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# Semantics for a basic relevant logic with intensional conjunction and disjunction (and some of its extensions)

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*In Memory of Sauro Tulipani*

This paper proposes a new relevant logic $\mathbf{B}^+_{\sqcap\sqcup}$, which is obtained by adding two binary connectives, *intensional conjunction* $\sqcap$ and *intensional disjunction* $\sqcup$, to Meyer–Routley minimal positive relevant logic $\mathbf{B}^+$, where $\sqcap$ and $\sqcup$ are weaker than fusion $\circ$ and fission $+$, respectively. We give Kripke-style semantics for $\mathbf{B}^+_{\sqcap\sqcup}$, with $\rightarrow$, $\sqcap$ and $\sqcup$ modelled by ternary relations. We prove the soundness and completeness of the proposed semantics. A number of axiomatic extensions of $\mathbf{B}^+_{\sqcap\sqcup}$, including negation-extensions, are also considered, together with the corresponding semantic conditions required for soundness and completeness to be maintained.

## 1. Introduction

With sufficiently strong relevant logics, there are two derivative connectives, $\circ$ and $+$, which may be defined as $A \circ B =_{df} \neg(A \rightarrow \neg B)$ and $A + B =_{df} \neg A \rightarrow B$ (Anderson and Belnap 1975). The former is called *fusion* and the latter *fission*. These two connectives may also be called *intensional conjunction* and *intensional disjunction*, since they may share some of the features classically attributed to extensional conjunction $\wedge$ and disjunction $\vee$, respectively. In general, $\wedge$ will be interpreted as a lattice 'meet' and $\vee$ as a lattice 'join'. But $\circ$ fails to have 'the lattice property' $A \circ B \rightarrow A$ or $A \circ B \rightarrow B$, so it is not $\wedge$; similarly, $+$ fails to satisfy $A \rightarrow A + B$ or $B \rightarrow A + B$, so it is not $\vee$.

By the above definitions, $\circ$ and $+$ are highly related to implication $\rightarrow$ and negation $\neg$. But in various applications in computer science and artificial intelligence such as automated theorem finding, knowledge discovery, reasoning rule generation, and so on, weaker versions of the standard intensional connectives $\circ$ and $+$ may play important roles (Cheng 2006). In order to axiomatise logics with weaker intensional conjunction and disjunction, we propose a basic relevant logic $\mathbf{B}^+_{\sqcap\sqcup}$, which is obtained by adding two binary connectives, $\sqcap$ and $\sqcup$, to the minimal positive relevant logic $\mathbf{B}^+$ proposed in Routley and Meyer (1972), where $\sqcap$ and $\sqcup$ are characterised in such a way that neither of them relies on the presence of negation $\neg$. That is, we adopt Dunn's approach (Dunn 1990) in which we assign to each of $\sqcap$ and $\sqcup$ a *distribution type*† such that $\sqcap$ shares the same distribution type with

† Dunn's general approach is algebraic, where each logical connective is characterised as an operation on distributive lattices, which 'distributes' in each of its places over at least one of $\wedge$ and $\vee$, leaving $\wedge$ or

$\circ$[†], and $\sqcup$ shares with $+$[‡]. Then, additional axioms or rules can be added to make $\sqcap$ coincide with $\circ$, and $\sqcup$ with $+$. This qualifies $\sqcap$ and $\sqcup$ as weaker versions of intensional conjunction and disjunction, respectively.

To give a semantics for $\mathbf{B}^+_{\sqcap\sqcup}$, we apply Dunn's strategy (Dunn 1990), that is, we use $n+1$-placed accessibility relations to model $n$-placed connectives. The semantics is defined by adapting and extending the traditional relational semantics for relevant logics. There are four ternary relations: $R_1$ and $R_2$ for $\rightarrow$; $S_1$ for $\sqcap$; and $S_2$ for $\sqcup$. To construct canonical models, as well as *theories*, we define *dualtheories* and *anti-dualtheories* such that $R_1, R_2, S_1, S_2$ are canonically defined as derivatives of operations on theories and anti-dualtheories. The crucial tools for completeness are extensions or reductions of a given theory or anti-dualtheory to a *prime theory*. Then, by well-known standard techniques, together with our extra definitions, we can establish the soundness and completeness of the proposed semantics for $\mathbf{B}^+_{\sqcap\sqcup}$. Furthermore, we consider a number of axiomatic extensions of $\mathbf{B}^+_{\sqcap\sqcup}$ (including negation-extensions with negation modelled by the Routley '$*$'-operation), together with the corresponding semantic conditions to ensure that soundness and completeness are maintained.

## 2. The basic system $\mathbf{B}^+_{\sqcap\sqcup}$

### 2.1. *An axiom system for* $\mathbf{B}^+_{\sqcap\sqcup}$

$\mathbf{B}^+_{\sqcap\sqcup}$ is expressed in a language $\mathscr{L}$, which has the two-place connectives $\rightarrow, \wedge, \vee, \sqcap$ and $\sqcup$, parentheses ( and ), and a stock of propositional variables $p, q, r, \ldots$ Formulas are defined recursively in the usual manner. We use the following scope conventions: the connectives are ranked $\sqcap$, $\sqcup$, $\wedge$, $\vee$, $\rightarrow$ in order of increasing scope (that is, $\sqcap$ binds more strongly than $\sqcup$, $\sqcup$ binds more strongly than $\wedge$, and so on), otherwise, association is to the left. $A, B, C, \ldots$ will be used to range over arbitrary formulas.

We begin by giving an axiom system for $\mathbf{B}^+$, which is defined in the same way as that of Priest and Sylvan (1992) and Restall (1993)[§]:

**Axioms**

**A1** $A \rightarrow A$

**A2** $A \rightarrow A \vee B$, $B \rightarrow A \vee B$

**A3** $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$

**A4** $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$

**A5** $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$

**A6** $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$

$\vee$ unchanged or switching it with its dual. More explicitly, let $\tau$ (with subscripts) range over $\{\wedge, \vee\}$, and associate with each $n$-ary operation $f$ a distribution type $(\tau_1, \ldots, \tau_i, \ldots, \tau_n) \longmapsto \tau$. Then, where $*$ and $\star$ is $\wedge$ or $\vee$ depending on the value of $\tau_i$ and $\tau$, respectively, $f(a_1, \ldots, b * c, \ldots, a_n) = f(a_1, \ldots, b, \ldots, a_n) \star f(a_1, \ldots, c, \ldots, a_n)$.

† When $\circ$ is defined as $A \circ B =_{df} \neg(A \rightarrow \neg B)$, its distribution type is $(\vee, \vee) \longmapsto \vee$, that is, $\circ$ satisfies $(A \vee B) \circ C \leftrightarrow (A \circ C) \vee (B \circ C)$ and $C \circ (A \vee B) \leftrightarrow (C \circ A) \vee (C \circ B)$.

‡ When $+$ is defined as $A + B =_{df} \neg A \rightarrow B$, its distribution type is $(\wedge, \wedge) \longmapsto \wedge$, that is, $+$ satisfies $(A \wedge B) + C \leftrightarrow (A + C) \wedge (B + C)$ and $C + (A \wedge B) \leftrightarrow (C + A) \wedge (C + B)$.

§ Here, disjunctive forms of rules are not given separately.

**Rules**

**R1** $A, A \to B \Rightarrow B$ (Modus Ponens)
**R2** $A, B \Rightarrow A \wedge B$ (Adjunction)
**R3** $A \to B, C \to D \Rightarrow (B \to C) \to (A \to D)$ (Affixing).

Thus $\mathbf{B}^+_{\sqcap\sqcup}$ is obtained by adding the following axioms and rules to $\mathbf{B}^+$:

**A7** $(A \vee B) \sqcap C \leftrightarrow (A \sqcap C) \vee (B \sqcap C)$,
$C \sqcap (A \vee B) \leftrightarrow (C \sqcap A) \vee (C \sqcap B)$
**A8** $(A \sqcup C) \wedge (B \sqcup C) \leftrightarrow (A \wedge B) \sqcup C$,
$(C \sqcup A) \wedge (C \sqcup B) \leftrightarrow C \sqcup (A \wedge B)$
**R4** $A \to B, C \to D \Rightarrow A \sqcap C \to B \sqcap D$
**R5** $A \to B, C \to D \Rightarrow A \sqcup C \to B \sqcup D$.

It may be noted that special cases of **R3** are:

$A \to B \Rightarrow (C \to A) \to (C \to B)$ (Prefixing)
$A \to B \Rightarrow (B \to C) \to (A \to C)$ (Suffixing)
$A \to B, B \to C \Rightarrow A \to C$ (Transitivity).

And special cases of **R4** and **R5** are, respectively:

$$A \to B \Rightarrow C \sqcap A \to C \sqcap B$$
$$A \to B \Rightarrow A \sqcap C \to B \sqcap C$$
$$A \to B \Rightarrow C \sqcup A \to C \sqcup B$$
$$A \to B \Rightarrow A \sqcup C \to B \sqcup C.$$

Note that **A7** and **A8** contain slight redundancies. **R4** and **R5**, together with the axioms and rules of $\mathbf{B}^+$, suffice to prove each of **A7** and **A8** in right-to-left direction.

Given a logical system **L**, we use $\vdash_L A$ to denote the fact that $A$ is a theorem of **L**. If **L** is obvious, the subscript '$_L$' will be omitted.

## 2.2. *Semantics for* $\mathbf{B}^+_{\sqcap\sqcup}$

Now we define semantics for $\mathbf{B}^+_{\sqcap\sqcup}$. We also give some notions for its extensions, that is, logics obtained by adding one or more axioms or rules to $\mathbf{B}^+_{\sqcap\sqcup}$. The semantics is an extension of the traditional semantics for $\mathbf{B}^+$ – see Routley *et al.* (1982, Chapter 4).

A $\mathbf{B}^+_{\sqcap\sqcup}$-*frame* (or *model structure*) is a 7-tuple $\mathscr{F} =< o, W, O, R_1, R_2, S_1, S_2 >$, where $W$ is a set (of worlds), $o \in W$ (the base world), $O$ is a nonempty subset of $W$, and $R_1, R_2, S_1$, and $S_2$ are ternary relations on $W$, such that definitions **d1**–**d4** apply and postulates **p1**–**p7** hold for every $a, b, c, d, e \in W^\dagger$.

