Relevant Logics:

Proof Theory and Model Theory

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Relevant Logics: Proof Theory and Model Theory

- Formal Language of Relevant Logics
- Hilbert Style Axiomatic Systems of Relevant Logics
- Various Properties of Relevant Logics
- Model Theory for Relevant Logics
- Natural Deduction Systems of Relevant Logics
- Sequent Calculus Systems of Relevant Logics
- Semantic Tableau Systems of Relevant Logics
- Bibliography

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Formal Language of Propositional Relevant Logics

- **Alphabet (Symbols)**

  $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, T, F, (,), p_1, p_2, ..., p_n, ... \}$

- **Propositional constants**

  $T,\ F$

- **Propositional variables (proposition symbols)**

  $p_1, p_2, ..., p_n, ...,$

Formal Language of Predicate Relevant Logics

- **Alphabet (Symbols)**

  $\{ \neg, \rightarrow, \wedge, \vee, \leftrightarrow, \Rightarrow, \otimes, \oplus, \Leftrightarrow, L, \forall, \exists, T, F, (,),$
  $x_1, x_2, ..., x_n, ..., c_1, c_2, ..., c_n, ...,$
  $f_1^1, ..., f_1^n, ..., f_2^1, ..., f_2^n, ..., f_k^1, ..., f_k^n, ...,$
  $p_0^1, ..., p_0^n, ..., p_1^1, ..., p_1^n, ..., p_2^1, ..., p_2^n, ..., p_k^1, ..., p_k^n, ... \}$

- **Propositional constants**

  $T,\ F$

- **Individual variables (variable symbols)**

  $x_1, x_2, ..., x_n, ...,$

- **(Individual) Constants (Names) (constant symbols, name symbols)**

  $c_1, c_2, ..., c_n, ...,$

- **(Individual) Functions (function symbols)**

  $f_1^1, ..., f_1^n, ..., f_2^1, ..., f_2^n, ..., f_k^1, ..., f_k^n, ...,$

- **(Individual) Predicates (Relations) (predicate symbols, relation symbols)**

  $p_0^1, ..., p_0^n, ..., p_1^1, ..., p_1^n, ..., p_2^1, ..., p_2^n, ..., p_k^1, ..., p_k^n, ...$

Formal Language of Relevant Logics

- **Primitive logical connectives**

  $\Rightarrow$: entailment (**primitive!** And therefore, **intensional!**)
  $\neg$: negation
  $\wedge$: extensional conjunction

- **Defined logical connectives**

  $\otimes$: intensional conjunction (fusion), $A \otimes B =_{df} \neg(A \Rightarrow \neg B)$
  $\oplus$: intensional disjunction, $A \oplus B =_{df} \neg A \Rightarrow B$
  $\Leftrightarrow$: intensional equivalence, $A \Leftrightarrow B =_{df} (A \Rightarrow B) \otimes (B \Rightarrow A)$
  $\vee$: extensional disjunction, $A \vee B =_{df} \neg(\neg A \wedge \neg B)$
  $\rightarrow$: material implication, $A \rightarrow B =_{df} \neg(A \wedge \neg B)$ or $\neg A \vee B$
  $\leftrightarrow$: extensional equivalence, $A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$
  L : necessity operator, $LA =_{df} (A \Rightarrow A) \Rightarrow A$

- **Terms and formulas**

  - Similar to that of classical propositional/predicate calculus.

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Hilbert Style Axiomatic Systems of Relevant Logics

- **Axiom schemata on entailment**

E1 $A \Rightarrow A$ (Self-Implication)
E2 $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$ (Prefixing)
E2′ $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ (Suffixing)
E3 $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (Contraction)
E3′ $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ (Self-Distribution)
E3″ $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$ (Permuted Self-Distribution)
E4 $(A \Rightarrow ((B \Rightarrow C) \Rightarrow D)) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow D))$ (Restricted Permutation)
E4′ $(A \Rightarrow B) \Rightarrow (((A \Rightarrow B) \Rightarrow C) \Rightarrow C)$ (Restricted Assertion)
E4″ $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$ (Specialized Assertion)
E4‴ $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (((A \Rightarrow C) \Rightarrow D) \Rightarrow D))$
E5 $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (Permutation)
E5′ $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ (Assertion)
E5″ $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$ (Demodalizer)

