The *indifference* price: an adjustment to market maker's theoretical price

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This technical note expands the framework for determining indifference prices [AS08], a heuristic method used to optimize the trading behavior of a market maker (e.g., pricing and risk management), into the options market-making system. Additionally, we integrate margin costs into the utility function. First, we review the basic model presented in [AS08]. The mid-price of the asset (e.g. a stock), S, follows the geometric Brownian motion,

$$\frac{dS(t)}{S} = \sigma dW(t), \tag{1}$$

where σ is the constant volatility and $W(t)$ is a standard Brownian motion, with initial state $S(0) = s$. We consider a market maker holds inventory q, cash x, the mean-variance utility function, evaluated at time t, can be written as,

$$\begin{aligned} V_t(x, s, q) &= \mathbf{E}\left[\overbrace{(x + qS(t))}^{\text{terminal wealth}} - \frac{\gamma}{2}\overbrace{(qS(t) - qs)^2}^{\text{variance}}\right] \\ &= x + qs - \frac{\gamma}{2}q^2 s^2 \overbrace{(e^{\sigma^2 t} - 1)}^{\sim\ \sigma^2 t}, \end{aligned} \tag{2}$$

where x is the initial wealth in dollars, we can set it to zero. The *reservation bid price* r^b is solved by $V_t(x - r^b, s, q + 1) = V_t(x, s, q)$, the *reservation ask price* r^a is solved by $V_t(x + r^a, s, q - 1) = V_t(x, s, q)$. This yields reservation prices of the form,

$$\begin{aligned} r_t^a(s, q) &= s - \gamma s^2 \sigma^2 t q + \frac{1}{2}\gamma s^2 \sigma^2 t, \\ r_t^b(s, q) &= s - \gamma s^2 \sigma^2 t q - \frac{1}{2}\gamma s^2 \sigma^2 t. \end{aligned} \tag{3}$$

Note that $s^2\sigma^2 t$ is the spot dollar variance, the reservation price is the mid-price adjusted by risk management shift (linear in q) and required spread (linear in variance).

We extend above approach to the options market-making system. We use black-scholes model, where option price is the function of spot price S and implied volatility σ. In general, we define option price, $f(Z_1, Z_2, ..., Z_m; k, \tau)$, as a pricing function of multiple market variables, $Z_1, ..., Z_m$, with strike k and time-to-maturity τ. Suppose the terminal time t is small, the option price can be written in it's expansion form,

$$\begin{aligned} f(Z_1(t), ..., Z_m(t)) \sim\ & f(Z_1(0), ..., Z_m(0)) + \sum_i (Z_i(t) - Z_i(0)) \frac{\partial f}{\partial Z_i}(Z_1(0), ..., Z_m(0)), \\ \text{in short,}\ \ & f(t) \sim f(0) + \sum_i \delta Z_i \frac{\partial f(0)}{\partial Z_i}. \end{aligned} \tag{4}$$

Furthermore, we take into account the margin cost of trading options (i.e., the financing cost). The margin might depend on the long/short position, for example, there is no margin for buying options (only need to pay option premiums), however, the margin of selling options is large. Thus, the `optionMargin` is a digital function of position. The margin cost of holding an option with inventory q is,

$$\overbrace{\left[-rt * \text{optionMargin} * \mathbb{1}_{\{q<0\}}\right]}^{\text{cost form, } c} q. \tag{5}$$

The cost form, c, is a indicator function, which can be approximated by a smooth sigmoid function. The discontinuity issue (around $q = 0$) is omitted in forming the utility function (see Eq. 6) by introducing an assumption on position variation range. We consider a portfolio of N options, the position vector is denoted by $\mathbf{q} \in \mathbb{R}^N$. Note that the bold symbols correspond to vector forms. The utility function has form,

$$V_t(x, \mathbf{f}(0), \mathbf{q}) = \mathbf{E}\left[\overbrace{(x + \mathbf{q}^T\mathbf{f}(t))}^{\text{terminal wealth}} - \frac{\gamma}{2}\overbrace{\left(\mathbf{q}^T\mathbf{f}(t) - \mathbf{q}^T\mathbf{f}(0)\right)^2}^{\text{variance}} - \phi \overbrace{\mathbf{q}^T\mathbf{c}(0)}^{\text{margin cost}}\right], \tag{6}$$

where the margin cost vector $\mathbf{c}(0) \in \mathbb{R}^N$ is a constant vector evaluated at initial state given position $\mathbf{q}$ and market state $\mathbf{Z}(0) \in \mathbb{R}^m$, the element of $\mathbf{c}(0)$ has form as Eq. 5. Substituting expansion form, Eq. 4, into the above utility function, we have closed form,

$$\begin{aligned} V_t(x, \mathbf{f}(0), \mathbf{q}) &= x + \mathbf{q}^T\mathbf{f}(0) - \frac{\gamma}{2}\mathbf{q}^T\Omega\mathbf{q} - \phi\mathbf{q}^T\mathbf{c}(0), \\ \text{where } \Omega &= \mathbf{E}\left[\sum_i\sum_j \delta Z_i \delta Z_j \mathbf{g}_i\mathbf{g}_j^T\right], \end{aligned} \tag{7}$$