**d1.** $a \leqslant b =_{df} \exists x(x \in O$ and $R_1xab)$
**d2.** $T_1(T_2ab)cd =_{df} \exists x(T_2abx$ and $T_1xcd)$
**d3.** $T_1a(T_2bc)d =_{df} \exists x(T_2bcx$ and $T_1axd)$
**d4.** $T(T_1ab)(T_2cd)e =_{df} \exists x, y(T_1abx, T_2cdy$ and $Txye)$,

$^\dagger$ Because the system $\mathbf{B}^+_{\sqcap\sqcup}$ is not strong enough, we can not use a 'reduced' frame, that is, to reduce $O$ to a single element, that is, the base world $o$. Actually, in non-reduced frames $O$ plays an important role to guarantee the soundness result.

where $T$, $T_1$ and $T_2$ represent any of $R_1, R_2, S_1$ and $S_2$. For **d2** and **d3**, if $T_1$ and $T_2$ coincide, we usually abbreviate $T_1(T_2ab)cd$ to $T_1(ab)cd$, and $T_1a(T_2bc)d$ to $T_1a(bc)d$. For example, $R_1(S_2ab)cd$ is defined as $\exists x(S_2abx$ and $R_1xcd)$, and $R_1(ab)cd$ as $\exists x(R_1abx$ and $R_1xcd)$†.

**p1.** $o \in O$
**p2.** $a \leqslant a$
**p3.** $R_1abc$ iff $R_2acb$
**p4.** if $R_1abc$ and $a' \leqslant a$, then $R_1a'bc$
**p5.** if $R_2abc$ and $a' \leqslant a$, then $R_2a'bc$
**p6.** if $S_1abc$ and $c \leqslant c'$, then $S_1abc'$
**p7.** if $S_2abc$ and $c' \leqslant c$, then $S_2abc'$.

In fact, definitions **d2–4** are only necessary for some extensions of $\mathbf{B}^+_{\sqcap\sqcup}$. We list them here for later use.

A $\mathbf{B}^+_{\sqcap\sqcup}$-*model* (or *interpretation*) is an 8-tuple $\mathscr{M} =< o, W, O, R_1, R_2, S_1, S_2, I >$, where $< o, W, O, R_1, R_2, S_1, S_2 >$ is a $\mathbf{B}^+_{\sqcap\sqcup}$-frame and $I$ is a function that assigns to each pair of a formula, $A$, and a world, $x$, a truth value $I(A, x) \in \{1, 0\}$ that satisfies the following condition and rules.

**Atomic Hereditary Condition.**

For a propositional variable $p$, if $I(p, x) = 1$ and $x \leqslant y$, then $I(p, y) = 1$.

**Evaluation Rules.**

$(\wedge)$ $I(A \wedge B, a) = 1$ iff $I(A, a) = 1$ and $I(B, a) = 1$
$(\vee)$ $I(A \vee B, a) = 1$ iff $I(A, a) = 1$ or $I(B, a) = 1$
$(\sqcap)$ $I(A \sqcap B, c) = 1$ iff $\exists a, b \in W$, $S_1abc$, $I(A, a) = 1$ and $I(B, b) = 1$
$(\sqcup)$ $I(A \sqcup B, c) = 1$ iff $\forall a, b \in W$, if $S_2abc$, then $I(A, a) = 1$ or $I(B, b) = 1$
$(\rightarrow_1)$ $I(A \rightarrow B, a) = 1$ iff $\forall b, c \in W$, if $R_1abc$ and $I(A, b) = 1$, then $I(B, c) = 1$
$(\rightarrow_2)$ $I(A \rightarrow B, a) = 1$ iff $\forall b, c \in W$, if $R_2abc$ and $I(A, c) = 1$, then $I(B, b) = 1$.

With **p3**, it is easy to see that the rules $(\rightarrow_1)$ and $(\rightarrow_2)$ are equivalent. Hence, if we omit $R_2$, a $\mathbf{B}^+_{\sqcap\sqcup}$-model is indeed an extension of a $\mathbf{B}^+$-model by adding definitions and postulates for $S_1$, $S_2$ and evaluation rules for $\sqcap$, $\sqcup$. But, the inclusion of $R_2$ makes it easier to give semantic conditions for additional axioms or rules involving $\sqcup$. So, we introduce relation $R_2$ and rule $(\rightarrow_2)$.

In this paper, we will usually use the following rules, which are equivalent to the above rules $(\sqcup)$ and $(\rightarrow_2)$, respectively:

$(\sqcup')$ $I(A \sqcup B, c) \neq 1$ iff $\exists a, b \in W$, $S_2abc$, $I(A, a) \neq 1$ and $I(B, b) \neq 1$;
$(\rightarrow_2')$ $I(A \rightarrow B, a) = 1$ iff $\forall b, c \in W$, if $R_2abc$ and $I(B, b) \neq 1$, then $I(A, c) \neq 1$.

Assuming **L** is a logic obtained by adding additional axioms or rules to $\mathbf{B}^+_{\sqcap\sqcup}$, an **L**-*frame* $\mathscr{F}$ or **L**-*model* $\mathscr{M}$ is obtained by adding corresponding conditions to a $\mathbf{B}^+_{\sqcap\sqcup}$-frame or $\mathbf{B}^+_{\sqcap\sqcup}$-model.

† Our representation is a little different from that in Routley *et al.* (1982), where, given a ternary relation $R$, we have $R^2abcd =_{df} \exists x(Rabx$ and $Rxcd)$, $R^2a(bc)d =_{df} \exists x(Rbcx$ and $Raxd)$, and $R^3ab(cd)e =_{df} \exists x(R^2abxe$ and $Rcdx)$.

Now, taking **L** to be any logic in this paper, we define:

— $A$ is *valid on an* **L**-*model* if $I(A, o) = 1$.
— $A$ *implies* $B$ *on an* **L**-*model* if for every $a \in W$, if $I(A, a) = 1$, then $I(B, a) = 1$.
— $A$ is *valid on an* **L**-*frame* if $A$ is valid on every **L**-model based on this frame.
— $A$ *implies* $B$ *on an* **L**-*frame* if $A$ implies $B$ on every **L**-model based on this frame.
— $A$ is **L**-*valid* if $A$ is valid on every **L**-frame.
— $A$ **L**-*implies* $B$ if $A$ implies $B$ on every **L**-frame.

The following lemmas will simplify the proof for soundness.

**Lemma 2.1 (Hereditary Condition).** For an arbitrary formula $A$, if $I(A, x) = 1$ and $x \leqslant y$, then $I(A, y) = 1$.

*Proof.* The proof is by an induction on the construction of $A$ with the Atomic Hereditary Condition as basis – note how **p4**–**7** are used. We just give proofs for $\sqcap$, $\sqcup$.

($\sqcap$) Suppose $I(A \sqcap B, x) = 1$ and $x \leqslant y$. Then, $\exists a, b \in W$ such that $S_1abx$, $I(A, a) = 1$ and $I(B, b) = 1$. By **p6**, we have $S_1aby$. So $I(A \sqcap B, y) = 1$ as required.

($\sqcup$) Suppose $I(A \sqcup B, y) \neq 1$ and $x \leqslant y$, to show $I(A \sqcup B, x) \neq 1$. Then $\exists a, b \in W$ such that $S_2aby$, $I(A, a) \neq 1$ and $I(B, b) \neq 1$. By **p7**, $S_2abx$. So $I(A \sqcup B, x) \neq 1$ as required. □

**Lemma 2.2 (Verification Lemma).**

(1) If $A$ implies $B$ on an **L**-model, then $A \to B$ is valid on this model.
(2) If $A$ implies $B$ on an **L**-frame, then $A \to B$ is valid on this frame.
(3) $A$ **L**-implies $B$ if and only if $A \to B$ is **L**-valid.

*Proof.* The proof is similar to that in Routley *et al.* (1982, pages 302–303). It is easy to show (1), (2) and the left-to-right direction of (3) by **d1**, **p1**, rule ($\to_1$) and Lemma 2.1. The converses of (1) and (2) fail, since there is no guarantee that $R_1oaa$ holds for every $a$ in an arbitrary **L**-frame or **L**-model.

We will now give the proof in full for the right-to-left direction of (3). Assume $A \to B$ is **L**-valid. Suppose $\mathscr{F}$ is an arbitrary **L**-frame with the base world $o$ in order to show $A$ implies $B$ on $\mathscr{F}$. Suppose also that $\mathscr{M}$ is an arbitrary **L**-model based on $\mathscr{F}$ with the assignment function $I$, and that $I(A, a) = 1$ for an arbitrary $a \in W$. Then it suffices to show $I(B, a) = 1$, since then $A$ implies $B$ on $\mathscr{M}$, and, furthermore, since $\mathscr{M}$ is arbitrary, $A$ implies $B$ on $\mathscr{F}$. Now by **d1** and **p2**, for some $o' \in W$, we have $o' \in O$ and $R_1o'aa$. Consider an **L**-frame $\mathscr{F}'$, which differs from $\mathscr{F}$ simply in a change in the base world brought about by selecting $o'$ as base in place of $o$. So $\mathscr{F}'$ is an **L**-frame, since no semantic condition depends on the choice of $o$ as base. We now define an assignment function $I'$ in $\mathscr{F}'$ such that $I'(C, x) = I(C, x)$ for every formula $C$ and every world $x$. Hence we obtain an **L**-model $\mathscr{M}'$ based on $\mathscr{F}'$ with the assignment function $I'$ such that $I'(A, a) = 1$ and $I'(B, a) = I(B, a)$. $\mathscr{M}'$ is indeed an **L**-model, since neither the Atomic Hereditary Condition nor the Evaluation Rules depend on the choice of $o$ as base, so both of them still hold. Since $A \to B$ is **L**-valid, $I'(A \to B, o') = 1$. But $R_1o'aa$, so it follows that $I'(B, a) = 1$ by rule ($\to_1$). Hence $I(B, a) = 1$ as required. □

Thus, for soundness, in order to show the validity of axioms with the form $A \to B$, we usually suppose for an arbitrary **L**-model $\mathscr{M}$, that $I(A,x) = 1$ ($I(B,x) \neq 1$) in order to show $I(B,x) = 1$ ($I(A,x) \neq 1$). Then, $A \to B$ is valid on $\mathscr{M}$ by Lemma 2.2 (1). Hence, since $\mathscr{M}$ is arbitrary, $A \to B$ is **L**-valid. Conversely, if $A \to B$ is **L**-valid, then for an arbitrary **L**-model $\mathscr{M}$, we will have $I(B,x) = 1$ ($I(A,x) \neq 1$) from $I(A,x) = 1$ ($I(B,x) \neq 1$) by Lemma 2.2 (3).