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Axiom schemata on conjunction**

C1 $(A\wedge B)\Rightarrow A$ (Conjunction Elimination)
C2 $(A\wedge B)\Rightarrow B$ (Conjunction Elimination)
C3 $((A\Rightarrow B)\wedge(A\Rightarrow C))\Rightarrow(A\Rightarrow(B\wedge C))$ (Conjunction Introduction)
C4 $(LA\wedge LB)\Rightarrow L(A\wedge B)$, where $LA =_{df} (A\Rightarrow A)\Rightarrow A$ (Distribution of Necessity over Conjunction)

♣ **Axiom schemata on disjunction**

D1 $A\Rightarrow(A\vee B)$ (Disjunction Introduction)
D2 $B\Rightarrow(A\vee B)$ (Disjunction Introduction)
D3 $((A\Rightarrow C)\wedge(B\Rightarrow C))\Rightarrow((A\vee B)\Rightarrow C)$ (Disjunction Elimination)

♣ **Distribution axiom schema**

DCD $(A\wedge(B\vee C))\Rightarrow((A\wedge B)\vee C)$ (Distribution of Conjunction over Disjunction)

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Axiom schemata on negation**

N1 $(A\Rightarrow(\neg A))\Rightarrow(\neg A)$ (Reduction)
N2 $(A\Rightarrow(\neg B))\Rightarrow(B\Rightarrow(\neg A))$ (Contraposition)
N3 $(\neg(\neg A))\Rightarrow A$ (Double Negation)

♣ **Mingle axiom schemata**

EM0 $(A\Rightarrow B)\Rightarrow((A\Rightarrow B)\Rightarrow(A\Rightarrow B))$
RM0 $A\Rightarrow(A\Rightarrow A)$

♣ **Axiom schemata on necessity**

*L*1 $LA\Rightarrow A$
*L*2 $L(A\Rightarrow B)\Rightarrow(LA\Rightarrow LB)$
*L*3 $(LA\wedge LB)\Rightarrow L(A\wedge B)$
*L*4 $LA\Rightarrow LLA$
*L*5 $LA\Rightarrow((LA\Rightarrow LA)\Rightarrow LA)$

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Axiom schemata on individual quantification**

IQ1 $\forall x(A\Rightarrow B)\Rightarrow(\forall xA\Rightarrow\forall xB)$
IQ2 $(\forall xA\wedge\forall xB)\Rightarrow\forall x(A\wedge B)$
IQ3 $\forall xAx\Rightarrow Ay$
IQ4 $\forall x(A\Rightarrow B)\Rightarrow(A\Rightarrow\forall xB)$ (x not free in A)
IQ5 $\forall x(A\vee B)\Rightarrow(A\vee\forall xB)$ (x not free in A)
IQ6 $\forall x_1 \dots \forall x_n (((A\Rightarrow A)\Rightarrow B)\Rightarrow B)$ $(n\geq 0)$ (for E and EM)
IQ7 $Ay\Rightarrow\exists xAx$
IQ8 $\forall x(A\Rightarrow B)\Rightarrow(\exists xA\Rightarrow B)$ (x not free in B)
IQ9 $(\exists xA\wedge B)\Rightarrow\exists x(A\wedge B)$ (x not free in B)

Axiom clause: if A is an axiom, so is $\forall xA$.

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Inference rules**

$\Rightarrow$E : From A and $A\Rightarrow B$ to infer B (Modus Ponens)

$\wedge$I : From A and B to infer $A\wedge B$ (Adjunction)

*L*I : If A is a theorem, so is LA

$\otimes$I : If $A\Rightarrow(B\Rightarrow C)$ is a theorem, so is $(A\otimes B)\Rightarrow C$

$\otimes$E : If $(A\otimes B)\Rightarrow C$ is a theorem, so is $A\Rightarrow(B\Rightarrow C)$

*T*I : If A is a theorem, so is $T\Rightarrow A$

*T*E : If $T\Rightarrow A$ is a theorem, so is A

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **The pure entailment (relevant implication) fragments of relevant logics**

