Blackwell Companions to Philosophy

A Companion to Philosophical Logic

Edited by

DALE JACQUETTE

Blackwell Publishing

© 2002, 2006 by Blackwell Publishing Ltd

BLACKWELL PUBLISHING
350 Main Street, Malden, MA 02148-5020, USA
9600 Garsington Road, Oxford OX4 2DQ, UK
550 Swanston Street, Carlton, Victoria 3053, Australia

The right of Dale Jacquette to be identified as the Author of the Editorial Material in this Work has been asserted in accordance with the UK Copyright, Designs, and Patents Act 1988.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, except as permitted by the UK Copyright, Designs, and Patents Act 1988, without the prior permission of the publisher.

First published 2002
First published in paperback 2006 by Blackwell Publishing Ltd

1 2006

Library of Congress Cataloging-in-Publication Data

A companion to philosophical logic/edited by Dale Jacquette.
p. cm. (Blackwell companions to philosophy ; 22)
Includes bibliographical references and index.
ISBN 0-631-21671-5 (alk. paper)
1. Logic. I. Jacquette, Dale. II. Series.
BC71.C65 2002
160—dc21
2001043236

ISBN-13: 978-0-631-21671-1 (hardback)
ISBN-13: 978-1-4051-4575-6 (paperback)
ISBN-10: 1-4051-4575-7 (paperback)

A catalogue record for this title is available from the British Library.

Set in 10 on 12½ pt Photina
by SNP Best-set Typesetter Ltd, Hong Kong
Printed and bound in the United Kingdom
by TJ International, Padstow, Cornwall

The publisher's policy is to use permanent paper from mills that operate a sustainable forestry policy, and which has been manufactured from pulp processed using acid-free and elementary chlorine-free practices. Furthermore, the publisher ensures that the text paper and cover board used have met acceptable environmental accreditation standards.

For further information on
Blackwell Publishing, visit our website:
www.blackwellpublishing.com

38

Relevance Logic

EDWIN D. MARES

1 Non-Sequiturs are Bad

Since 1993, when Andrew Wiles completed his difficult proof of Fermat's Last Theorem, mathematicians have wanted a shorter, easier proof. Suppose when someone addressing a conference of number theorists suggests the following proof of the theorem:

The sky is blue.

$\therefore$ There is no integer n greater than or equal to 3 such that for any non-zero integers x, y, z, $x^n = y^n + z^n$.

This proof would not be well received. But it is valid, in fact sound, on the classical logicians' definition. The premise cannot be true in any possible circumstance in which the conclusion is false. For the conclusion is necessarily true. And the premise is true. Thus the argument is sound and known to be sound.[1]

The classical notion of validity does not agree with our pre-logical intuitions about where the division between good arguments and *non-sequiturs* should be. Classical logic allows connections between premises and conclusions in valid arguments that are extremely loose. There needs to be more of a connection between the content of the premises and conclusion in an argument that we are prepared to call 'valid.'

Some classical logicians have defined content semantically, usually using possible worlds, in such a way as to vindicate arguments like our proof of Fermat's Last Theorem (see e.g. Lewis 1988). On these views, there is a real relationship between the content of the premises and that of the conclusion. I don't want to argue in detail against such attempts here. Such notions of content may be fine for some purposes. But, since they approve of arguments like our proof, they do not coincide with the intuitions that we usually apply when considering whether a proof is good or bad.

Another line of reply is that our notion of good proof is not completely logical, but rather it is partly pragmatic.[2] There is probably some truth to this claim, but we should resist the temptation to push this problem completely into pragmatics. Theories of pragmatics are notoriously vague. They tell us, for example, to reject the above argument because it violates the Gricean maxim to 'be relevant.' What counts as relevant is left unsaid in Grice's theory. Surely, if there is a theory of relevance that is more rigorous than this, it would be better, all things being equal, to appeal to the more rigorous theory. And relevant logic does provide a very specific view about what counts as a relevant deduction.

The plan of this chapter is to use the natural deduction system for the relevance logic **R** as a guide to the various elements of relevance logic – its proof theory, its semantics, and their interpretation. Later we will introduce weaker relevance logics and two applications of relevance logics: one to the problem of conditionals and the other to the theory of properties.

2 The Real Use of Premises

The problem with *non-sequiturs* like the one given above is that the premises of the inference appear to have nothing to do with the conclusion. Relevance logic attempts to repair this problem, in part, by forcing a constraint on proofs that the *premises really be used in the derivation of the conclusion*.

We will present this idea in the context of Anderson and Belnap's natural deduction system for the logic **R**. The idea is pretty simple. Each premise, or rather hypothesis, in a proof is indexed by a number. The various steps in a proof are indexed by the numbers of the premises which are used to derive the steps. For example, the following is a valid argument in this system:

1.	$A \rightarrow B_{\{1\}}$	*hyp.*
2.	$A_{\{2\}}$	*hyp.*
3.	$B_{\{1,2\}}$	$(1)(2) \times (\rightarrow E)$

where '$\rightarrow E$' is the rule of modus ponens or implication elimination. The numbers in parentheses are the indices.

Throughout this chapter, we will be using the natural deduction system for the logic **R**. This system allows free repetition of premises and the free reiteration of steps into subproofs.

3 Implication

In natural deduction systems we do not usually merely display proofs with premises. We discharge premises to prove theorems of a system. The key rule that we will use here is the rule of conditional proof, or $\rightarrow I$ (implication introduction), *viz*.,:[3]

$A_{\{k\}}$	*hyp.*
$\vdots$	
B_{α}	
$A \rightarrow B_{\alpha-\{k\}}$	$(\rightarrow I)$

where k occurs in α. The proviso that k occur in α is essential. It ensures that, in this case, A is really used in the derivation of B.

