$ and $\wedge$
(b) the non-empty theories of this logic (with T) constitute a model for **CL**
(c) the fusion of theories models functional application in **CL**
(d) combinators are the theories determined by their 'types'
(e) all combinator laws hold as equalities in the calculus of theories.

Much has been made by **CL** and **LC** theorists about the interpretations of *formulas-as-types*. Relevant logics turn this so-called *Curry–Howard isomorphism* on its head [see chapter 11]. We interpret *types-as-formulas*. And our formulas really *are* formulas – the formal sentences of some logical language. Aggregations of formulas are bound into theories by conjunction and entailment. And it turns out, as Fine (1992b [1974]) also emphasized, both that

(i) whole theories are the underlying ingredients of relevant semantical analysis, and
(ii) the shape of the semantics for any particular relevant logic will be determined by the combinators that correspond to its axioms.

Passing now to the analysis of further logical particles beyond $\rightarrow$ and $\wedge$, and stronger relevant logics than **B** and its kin, the disjunction $\vee$ poses immediate problems. One prefers (and *ought* to prefer) *prime* theories x, which satisfy, for all A, B, the primeness condition:

Primeness Condition If $A \vee B \in x$ then $A \in x$ or $B \in x$.

This is wanted because it corresponds to the truth-condition on $\vee$, on which the truth at x of a disjunction requires the truth at x of a disjunct. But prime theories are not always easy to come by. Worse, even when each of the theories x, y is prime, there is no guarantee that their fusion xy will be prime.

Nonetheless, there remains a strong relationship between combinators and the relational semantics. For the *canonical* ternary accessibility relation is definable on the structure of prime theories as

$$Rxyz \quad \text{iff} \quad xy \subseteq z$$

And it turns out that, using the combinator facts about the calculus of theories (for a given logic **L**), the necessary semantical postulates on the relation R almost suggest themselves. So, think briefly (but only by example) about what makes the relational semantics sound and complete. Suppose the familiar example, the **C** scheme $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$, is an axiom scheme of the logic **L**. Its mated relational (Pasch) postulate is $Rxyzw \Rightarrow Rxzyw$, whose correspondence to the combinator $\mathbf{C}$ has also been observed. To show that the postulate suffices to verify the axiom – hence the closure of the theory structure of **L** under $\mathbf{C}$ – one simply applies the SemEnt fact of section 13.5. (This is left for the reader to check.)

This illustrates the *soundness* of the ternary relational semantics for logics **L**. On the side of *completeness*, the converse for the same case will be shown. Assuming that all instances of the permutation scheme are theorems of **L**, – equivalently, that the structure of theories of **L** is closed under C – it will be shown that the Pasch postulate holds for the canonical ternary relation R defined above on the structure of prime theories of **L**.

In the first place, closure of *all* theories x of **L** under provable entailment still means that $Cx = x$, in the presence of permutation as a theorem scheme. Hence, $xyz = Cxyz = xzy$ for all theories x, y, z of **L**. (The presence of $\vee$, or even $\sim$, in the vocabulary makes no difference to this situation.) But it needs to be shown that permutation forces Pasch for the structure of prime theories of **L**. This, however, follows from the squeezing lemma which holds for any of our logics **L** (Anderson et al., 1992; Routley and Meyer, 1982a [1973]; Routley et al., 1982):

> **Squeezing Lemma** Let x, y be theories, and let z' be a prime theory of **L**. Suppose $xy \subseteq z'$. Then there exist prime theories x', y' of **L**, such that $x'y \subseteq z'$ and $xy' \subseteq z'$, where $x \subseteq x'$ and $y \subseteq y'$

Then, to verify Pasch, given that the structure of theories of **L** is closed under C, assume that there are prime theories x', y', a', z', w' such that

(i) $x'y' \subseteq a'$ and
(ii) $a'z' \subseteq w'$.