$T_{\Rightarrow}$ = {E1, E2, E2′, E3 | E3″} + $\Rightarrow$E

$E_{\Rightarrow}$ = {E1, E2 | E2′, E3 | E3′, E4 | E4′} + $\Rightarrow$E
$E_{\Rightarrow}$ = {E2′, E3, E4″} + $\Rightarrow$E
$E_{\Rightarrow}$ = {E1, E3, E4‴} + $\Rightarrow$E
$E_{\Rightarrow} = T_{\Rightarrow}$ + E4 [E4: $(A\Rightarrow((B\Rightarrow C)\Rightarrow D))\Rightarrow((B\Rightarrow C)\Rightarrow(A\Rightarrow D))$]
$E_{\Rightarrow} = T_{\Rightarrow}$ + E4′ [E4′: $(A\Rightarrow B)\Rightarrow(((A\Rightarrow B)\Rightarrow C)\Rightarrow C)$]
$E_{\Rightarrow} = T_{\Rightarrow}$ + E4″ [E4″: $((A\Rightarrow A)\Rightarrow B)\Rightarrow B$]

$R_{\Rightarrow}$ = {E1, E2 | E2′, E3 | E3′, E5 | E5′} + $\Rightarrow$E
$R_{\Rightarrow} = E_{\Rightarrow}$ + E5″ [E5″: $A\Rightarrow((A\Rightarrow A)\Rightarrow A) = A\Rightarrow LA$]

♣ **Note**

"A | B" means that one can choose any one of the two axiom schemata A and B.

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **The entailment (relevant implication) with negation fragments of relevant logics**

$T_{\Rightarrow,\neg} = T_{\Rightarrow}$ + {N1, N2, N3}

$E_{\Rightarrow,\neg} = E_{\Rightarrow}$ + {N1, N2, N3}

$R_{\Rightarrow,\neg} = R_{\Rightarrow}$ + {N2, N3}

♣ **The positive (negation-free) fragments of relevant logics**

$T_{+} = T_{\Rightarrow}$ + {C1~C3, D1~D3, DCD} + $\wedge$I

$E_{+} = E_{\Rightarrow}$ + {C1~C4, D1~D3, DCD} + $\wedge$I
$E_{+} = T_{+}$ + {E4 | E4′ | E4″, C4}

$R_{+} = R_{\Rightarrow}$ + {C1~C3, D1~D3, DCD} + $\wedge$I
$R_{+} = E_{+}$ + E5″

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Propositional relevant logics**

T = $T_{\Rightarrow,\neg}$ + {C1~C3, D1~D3, DCD} + ∧I

E = $E_{\Rightarrow,\neg}$ + {C1~C4, D1~D3, DCD} + ∧I
E = T + { E4 | E4′ | E4″, C4}

R = $R_{\Rightarrow,\neg}$ + {C1~C3, D1~D3, DCD} + ∧I
R = E + $A \Rightarrow LA$,   LR = R – DCD

EM = E + EM0 (semi-relevant logic)

RM = R + RM0 (semi-relevant logic)
RM = EM + $A \Rightarrow LA$ (semi-relevant logic)

E^L = E + {$L1$~$L5$} + LI

R^L = R + {$L1$~$L4$} + LI

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Predicate relevant logics**

$S^{\forall\exists x}$ = S + {IQ1, IQ3, IQ4, IQ7, IQ8}
where S = $T_{\neg}$, $T_{\Rightarrow,\neg}$, $R_{\neg}$, $R_{\Rightarrow,\neg}$

$S^{\forall\exists x}$ = S + {IQ1~IQ5, IQ7~IQ9}
where S = T_{+} , T , R_{+} , R , RM

$S^{\forall\exists x}$ = S + {IQ1, IQ3, IQ4, IQ6~IQ8}
where S = $E_{\neg}$, $E_{\Rightarrow,\neg}$

$S^{\forall\exists x}$ = S + {IQ1~IQ9}
where S = E_{+} , E , EM

Hilbert Style Axiomatic Systems of Relevant Logics

♣ **Axiom schemata on conjunction**

C5 $(A \wedge A) \Rightarrow A)$
C6 $(A \wedge B) \Rightarrow (B \wedge A)$
C7 $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
C8 $(A \wedge (A \Rightarrow B)) \Rightarrow B)$
C9: ¬(A∧¬A)
C10: $A \Rightarrow (B \Rightarrow (A \wedge B))$