### 2.3. *Soundness*

In this section we demonstrate the soundness of the semantics for $\mathbf{B}^+_{\sqcap\sqcup}$.

**Theorem 2.3.** If $A$ is a theorem of $\mathbf{B}^+_{\sqcap\sqcup}$, then $A$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid.

*Proof.* The proof is by a simple induction over the length of proofs. It suffices to prove that all axioms are $\mathbf{B}^+_{\sqcap\sqcup}$-valid and all rules preserve validity. We just give proofs for one of **A8** (in one direction) and **R4**.

**A8** Suppose $I((A \wedge B) \sqcup C, c) \neq 1$ in order to show $I((A \sqcup C) \wedge (B \sqcup C), c) \neq 1$. Then $\exists a, b \in W$ such that $S_2abc$, $I(A \wedge B, a) \neq 1$ and $I(C, b) \neq 1$. So $I(A, a) \neq 1$ or $I(B, a) \neq 1$. Since $S_2abc$, we have $I(A \sqcup C, c) \neq 1$ or $I(B \sqcup C, c) \neq 1$ by rule ($\sqcup$). Hence $I((A \sqcup C) \wedge (B \sqcup C), c) \neq 1$ as required.

**R4** Suppose $A \to B$ and $C \to D$ are $\mathbf{B}^+_{\sqcap\sqcup}$-valid in order to show that $A \sqcap C \to B \sqcap D$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid. Suppose also that $I(A \sqcap C, c) = 1$. It suffices to show $I(B \sqcap D, c) = 1$. Then $\exists a, b \in W$ such that $S_1abc$, $I(A, a) = 1$ and $I(C, b) = 1$. Since $A \to B$ and $C \to D$ are $\mathbf{B}^+_{\sqcap\sqcup}$-valid, we have $I(B, a) = 1$ and $I(D, b) = 1$ by Lemma 2.2 (3). But $S_1abc$, so by rule ($\sqcap$), we have $I(B \sqcap D, c) = 1$ as required. □

### 2.4. *Key notions for completeness*

We establish completeness in the usual way. For any non-theorem $A$, we design a canonical model that refutes $A$. Most of the techniques come from Routley *et al.* (1982, Chapter 4), and Brady (2003, Chapter 8). In this section, we will give some definitions for any logic **L** in this paper.

First, we establish some conventions. With $\Sigma$ the set of all formulas, we have for every $V, U \subseteq \Sigma$:

(1) $U$ is **L**-*derivable* from $V$, written $V \vdash_L U$, if and only if for some $A_1, \ldots, A_n$ in $V$ and some $B_1, \ldots, B_m$ in $U$, we have $\vdash_L A_1 \wedge \ldots \wedge A_n \to B_1 \vee \ldots \vee B_m$.

(2) An **L**-*derivation* of $A$ from $V$, written $V \vdash_L A$, is a finite sequence of formulas $A_1, \ldots, A_n$, with $A_n = A$ such that each member $A_i$ of the sequence either belongs to $V$ or is obtained from predecessors in the sequence by adjunction or a provable **L**-implication (that is, in the latter case, $A_i$ is obtained from $A_j$ ($j < i$) since $\vdash_L A_j \to A_i$).

(3) An **L**-*derivation* of $U$ from $V$ is an **L**-derivation of some disjunction $B_1 \vee \ldots \vee B_m$ of formulas $B_1, \ldots, B_m$ of $U$ from $V$. Hence, $U$ is **L**-derivable from $V$ if and only if there is an **L**-derivation of $U$ from $V$.

(4) $< V, U >$ is an **L**-*maximal pair* if and only if:

— $V \cup U = \Sigma$

— $V \nvdash_L U$.

It is immediate that if $< V, U >$ is an **L**-maximal pair, then $V \cap U = \varnothing$.

Next, it is easy to see that if $a \subseteq \Sigma$ and $b = \Sigma - a$, then $a$ satisfies the following **a1**, **a2**, **a3** separately if and only if $b$ satisfies **b1**, **b2**, **b3** separately.

**a1.** If $\vdash_L A \to B$ and $A \in a$, then $B \in a$.
**a2.** If $A \in a$ and $B \in a$, then $A \wedge B \in a$.
**a3.** If $A \vee B \in a$, then $A \in a$ or $B \in a$.
**b1.** If $\vdash_L A \to B$ and $B \in b$, then $A \in b$.
**b2.** If $A \wedge B \in b$, then $A \in b$ or $B \in b$.
**b3.** If $A \in b$ and $B \in b$, then $A \vee B \in b$.

Then we define, for every $a, b \subseteq \Sigma$:

(1) $a$ is an **L**-*theory* if and only if it satisfies **a1** and **a2**.
(2) An **L**-theory $a$ is *prime* if and only if it satisfies **a3** also.
(3) $a$ is an **L**-*anti-dualtheory* if and only if it satisfies **a1** and **a3**.
(4) An **L**-anti-dualtheory $a$ is *prime* if and only if it satisfies **a2** also.
(5) $b$ is an **L**-*dualtheory* if and only if it satisfies **b1** and **b3**.
(6) An **L**-dualtheory $b$ is *prime* if and only if it satisfies **b2** also.

Thus, if $a \subseteq \Sigma$ and $b = \Sigma - a$:

— $a$ is a prime **L**-theory if and only if $a$ is a prime **L**-anti-dualtheory.
— $a$ is an **L**-anti-dualtheory if and only if $b$ is an **L**-dualtheory.
— $a$ is a prime **L**-theory if and only if $b$ is a prime **L**-dualtheory.

It is obvious that the set of all theorems of **L** is a theory. We will use $l$ to denote this particular theory. In addition, an **L**-theory $a$ is *regular* if and only if $l \subseteq a$, that is, whenever $\vdash_L A$, $A \in a$.

In the following, the subscript '$_L$' and the prefix '**L**-' will be omitted if system **L** is obvious.

Now we define four operations[†] on sets of formulas. For every $a, b \subseteq \Sigma$:

$$
\begin{aligned}
a \oplus b &= \{B : \exists A \in b, A \to B \in a\} \\
a \otimes b &= \{A : \forall B, A \to B \in a \Rightarrow B \in b\}^{\ddagger} \\
a \ominus b &= \{C : \exists A \in a, \exists B \in b, \vdash_L A \sqcap B \to C\} \\
a \oslash b &= \Sigma - \{C : \exists A \notin a, \exists B \notin b, \vdash_L C \to A \sqcup B\}.
\end{aligned}
$$

We now give some propositions for $\oplus, \otimes, \ominus$ and $\oslash$.

**Proposition 2.4.**

(1) If $a, b$ are **L**-theories, then $a \oplus b$ is an **L**-theory.
(2) If $a$ is an **L**-theory and $b$ is an **L**-anti-dualtheory, then $a \otimes b$ is an **L**-anti-dualtheory.
(3) If $a, b$ are **L**-theories, then $a \ominus b$ is an **L**-theory.
(4) If $a, b$ are **L**-anti-dualtheories, then $a \oslash b$ is an **L**-anti-dualtheory.

[†] The notation '$\oplus$' and '$\otimes$' come from Brady (2003).

*Proof.* We will just show (4) as an example. First, suppose $C_2 \notin a \oslash b$ and $\vdash C_1 \to C_2$ in order to show $C_1 \notin a \oslash b$. Then $\exists A \notin a, \exists B \notin b$ such that $\vdash C_2 \to A \sqcup B$. So $\vdash C_1 \to A \sqcup B$. Hence $C_1 \notin a \oslash b$ as required.

Now suppose $C_1, C_2 \notin a \oslash b$ in order to show $C_1 \vee C_2 \notin a \oslash b$. Then $\exists A_1, A_2 \notin a, \exists B_1, B_2 \notin b$ such that $\vdash C_1 \to A_1 \sqcup B_1$ and $\vdash C_2 \to A_2 \sqcup B_2$. Since $\vdash A_1 \sqcup B_1 \to (A_1 \sqcup B_1) \vee (A_2 \sqcup B_2)$, we have $\vdash C_1 \to (A_1 \sqcup B_1) \vee (A_2 \sqcup B_2)$. Similarly, $\vdash C_2 \to (A_1 \sqcup B_1) \vee (A_2 \sqcup B_2)$. So $\vdash C_1 \vee C_2 \to (A_1 \sqcup B_1) \vee (A_2 \sqcup B_2)$. Now, since $\vdash A_1 \to A_1 \vee A_2$ and $\vdash B_1 \to B_1 \vee B_2$, we have $\vdash A_1 \sqcup B_1 \to (A_1 \vee A_2) \sqcup (B_1 \vee B_2)$ by **R5**. Similarly, $\vdash A_2 \sqcup B_2 \to (A_1 \vee A_2) \sqcup (B_1 \vee B_2)$. So $\vdash (A_1 \sqcup B_1) \vee (A_2 \sqcup B_2) \to (A_1 \vee A_2) \sqcup (B_1 \vee B_2)$. Thus $\vdash C_1 \vee C_2 \to (A_1 \vee A_2) \sqcup (B_1 \vee B_2)$. Since $a$ and $b$ are **L**-anti-dualtheory, $A_1 \vee A_2 \notin a$ and $B_1 \vee B_2 \notin b$. So $C_1 \vee C_2 \notin a \oslash b$ as required. □

Thus, if $a$ and $b$ are prime **L**-theories, then $a \oplus b$ and $a \ominus b$ are **L**-theories; $a \otimes b$ and $a \oslash b$ are **L**-anti-dualtheories.

**Proposition 2.5.** For every $a, b, c \subseteq \Sigma$, we have $a \oplus b \subseteq c$ if and only if $b \subseteq a \otimes c$.

*Proof.* For the left-to-right direction, suppose $A \in b$, but $A \notin a \otimes c$. Then $\exists B \notin c$ such that $A \to B \in a$. Since $A \in b$, we have $B \in a \oplus b$; and since $a \oplus b \subseteq c$, we have $B \in c$, giving a contradiction. Thus $A \in a \otimes c$.

For the right-to-left direction, suppose $B \in a \oplus b$ in order to show $B \in c$. Then $\exists A \in b$ such that $A \to B \in a$. Since $b \subseteq a \otimes c$, we have $A \in a \otimes c$, so $B \in c$ as required. □

We now define ternary relations $R_1, R_2, S_1, S_2$ on sets of formulas. For every $a, b, c \subseteq \Sigma$:

— $R_1abc$ if and only if $a \oplus b \subseteq c$, that is, for every $A, B$, if $A \to B \in a$ and $A \in b$, then $B \in c$.