We can think of the problem of logical relevance in terms of inference, as we have been doing, but also in terms of implication. Relevance logic was developed in part to avoid the so-called paradoxes of material implication. These are formulae that are theorems of classical logic, but are counterintuitive when we think of the arrow as meaning 'implication' in any ordinary sense, or pre-logical philosophical sense, of that term. Among these paradoxes are the following (with names given where they exist):

1. $A \to (B \to A)$ (positive paradox)
2. $A \to (\sim A \to B)$ (negative paradox)
3. $(A \to B) \vee (B \to A)$
4. $(A \to B) \vee (B \to C)$
5. $(A \wedge \sim A) \to B$ (*ex falso quodlibet* (EFQ))
6. $A \to (B \to B)$
7. $A \to (B \vee \sim B)$

Consider, for example, the positive paradox. It says that any true formula is implied by any formula at all. Implication is usually thought to indicate a tighter relationship than one that can exist between any proposition and a true proposition, merely because the latter is true. Similarly, negative paradox says that any proposition is implied by a false proposition. Again, we have an indication that the material conditional (which always makes a negative paradox true) is too loose a connection to capture the intuitive sense of 'implication.'

Relevant logics were introduced to avoid the paradoxes of implication. Now, this does not mean that the semantics of relevant logics will *show* that all such paradoxes are false in every circumstance. Rather, relevant logicians have developed semantic and proof-theoretic techniques that do not force the paradoxes to be true. Thus, there are at least two notions of relevance at play in relevance logic: (a) The system of proof forces us actually to use every premise in a deduction; (b) the proof theory and semantics do not force us to accept the paradoxes of material implication.

Returning to our natural deduction system, consider the following attempt at a proof of positive paradox:

1.	$A_{\{1\}}$	*hyp.*
2.	$B_{\{2\}}$	*hyp.*
3.	$A_{\{1\}}$	$(1) \times$ (*reiteration*)
4.	$B \to A_{\{1\}}$	$(2) - (3) \times (\to I)$
5.	$A \to (B \to A)_{\emptyset}$	$(1) - (4) \times (\to I)$

The illegitimate move here is the use of an implication introduction in the fourth step. 2 does not belong to $\{1\}$ and so we cannot discharge the second hypothesis here. The other paradoxes are avoided in similar ways.

4 From Proof Theory to Semantics

In 1972 Alasdair Urquhart presented a semantic interpretation of relevance numerals. He begins with the notion of a 'piece of information.' A piece of information is a concept which encompasses but is more general than that of a possible world or an evidential situation (the latter is from Kripke's semantics for intuitionist logic).

Pieces of information satisfy statements. To take an example from Urquhart (1992), if we have a piece of information a that consists of the fact that *Harry is taller than Fred* and the fact that *Jim is taller than Harry*, then

a $\models$ *Jim is taller than Fred.*

The satisfaction relation ($\models$) holds between pieces of information and basic statements of a language by virtue of the meanings of those basic statements. Here the meaning of 'is taller than' includes its transitivity.

Among those facts that can be satisfied by pieces of information are what we might call 'informational links.'[4] Among informational links are laws of nature – such as the law that all pieces of matter attract all other pieces of matter – and convention connections – such as the fact that all objects that are exactly a metre in length are the same length as a particular bar in Paris. These informational links provide the truthmakers for implicational statements.

But the truth-making relation between implicational statements and informational links is not very straightforward. For example, if it is a law of nature (or, rather, an instance of a law of nature) at a that $A \to B$ obtains at piece of information a and there is a conventional link in a such that $B \to C$ holds in a, then it would also seem that $a \models A \to C$. But there is no informational link in a which directly makes true $A \to C$. Like 'taller than,' implication seems to be transitive by virtue of its meaning.

To enforce this and other features of implication on the model, Urquhart devised a truth condition for implication using what is now called 'fusion.' Pieces of information can be combined or 'fused' together. The fusion of two pieces of information a and b is written '$a \circ b$'. $a \circ b$ is itself a piece of information.

When we fuse two pieces of information a and b together, we apply the informational links from a to the information in b. Thus, for example, suppose that it is a law in a that all material objects attract all other material objects and among the facts in b are *i is a material object* and *j is a material object*. Thus, in $a \circ b$ we have the fact that *i and j attract one another*.

So now we have a connection between informational links and fusion and a connection between informational links and implication. Putting these together, we can derive the following truth condition for implication:

$$a \models A \to B \quad \text{if and only if} \quad \forall b(b \models A \Rightarrow a \circ b \models B).$$

An implication is true at a piece of information, if whenever that piece of information is fused with a piece of information which satisfies the antecedent the fusion satisfies the consequent.

There is a tidy connection between fusion and the natural deduction system. Here is an instance of our $\rightarrow E$ rule:

1.	$A \rightarrow B_{\{1\}}$	*hyp.*
2.	$A_{\{2\}}$	*hyp.*
3.	$B_{\{1,2\}}$	$(1)(2) \times (\rightarrow E)$

We can think of the subscripts as names of pieces of information. In line one, we have the statement that $A \rightarrow B$ is true in a piece of information 1. In line two, we have A holding in piece of information 2. And in line three we have B obtaining in $1 \circ 2$. Thus, Urquhart's semantics provides us with a semantic understanding of the indices used in our natural deduction system.

The connection between informational links and implication gives us a means of interpreting relevant implication. Implications are made true by informational links or the results of implicational links under certain closure principles, like transitivity. We take from Devlin (1991) and Israel and Perry (1990) the idea that it is because of informational links and their closures that facts carry the information that other facts obtain.[5] We follow a popular tradition in philosophy of language and hold that statements express their truth conditions. We say that $A \rightarrow B$ means that *A carries the information that B* since this formula expresses the 'fact' that there is an informational connection between A and B or the result of one or more informational connection and certain closure principles.