♣ **Strong relevant logics**

Tc $=_{df}$ $T_{\Rightarrow,\neg}$ + {C3, C5~C10}   TcQ $=_{df}$ Tc + {IQ1~IQ5} + ∀I

Ec $=_{df}$ $E_{\Rightarrow,\neg}$ + {C3~C10}   EcQ $=_{df}$ Ec + {IQ1~IQ5} + ∀I

Rc $=_{df}$ $R_{\Rightarrow,\neg}$ + {C3, C5~C10}   RcQ $=_{df}$ Rc + {IQ1~IQ5} + ∀I

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The Notion of Degree

♣ **The degree of a formula with implicational connective**

- The *degree* of a formula with implicational connective is the largest number of nesting of implicational connective, e.g. ⇒, that represents the notion of conditional within it.

♣ **Zero degree formula**

- A formula is called a *zero degree formula* (zdf) if and only if there is no occurrence of ⇒ in it.

♣ **First degree conditional (entailment)**

- A formula of the form $A \Rightarrow B$ is called a *first degree conditional* (fdc) (*entailment*, fde) if and only if both A and B are zero degree formulas.

The Notion of Degree

♣ **First degree formula**

- A formula A is called a *first degree formula* (fdf) if and only if it satisfies the one of the following conditions:
(1) A is a first degree conditional,
(2) A is in the form + B (+ is a one-place connective such as negation and so on) where B is a first degree formula, and
(3) A is in the form $B * C$ (* is a non-implicational two-place connective such as conjunction or disjunction and so on) where both of B and C is first degree formulas, or one of B and C is a first degree formula and the another is a zero degree formula.

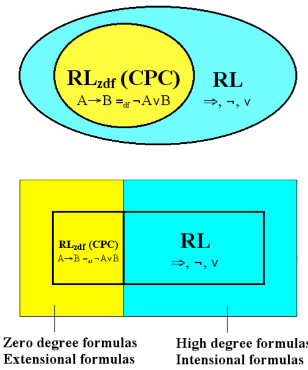
The Notion of Degree

- k^{th} **degree conditional (entailment)**
  - Let k be a natural number. A formula of the form $A \Rightarrow B$ is called a *k^{th} degree conditional* (kdc) (*entailment*, kde) if and only if both A and B are $(k-1)^{th}$ degree formulas, or one of A and B is a $(k-1)^{th}$ degree formula and another is a j^{th} $(j<k-1)$ degree formula.
- k^{th} **degree formula**
  - Let k be a natural number. A formula A is called a *k^{th} degree formula* (kdf) if and only if it satisfies the one of the following conditions:
    (1) A is a k^{th} degree conditional,
    (2) A is in the form $+B$ where B is a k^{th} degree formula, and
    (3) A is in the form $B*C$ where both of B and C is k^{th} degree formulas, or one of B and C is a k^{th} degree formula and another is a j^{th} $(j<k)$ degree formula.

Zero Degree Fragments of Relevant Logics: CPC

- **Classical Propositional Calculus (CPC) is contained in E (R, T)** [A&B-E1-75]
  - Theorem: All tautologies (theorems) of CPC are provable in E (R, T), i.e., E (R, T) is complete with respect to CPC.
  - Theorem: Only tautologies (theorems) of CPC among the zero degree formulas of E (R, T) are provable in E (R, T), i.e., E (R, T) is conservative with respect to CPC.
- **Classical Propositional Calculus (CPC) is the zero degree fragment of E (R, T)** [A&B-E1-75]
  - CPC is in exactly the right sense contained in E (R, T), i.e., it is exactly the zero degree (extensional) fragment of E (R, T).

Zero Degree Fragments of Relevant Logics: CPC



Primitive Entailments

- **Atom**
  - An *atom* is a propositional variable or the negate of such, i.e., an atom has either the form p or the form $\neg p$.
- **Primitive conjunction and disjunction**
  - A *primitive conjunction* is a conjunction $A_1 \wedge A_2 \wedge \ldots \wedge A_m$ $(m \geq 1)$ where each A_i is an atom.
  - A *primitive disjunction* is a disjunction $B_1 \vee B_2 \vee \ldots \vee B_n$ $(n \geq 1)$ where each B_i is an atom.
- **Primitive entailment**
  - $A \Rightarrow B$ is a *primitive entailment* if A is a primitive conjunction and B is a primitive disjunction, i.e., $A \Rightarrow B = A_1 \wedge A_2 \wedge \ldots \wedge A_m \Rightarrow B_1 \vee B_2 \vee \ldots \vee B_n$.