the risk sensitivity shape with respect to risk factor Z_i, $\partial\mathbf{f}(0)/\partial Z_i \in \mathbb{R}^N$, is denoted by $\mathbf{g}_i$. We assume there is no correlation between any pair of market risk factors, $\delta Z_i \perp\!\!\!\perp \delta Z_j$ for any $i \neq j$, the covariance structure is simplified to

$$\Omega = \sum_i \lambda_i \mathbf{g}_i\mathbf{g}_i^T,\ \Omega \in \mathbb{R}^{N\times N} \tag{8}$$

where $\lambda_i = \mathbf{E}[\delta Z_i^2]$ is the variance of risk factor δZ_i, it depends on the trading horizon (t), the variance gets larger for longer time. We find reservation bid price, $\mathbf{f}_b$, by solving the equation $V_t(x - \mathbf{f}_b\mathbf{h}^T, \mathbf{f}(0), \mathbf{q} + \mathbf{h}) = V_t(x, \mathbf{f}(0), \mathbf{q})$, where $\mathbf{h}$ is a position incremental vector. The position incremental vector should be small, so that the margin cost remains same for all contracts after adding a small position change $\mathbf{h}$, e.g. $\text{sign}(\mathbf{q}) = \text{sign}(\mathbf{q} + \mathbf{h})$. We solve for $\mathbf{f}_b$ as follows,

$$\begin{aligned} x - \mathbf{f}_b\mathbf{h}^T + (\mathbf{q}+\mathbf{h})^T\mathbf{f}(0) - \frac{\gamma}{2}(\mathbf{q}+\mathbf{h})^T\Omega(\mathbf{q}+\mathbf{h}) - \phi(\mathbf{q}+\mathbf{h})^T\mathbf{c}(0) &= x + \mathbf{q}^T\mathbf{f}(0) - \frac{\gamma}{2}\mathbf{q}^T\Omega\mathbf{q} - \phi\mathbf{q}^T\mathbf{c}(0), \\ -\mathbf{f}_b\mathbf{h}^T + \mathbf{h}^T\mathbf{f}(0) - \frac{\gamma}{2}\left(2\mathbf{h}^T\Omega\mathbf{q} + \mathbf{h}^T\Omega\mathbf{h}\right) - \phi\mathbf{h}^T\mathbf{c}(0) &= 0, \\ -\mathbf{h}^T\left(\gamma\Omega\mathbf{q} + \frac{\gamma}{2}\Omega\mathbf{h} + \phi\mathbf{c}(0)\right) &= \mathbf{h}^T(\mathbf{f}_b - \mathbf{f}(0)), \\ \mathbf{f}_b &= \mathbf{f}(0) - \gamma\Omega\mathbf{q} - \frac{\gamma}{2}\Omega\mathbf{h} - \phi\mathbf{c}(0). \end{aligned} \tag{9}$$

Similarly, we have form of $\mathbf{f}_a$,

$$\mathbf{f}_a = \mathbf{f}(0) - \overbrace{\gamma\Omega\mathbf{q}}^{\text{lean on risk factors}} + \overbrace{\frac{\gamma}{2}\Omega\mathbf{h}}^{\text{risk spread}} - \overbrace{\phi\mathbf{c}(0)}^{\text{margin adjust}}. \tag{10}$$

Note that

$$\Omega\mathbf{q} = \sum_i \lambda_i \overbrace{(\mathbf{q}^T\mathbf{g}_i)}^{\text{factor } i\text{'s risk position}} \mathbf{g}_i, \tag{11}$$

this is equivalent to the risk management model in our market making system, we adjust the factor value Z_i proportional to it's risk position. To see this, we write the adjustment on factor Z_i in terms of it's risk position, $\Delta Z_i = \beta_i\mathbf{q}^T\mathbf{g}_i$, where β_i is a control parameter. Next, we calculate the option price change vector, $\mathbf{f}(\mathbf{Z} + \Delta Z_i\mathbf{1}_i) - \mathbf{f}(\mathbf{Z}) \sim \ \Delta Z_i\mathbf{g}_i$, where $\mathbf{1}_i$ is the zero-one coded vector with value one at index i. This form recovers the *penalty* function in our system. For example, if we sell a million cash delta, we would increase the underlying price by one tick.

The second term, *risk spread*, is linear in trade size vector $\mathbf{h}$, we require more spread (per lot) for large trade, *i.e.,* size-edge relationship. The third term, $\phi\mathbf{c}(0)$, accounts for the margin cost adjustment, it's not depending on position.

References

[AS08] Marco Avellaneda and Sasha Stoikov. High-frequency trading in a limit order book. *Quantitative Finance*, 8(3):217–224, 2008.