5 Adding Conjunction

Let's move to discuss another connective. The truth condition for conjunction in this semantics is quite straightforward. That is,

$$a \models A \wedge B \quad \text{if and only if} \quad a \models A \text{ and } \alpha \models B.$$

This is merely the same truth condition for conjunction that one finds in Kripke's semantics for modal and intuitionist logic.

As we saw in the previous section, Urquhart's semantics gives us a clear relationship between pieces of information and indices in the natural deduction system. If we apply this relationship to derive rules for conjunction, the truth condition given above yields both introduction and elimination rules for that connective. First the introduction rule:

From A_α and B_α, infer $A \wedge B_\alpha$.

And now the elimination rules:

From $A \wedge B_\alpha$, infer A_α

and

From $A \wedge B_\alpha$, infer B_α.

Note that in the introduction rule the subscript on the two formulae to be conjoined is the same. Before we can conjoin two formulae we have to know that they are true at the same piece of information.

The restriction in this rule that the subscript must remain the same allows us to avoid admitting a well-known proof for positive paradox (see Lemmon 1965). For if we were to allow formulae with different subscripts to be conjoined, we would allow the following proof:

1.	$A_{\{1\}}$	*hyp.*
2.	$B_{\{2\}}$	*hyp.*
3.	$A \wedge B_{\{1,2\}}$	$(1)(2) \times (\wedge I)$
4.	$A_{\{1,2\}}$	$(3) \times (\wedge E)$
5.	$B \rightarrow A_{\{1\}}$	$(2) - (4) \times (\rightarrow I)$
6.	$A \rightarrow (B \rightarrow A)_{\emptyset}$	$(1) - (5) \times (\rightarrow I)$

The illegitimate step here is step 3. It conjoins two formulae that do not have the same index.

6 The Problem of Disjunction

The use of Urquhart's semantics for relevant logic depends crucially on the notion of a theory. We show that Urquhart's semantics, given certain additional semantic postulates, is a semantics for a relevant logic by proving a completeness theorem. In a completeness proof for Urquhart's semantics, we construct a model using theories as pieces of information. A theory is a set of formulae closed under conjunction and provable implication. Let us suppose that Γ is a theory. Then, if $A \in \Gamma$ and $B \in \Gamma$ then $A \wedge B \in \Gamma$. Also, if $A \in \Gamma$ and $A \rightarrow B$ is a theorem of the logic we are using then $B \in \Gamma$. To represent fusion in our model, we take $a \circ b$ to be the set of formulae B such that there is some formula A such that $A \rightarrow B$ is in a and A is in b. In other words, in constructing $a \circ b$ we take major premises from a and minor premises from b and perform *modus ponens* on them. The result is $a \circ b$. This construction crucially depends upon the result of a fusion between two theories itself being a theory. As we shall see, this is a very important fact for Urquhart's semantics.

Disjunction adds a new and difficult dimension to the semantics. The natural truth condition for disjunction is the following:

$$a \models A \vee B \quad \text{iff} \quad a \models A \text{ or } a \models B.$$

But theories, in general, do *not* meet the corresponding inclusion condition for disjunction, *viz.*,

$$A \vee B \in \Gamma \quad \text{iff} \quad A \in \Gamma \text{ or } B \in \Gamma.$$

Theories that do meet this condition are called *prime* theories. The Urquhart semantics, however, does not work if we restrict ourselves to using prime theories. For the fusion of two prime theories is not always a prime theory.

Thus, either we are forced to modify the standard truth condition for disjunction or abandon the use of fusion in the semantics. Relevant logicians have tried both alternatives, each with success. Kit Fine (1974) has extended the Urquhart semantics to include a treatment of disjunction by altering its truth condition. Richard Routley and Robert Meyer, on the other hand, have retained the standard truth condition for disjunction and relinquished the use of fusion in their series of papers (1973, 1972a, 1972b); for a detailed discussion of this semantics see Routley *et al.* (1982). In the following section, I give a brief overview of this semantics, which is also known as the *relational semantics*.

7 Routley and Meyer's Ternary Relation

To distinguish between the elements of Urquhart's semantics and those of Routley and Meyer's relational semantics, let us call the latter *situations*, although in the literature they are also called 'worlds' and 'setups.' The Routley–Meyer semantics replaces fusion with a three-place accessibility relation on situations, R. The truth condition for implication is correspondingly changed:

$$a \models A \to B \quad \text{iff} \quad \forall x \forall y ((Raxy \;\&\; x \models A) \Rightarrow y \models B)$$

This truth condition might look like a 'bolt from the blue,' but it is actually a generalization of the truth condition for necessity from Kripke's semantics for modal logic. Whereas Kripke uses a binary relation to interpret a monadic connective (necessity), Routley and Meyer use a ternary relation to interpret a binary connective (implication).

We can view the relational semantics as generalizing fusion. Recall that we said that $a \circ b$ results when the informational links from a are applied to the information in b. Adapting this idea to the ternary relation is quite easy. We say that $Rabc$ obtains when the information that *results* from the application of the links in a to the information in b is *contained in* c. To return to our previous example, suppose that a contains the law that *all matter attracts all other matter* and b contains the information that *i and j are material objects*. Then c contains the information that *i and j attract one another*.

We can force implication to have the various properties that we want by accepting certain postulates in our model theory. For example, if we want the statements satisfied by situations to be closed under *modus ponens*, we adopt the following postulate. For all situations a,

$$Raaa.$$

For suppose that $a \models A \to B$ and $a \models A$. By the truth condition for implication, for all b such that $Raab$, $b \models B$. But $Raaa$, by the above postulate. So, $a \models B$. Thus, a is closed under *modus ponens* as we suggested. There are similar postulates that are required to satisfy transitivity and other properties that one might desire implication to have.

8 Rules for Disjunction

Now we return to our natural deduction system and add some rules for disjunction. The introduction rules are quite obvious:

From A_α, to infer $A \vee B_\alpha$.

and

From B_α, to infer $A \vee B_\alpha$.