To prove that there is a prime theory u' such that $x'z' \subseteq u'$ and $u'y' \subseteq w'$:

1	$x'y' \subseteq a'$	Hypothesis (i)
2	$x'y'z' \subseteq a'z' \subseteq w'$	1, Fusion fact, Hypothesis (ii), Transitivity of $\subseteq$
3	$x'z'y' = x'y'z' \subseteq w'$	2, Closure under C
4	Set $u = x'z'$	Definition (but u may *not* be prime)
5	$uy' \subseteq w'$	3, 4

But then, by the Lemma, there exists a prime theory u' of **L**, $u \subseteq u'$, such that

6	$u'y' \subseteq w'$	5, Squeezing lemma
7	$x'z' = u \subseteq u'$	4

The conjunction of steps 7 and 6 verifies the conclusion of the Pasch postulate, on the antecedent hypotheses (i) and (ii). So, given that **L** provides permutation, the canonical ternary relation R on prime theories delivers Pasch, as promised. Other cases are similar.

Thus there is a mathematically elegant relationship between combinatory logic and relational semantics. This, however, is a point about the relational semantics in general, rather than a feature of relevant logics. For one can give a ternary relational semantics for non-relevant logics (Routley and Meyer, 1976). For example, if a structure of theories is closed under ***K***, this is not very relevant!

13.9. Relevant Results

To understand a logic, one needs to understand a few of its mathematical properties. This section presents, in a non-technical way, some technical results about relevant logics.

13.9.1. The admissibility of gamma

When Anderson and Belnap (1975) reformulated Ackermann's logic Π' (1956) as their system **E**, they omitted Ackermann's third rule of inference, named γ:

$$\frac{\begin{array}{l}\vdash {\sim}A \vee B\\ \vdash A\end{array}}{\vdash B}$$

Anderson and Belnap argue that a logic should not include a rule unless it includes the corresponding theorem scheme. In this case, the corresponding theorem scheme is

$$(({\sim}A \vee B) \circ A) \rightarrow B \qquad \text{or} \qquad (({\sim}A \vee B) \wedge A) \rightarrow B$$

Adding the latter to **E**, by virtue of the Lewis argument given in section 13.1 above, makes XP– valid in **E**. And so adding it would remove **E** from the class of relevant logics. Luckily, as Meyer has shown with Dunn (1975 [1969]) by algebraic means and then on his own by a technique called 'metavaluations' (Meyer, 1975 [1976]), the theorems of **E** are closed under γ. Thus, Anderson and Belnap's **E** has the same theorems as Ackermann's logic.

The importance of the admissibility of γ goes far beyond proving the coincidence of **E** and Π'. It shows that a logic is characterized by its class of 'normal' theories. A ***normal theory*** is a theory that contains all the theorems of the logic, is consistent and is prime.

Sometimes admitting γ also shows that a relevant logic contains the corresponding logic based on the classical propositional calculus. Using γ, one can show that **R** and **E** contain all of classical propositional logic (phrased in terms of negation, conjunction and disjunction) and that various modal relevant logics contain all the theorems of the corresponding classically based logics (Mares and Meyer, 1992). Unfortunately, Meyer's relevant Peano arithmetic (the system **R#**) was shown by Friedman and Meyer (1992) not to admit γ. In the process, it was also shown not to contain all of the theorems of classical Peano arithmetic.

13.9.2. *The undecidability of* **R** *and* **E**

Urquhart (1992b [1984]) proved that the logics **E**, **R** and **T** are undecidable. This result is important and the proof is very clever. Most philosophically motivated propositional logics are decidable – for instance, classical propositional logic, intuitionist logic and the standard normal modal logics. In fact **E**, **R** and **T** are the first philosophically motivated propositional logics to have been proven undecidable.

The proof of undecidability is an extraordinary piece of work. Urquhart shows that there is an interesting and important link between the relational semantics for these relevant logics and projective spaces (of projective geometry). He then uses the fact that the word problem for a particular class of infinite-dimensional projective spaces is unsolvable to prove that the logics are undecidable.