— $R_2abc$ if and only if $c \subseteq a \otimes b$, that is, for every $A, B$, if $A \to B \in a$ and $B \notin b$, then $A \notin c$.

— $S_1abc$ if and only if $a \ominus b \subseteq c$, that is, for every $A, B, C$, if $A \in a, B \in b$ and $\vdash_L A \sqcap B \to C$, then $C \in c$.

— $S_2abc$ if and only if $c \subseteq a \oslash b$, that is, for every $A, B, C$, if $A \notin a$, $B \notin b$ and $\vdash_L C \to A \sqcup B$, then $C \notin c$.

Thus, since $a \oplus b \subseteq c$ if and only if $b \subseteq a \otimes c$, it is immediate that $R_1abc$ if and only if $R_2acb$.

Please note that since $\vdash_L A \sqcap B \to A \sqcap B$, $\vdash_L A \sqcup B \to A \sqcup B$, it is easy to see that for every $a, b, c \subseteq \Sigma$, and every formula $A, B$:

— If $S_1abc$, $A \in a$ and $B \in b$, then $A \sqcap B \in c$.

— If $S_2abc$, $A \notin a$ and $B \notin b$, then $A \sqcup B \notin c$.

### 2.5. *Lemmas for completeness*

We begin by giving some results (Lemmas 2.6–2.8), which are either proved in Routley *et al.* (1982, Pages 307–308)) or are easy to obtain.

**Lemma 2.6.** If $< V, U >$ is an **L**-maximal pair, then $V$ is a prime **L**-theory, and $U$ is a prime **L**-dualtheory.

**Lemma 2.7 (Extension Lemma).** Let $V$ and $U$ be sets of formulas such that $V \nvdash_L U$. Then there is an **L**-maximal pair $< V', U' >$ with $V \subseteq V'$ and $U \subseteq U'$.

**Lemma 2.8 (Priming Lemma 1).** Let $V$ be an **L**-theory, $U$ be closed under disjunction, and $V \cap U = \emptyset$. Then there is an **L**-theory $V'$ such that:

(1) $V \subseteq V'$;
(2) $V' \cap U = \emptyset$; and
(3) $V'$ is prime.

We also have Priming Lemma 2, which is similar to Priming Lemma 1.

**Lemma 2.9 (Priming Lemma 2).** Let $V$ be closed under conjunction, $U$ be an **L**-dualtheory, and $V \cap U = \emptyset$. Then there is an **L**-dualtheory $U'$ such that:

(1) $U \subseteq U'$;
(2) $V \cap U' = \emptyset$; and
(3) $U'$ is prime.

*Proof.* $V \nvdash_L U$ as otherwise there would be $A_1, \ldots, A_n \in V$ such that $A_1 \wedge ... \wedge A_n \in V \cap U$, since $U$ is an **L**-dualtheory. By Lemma 2.7, there is an **L**-maximal pair $< V', U' >$ with $V \subseteq V'$ and $U \subseteq U'$, and the result then follows by Lemma 2.6. □

The following results are proved in Routley *et al.* (1982, Page 309)†.

**Corollary 2.10.**

(1) If $A$ is a non-theorem of **L**, then there is a prime regular **L**-theory $o_c$ such that $A \notin o_c$.
(2) If $a, b$ are **L**-theories, $c$ is an **L**-anti-dualtheory and $R_1abc$, then there is a prime **L**-theory $a'$ such that $a \subseteq a'$ and $R_1a'bc$.
(3) If $a, b$ are **L**-theories, $c$ is an **L**-anti-dualtheory and $R_1abc$, then there is a prime **L**-theory $b'$ such that $b \subseteq b'$ and $R_1ab'c$.
(4) If $a, b, c$ are **L**-theories, $R_1abc$ and $C \notin c$, then there are prime **L**-theories, $b'$, $c'$, such that $b \subseteq b'$, $C \notin c'$ and $R_1ab'c'$.

We now prove several corollaries of Lemmas 2.8 and 2.9.

**Corollary 2.11.**

(1) If $a, c$ are **L**-theories, $b$ is an **L**-anti-dualtheory and $R_2abc$, then there is a prime **L**-theory $a'$ such that $a \subseteq a'$ and $R_2a'bc$.
(2) If $a, c$ are **L**-theories, $b$ is an **L**-anti-dualtheory and $R_2abc$, then there is a prime **L**-theory $b'$ such that $b' \subseteq b$ and $R_2ab'c$.

† Note that the form of (2) and (3) in Corollary 2.10 is a little different from that given in Routley *et al.* (1982), where $c$ is required to be a prime **L**-theory. In fact, it is sufficient to require that $c$ is only an **L**-anti-dualtheory for the proof to go through.

*Proof.* Since $R_2abc$ if and only if $R_1acb$, (1) here is equivalent to Corollary 2.10 (2). Hence, we just give the proof for (2).

Set $V = \{B : \exists A \in c, A \to B \in a\}$. We want to prove:

(a) $V$ is closed under conjunction.
(b) $\Sigma - b$ is disjoint from $V$.

(a) Suppose $B_1, B_2 \in V$. Then $\exists A_1, A_2 \in c$, $A_1 \to B_1 \in a$ and $A_2 \to B_2 \in a$. Since $\vdash A_1 \wedge A_2 \to A_1$, we have $\vdash (A_1 \to B_1) \to (A_1 \wedge A_2 \to B_1)$, so $A_1 \wedge A_2 \to B_1 \in a$. Similarly, $A_1 \wedge A_2 \to B_2 \in a$, so $(A_1 \wedge A_2 \to B_1) \wedge (A_1 \wedge A_2 \to B_2) \in a$. By

$$\vdash (A_1 \wedge A_2 \to B_1) \wedge (A_1 \wedge A_2 \to B_2) \to (A_1 \wedge A_2 \to B_1 \wedge B_2),$$

we have $A_1 \wedge A_2 \to B_1 \wedge B_2 \in a$. Since $c$ is an **L**-theory, $A_1 \wedge A_2 \in c$. So $B_1 \wedge B_2 \in V$ as required.

(b) To show a contradiction, suppose $\exists B \in \Sigma - b$, that is, $B \notin b$ and $B \in V$. Then $\exists A \in c$ such that $A \to B \in a$. But $R_2abc$, so $A \notin c$, which gives a contradiction.

Since $b$ is an **L**-anti-dualtheory, $\Sigma - b$ is an **L**-dualtheory. Hence by (a) and (b), Lemma 2.9 applies to provide a prime **L**-dualtheory $b''$ disjoint from $V$ with $\Sigma - b \subseteq b''$. Let $b' = \Sigma - b''$. Then $b'$ is a prime **L**-anti-dualtheory, that is, a prime **L**-theory, and $b' \subseteq b$. Next we prove $R_2ab'c$. Suppose $A \to B \in a$ and $B \notin b'$, that is, $B \in b''$. Since $b''$ is disjoint from $V$, we have $A \notin c$, so $R_2ab'c$. □

**Corollary 2.12.**

(1) If $a, b$ are **L**-theories, $c$ is an **L**-anti-dualtheory and $S_1abc$, then there is a prime **L**-theory $a'$ such that $a \subseteq a'$ and $S_1a'bc$.
(2) If $a, b$ are **L**-theories, $c$ is an **L**-anti-dualtheory and $S_1abc$, then there is a prime **L**-theory $b'$ such that $b \subseteq b'$ and $S_1ab'c$.

*Proof.* We give the proof for (1); the proof for (2) is similar.

Set $U = \{A : \exists B \in b, \exists C \notin c, \vdash A \sqcap B \to C\}$. We want to prove:

(a) $U$ is closed under disjunction.
(b) $a$ is disjoint from $U$.

(a) Suppose $A_1, A_2 \in U$. Then $\exists B_1, B_2 \in b$, $\exists C_1, C_2 \notin c$ such that $\vdash A_1 \sqcap B_1 \to C_1$ and $\vdash A_2 \sqcap B_2 \to C_2$. Since $\vdash B_1 \wedge B_2 \to B_1$, we have $\vdash A_1 \sqcap (B_1 \wedge B_2) \to A_1 \sqcap B_1$ by **R4**. So $\vdash A_1 \sqcap (B_1 \wedge B_2) \to C_1$. Since $\vdash C_1 \to C_1 \vee C_2$, we have $\vdash A_1 \sqcap (B_1 \wedge B_2) \to C_1 \vee C_2$. Similarly, $\vdash A_2 \sqcap (B_1 \wedge B_2) \to C_1 \vee C_2$. So

$$\vdash (A_1 \sqcap (B_1 \wedge B_2)) \vee (A_2 \sqcap (B_1 \wedge B_2)) \to C_1 \vee C_2\,.$$

By **A7**,

$$\vdash (A_1 \vee A_2) \sqcap (B_1 \wedge B_2) \to (A_1 \sqcap (B_1 \wedge B_2)) \vee (A_2 \sqcap (B_1 \wedge B_2))\,.$$

So $\vdash (A_1 \vee A_2) \sqcap (B_1 \wedge B_2) \to C_1 \vee C_2$. Since $b$ is an **L**-theory, $B_1 \wedge B_2 \in b$. And since $c$ is an **L**-anti-dualtheory, $C_1 \vee C_2 \notin c$. Hence $A_1 \vee A_2 \in U$.

(b) To show a contradiction, suppose $\exists A \in a$ and $A \in U$. Then $\exists B \in b$, $\exists C \notin c$ such that $\vdash A \sqcap B \to C$. But $S_1abc$, so $C \in c$, which gives a contradiction.

Hence, by (a) and (b), Lemma 2.8 applies to provide a prime **L**-theory $a'$ disjoint from $U$ with $a \subseteq a'$. Next we prove $S_1a'bc$. Suppose $A \in a'$, $B \in b$ and $\vdash A \sqcap B \to C$. Since $a'$ is disjoint from $U$, we have $C \in c$. So $S_1a'bc$. □

**Corollary 2.13.**

(1) If $a, b$ are **L**-anti-dualtheories, $c$ is an **L**-theory and $S_2abc$, then there is a prime **L**-anti-dualtheory $a'$ such that $a' \subseteq a$ and $S_2a'bc$.

(2) If $a, b$ are **L**-anti-dualtheories, $c$ is an **L**-theory and $S_2abc$, then there is a prime **L**-anti-dualtheory $b'$ such that $b' \subseteq b$ and $S_2ab'c$.