The elimination rule is a little more complicated:[6]

From $A \vee B_\alpha$, $A \to C_\beta$, and $B \to C_\beta$,
infer $C_{\alpha\cup\beta}$.

There is also a rule that ensures the distribution of conjunction over distribution (it simply states that one can infer from $A \wedge (B \vee C)_\alpha$ to $(A \wedge B) \vee (A \wedge C)_\alpha$).[7]

9 The Semantics of Negation

The treatment of negation is one of the most controversial elements of relevant logic (see Copeland 1979). The key here is that in order to block *ex falso quodlebet* we need situations in our semantics that make contradictions true. In addition, in order to reject the paradox $A \to (B\vee \sim B)$ we need situations at which bivalence fails. Thus, we need a semantics for negation that does not force bivalence or consistency on us.

Routley and Meyer's model theory incorporates a device from a semantics developed originally for relevant logic by Richard and Val Routley in 1972. This device is an operator on situations, which has become known as the *Routley star*. For each situation in the semantics, there is a situation that is its 'star.' The star of a situation a is a^*. Some situations (in some models) are identical with their stars, but not all. The truth condition for negation then becomes:

$a \models_\upsilon \sim A \quad \text{iff} \quad a^* \not\models A.$

The relative independence of a situation and its star allows inconsistencies and failures of bivalence.

On the other hand, Routley and Meyer relate worlds to their pairs in order to satisfy the various postulates governing negation. For example, they set $a = a^*$ in order to satisfy $A \to \sim\sim A$ and $\sim\sim A \to A$.

The problem with the Routley star is that many philosophers have had trouble understanding what it is and what it is supposed to do with negation. Here we will use an explanation due to J. M. Dunn (1993). Dunn does not begin with the Routley star. Instead, he postulates a binary relation C on situations that is supposed to relate situations to situations with which they are 'compatible.' For example, suppose there is information in the present situation that a particular table is completely red. In another situation, there is the information that it is completely green. These two situations are incompatible with each other. If there are no such conflicts, then situations are compatible.

Note that situations need not be compatible with themselves. Situations are abstract ('ersatz') entities and can contain conflicting information (e.g. that the table is red and that it is green all over).

Now we have the following truth clause for negation:

$$a \models_{\omega} {\sim}A \quad \text{iff} \quad \forall b(Cab \supset b \not\models A)$$

Now we can use the compatibility relation to define the star operator. a^* is the largest situation that is compatible with a. a^* is largest in the sense that it contains more information than any other situation compatable with a.[8]

10 Rules for Negation

The introduction rule for negation is a form of the *reductio* rule:

From $A \rightarrow {\sim}A_{\alpha}$, infer ${\sim}A_{\alpha}$. (${\sim}I$)

The elimination rule is a form of *modus tollens*:

From ${\sim}B_{\alpha}$ and $A \rightarrow B_{\beta}$ infer ${\sim}A_{\alpha\cup\beta}$. (${\sim}E$)

We also add two double negation rules: From ${\sim\sim}A_{\alpha}$ to infer A_{α} and the converse of this rule.

11 Disjunctive Syllogism

The rule of disjunctive syllogism (DS) is

From $A \vee B_{\alpha}$ and ${\sim}A_{\alpha}$, infer B_{α}.

This is an intuitive rule of inference. We use it to 'deduce' the identity of the murderer when reading mystery novels – we eliminate all but the guilty party. We use it when determine who sits on university committees ('X are going on leave. Y will say crazy things. So, it has to be Z.'). In fact, we use DS all the time. But, if we add DS we get back one of the paradoxes of implication. The following proof is due to C. I. Lewis (see Lewis and Langford 1959):

1.	$A \wedge {\sim} A_{\{1\}}$	*hyp.*
2.	$A_{\{1\}}$	$(1) \times (\wedge E)$
3.	$A \vee B_{\{1\}}$	$(2) \times (\vee I)$
4.	${\sim} A_{\{1\}}$	$(1) \times (\wedge E)$
5.	$B_{\{1\}}$	$(3)(4) \times (DS)$
6.	$(A \wedge {\sim} A) \rightarrow B_{\emptyset}$	$(1) - (5) \times (- I)$

So, adding DS gets us EFQ.

Reactions to this proof are varied. Some, like Stephen Read (1988), claim that the problem is not with disjunctive syllogism, but rather with our understanding of disjunction in natural language. We should interpret natural language disjunctions as an intensional disjunction. In particular, we should treat it as *fission*, sybolised $\oplus$. In **R** and closely related relevant logics, fission can be defined as

$$A \oplus B =_{df} {\sim} A \rightarrow B$$

Clearly, disjunctive syllogism is valid for fission. It is just a form of modus ponens ($\rightarrow E$). What is not valid for fission is addition. We cannot infer from A to $A \oplus B$. This blocks the step from lines 2 to 3 in the proof above. Thus the proof is blocked and so we can create a form of disjunctive syllogism without buying into a paradox of implication.

Read's solution has definite merits. I have found that students, who have not yet become accustomed to the quirks of classical logic, find $\vee I$ a rather strange rule. When I have shown Lewis's proof during seminars, I have the audience vote to decide which rule to reject. $\vee I$ is always the one they choose.

But there are problems with taking natural language disjunction to be intensional. In particular, the truth condition for intensional disjunction does not look like anything we would identify with natural language disjunction. That is,

$$a \models A \oplus B \quad \text{if and only if} \quad \forall b \forall c (Rbca \supset (b \models A \textit{ or } c \models B)).$$

Many relevant logicians have been reluctant to accept Read's view because they do not think that this truth condition looks like it explicates the meaning of natural language disjunction.

There are other ways of dealing with the apparent validity of DS. We'll take a look at one due to Chris Mortensen (1986), with some minor changes made to fit the current chapter.