13.9.3. *The failure of interpolation in* **R** *and* **E** *(and a host of other systems)*

Another difficult and interesting proof due to Urquhart is his theorem that interpolation fails in **E** and **R** as well as in **T** and a range of other logics.

What is interesting about interpolation from a relevant point of view is that some relevant logics satisfy what Anderson and Belnap call the 'Perfect Interpolation Theorem.' Consider Craig's interpolation theorem as stated for classical propositional logic [see chapter 1, page 31]:

Suppose that C is derivable from A. Then,

(Cop-out) if A is not a contradiction and C is not a tautology

there is some formula B such that

(a) B contains only propositional variables that occur in both A and C;
(b) B is derivable from A; and
(c) C is derivable from B.

The Perfect Interpolation Theorem is the same as Craig's theorem with the omission of the qualification Cop-out.

Some relevant logics do satisfy the Perfect Interpolation Theorem. For example, McRobbie (1979) showed that the system **OR** (which is **R** without the distribution axiom, Axiom 8, given on page 292) is perfectly interpolable. But Urquhart (1993) again used the relationship between projective geometry and the relational semantics to show that a range of relevant logics around and including **E**, **R** and **T** do not interpolate.[17]

13.9.4. *Boolean conservative extension results*

Meyer and Routley (1982 [1973]) show that one can add a second, Boolean, negation to certain relevant logics without altering the stock of theorems that the logic has in the old vocabulary. This is what is called a *conservative extension result.* Boolean negation, $\neg$, is governed by some very unrelevant looking principles, such as

$$(\neg A \wedge A) \to B$$

$$B \to (\neg A \vee A)$$

This extension is interesting for a variety of reasons, mathematical and philosophical. First, it allows one to use what is sometimes called 'denial negation' (a negation that expresses the failure of something to be true) in relevant logic. Second, the conservative extension result has enabled Belnap (1992b [1982]) to prove the correctness of his elegant proof theory – Display Logic – for a range of relevant logics.

The conservative extension result holds for a wide range of logics, including **B** and **R**. But it does not hold for **E** (Mares, 2000) nor for **NR**, mentioned above (Meyer and Mares, 1993). (Recently Ross Brady has proved that quantified **R** is conservatively extended by the addition of Boolean negation. As of the writing of this chapter, he has not yet published this result.)

13.9.5. *The consistency of relevant set theories*

Brady (1983) showed that a class theory with a naïve comprehension axiom, based on a weak relevant logic, is consistent. The comprehension axiom is

$$\exists Y \forall X (X \in Y \leftrightarrow A)$$

where X and Y are variables ranging over classes. This says that for each open sentence A, there is a set that is its extension. This axiom was restricted in classical set theory because it enabled the derivation of Russell's paradox [see chapter 3]. But, using his logic, $\mathbf{DJ^dQ}$, as a base, Brady (1989) shows that a theory of classes – and in Brady (2001), a theory of sets – that incorporates naïve comprehension is consistent.

13.9.6. *The completeness and incompleteness of quantified relevant logic*

Fine (1992a [1989]) shows that the quantified relevant logic, **RQ**, is not complete over its *constant domain semantics.* According to the constant domain semantics, each world has the same stock of individuals and the truth condition for the universal quantifier is the standard clause from modal logic [see chapter 7], namely,

$I_v(\forall x A, a) = \mathit{true}$ iff $I_u(A, a) = \mathit{true}$, for every u, x-variant of v

Fine shows that there is a thesis valid over the class of constant domain models for **R** that is not provable in the logic **RQ**. How to axiomatize a logic complete over the constant domain semantics is still an open question.