Explicitly Tautological Entailments

- **Explicitly tautological entailments**
  - A primitive entailment $A \Rightarrow B$ is said to be *explicitly tautological* if some (conjoined) atom of A is identical with some (disjoined) atom of B, i.e., $A_i = B_j$, for some i and j.
  - Explicitly tautological is both necessary and sufficient for the (weak-relevant!) validity of a primitive entailment.
- **Note**
  - Explicitly tautological entailments satisfy the von Wright-Geach-Smiley criterion for entailment: every explicitly tautological entailment answers to a material "implication" which is a substitution instance of a tautologous material "implication" with non-contradictory antecedent and non-tautologous consequent; and evidently we may ascertain the truth of the entailment without coming to know the truth of the consequent or the falsity of the antecedent.

Explicitly Tautological Entailments in Normal Form

- **Entailments in normal form**
  - An entailment $A \Rightarrow B$ is said to be *in normal form* if it has the form $A_1 \vee A_2 \vee \ldots \vee A_m \Rightarrow B_1 \wedge B_2 \wedge \ldots \wedge B_n$ $(m \geq 1, n \geq 1)$ where each A_i is a primitive conjunction and each B_j is a primitive disjunction.
  - An entailment $A \Rightarrow B$ in normal form is (weak-relevantly!) valid just in case each $A_i \Rightarrow B_j$ $(m \geq i \geq 1, n \geq j \geq 1)$ is explicitly tautological.
- **Explicitly tautological entailments in normal form**
  - An entailment $A_1 \vee A_2 \vee \ldots \vee A_m \Rightarrow B_1 \wedge B_2 \wedge \ldots \wedge B_n$ $(m \geq 1, n \geq 1)$ in normal form is said to be *explicitly tautological* if and only if for every A_i and B_j, $A_i \Rightarrow B_j$ $(m \geq i \geq 1, n \geq j \geq 1)$ is explicitly tautological.

Tautological Entailments

- **Tautological entailments**
  - An entailment $A_1 \lor A_2 \lor \ldots \lor A_m \Rightarrow B_1 \land B_2 \land \ldots \land B_n$ ($m \geq 1$, $n \geq 1$) in normal form is called a ***tautological entailment*** if and only if it is explicitly tautological, i.e., for every A_i and B_j, $A_i \Rightarrow B_j$ ($m \geq i \geq 1$, $n \geq j \geq 1$) is explicitly tautological.
  - Explicitly tautological is both necessary and sufficient for the (weak-relevant!) validity of a first degree entailment.
- **Fundamental question**
  - Can we construct a formal calculus of tautological entailments?
  - Answer: YES

Tautological Entailment Fragments of Relevant Logics

- **E_{fde} (R_{fde}, T_{fde}): First degree entailment fragment of E (R, T)**
  - Entailment:
    Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$
  - Conjunction:
    Axiom schemata: $(A \land B) \Rightarrow A$ (C1), $(A \land B) \Rightarrow B$ (C2)
    Inference rule: from $A \Rightarrow B$ and $A \Rightarrow C$ to infer $A \Rightarrow (B \land C)$
  - Disjunction:
    Axiom schemata: $A \Rightarrow (A \lor B)$ (D1), $B \Rightarrow (A \lor B)$ (D2)
    Inference rule: from $A \Rightarrow C$ and $B \Rightarrow C$ to infer $(A \lor B) \Rightarrow C$
  - Distribution:
    Axiom schema: $(A \land (B \lor C)) \Rightarrow ((A \land B) \lor C)$ (DCD)
  - Negation:
    Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
    Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$