Mortensen holds that we can use DS with extensional conjunction under certain circumstances. According to Mortensen, the problem with DS is that we cannot use it when reasoning about inconsistent situations. When we reason about consistent situations, we can use it. For, the following argument is valid (Mortensen 1986: 196):

1. If a situation a is consistent, $a \models A \vee B$ and $a \models {\sim} A$, then $a \models B$. (hypothesis)
2. a is consistent hypothesis
3. $a \models A \vee B$ hypothesis

4. $a \models \sim A$ hypothesis
5. $a \models B$ (1)(2)(3)(4) × (*modus ponens*)

There are at least two interesting features of this argument. First, it is done within the semantic metalanguage. Second, the nature of the first premise is worth nothing. It tells us that consistent situations are closed under disjunctive syllogism. Note that this is a premise – it is not proven by the argument. Nor does Mortensen establish it in any other way. Mortensen says that it is justified by intuition (Mortensen 1986).

I do not have space here to argue the merits of either of these treatments of DS, nor do I have space to discuss other alternative approaches. I direct the interested reader to the bibliographies in Anderson *et al*. (1992) and Read (1988) for readings on this topic.

12 Logics Stronger than R

So far, we have been motivating the logic **R**. But there are many other relevant logics that have been studied.

Logics stronger than (that contain more theorems than) but close to **R**, tend to be only marginally relevant. For example, the logic **R** Mingle (**RM**) of Dunn and Storrs McCall contains all the axioms and rules of **R**, plus the mingle axiom:

$$A \rightarrow (A \rightarrow A)$$

For a logic to be counted as relevant it must have the variable sharing property. That is, if a formula $A \rightarrow B$ is a theorem, then formulae A and B must contain at least one propositional variable in common. **RM** does not have the variable sharing property, but does have a property very close to it, *viz*.,[9]

If $A \rightarrow B$ is a theorem of RM, then either
  A and B share a propositional variable
or $\sim A$ and B are both theorems of RM.

RM contains theorems that seem paradoxical from a relevant point of view. Among these is '$(A \rightarrow B) \vee (B \rightarrow A)$'. Despite the semi-relevance of **RM**, it has its attractions and it and systems very similar to it have their advocates (see, e.g. Avron 1990a, 1990b).

Slightly stronger than **RM** is the elegant logic **RM3**. **RM3** is characterized by three-valued truth tables. The three values that we will use are T, F, and B. T and F are true and false respectively, and B is the value 'both true and false.' Dialetheists are philosophers that hold that some sentences can be both true and false. Even if we are not dialetheists, we can make sense of this truth-value by thinking about inconsistent fictional stories. We can understand such stories by taking the inconsistencies in them to be both true and false of the story. Both T and B are designated values in this semantics. That is, a statement is considered to be (at least) true if it is T (perhaps best thought of as 'merely true') or B.

The truth tables for **RM3** are the following:

$\wedge$	T	B	F
T	T	B	F
B	B	B	F
F	F	F	F

$\vee$	T	B	F
T	T	T	T
B	T	B	B
F	T	B	F

$\rightarrow$	T	B	F
T	T	B	F
B	T	T	B
F	T	T	T

$\sim$	
T	F
B	B
F	T

This is a simple, elegant logic that comes very close to being relevant.

13 Logics Weaker than **R**

On the other hand, many relevant logicians have argued for systems weaker than **R**. In fact, the preferred logic of the *Entailment* volumes (Anderson and Belnap (1975) and Anderson *et al.* (1992)) is the logic **E** of relevant entailment. **E** is meant to capture a *strict* relevant implication. Given this, it was conjectured that **E** would be captured by **R** extended by the addition of a necessity operator and some (S4-ish) axioms governing that operator. Unfortunately, **NR** or $\mathbf{R}^{\Box}$, as the resulting logic was called, was shown to be somewhat stronger than **E**. As a result, relevant logicians could not accept both that **R** is the logic of relevant implication and that **E** is the logic of necessary relevant implication. (See Anderson and Belnap (1975) for a more detailed history.)

Other relevant logicians have given various reasons for adopting weaker logics, but we will only look at one such motivation.

Some logicians have used relevant logics to develop naïve theories of truth and naïve set theories. A naïve theory of truth both contains its own truth predicate and admits Tarski's truth schema, *viz*.,

$$\mathit{True}(\ulcorner A \urcorner) \leftrightarrow A.$$

As we shall soon see, if we base a naïve theory of truth on the logic **R**, we end up with a trivial system. That is, the resulting logic can prove every proposition.

A similar state of affairs holds for naïve theory of sets. A naïve set theory contains an unrestricted comprehension principle, such as, for each formula A,

$$\exists x \forall y (A(y) \leftrightarrow y \in x).$$

Again, adding this principle (along with other standard principles of set theory) to **R** yields a trivial theory.

Let's use contraction to prove 'Curry's paradox' in **R** with the naïve theory of truth. My proof follows Meyer *et al.* (1979), and is done in the Hilbert-style axiom system.

We start with a definition of a proposition C:

$$C =_{\mathrm{df}} Tr(\ulcorner C \urcorner) \rightarrow p$$

where p is any arbitrary proposition. We also define a biconditional ($A \leftrightarrow B =_{df} (A \rightarrow B) \wedge (B \rightarrow A)$).