Fine (1992c [1988]) developed a variable domain semantics for **RQ**. The semantics is quite complicated and appeals to a special notion of arbitrary objects; but one can make contact with more familiar terrain by linking Fine's ideas to central ones from Kripke's model theory for intuitionistic logic [see chapter 11]. Altering Fine's notation slightly, each world is linked to other worlds with larger domains of individuals by a relation, Q. Thus, Qab only if $D(a) \subseteq D(b)$. Fine's truth condition for the quantifier may be put as

$$I_v(\forall xA,\ a) = \textit{true} \text{ iff } I_u(A,\ b) = \textit{true}, \text{ for all } b \text{ that } Qab \text{ and every } u,\ x\text{-variant of } v$$

In effect, this says that a universally quantified sentence, $\forall xA$, is true at a world, a, iff the open sentence A is true of everything in every world b larger than a.

13.9.7. *The* **P–W** *problem*

The logic **P–W**, as it is usually called (or $\mathbf{T}_{\rightarrow}$–**W** as it should be called) has implication as its sole connective. It contains as axiom schemes the type schemes for the combinators ***B***, ***B′*** and ***I*** and is closed under ***modus ponens***. Belnap had conjectured that, for formulas A and B, the only cases in which both $A \rightarrow B$ and $B \rightarrow A$ are provable in **P–W** are those in which A and B are the same formula. In the late 1960s, Larry Powers showed that this conjecture is equivalent to the conjecture that no formula of the form $A \rightarrow A$ can be proved in the logic **S**, which is the closure under ***modus ponens*** of the schemes ***B*** and ***B′*** ***alone***. Martin (1992 [1978]), and Martin and Meyer (1982) proved Powers' **S** conjecture. So Martin solved the **P–W** problem (which had been shaping up as the *Fermat's Last Theorem* of the area).

Martin's solution of the **P–W** problem is a wonderful example of technical ingenuity and philosophical insight going hand in hand in the advancement of relevant logic. For one now has a logic that does not make valid any form of circular reasoning. It has been thought since the beginning of logic that deriving a proposition from itself is not only useless but also fallacious. In **S** one has a logic that rejects (root and branch) all forms of circular reasoning. Thus it serves as a test-bed for ideas about circularity and how to avoid it.[18]

13.10. But There's So Much More to Say

This chapter has provided the reader with a brief look at the motivation for and some of the technical and philosophical aspects of relevant logic. But it has only scratched the surface of this vibrant field of logic that has been the focus of fairly intense mathematical scrutiny and philosophical debate in the past four decades. It has not touched on the use of relevant logic in automated theorem proving, (Thistlewaite et al., 1988), or its relationship to linear logic or to computing more generally. Nor has it discussed the sometimes heated debate over the status of disjunctive syllogism (Read, 1988). And there is so much more. There are areas in relevant logic that are just now beginning to be explored: The relationship between these logics and logics of natural language conditionals, the use of relevant logic in non-monotonic reasoning, among others. Hopefully, this chapter will inspire readers to delve into these areas on their own, sadly without our guidance.

Suggested further reading

The best detailed introduction to relevant logic is Dunn (1984), which has recently been updated (Dunn and Restall, 2001). For the philosophical debates surrounding relevant logic, Routley, et al., (1982) and Read (1988) are good places to start. A fine and very readable introduction to substructural logics, with much about relevant logics, is Restall (2000). Mares (2002) introduces relevant logic through natural deduction. For the reader who wants technical details and proofs of theorems, Anderson and Belnap (1975) and Anderson et al. (1992) are excellent sources. Anderson et al. (1992) contains a detailed, although now out of date, bibliography of work on relevant logic, compiled by Robert G. Wolf.[19]