*Proof.* We just give proof for (1); the proof for (2) is similar.

Set $V = \{A : \exists B \notin b, \exists C \in c, \vdash C \to A \sqcup B\}$. We want to prove:

(a) $V$ is closed under conjunction.

(b) $\Sigma - a$ is disjoint from $V$.

(a) Suppose $A_1, A_2 \in V$. Then $\exists B_1, B_2 \notin b$, $\exists C_1, C_2 \in c$ such that $\vdash C_1 \to A_1 \sqcup B_1$ and $\vdash C_2 \to A_2 \sqcup B_2$. Since $\vdash B_1 \to B_1 \vee B_2$, we have $\vdash A_1 \sqcup B_1 \to A_1 \sqcup (B_1 \vee B_2)$ by **R5**. So $\vdash C_1 \to A_1 \sqcup (B_1 \vee B_2)$. Since $\vdash C_1 \wedge C_2 \to C_1$, we have $\vdash C_1 \wedge C_2 \to A_1 \sqcup (B_1 \vee B_2)$. Similarly, $\vdash C_1 \wedge C_2 \to A_2 \sqcup (B_1 \vee B_2)$. So $\vdash C_1 \wedge C_2 \to (A_1 \sqcup (B_1 \vee B_2)) \wedge (A_2 \sqcup (B_1 \vee B_2))$. By **A8**,

$$\vdash (A_1 \sqcup (B_1 \vee B_2)) \wedge (A_2 \sqcup (B_1 \vee B_2)) \to (A_1 \wedge A_2) \sqcup (B_1 \vee B_2).$$

So $\vdash C_1 \wedge C_2 \to (A_1 \wedge A_2) \sqcup (B_1 \vee B_2)$. Since $b$ is an **L**-anti-dualtheory, $B_1 \vee B_2 \notin b$. And since $c$ is an **L**-theory, $C_1 \wedge C_2 \in c$. Hence $A_1 \wedge A_2 \in V$.

(b) To show a contradiction, suppose $\exists A \in \Sigma - a$, that is, $A \notin a$ and $A \in V$. Then $\exists B \notin b$, $\exists C \in c$ such that $\vdash C \to A \sqcup B$. But $S_2abc$, so $C \notin c$, which gives a contradiction.

Since $a$ is an **L**-anti-dualtheory, $\Sigma - a$ is an **L**-dualtheory. Hence by (a) and (b), we can use Lemma 2.9 to provide a prime **L**-dualtheory $a''$ disjoint from $V$ with $\Sigma - a \subseteq a''$. Let $a' = \Sigma - a''$. Then $a'$ is a prime **L**-theory, that is, prime **L**-anti-dualtheory, and $a' \subseteq a$. Next we prove $S_2a'bc$. Suppose $A \notin a'$, that is, $A \in a''$, $B \notin b$ and $\vdash C \to A \sqcup B$. Since $a''$ is disjoint from $V$, we have $C \notin c$, so $S_2a'bc$. □

**Lemma 2.14.**

(1) Let $a$ be a prime **L**-theory and $A \to B \notin a$. Then there are prime **L**-theories, $b', c'$, such that $R_1ab'c'$, $A \in b'$, and $B \notin c'$.

(2) Let $c$ be a prime **L**-theory and $A \sqcap B \in c$. Then there are prime **L**-theories, $a', b'$, such that $S_1a'b'c$, $A \in a'$, and $B \in b'$.

(3) Let $c$ be a prime **L**-theory and $A \sqcup B \notin c$. Then there are prime **L**-theories, $a', b'$, such that $S_2a'b'c$, $A \notin a'$, and $B \notin b'$.

*Proof.*

(1) Suppose $a$ is a prime **L**-theory such that $A \to B \notin a$. Let $b = \{A' : \vdash A \to A'\}$. We will show that $b$ is an **L**-theory. Suppose that $\vdash A'_1 \to A'_2$ and $A'_1 \in b$. Then $\vdash A \to A'_1$, so

$\vdash A \to A_2'$. Hence $A_2' \in b$. Now suppose $A_1', A_2' \in b$. Then $\vdash A \to A_1'$ and $\vdash A \to A_2'$. Thus $\vdash A \to A_1' \wedge A_2'$. So $A_1' \wedge A_2' \in b$. Now let $c = a \oplus b$. Then $c$ is an **L**-theory by Proposition 2.4. Also, $R_1abc$. It is obvious that $A \in b$. Moreover, $B \notin c$. Otherwise, $\exists A' \in b$ and $A' \to B \in a$. Then, $\vdash A \to A'$, so $\vdash (A' \to B) \to (A \to B)$. So $A \to B \in a$, which gives a contradiction. Hence, by Corollary 2.10 (4), there are prime **L**-theories $b', c'$ such that $A \in b'$, $B \notin c'$ and $R_1ab'c'$.

(2) Suppose $c$ is a prime **L**-theory such that $A \sqcap B \in c$. Let $a = \{A' :\vdash A \to A'\}$ and $b = \{B' :\vdash B \to B'\}$. Then $a, b$ are **L**-theories by the same proof as in (1). It is immediate that $A \in a$ and $B \in b$. To show $S_1abc$, suppose $A' \in a, B' \in b$ and $\vdash A' \sqcap B' \to C$. Then, $\vdash A \to A'$ and $\vdash B \to B'$. So $\vdash A \sqcap B \to A' \sqcap B'$ by **R4**. Then $\vdash A \sqcap B \to C$. But $A \sqcap B \in c$, so $C \in c$. Thus $S_1abc$. By Corollary 2.12, there are prime **L**-theories $a', b'$ such that $A \in a'$, $B \in b'$ and $S_1a'b'c$.

(3) Suppose $c$ is a prime **L**-theory such that $A \sqcup B \notin c$. Let $a'' = \{A' :\vdash A' \to A\}$ and $b'' = \{B' :\vdash B' \to B\}$. Then $a'', b''$ are **L**-dualtheories. For $a''$, suppose that $\vdash A_1' \to A_2'$ and $A_2' \in a''$. Then $\vdash A_2' \to A$. So $\vdash A_1' \to A$. Hence, $A_1' \in a''$. Now suppose $A_1', A_2' \in a''$. Then $\vdash A_1' \to A$ and $\vdash A_2' \to A$. Hence $\vdash A_1' \vee A_2' \to A$. So $A_1' \vee A_2' \in a''$. Thus $a''$ is an **L**-dualtheory. Similarly, $b''$ is also an **L**-dualtheory. Let $a = \Sigma - a''$ and $b = \Sigma - b''$. Then, $a, b$ are **L**-anti-dualtheories. It is immediate that $A \notin a$ and $B \notin b$. To show $S_2abc$, suppose $A' \notin a$, $B' \notin b$ and $\vdash C \to A' \sqcup B'$. Then $\vdash A' \to A$ and $\vdash B' \to B$. So $\vdash A' \sqcup B' \to A \sqcup B$ by **R5**, and thus $\vdash C \to A \sqcup B$. But $A \sqcup B \notin c$, so $C \notin c$. Thus $S_2abc$. Then, by Corollary 2.13, there are prime **L**-theories $a', b'$ such that $A \notin a'$, $B \notin b'$ and $S_2a'b'c$. □

### 2.6. *Completeness*

For any non-theorem $A$ of **L**, by Corollary 2.10 (1), there is a prime regular **L**-theory $o_c$ such that $A \notin o_c$. Thus we design a *canonical model* for **L**,

$$< o_c, W_c, O_c, R_1, R_2, S_1, S_2, I >$$

where $W_c$ is the class of all prime **L**-theories, that is, the class of all prime **L**-anti-dualtheories; $O_c$ is the subset of $W_c$ such that $x \in O_c$ if and only if $x$ is regular; $R_1$, $R_2$, $S_1$ and $S_2$ are canonically defined as above (restricted to $W_c$); and $I$ is defined, for every prime theory $x$ and formula $A$, as $I(A, x) = 1$ if and only if $A \in x$.

**Theorem 2.15.** If $A$ is $\mathbf{B}^+_{\sqcap\sqcup}$-valid, then $A$ is a theorem of $\mathbf{B}^+_{\sqcap\sqcup}$.

*Proof.* We prove the contrapositive. Given a non-theorem $A$, there is a canonical model $< o_c, W_c, O_c, R_1, R_2, S_1, S_2, I >$ for $\mathbf{B}^+_{\sqcap\sqcup}$. We show that the canonical model is really a $\mathbf{B}^+_{\sqcap\sqcup}$-model. It suffices to show that **p1**–**7** hold, and that $I$ satisfies the Atomic Hereditary Condition and the Evaluation Rules. **p1** and the Atomic Hereditary Condition are immediate. By the corresponding proof in Routley *et al.* (1982, Page 312), we can prove that $a \leqslant b$ if and only if $a \subseteq b$. Thus **p2** is obvious. **p3** was shown by Proposition 2.5. Finally, **p4**–**7** are immediate from the canonical definitions of $R_1$, $R_2$, $S_1$ and $S_2$.

Now we show $a \leqslant b$ if and only if $a \subseteq b$.

For the left-to-right direction, since $a \leqslant b$, there is an $x$ such that $x \in O_c$, that is, $x$ is regular, and $R_1xab$. Hence, since $A \to A \in x$, by the canonical definition of $R_1$, $A \in b$ whenever $A \in a$. So $a \subseteq b$.

For the right-to-left direction, suppose $a \subseteq b$. Then it is easy to see that $R_1lab$. Since $l$ is an **L**-theory, by Corollary 2.10 (2), $l$ can be replaced by a prime theory $x$ such that $l \subseteq x$ and $R_1xab$. Thus $x$ is regular, that is, $x \in O_c$. So $a \leqslant b$.

Next we show that $I$ satisfies the Evaluation Rules, and hence the canonical model is a $\mathbf{B}^+_{\sqcap\sqcup}$-model. It follows that $A$ is not valid on $< o_c, W_c, O_c, R_1, R_2, S_1, S_2, I >$. Hence $A$ is not $\mathbf{B}^+_{\sqcap\sqcup}$-valid. The cases for $\wedge$ and $\vee$ are immediate from the definition of a prime theory. Here we will just give proofs for the connectives $\to$, $\sqcap$ and $\sqcup$.

($\to$) It suffices to prove that $A \to B \in a$ if and only if for every $b, c \in W_c$, if $R_1abc$ and $A \in b$, then $B \in c$. But this is guaranteed by Lemma 2.14 and the canonical definition of $R_1$.