1.	$C \leftrightarrow (Tr(\ulcorner C \urcorner) \rightarrow p)$	definition of C
2.	$C \rightarrow (Tr(\ulcorner C \urcorner) \rightarrow p)$	(1) × (simplification)
3.	$Tr(\ulcorner C \urcorner) \leftrightarrow C$	(T-schema)
4.	$C \rightarrow (C \rightarrow \mathrm{p})$	(2)(3) × (replacement of equivalents)
5.	$\vdash_{\mathbf{R}} (C \rightarrow (C \rightarrow p)) \rightarrow (C \rightarrow p)$	(contraction)
6.	$C \rightarrow p$	(4)(5) × (*modus ponens*)
7.	$(Tr(\ulcorner C \urcorner) \rightarrow p) \rightarrow C$	(1) × (simplification)
8.	$(C \rightarrow p) \rightarrow C$	(7)(3) × (replacement of equivalents)
9.	C	(6)(8) × (*modus ponens*)
10.	p	(6)(9) × (*modus ponens*)

Thus, we can prove any arbitrary proposition in **R** with the naïve theory of truth. A very similar proof can be used to show that every proposition is provable in **R** with a naïve set theory.

Some relevant logicians hold that *one* problem with **R** is that it contains the principle of *contraction*, used in step three in the argument. In schematic form, contraction is

$$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B).$$

Here is a proof of contraction in our natural deduction system:

1.	$A \rightarrow (A \rightarrow B)_{\{1\}}$	*hyp.*
2.	$A_{\{2\}}$	*hyp.*
3.	$A \rightarrow B_{\{1,2\}}$	(1)(2) × ($\rightarrow E$)
4.	$B_{\{1,2\}}$	(2)(3) × ($\rightarrow E$)
5.	$A \rightarrow B_{\{1\}}$	(2) – (4) × ($\rightarrow I$)
6.	$(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)_{\emptyset}$	(1) – (5) × ($\rightarrow I$)

How should we block this proof? Note that hypothesis 2 is used in *modus ponens* at both lines 3 and 4. One step towards rejecting contraction in a natural deduction system is to restrict the use of premises in a proof. That is, we allow each hypothesis to be discharged only once. This blocks the proof. (Of course, in setting up a natural deduction system that does not prove contraction, we should be sure that there is no other way to prove that thesis.)

There has been some success in developing naïve theories using weak relevant logics. Ross Brady has shown that a weak relevant logic can support a consistent naïve class theory and a consistent naïve set theory (see Brady 1983, 1989, and forthcoming).

14 Relevant Logics and Natural Language Conditionals

In this and the next section, we will look at two applications of relevant logic and its semantics.

As Dunn (1986) argues, we need a relevant theory of conditionals. For example, consider the following example:

> If you pick up a pregnant guinea pig by the tail,
> all her babies will be born without tails.[10]

Intuitively, this statement is false. Taking the conditional to be a material implication, however, makes it true.

Probabilistic treatments of the conditional, quite popular at the moment, do not capture relevance phenomena adequately either. These treatments, for the most part, accept *Adam's thesis*. This thesis says that a conditional $A \Rightarrow B$ is assertable if and only if the conditional probability $\Pr(B/A)$ is high. But there are cases in which the consequent of a conditional has a high probability independently of the probability of the antecedent. For example,

> If John dropped this piece of chalk, Einstein's theory of gravity holds.

There is clearly something wrong with this conditional. It makes it seem as if the truth of Einstein's theory of gravity was caused by John's dropping this piece of chalk. Conditional probability $\Pr(E/C)$ is high because the probability of E is high and is independent of the probability of C. On the probabilistic theory of conditionals, this makes the conditional assertable.

These brief, and rather dogmatic, remarks motivate a relevant treatment of the conditional. But we cannot merely take the conditional to be a form of relevant implication. For the implications of the various logics we have seen have properties that the natural language conditional does not have. For instance, the principle of strengthening the antecedent is valid for all these conditionals, that is, we can infer from the truth of $(A \rightarrow C)$ to $((A \wedge B) \rightarrow C)$. But we cannot infer from 'If Ramsey got a new chew toy this afternoon, he is now happy' to 'If Ramsey got a new chew toy this afternoon and he had a bath, he is now happy'.[11]

Instead, we can formulate another connective, which shares some properties with relevant implication. The view that I present here is a simplification of that of Mares (forthcoming), which in turn is a development of the theory of Dov Gabbay (1972, 1976). Here I develop this idea semantically.

In addition to the elements from the Routley–Meyer semantics, we add a four-place accessibility relation, I. This relation holds between two propositions and three situations. Here a proposition will merely be a set of situations. Thus, $|A|$ is the set of situations at which the formula A is true. The conditional is represented by the symbol $\Rightarrow$.

Our truth condition for the conditional mirrors the Routley–Meyer truth condition for implication.

$$a \models A \Rightarrow B \quad \text{iff} \quad \forall x \forall y((I|A||B|axy \;\&\; x \models A) \supset y \models B)$$

The difference here, of course, is the insertion of the antecedent and consequent propositions into the accessibility relation. Why we need the consequent proposition represented here will be treated later. The inclusion of the antecedent proposition allows us to block unwanted inferences such as strengthening of the antecedent. For we cannot infer on this semantics from $I|A \wedge B||C|abc$ to $I|A||C|abc$. This blocks the inference from $a \models A \Rightarrow C$ to $a \models (A \wedge B) \Rightarrow C$.

There are some inferences, however, that we do want to hold of the conditional. One way of understanding implication is as an idealization of the conditional. An analogy might help here. Recall that we interpreted the implication of **R** in terms of *universal* laws of nature, *sufficient* causal statements, and so on. Yet our standard laws of science and causal statements are not universal or sufficient. They have *ceteris paribus* clauses built into them. I suggest that the relationship between implication and the conditional is akin to that between universal laws of nature and sufficient causal statements, on the one hand, and *ceteris paribus* laws and normal causal statements on the other. The latter have built into their interpretation unstated conditions that indicate where they do and do not apply. Similarly, standard conditionals have unstated restrictions of these sorts built into their interpretation. Thus, in the semantics of conditionals, the conditional $I|A||B|abc$ only holds when b and c satisfy the restrictions associated with the proposition $|A|$ and $|B|$ at the situation a.