First Degree Entailment Fragments of Relevant Logics

- **E_{fde} (R_{fde}, T_{fde}): First degree entailment fragment of E (R, T)**
  - Entailment:
    Inference rule: from $A \Rightarrow B$ and $B \Rightarrow C$ to infer $A \Rightarrow C$
  - Conjunction:
    Axiom schemata: $(A \land B) \Rightarrow A$ (C1), $(A \land B) \Rightarrow B$ (C2)
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  - Distribution:
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  - Negation:
    Axiom schema: $A \Rightarrow (\neg(\neg A))$, $(\neg(\neg A)) \Rightarrow A$ (N3)
    Inference rule: from $A \Rightarrow B$ to infer $\neg B \Rightarrow \neg A$

First Degree Formula Fragments of Relevant Logics

- **E_{fdf} and tautological entailments**
  - All logical theorems of E_{fdf} are first degree entailments.
  - Every tautological entailment is provable in E_{fdf}.
  - Only tautological entailments are provable in E_{fdf}.
  - Therefore, E_{fdf} is a formalization of tautological entailments.
- **Perfect interpolation theorem**
  - If $A \Rightarrow C$ is provable in E_{fdf}, then there is an "interpolation formula" B such that (1) $A \Rightarrow B$ is provable in E_{fdf}, (2) $B \Rightarrow C$ is provable in E_{fdf}, and (3) B has no variables not in both A and C.

Antecedent and Consequent Parts of Formulas

- A is a consequent part of A
- If $\neg B$ is a ***consequent*** {***antecedent***} ***part*** of A, then B is an antecedent part {consequent part} of A
- If $B \Rightarrow C$ is a consequent {antecedent} part of A, then B is an antecedent {consequent} part of A, and C is a consequent {antecedent} part of A
- If either $B \land C$ or $B \lor C$ is a consequent {antecedent} part of A, then both B and C are consequent {antecedent} parts of A

Theorems of Variable-Sharing in Relevant Logics

- **Variable-Sharing in $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$)** [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$), then A and B share a sentential variable.
  - If A is a theorem of $E_{\Rightarrow}$ ($R_{\Rightarrow}$, $T_{\Rightarrow}$), then every sentential variable occurring in A occurs at least once as an antecedent part and at least once as a consequent part of A.
- **Variable-Sharing in $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$)** [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$), then A and B share a sentential variable.
  - If A is a theorem of $E_{\Rightarrow,\neg}$ ($R_{\Rightarrow,\neg}$, $T_{\Rightarrow,\neg}$), then every sentential variable occurring in A occurs at least once as an antecedent part and at least once as a consequent part of A (Note: This is NOT true for E (or for R, or for T).

Theorems of Variable-Sharing in Relevant Logics

- **Variable-Sharing in E (R, T)** [A&B-E1-75]
  - If $A \Rightarrow B$ is provable in E (R, T), then A and B share a sentential variable.
  - If $A \Rightarrow B$ is a theorem of E, then some sentential variable occurs as an antecedent part of both A and B, or else as a consequent part of both A and B.
  - If A is a theorem of E containing no conjunctions as antecedent parts and no disjunctions as consequent parts, then every sentential variable in A occurs at least once as an antecedent part and at least once as a consequent part [Maksimova, 1967].

Theorems of Variable-Sharing in Relevant Logics

- **Variable-Sharing in CPC** [A&B-E1-75]
  - If $A \rightarrow B$ is provable in CPC, then either (1) A and B share a sentential variable or (2) either $\neg A$ or B is provable in CPC.
- **Variable-Sharing in RM** [A&B-E1-75]
  - If $A \rightarrow B$ is provable in RM, then either (1) A and B share a sentential variable or (2) both $\neg A$ and B are provable in RM.

Facts in Relevant Logics

- **Why the inference rule of adjunction?**
  - $A \Rightarrow (B \Rightarrow (A \wedge B))$ is not a logical theorem of E (R, T).
  - $A \Rightarrow (B \Rightarrow (A \wedge B))$ is a familiar axiom for intuitionistic and classical logic, but it is only a hair's breadth away from positive paradox $A \Rightarrow (B \Rightarrow A)$, and indeed yields it given $(A \wedge B) \Rightarrow A$ and $(A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$.
- **The disjunctive syllogism and the inference rule γ**
  - $A \wedge (\neg A \vee B) \Rightarrow B$ $(\neg A \wedge (A \vee B) \Rightarrow B)$ is not a logical theorem of either E (R, T). This is the most notorious feature of relevant logic.
  - The inference rule γ, i.e., suppose $\vdash A$ and $\vdash \neg A \vee B$, then $\vdash B$, is admissible in E, R, and many others.