Notes

1 For the pre-history and history of relevant logics, a good source is Read (1988).
2 Relevant logic locates the fallacy at line 5, denying the entailment $(\sim A \wedge (A \vee B)) \rightarrow B$. See Anderson and Belnap (1975, section 16.1) or Read (1988) for discussion of this issue.
3 This holds for propositional relevant logics without so-called Ackermann constants t and f or Church constants T and F (see section 13.8).
4 We are very grateful to Nicholas Griffin for pointing out this passage to us.
5 For a clear presentation of the relevant deduction theorem, see Dunn (1984).
6 There are several good introductions to natural deduction for relevant logics, such as Anderson and Belnap (1975), Dunn (1984), and Mares (2002).
7 As for other properties of fusion, these vary among relevant logics. Not all relevant logics allow fusions to commute. That is, there are relevant logics in which $A \circ B$ is not equivalent to $B \circ A$, but there are others, like **R**, where $\vdash A \circ B \rightarrow B \circ A$. Fusion is not idempotent in *most* relevant logic, i.e., in systems like **R** and **E**, $A \circ A$ is not equivalent to A. And so on.
8 This is what Meyer and Martin (1986) call the *Australian plan* of relevant semantical analysis. There is a contrasting *American Plan*, developed by Dunn (1992 [1976], 1969) and championed and augmented by Belnap (1977, 1992a [1977]), that utilizes a natural four-valued semantics to refute XP–. For simplicity's sake, in what follows we discuss only the two-value semantics with the Routley star. Interested readers should consult Routley et al. (1982) for the technical details of the completed American plan.
9 There is another interpretation of negation that uses the compatibility relation, but for it this relation is not a semantical primitive. This is the implicational interpretation of negation of Mares (1995). On this semantics, there is a falsum, f, that is taken to be true at all and only impossible worlds (impossible worlds too can be defined in terms of other primitives). On this semantics, one can define Cab to hold iff there is some world c, such that $Rabc$, and c is not impossible. In other words, two worlds are taken to be compatible if they can be combined (in the sense of fusion) in a possible world.

10 We do not try to give a full account of combinatory logic here. We just want to give the reader the flavor of the theory. For a more detailed introduction, see Hindley and Seldin (1986), *q.v.* for further bibliographical references to **CL** and **LC**.

11 Curry was anticipated in 1924 by Schönfinkel. Church invented the kindred **LC** *circa* 1932.

12 Strictly speaking we should distinguish the general reduction relation >, which is reflexive, transitive and satisfies positive monotonic replacement, from $>_1$ of immediate (one-step) reducibility. Equality, '=,' is the symmetric transitive closure of >.

13 For an introduction to types, with references and especially in **LC**, see Takahashi et al. (1998).

14 For technical reasons, we will add a (top) Church constant T below; it is a member of all *non-empty* theories. Then the fusion of two non-empty theories in $\rightarrow$, $\wedge$, T (at least) is also a non-empty theory.

15 But it does correspond to the constant ω which is the *whole domain* of the models of **LC** discussed in Barendregt et al. (1983) and Dezani et al. (1998). While we prefer the more natural *Ackermann constant t* in relevant logics, T continues to make some sense as the trivial truth implied by absolutely everything. Ackermann t, when present, admits the 2-sided rule $\vdash A$ iff $\vdash t \rightarrow A$. Think of t as a *conjunction* of truths (interesting) but T as a corresponding *disjunction* (boring, but that many logics *confuse t* and T!).

16 In essence, the **LC** investigators had rediscovered the basic positive relevant logic of, e.g., Meyer and Routley (1972). Alas, they thought that they had invented a *type theory*, and were not aware that they had stumbled on a *relevant logic*. When Meyer first met Barendregt in 1990, he pointed this out, Barendregt conceded that, while he had not previously thought much of relevant logics, perhaps it was time to change his tune; for now it was clear that he had been involved (with members of the Torino group) in the (re)invention of one.

17 This means, in McRobbie's lingo, that **E** is *not* reasonable in the sense of Anderson and Belnap.

18 More accurately we have such a logic in $\mathbf{S}_{\rightarrow}$ (and, thanks to more joint work by Fine and Martin (2001), in $\mathbf{S}_{\rightarrow\wedge}$ as well). Further particles (and, it may be, further axioms) await. We add that it is a pity that the elegantly combinatorial Martin insights are as yet insufficiently appreciated.

19 Thanks are due to Neil Leslie, who commented extensively on an earlier draft, as did Lou Goble, Katalin Bimbó, Chris Mortensen and Beate Elsner.

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