The inclusion of the consequent proposition is supposed to prevent certain irrelevances for appearing. For example, when I talk about what happens if a piece of chalk is dropped, one of the background assumptions that I make is that the laws of gravity will hold when I drop the chalk. But as we have said, we don't want to accept the conditional, 'If John dropped this piece of chalk, Einstein's theory of gravity holds.'[12] Rather, on the present view the consequent cannot count as a background assumption. Thus the consequent helps to determine which situations are used in the evaluation of the conditional.

We can represent the idea that the conditional is a relevant implication plus some restrictions by the following principle:

$$\frac{I|A||B|abc}{\therefore\ Rabc}.$$

This principle makes valid the thesis below:

$$(A \rightarrow B) \rightarrow (A \Rightarrow B)$$

We can also force the conditional to satisfy *modus ponens* by adding the principle that $I|A||B|aaa$, for all propositions $|A|$ and $|B|$ and all situations a. Given the previous principle, it makes sense to have the conditional satisfying modus ponens if the corresponding implication also satisfies *modus ponens*.

15 Theory of Properties

The doctrine of relevant predication is due to Dunn (1987, 1990a, 1990b). Philosophers distinguish between those properties that an object really has from those that it has in a rather tenuous way. Consider the problem of Cambridge change. Two hours ago, in Toronto (on the other side of the world from me), it was raining. Now it has stopped. So we have a change from

Ed is such that it is raining in Toronto

to

Ed is such that it is not raining in Toronto.

Here there is a change, but not a real change in me. The change doesn't really affect me.

The standard treatment of lambda abstraction in logic does not distinguish between those properties that an object really has from those that it has only in this incidental way. For '$\lambda x(Raining(toronto))ed$' is usually treated as a legitimate predication.

Dunn (1987, 1990a, 1990b) distinguish between ordinary predication and relevant predication. A property φ is had by an entity i relevantly if a thing's being i implies that it has φ. In Dunn's formalism,

$$(\rho x \varphi x)i =_{\mathrm{df}} \forall x(x = i \rightarrow \varphi x).$$

The idea is that the property here is had by the thing by virtue of its being that thing. Relevant implication is used to formalise the 'in virtue of' here.

As Dunn puts it in (1990b), if i has a φ relevantly, then for anything x,

$$\sim\varphi x \rightarrow x \neq i.$$

So not to have φ is to make a thing, or in our own terms, to carry the information that it is, not i.

To take an example, consider

Ronald Reagan is such that Socrates is wise.

We can ask what this 'is such that' is doing here. If it is there to indicate that 'Socrates is wise' is relevantly predicated of Reagan it would seem to be false, for

$$(\rho x S)ron \leftrightarrow \forall x(x = ron \rightarrow S).$$

There is no reason to believe that $x = ron \rightarrow S$ for any x here, since just because S is true does not mean that arbitrary statements imply it. And the sentence '$x = ron$' seems to have nothing to do with S. Thus, we can reject '$(\rho x S)ron$.'

16 Summary

This has been a rather opinionated introduction to relevant logic. I have used the natural deduction system for **R** as a guide, since it is rather elegant and because *I* like it. And I have used a particular reading of the semantics in order to give the philosophically inclined reader a way of understanding the system. There are many other ways of understanding relevant logic, but in a short chapter one cannot cover them all. So I have decided to treat only one interpretation at some length.

Acknowledgements

An early version of this chapter was presented to a group of staff and students at Monash University. I am grateful for their comments.

Notes

1 It won't help to argue that in a proof, all the steps in the proof must be transparent to the people to whom the proof is presented, for even after it is pointed out to us that from it is a theorem that A we can infer that B, therefore A, we still feel quite cheated by the so-called proof.

2 For a very clear version of this approach, see, Robert Fogelin (1978).

3 Anderson and Belnap (1975) present their rules in 'horizontal' form: From $A_1, \ldots, A_n$ infer B. I will use both their method of presentation and the current 'vertical' method, depending on which is easier in a given context.

4 Here I am not presenting Urquhart's theory. Rather, the present theory, in effect, is that of Mares (1996). This view is an elaboration of Meyer's interpretation of Urquhart's semantics. Meyer (in conversation) takes fusion to be the application of the 'laws' of one piece of information to the facts in another piece of information.

5 Note that for Devlin and Israel and Perry (as for Mares 1996) the list of sorts of informational links is much longer than the one that I gave earlier. A reasonably good list is at Devlin (1991: 12). I don't agree with all the types of links that Devlin includes, especially empirical generalizations, but the reader can get the general idea of what I am talking about from that list.

6 I use what Anderson and Belnap (1975) call $\vee E^s$, because it is simpler than the usual rule.

7 Ross Brady has developed a natural deduction system that is supposed to eliminate the need for this additional rule for distribution. It has not yet been published.

8 In the full Routley–Meyer semantics there is a binary relation $\leq$ on situations. If Cab, then either $b < a^*$ or $b = a$.

9 This is stronger than the closest property had by classical logic. For classical logic, if $A \rightarrow B$ is a theorem, then either A and B share a variable or $\sim A$ is a theorem *or* B is a theorem. This property is called 'Halldén reasonableness.'

10 This example is courtesy of A. R. Anderson, but comes originally from a children's story. If any reader knows the title and author of this story, I would appreciate the information.

11 Ramsey is a dog.

12 If we add 'still' in the consequent of the above conditional, it becomes acceptable; 'still' and 'even' would seem to be 'relevance breakers' and a non-relevant analysis of them is appropriate.

References

Anderson, Alan R. and Nuel D. Belnap, Jr. (1975) *Entailment: Logic of Relevance and Necessity*, vol. I. Princeton, NJ: Princeton University Press.

Anderson, Alan R., Nuel D. Belnap, Jr. and Dunn, J. M. (1992) *Entailment: Logic of Relevance and Necessity*, vol. II. Princeton, NJ: Princeton University Press.