($\sqcap$) It suffices to prove that $A \sqcap B \in c$ if and only if for some $a, b \in W_c$, we have $S_1abc$, $A \in a$ and $B \in b$. But this is guaranteed by Lemma 2.14 and the canonical definition of $S_1$.

($\sqcup$) It suffices to prove that $A \sqcup B \notin c$ if and only if for some $a, b \in W_c$, we have $S_2abc$, $A \notin a$ and $B \notin b$. But this is guaranteed by Lemma 2.14 and the canonical definition of $S_2$.

Hence the result is proved. □

### 3. Extensions of $\mathbf{B}^+_{\sqcap\sqcup}$

The following are some additional axioms and rules that can be added to $\mathbf{B}^+_{\sqcap\sqcup}$ to obtain stronger systems. For a given postulate **S*i***, **s*i*** is the corresponding semantic condition on models.

| | |
|---|---|
| **S1** $A \sqcap B \to B \sqcap A$ | **s1** $S_1abc \Rightarrow S_1bac$ |
| **S2** $A \sqcup B \to B \sqcup A$ | **s2** $S_2abc \Rightarrow S_2bac$ |
| **S3** $(A \to B) \to (A \sqcap C \to B \sqcap C)$ | **s3** $R_1a(S_1de)c \Rightarrow S_1(R_1ad)ec$ |
| **S4** $(A \to B) \to (C \sqcap A \to C \sqcap B)$ | **s4** $R_1a(S_1de)c \Rightarrow S_1d(R_1ae)c$ |
| **S5** $(A \to B) \to (A \sqcup C \to B \sqcup C)$ | **s5** $R_2a(S_2de)c \Rightarrow S_2(R_2ad)ec$ |
| **S6** $(A \to B) \to (C \sqcup A \to C \sqcup B)$ | **s6** $R_2a(S_2de)c \Rightarrow S_2d(R_2ae)c$ |
| **S7** $(A \to C) \sqcap (B \to D) \to (A \sqcap B \to C \sqcap D)$ | **s7** $R_1(S_1ab)(S_1fg)e \Rightarrow S_1(R_1af)(R_1bg)e$ |
| **S8** $(A \to C) \sqcap (B \to D) \to (A \sqcup B \to C \sqcup D)$ | **s8** $R_2(S_1ab)(S_2fg)e \Rightarrow S_2(R_2af)(R_2bg)e$ |
| **S9** $(A \to B) \sqcap (B \to C) \to (A \to C)$ | **s9** $R_1(S_1ab)de \Rightarrow R_1b(ad)e$ |
| **S10** $(B \to C) \sqcap (A \to B) \to (A \to C)$ | **s10** $R_1(S_1ab)de \Rightarrow R_1a(bd)e$ |
| **S11** $A \sqcap (A \to B) \to B$ | **s11** $S_1abc \Rightarrow R_1bac$ |
| **S12** $A \to (B \to A \sqcap B)$ | **s12** $R_1abc \Rightarrow S_1abc$ |
| **S13** $(A \to (B \to C)) \to (A \sqcap B \to C)$ | **s13** $R_1a(S_1de)c \Rightarrow R_1(ad)ec$ |
| **S14** $(A \sqcap B \to C) \to (A \to (B \to C))$ | **s14** $R_1(ab)de \Rightarrow R_1a(S_1bd)e$ |
| **S15** $A \wedge B \to A \sqcap B$ | **s15** $S_1aaa$ |
| **S16** $A \sqcup B \to A \vee B$ | **s16** $S_2aaa$ |
| **S17** $A \sqcap B \to A \sqcup B$ | **s17** $\exists x(S_1abx$ and $S_2dex) \Rightarrow a \leqslant d$ or $b \leqslant e$ |

**S18** $A \to (B \to C) \Longrightarrow A \sqcap B \to C$ **s18**$S_1abc \Rightarrow R_1abc$
**S19** $A \sqcap B \to C \Longrightarrow A \to (B \to C)$ **s19**$R_1abc \Rightarrow S_1abc$.

**Theorem 3.1.** For each row in the list above, the extension of $\mathbf{B}^+_{\sqcap\sqcup}$ obtained by adding axiom or rule **S*i*** is sound and complete with respect to the class of $\mathbf{B}^+_{\sqcap\sqcup}$-models

$$< o, W, O, R_1, R_2, S_1, S_2, I >$$

that satisfy **s*i***.

*Proof.* For soundness, we take an arbitrary model and assume that it satisfies **s*i***. Then we demonstrate that **S*i*** (as an axiom) is valid or **S*i*** (as a rule) preserves validity in this model. Completeness is proved by showing that the canonical model for an extension with **S*i*** must satisfy **s*i***. We will give proofs for some rows as examples.

We will sketch the approach for completeness. In many cases, we search for prime **L**-theories satisfying some specific conditions. In general, we first construct appropriate **L**-theories or **L**-anti-dualtheories using operations $\oplus$, $\otimes$, $\ominus$ or $\oslash$, from given prime **L**-theories, and then apply Corollaries 2.10–13 to obtain the required prime **L**-theories.

**1.** For soundness, suppose $I(A \sqcap B, c) = 1$ in order to show $I(B \sqcap A, c) = 1$. Then $\exists a, b \in W$ such that $S_1abc$, $I(A, a) = 1$ and $I(B, b) = 1$. Since $S_1abc$, we have $S_1bac$ by **s1**. So $I(B \sqcap A, c) = 1$ as required.
For completeness, assume that **S1** holds. Let $S_1abc$ in order to show $S_1bac$. Suppose $B \in b$, $A \in a$ and $\vdash B \sqcap A \to C$. It suffices to show $C \in c$. By **S1**, we have $\vdash A \sqcap B \to C$. But $S_1abc$, so $C \in c$ as required.

**5.** For soundness, suppose $I(A \to B, a) = 1$ in order to show $I(A \sqcup C \to B \sqcup C, a) = 1$. Suppose also that $R_2abc$ and $I(B \sqcup C, b) \neq 1$. It suffices to show $I(A \sqcup C, c) \neq 1$. Then $\exists d, e \in W$ such that $S_2deb$, $I(B, d) \neq 1$ and $I(C, e) \neq 1$. Since $S_2deb$ and $R_2abc$, we have $R_2a(S_2de)c$. So, by **s5**, we have $S_2(R_2ad)ec$, that is, $\exists x \in W$ such that $R_2adx$ and $S_2xec$. Since $R_2adx$, we have $I(A, x) \neq 1$. And since $S_2xec$, we have $I(A \sqcup C, c) \neq 1$ as required.
For completeness, assume that **S5** holds. Suppose that $R_2a(S_2de)c$ in order to show $S_2(R_2ad)ec$. Then $\exists x \in W_c$ such that $S_2dex$ and $R_2axc$. Let $y = a \otimes d$. It is immediate that $R_2ady$ and that $y$ is an **L**-anti-dualtheory. We show $S_2yec$. Suppose that $A \notin y$, $C \notin e$ and $\vdash E \to A \sqcup C$. It suffices to show $E \notin c$. Since $A \notin y$, we have $\exists B \notin d$ such that $A \to B \in a$. Then $A \sqcup C \to B \sqcup C \in a$ by **S5**. Since $S_2dex$, we have $B \sqcup C \notin x$; and since $R_2axc$, we have $A \sqcup C \notin c$. So $E \notin c$ as required. Thus $S_2yec$. Now we can use Corollary 2.13 to provide a prime **L**-theory $y'$ such that $y' \subseteq y$ and $S_2y'ec$. It is immediate that $R_2ady'$. So we have $S_2(R_2ad)ec$.

**7.** For soundness, suppose $I((A \to C) \sqcap (B \to D), c) = 1$ in order to show $I(A \sqcap B \to C \sqcap D, c) = 1$. Then $\exists a, b \in W$ such that $S_1abc$, $I(A \to C, a) = 1$ and $I(B \to D, b) = 1$. Suppose also that $R_1cde$ and $I(A \sqcap B, d) = 1$. It suffices to show $I(C \sqcap D, e) = 1$. Then $\exists f, g \in W$ such that $S_1fgd$, $I(A, f) = 1$ and $I(B, g) = 1$. Since $S_1abc$, $S_1fgd$ and $R_1cde$, we have $R_1(S_1ab)(S_1fg)e$. So, by **s7**, we have $S_1(R_1af)(R_1bg)e$, that is, $\exists x, y \in W$ such that $R_1afx$, $R_1bgy$ and $S_1xye$. Since $R_1afx$, we have $I(C, x) = 1$. And since $R_1bgy$, we have $I(D, y) = 1$. Finally, since $S_1xye$, we have $I(C \sqcap D, e) = 1$ as required.

For completeness, assume that **S7** holds. Suppose $R_1(S_1ab)(S_1fg)e$ in order to show $S_1(R_1af)(R_1bg)e$. Then $\exists x_1, x_2 \in W_c$ such that $S_1abx_1$, $S_1fgx_2$ and $R_1x_1x_2e$. Let $y_1 = a \oplus f$ and $y_2 = b \oplus g$. It is immediate that $R_1afy_1$ and $R_1bgy_2$, and that $y_1, y_2$ are **L**-theories. We show $S_1y_1y_2e$. Suppose that $C \in y_1$, $D \in y_2$ and $\vdash C \sqcap D \to E$. It suffices to show $E \in e$. Since $C \in y_1$, we have $\exists A \in f$ such that $A \to C \in a$; and since $D \in y_2$, we have $\exists B \in g$ such that $B \to D \in b$. Then, since $S_1abx_1$, we have $(A \to C) \sqcap (B \to D) \in x_1$. And since $S_1fgx_2$, we have $A \sqcap B \in x_2$. So $A \sqcap B \to C \sqcap D \in x_1$ by **S7**. Since $R_1x_1x_2e$, we have $C \sqcap D \in e$, so $E \in e$ as required. Thus $S_1y_1y_2e$. Now we can use Corollary 2.12 to provide prime **L**-theories $y_1', y_2'$ such that $y_1 \subseteq y_1'$, $y_2 \subseteq y_2'$ and $S_1y_1'y_2'e$. It is immediate that $R_1afy_1'$ and $R_1bgy_2'$. So we have $S_1(R_1af)(R_1bg)e$.