Facts in Relevant Logics

- **The relationship between relevant logics and other logics**

S5 ---> S4 ---> S3 ---> EM

CML --->   ---> E ---> T

RM ----------------------> R

(S3, S4, S5: Lewis's modal systems)

- **The relationship between relevant logics and their first degree entailment fragments**
  - If $A \Rightarrow B$ is provable in E_{fde} (R_{fde}, T_{fde}), then it is also provable in E (R, T) [A&B-E1-75].

Facts in Relevant Logics

- **Conservative extensions**
  - Let S′ is an extension of S in the sense that S′ has some new language components, e.g., connectives, or axioms, or inference rules. S′ is called to be a *conservative extension* of S if for any formula A in the notation of S, if A is provable in S′ then A is also provable in S.
- **Conservative extensions in relevant logics**
  - Both E and $E_{\Rightarrow,\neg}$ are conservative extensions of $E_{\Rightarrow}$
  - Both R and $R_{\Rightarrow,\neg}$ are conservative extensions of $R_{\Rightarrow}$

Facts in Relevant Logics

- **Conservative extensions in relevant logics**
  - RM is not a conservative extension of $RM0_{\Rightarrow}$ ($RM0_{\Rightarrow} = R_{\Rightarrow}$ + RM0)
  - $RM0_{\Rightarrow}$ is not the pure implicational fragment of RM
  - RM does not satisfy the relevance principle but it does satisfy the weaker relevance principle that $A \rightarrow B$ is a theorem of RM only if either A and B share a sentential variable or both $\neg A$ and B are theorems.

Facts in Relevant Logics

♣ Necessity

- In E, the necessity operator *L* can be defined as $LA =_{df} (A \Rightarrow A) \Rightarrow A$, but this is impossible in R because R has $A \Rightarrow ((A \Rightarrow B) \Rightarrow B)$ as an axiom scheme.
- E is both a relevant logic and a modal logic but R is only a relevant logic.
- The rule of necessity (if *A* is provable in E, then *LA* is also provable in E) is naturally holds in E, and therefore no new logical primitives need be introduced to get the desired effect.

Why R Is Interesting?

♣ Major reason 1: Age

- The pure implicational fragment R⇒ of R, first considered by Moh Shaw-Kwei in 1950 and by A. Church in 1951, was regarded to be the oldest of the relevant logics [A&B-E1-75].
- The implication-negation fragment $R_{\Rightarrow,\neg}$ of R, given by I. E. Orlov in 1928, is the oldest of the relevant logics [Dosen, 1990].

♣ Major reason 2: Isolating relevance

- In R one has an even clearer view of relevance than in E, just because of the absence of modal complications.

♣ Other reasons

- Stability, Richness, Easy proof theory, Fragments, Applicability, Extensibility.

Decision Problems in Relevant Logic

- Both $E_{\Rightarrow}$ and $R_{\Rightarrow}$ are decidable [Kripke, 1959].
- LR is decidable [Mayer, 1966].
- E_{fde} (R_{fde}, T_{fde}) is decidable [Anderson and Belnap, 1975].
- E_{fdf} is decidable [Anderson and Belnap, 1975].
- Both $E_{\Rightarrow,\neg}$ and $R_{\Rightarrow,\neg}$ are decidable [Anderson and Belnap, 1975].
- RM is decidable [Anderson and Belnap, 1975].
- E_{+}, R_{+}, and T_{+} are undecidable [Urquhart, 1982].
- E, R, and T are undecidable [Urquhart, 1982].
- The decision problem of $T_{\Rightarrow}$ (or of $T_{\Rightarrow,\neg}$) is open.

Relevant Logics: Proof Theory and Model Theory

- Formal Language of Relevant Logics
- Hilbert Style Axiomatic Systems of Relevant Logics
- Various Properties of Relevant Logics
- Model Theory for Relevant Logics
- Natural Deduction Systems of Relevant Logics
- Sequent Calculus Systems of Relevant Logics
- Semantic Tableau Systems of Relevant Logics
- Bibliography