Avron, Arnon (1990a) Relevance and paraconsistency – a new approach. *Journal of Symbolic Logic*, 55, 707–32.

Avron, Arnon (1990b) Relevance and paraconsistency – a new approach II: the formal systems. *Notre Dame Journal of Formal Logic*, 31, 168–202.

Brady, Ross T. (1983) The simple consistency of a set theory based on the logic CSQ. *Notre Dame Journal of Formal Logic*, 24, 431–49.

Brady, Ross T. (1989) The non-triviality of dialectical set theory. In G. Priest, R. Routley and J. Norman (eds.), *Paraconsistent Logic* (pp. 437–71). Munich: Philosophia Verlag.

Brady, Ross T. (forthcoming) *Universal Logic.* Stanford, CA: CSLI.

Copeland, B. J. (1979) On when a semantics is not a semantics: some reasons for disliking the Routley–Meyer semantics for relevance logic. *Journal of Philosophical Logic*, 8, 399–413.

Devlin, Keith (1991) *Logic and Information.* Cambridge: Cambridge University Press.

Dunn, J. M. (1986) Relevance logic and entailment. In D. Gabbay and F. Geunthner (eds.), *Handbook of Philosophical Logic III* (pp. 117–224). Dordrecht: Reidel.

Dunn, J. M. (1987) Relevant predication I: the formal theory. *Journal of Philosophical Logic*, 16, 347–81.

Dunn, J. M. (1990a) Relevant predication II: intrinsic properties and internal relations. *Philosophical Studies*, 60, 177–206.

Dunn, J. M. (1990b) Relevant predication III: essential properties. In J. M. Dunn and A. Gupta (eds.), *Truth or Consequences: Essays in Honour of Nuel Belnap* (pp. 77–95). Dordrecht: Kluwer.

Dunn, J. M. (1993) Star and perp. *Philosophical Perspectives*, 7, 331–57.

Fine, Kit (1974) Models for entailment. *Journal of Philosophical Logic*, 3, 347–72.

Fogelin, Robert (1978) *Understanding Arguments: An Introduction to Informal Logic*. New York: Harcourt, Brace, Jovanovich.

Gabbay, Dov (1972) A general theory of the conditional in terms of a ternary operator. *Theoria*, 38, 97–104.

Gabbay, Dov (1976) *Investigations in Modal and Tense Logics with Applications to Problems in Philosophy and Linguistics.* Dordrecht: Reidel.

Israel, David and John Perry (1990) What is information? In P. P. Hanson (ed.), *Information, Language, and Cognition* (pp. 1–19). Vancouver: University of British Columbia Press.

Lemmon, E. J. (1965) *Beginning Logic.* London: Nelson.

Lewis, C. I. and Langford, C. H. (1959) *Symbolic Logic*, 2nd end. New York: Dover.

Lewis, David K. (1988) Relevant implication. *Theoria*, 54, 161–74. Reprinted in D. K. Lewis, *Papers in Philosophical Logic*. Cambridge: Cambridge University Press, 1998, 111–24.

Mares, Edwin D. (1996) Relevant logic and the theory of information. *Synthese*, 109, 345–60.

Mares, Edwin D. (forthcoming) Relevant implication and the indicative conditional. *Synthese.*

Mares, Edwin D. and Robert K. Meyer (2001) Relevant logics – an overview, in L. Goble (ed.), *A Guide to Philosophical Logic*. Oxford: Blackwell.

Meyer, Robert K., Richard Routley and Dunn, J. M. (1979) Curry's Paradox. *Analysis*, n.s. 39, 124–8.

Mortensen, Chris (1986) Reply to Burgess and to Read. *Notre Dame Journal of Formal Logic*, 27, 195–200.

Restall, Greg (2000) *An Introduction to Substructural Logics*. London: Routledge.
Read, Stephen (1988) *Relevant Logic*. Oxford: Blackwell.
Routley, Richard and Robert K. Meyer (1973) Semantics of entailment. In H. Leblanc (ed.), *Truth, Syntax, and Modality* (pp. 199–243). Amsterdam: North Holland.
Routley, Richard and Robert K. Meyer (1972a) Semantics of entailment II. *Journal of Philosophical Logic*, 1, 53–73.
Routley, Richard and Robert K. Meyer (1972b) Semantics of Entailment III. *Journal of Philosophical Logic*, 1, 192–208.
Routley, Richard, Robert K. Meyer, Ross T. Brady and Val Plumwood (1982) *Relevant Logics and its Rivals*, vol. 1. Atascardero: Ridgeview.
Routley, Richard and Val Routley (1972) Semantics for first degree entailment. *Noûs*, 6, 335–59.
Urquhart, Alasdair (1972) Semantics for relevant logics. *Journal of Symbolic Logic*, 37, 159–69. Reprinted in Anderson, Belnap, and Dunn (1992).

Further Reading

A longer, more technical but very readable introduction is J. M. Dunn (1986). Dunn and Greg Restall have rewritten this article for a new addition of the *Handbook of Philosophical Logic* which is forthcoming. Greg Restall has also written a textbook on substructural logics (2000), among which are relevant logics. Mares and Meyer (2001) is a longer introduction than the current piece and somewhat more detailed (although it does not deal with all the topics treated here).

For the more advanced reader, Anderson and Belnap (1975) and Anderson *et al.* (1992), as well as Routley *et al.* (1982) contain most of the formal material on these logics. Of course, there have been technical developments since these books were published.

On the more philosophical side, Read (1988) is an interesting attempt to interpret and defend relevant logic. His work is idiosyncratic, but this is necessarily the case in relevant logic. Relevant logic, unlike intuitionist logic for example, was not developed with the aim of articulating a philosophical position. And relevant logic fits with many philosophical perspectives. It is my experience that there are as many interpretations of relevant logic as there are relevant logicians. For a very different philosophical outlook, see Routley *et al.* (1982).