**9.** For soundness, suppose $I((A \to B) \sqcap (B \to C), c) = 1$ in order to show $I(A \to C, c) = 1$. Then $\exists a, b \in W$ such that $S_1abc$, $I(A \to B, a) = 1$ and $I(B \to C, b) = 1$. Suppose also that $R_1cde$ and $I(A, d) = 1$. It suffices to show $I(C, e) = 1$. Since $S_1abc$ and $R_1cde$, we have $R_1(S_1ab)de$. So, by **s9**, $R_1b(ad)e$, that is, $\exists x \in W$ such that $R_1adx$ and $R_1bxe$. Since $R_1adx$, we have $I(B, x) = 1$; and since $R_1bxe$, we have $I(C, e) = 1$ as required.
For completeness, assume that **S9** holds. Suppose that $R_1(S_1ab)de$ in order to show $R_1b(ad)e$. Then $\exists x \in W_c$ such that $S_1abx$ and $R_1xde$. Let $y = a \oplus d$. It is immediate that $R_1ady$ and that $y$ is an **L**-theory. We show $R_1bye$. Suppose that $B \to C \in b$ and $B \in y$. It suffices to show $C \in e$. Since $B \in y$, we have $\exists A \in d$ such that $A \to B \in a$. Then, by $S_1abx$, we have $(A \to B) \sqcap (B \to C) \in x$. So $A \to C \in x$ by **S9**; and since $R_1xde$, we have $C \in e$ as required. Thus $R_1bye$. Now we can use Corollary 2.10 to provide a prime **L**-theory $y'$ such that $y \subseteq y'$ and $R_1by'e$. It is immediate that $R_1ady'$. So we have $R_1b(ad)e$.

**12.** For soundness, suppose $I(A, a) = 1$ in order to show $I(B \to A \sqcap B, a) = 1$. Suppose also that $R_1abc$ and $I(B, b) = 1$. It suffices to show $I(A \sqcap B, c) = 1$. Since $R_1abc$, we have $S_1abc$ by **s12**. So $I(A \sqcap B, c) = 1$ as required.
For completeness, assume that **S12** holds. Let $R_1abc$ in order to show $S_1abc$. Suppose also that $A \in a$, $B \in b$ and $\vdash A \sqcap B \to C$. It suffices to show $C \in c$. Then $\vdash (B \to A \sqcap B) \to (B \to C)$. Since $A \in a$, we have $B \to A \sqcap B \in a$ by **S12**. So $B \to C \in a$. But $R_1abc$, so $C \in c$ as required.

**14.** For soundness, suppose $I(A \sqcap B \to C, a) = 1$ in order to show $I(A \to (B \to C), a) = 1$. Suppose also that $R_1abc$ and $I(A, b) = 1$ in order to show $I(B \to C, c) = 1$. Finally, suppose $R_1cde$ and $I(B, d) = 1$. It suffices to show $I(C, e) = 1$. Since $R_1abc$ and $R_1cde$, we have $R_1(ab)de$. So, by **s14**, we have $R_1a(S_1bd)e$, that is, $\exists x \in W$ such that $S_1bdx$ and $R_1axe$. Since $S_1bdx$, we have $I(A \sqcap B, x) = 1$. And since $R_1axe$, we have $I(C, e) = 1$ as required.
For completeness, assume that **S14** holds. Suppose $R_1(ab)de$ in order to show $R_1a(S_1bd)e$. Then $\exists x \in W_c$ such that $R_1abx$ and $R_1xde$. Let $y = b \ominus d$. It is immediate that $S_1bdy$ and that $y$ is an **L**-theory. We will show $R_1aye$. Suppose that $D \to C \in a$ and $D \in y$. It suffices to show $C \in e$. Since $D \in y$, we have $\exists A \in b$ and $\exists B \in d$ such that $\vdash A \sqcap B \to D$. Then $\vdash (D \to C) \to (A \sqcap B \to C)$. So $A \sqcap B \to C \in a$, and thus $A \to (B \to C) \in a$ by **S14**. Since $R_1abx$, we have $B \to C \in x$, and since $R_1xde$, we have $C \in e$ as required. Thus $R_1aye$. Now we can use Corollary 2.10 to provide a

prime **L**-theory $y'$ such that $y \subseteq y'$ and $R_1ay'e$. It is immediate that $S_1bdy'$. So we have $R_1a(S_1bd)e$.

**15.** For soundness, suppose $I(A \wedge B, a) = 1$ in order to show $I(A \sqcap B, a) = 1$. Then $I(A, a) = 1$ and $I(B, a) = 1$. Since $S_1aaa$, we have $I(A \sqcap B, a) = 1$ as required.
For completeness, assume that **S15** holds. Suppose $A, B \in a$ and $\vdash A \sqcap B \to C$. It suffices to show $C \in a$. By **S15**, we have $\vdash A \wedge B \to A \sqcap B$. So $\vdash A \wedge B \to C$. Since $A, B \in a$, we have $A \wedge B \in a$. Hence $C \in a$ as required.

**17.** For soundness, suppose $I(A \sqcap B, x) = 1$, but $I(A \sqcup B, x) \neq 1$. Then $\exists a, b \in W$ such that $S_1abx$, $I(A, a) = 1$ and $I(B, b) = 1$. Also, $\exists d, e \in W$ such that $S_2dex$, $I(A, d) \neq 1$ and $I(B, e) \neq 1$. By **s17**, we have $a \leqslant d$ or $b \leqslant e$. So $I(A, d) = 1$ or $I(B, e) = 1$ by Lemma 2.1. But this gives a contradiction. Hence $I(A \sqcup B, x) = 1$.
For completeness, assume that **S17** holds. Let $S_1abx$ and $S_2dex$ in order to show $a \subseteq d$ or $b \subseteq e$. Suppose $a \not\subseteq d$ and $b \not\subseteq e$. Then $\exists A \in a$ but $A \notin d$, and $\exists B \in b$ but $B \notin e$. Since $S_1abx$, we have $A \sqcap B \in x$. And since $S_2dex$, we have $A \sqcup B \notin x$. But this gives a contradiction by **S17**. Hence $a \subseteq d$ or $b \subseteq e$.

**18.** For soundness, suppose $A \to (B \to C)$ is **L**-valid in order to show $A \sqcap B \to C$ is **L**-valid also. Suppose also that $I(A \sqcap B, c) = 1$. It suffices to show $I(C, c) = 1$. Then $\exists a, b \in W$ such that $S_1abc$, $I(A, a) = 1$ and $I(B, b) = 1$. Since $A \to (B \to C)$ is **L**-valid, we have $I(B \to C, a) = 1$ by Lemma 2.2 (3). But since $S_1abc$, we have $R_1abc$ by **s18**, so $I(C, c) = 1$ as required.
For completeness, assume that **S18** holds. Let $S_1abc$ in order to show $R_1abc$. Suppose $A \to B \in a$ and $A \in b$ in order to show $B \in c$. Since $\vdash (A \to B) \to (A \to B)$, we have $\vdash (A \to B) \sqcap A \to B$ by **S18**. But $S_1abc$, so $B \in c$ as required. □

It is easy to see that in any extension of $\mathbf{B}^+_{\sqcap\sqcup}$ with the rules **S18** and **S19**, $\to$ is the residual of $\sqcap$ such that $S_1$ collapses to $R_1$ in models.

## 4. Negation

### 4.1. *The systems* $\mathbf{BM}_{\sqcap\sqcup}$ *and* $\mathbf{B}_{\sqcap\sqcup}$

For a basic negation-extension of $\mathbf{B}^+_{\sqcap\sqcup}$, we add the De Morgan Laws **A9**, **A10** and Contraposition **R6**:

**A9** $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$
**A10** $\neg A \wedge \neg B \leftrightarrow \neg(A \vee B)$
**R6** $A \to B \Rightarrow \neg B \to \neg A$.

We call this system $\mathbf{BM}_{\sqcap\sqcup}$[†]. **A9** and **A10** also contain redundancies. We can prove each of **A9** and **A10** in the right-to-left direction using Contraposition and the positive axioms.

A $\mathbf{BM}_{\sqcap\sqcup}$-frame $\mathscr{F}$ is an 8-tuple $< o, W, O, R_1, R_2, S_1, S_2, * >$, where $*$ is a one-place function from $W$ to $W$, and the other elements are as before, such that postulate **p8** holds

[†] The system **BM** is a negation-extension of $\mathbf{B}^+$ by the addition of the De Morgan Laws and Contraposition. We can also obtain $\mathbf{BM}_{\sqcap\sqcup}$ by adding $\sqcap, \sqcup$ to **BM**.

for every $a, b \in W$:

**p8.** If $a \leqslant b$, then $b^* \leqslant a^*$.

Note that **p8** is required for the Hereditary Condition.

A $\mathbf{BM}_{\sqcap\sqcup}$-model $\mathscr{M}$ is a 9-tuple

$$< o, W, O, R_1, R_2, S_1, S_2, *, I >$$

where

$$< o, W, O, R_1, R_2, S_1, S_2, * >$$

is a $\mathbf{BM}_{\sqcap\sqcup}$-frame, and $I$ is as before, such that the evaluation rule for negation is as follows:

$(\neg)$ $I(\neg A, a) = 1$ if and only if $I(A, a^*) \neq 1$.

It is easy to verify that the Hereditary Condition holds as before, and hence that $\mathbf{BM}_{\sqcap\sqcup}$ is sound with respect to the class of $\mathbf{BM}_{\sqcap\sqcup}$-models. For completeness, we define $*$ on a set of formulas $a$ by $a^* = \{A : \neg A \notin a\}$. The canonical model for $\mathbf{BM}_{\sqcap\sqcup}$ is now $< o_c, W_c, O_c, R_1, R_2, S_1, S_2, *, I >$. By the De Morgan Laws and Contraposition, it can be shown that:

— If $a$ is a theory, then $a^*$ is an anti-dualtheory.
— If $a$ is an anti-dualtheory, then $a^*$ is a theory.

Hence, if $a$ is a prime theory, $a^*$ is also, that is, $*$ is well-defined. Also, **p8** is easy to verify. Finally, rule $(\neg)$ holds well in the canonical model. Thus the canonical model is indeed a $\mathbf{BM}_{\sqcap\sqcup}$ model.

The system $\mathbf{B}_{\sqcap\sqcup}$ is obtained from $\mathbf{BM}_{\sqcap\sqcup}$ by adding Double Negation†:

**A11.** $A \leftrightarrow \neg\neg A$

Then, a $\mathbf{B}_{\sqcap\sqcup}$-model is a $\mathbf{BM}_{\sqcap\sqcup}$-model satisfying $a^{**} = a$ for all $a \in W$. The soundness and completeness results are easy to prove.

### 4.2. *Negation extensions*

We now give some extensions of $\mathbf{BM}_{\sqcap\sqcup}$ and $\mathbf{B}_{\sqcap\sqcup}$.

| | |
|---|---|
| **S20** $\neg A \to (A \sqcup B \to B)$ | **s20** $R_2abc \Rightarrow S_2a^*bc$ |
| **S21** $(A \to (B \to C)) \to (B \to \neg A \sqcup C)$ | **s21** $R_2a(S_2de)c \Rightarrow R_2(R_1ad^*)ec$ |
| **S22** $(A \to B \sqcup C) \to (\neg B \to (A \to C))$ | **s22** $R_2(R_1ab)de \Rightarrow R_2a(S_2b^*d)e$ |
| **S23** $\neg(A \sqcap B) \to \neg A \sqcup \neg B$ | **s23** $S_2abc \Rightarrow S_1a^*b^*c^*$ |
| **S24** $\neg A \sqcap \neg B \to \neg(A \sqcup B)$ | **s24** $S_1abc \Rightarrow S_2a^*b^*c^*$ |
| **S25** $A \sqcup \neg A$ | **s25** $S_2abo \Rightarrow b^* \leqslant a$ |
| **S26** $A \to (B \to C) \Rightarrow B \to \neg A \sqcup C$ | **s26** $S_2abc \Rightarrow R_2a^*bc$ |
| **S27** $A \to B \sqcup C \Rightarrow \neg B \to (A \to C)$ | **s27** $R_2abc \Rightarrow S_2a^*bc$. |

† The system **B** is an extension of **BM** by the addition of Double Negation. We can also obtain $\mathbf{B}_{\sqcap\sqcup}$ by adding $\sqcap, \sqcup$ to **B**.

**Theorem 4.1.** For each row in the list above, the extension of $\mathbf{BM}_{\sqcap\sqcup}/\mathbf{B}_{\sqcap\sqcup}$ obtained by adding axiom or rule **S*i*** is sound and complete with respect to the class of $\mathbf{BM}_{\sqcap\sqcup}/\mathbf{B}_{\sqcap\sqcup}$-models $< o, W, O, R_1, R_2, S_1, S_2, *, I >$ that satisfy **s*i***.

*Proof.* These are proved in the same way as the positive extensions. We will just give proofs for some rows as examples.

**21.** For soundness, suppose $I(A \to (B \to C), a) = 1$ in order to show $I(B \to \neg A \sqcup C, a) = 1$. Suppose also that $R_2abc$ and $I(\neg A \sqcup C, b) \neq 1$. It suffices to show $I(B, c) \neq 1$. Then $\exists d, e \in W$ such that $S_2deb$, $I(\neg A, d) \neq 1$ and $I(C, e) \neq 1$. Since $S_2deb$ and $R_2abc$, we have $R_2a(S_2de)c$. So, by **s21**, $R_2(R_1ad^*)ec$, that is, $\exists x \in W$ such that $R_1ad^*x$ and $R_2xec$. Since $I(\neg A, d) \neq 1$, we have $I(A, d^*) = 1$. So $I(B \to C, x) = 1$ by $R_1ad^*x$. Hence $I(B, c) \neq 1$ by $R_2xec$.
For completeness, assume **S21** holds. Let $R_2a(S_2de)c$ in order to show $R_2(R_1ad^*)ec$. Then $\exists x \in W_c$ such that $S_2dex$ and $R_2axc$. Let $y = a \oplus d^*$. It is immediate that $R_1ad^*y$ and that $y$ is an **L**-theory. We show $R_2yec$. Suppose that $B \to C \in y$ and $C \notin e$. It suffices to show $B \notin c$. Since $B \to C \in y$, we have $\exists A \in d^*$ such that $A \to (B \to C) \in a$. So $B \to \neg A \sqcup C \in a$ by **S21**. Since $A \in d^*$, we have $\neg A \notin d$. So $\neg A \sqcup C \notin x$ by $S_2dex$. Hence $B \notin c$ by $R_2axc$. Thus $R_2yec$. Now we can use Corollary 2.11 to provide a prime **L**-theory $y'$ such that $y \subseteq y'$ and $R_2y'ec$. It is immediate that $R_1ad^*y'$. So we have $R_2(R_1ad^*)ec$.

**22.** For soundness, suppose $I(A \to B \sqcup C, a) = 1$ in order to show $I(\neg B \to (A \to C), a) = 1$. Suppose also that $R_1abc$ and $I(\neg B, b) = 1$ in order to show $I(A \to C, c) = 1$. Finally, suppose $R_2cde$ and $I(C, d) \neq 1$. It suffices to show $I(A, e) \neq 1$. Since $R_1abc$ and $R_2cde$, we have $R_2(R_1ab)de$. So, by **s22**, $R_2a(S_2b^*d)e$, that is, $\exists x \in W$ such that $S_2b^*dx$ and $R_2axe$. Since $I(\neg B, b) = 1$, we have $I(B, b^*) \neq 1$. So $I(B \sqcup C, x) \neq 1$ by $S_2b^*dx$. Hence $I(A, e) \neq 1$ by $R_2axe$.
For completeness, assume **S22** holds. Let $R_2(R_1ab)de$ in order to show $R_2a(S_2b^*d)e$. Then $\exists x \in W_c$ such that $R_1abx$ and $R_2xde$. Let $y = b^* \oslash d$. It is immediate that $S_2b^*dy$ and that $y$ is an **L**-anti-dualtheory. We will show $R_2aye$. Suppose that $A \to D \in a$ and $D \notin y$. It suffices to show $A \notin e$. Since $D \notin y$, we have $\exists B \notin b^*, \exists C \notin d$ such that $\vdash D \to B \sqcup C$. Hence $\vdash (A \to D) \to (A \to B \sqcup C)$. So $A \to B \sqcup C \in a$, and thus $\neg B \to (A \to C) \in a$ by **S22**. Since $B \notin b^*$, we have $\neg B \in b$. So $A \to C \in x$ by $R_1abx$, and thus $A \notin e$ by $R_2xde$. So $R_2aye$. Now we can use Corollary 2.11 to provide a prime **L**-theory $y'$ such that $y' \subseteq y$ and $R_2ay'e$. It is immediate that $S_2b^*dy'$. So we have $R_2a(S_2b^*d)e$.

**23.** For soundness, suppose $I(\neg A \sqcup \neg B, c) \neq 1$ in order to show $I(\neg(A \sqcap B), c) \neq 1$. Thus $\exists a, b \in W$ such that $S_2abc$, $I(\neg A, a) \neq 1$ and $I(\neg B, b) \neq 1$. So $I(A, a^*) = 1$ and $I(B, b^*) = 1$. Since $S_2abc$, we have $S_1a^*b^*c^*$ by **s23**. So $I(A \sqcap B, c^*) = 1$. Hence $I(\neg(A \sqcap B), c) \neq 1$ as required.
For completeness, assume that **S23** holds. Let $S_2abc$ in order to show $S_1a^*b^*c^*$. Suppose $A \in a^*$, $B \in b^*$ and $\vdash A \sqcap B \to C$. It suffices to show $C \in c^*$. Then $\neg A \notin a$ and $\neg B \notin b$. Since $S_2abc$, we have $\neg A \sqcup \neg B \notin c$. By **S23**, we have $\neg(A \sqcap B) \notin c$, that is, $A \sqcap B \in c^*$. Hence $C \in c^*$ as required.

**26.** For soundness, suppose $A \to (B \to C)$ is **L**-valid in order to show that $B \to \neg A \sqcup C$ is **L**-valid also. So suppose $I(\neg A \sqcup C, c) \neq 1$ in order to show $I(B, c) \neq 1$. Then $\exists a, b \in W$ such that $S_2abc$, $I(\neg A, a) \neq 1$ and $I(C, b) \neq 1$. So $I(A, a^*) = 1$. Since $A \to (B \to C)$ is **L**-valid, we have $I(B \to C, a^*) = 1$ by Lemma 2.2 (3). But since $S_2abc$, we have $R_2a^*bc$ by **s26**. Hence $I(B, c) \neq 1$ as required.

For completeness, assume that **S26** holds. Let $S_2abc$ in order to show $R_2a^*bc$. Suppose $B \to C \in a^*$ and $C \notin b$, but $B \in c$. Since $\vdash (B \to C) \to (B \to C)$, we have $\vdash B \to \neg(B \to C) \sqcup C$ by **S26**. Hence $\neg(B \to C) \sqcup C \in c$. But $B \to C \in a^*$, that is, $\neg(B \to C) \notin a$, so $\neg(B \to C) \sqcup C \notin c$ by $S_2abc$. This gives a contradiction. Hence $B \notin c$. Thus $R_2a^*bc$. □

## 5. Conclusions and future work

This paper has introduced and investigated a basic relevant logic $\mathbf{B}^+_{\sqcap\sqcup}$, which is obtained by adding two binary connectives $\sqcap$ and $\sqcup$ to the minimal positive relevant logic $\mathbf{B}^+$. The connectives $\sqcap$ and $\sqcup$ are axiomatised by Dunn's approach for Gaggle Theory, and can be seen as weaker versions of intensional conjunction and disjunction. Accordingly, the semantics for $\mathbf{B}^+_{\sqcap\sqcup}$ is an extension of the well-known relational semantics for $\mathbf{B}^+$, with $\to$, $\sqcap$, $\sqcup$ modelled by ternary relations: $R_1$ and $R_2$ for $\to$, $S_1$ for $\sqcap$, and $S_2$ for $\sqcup$. The soundness and completeness of our semantics were proved by adaptations of familiar methods for relevant logics. Finally, a number of additional axioms and rules were given, each with the corresponding semantic conditions required for maintaining soundness and completeness.

In order to construct the canonical model, we defined $R_1, R_2, S_1, S_2$ as derivatives of operations $\oplus, \otimes, \ominus, \oslash$ on theories and anti-dualtheories, respectively. This technique was mainly inspired by the operational treatments for $\to$ in Fine (1974) and Brady (2003). It seems that the method can be generalised to $n$-placed connectives such that an $n$-placed connective can be modelled by several $n$-placed operations. In addition, since an anti-dualtheory $a$ satisfies $A \vee B \in a$ if and only if $A \in a$ or $B \in a$, we expect that a method for using anti-dualtheories to model $\vee$ canonically can be developed, just as with theories for $\wedge$. Then it turns out that $\wedge$ and $\vee$ can be dealt with separately without regard to distribution. Based on the above notions, we will investigate operational semantics for various logics with or without distribution. The further work will be presented in other papers.

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