傅立葉轉換

20151204

資訊系普物

Chapter 17 Sound -- Fourier series

Harris Benson

University Physics, 2nd revised edition

2013

17.7 FOURIER SERIES (Optional)

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

$$B_z = B_0 \sin(kx - \omega t)$$

$$E_y = E_0 \sin(kx - \omega t)$$

The sinusoidal wave functions we have employed so far are rarely encountered in practice. An almost purely sinusoidal wave is formed by the pressure variations produced by a tuning fork, as shown in Fig. 17.15*a*. In general, however, periodic wave functions have complex shapes. Figure 17.15*b* shows the pressure variations associated with a musical instrument playing a musical note of the same fundamental frequency as the tuning fork. Both would have the same apparent musical pitch, but the instrument has a "richer" and perhaps more pleasing sound. The richness arises from the fact that the wave function is the result of the superposition of a large number of harmonic waves of different amplitudes and frequencies. In 1807, Joseph Fourier showed that any reasonably well-behaved periodic function may be generated by the superposition of a sufficient number of sine or cosine functions. According to **Fourier's theorem,** the function is represented by the infinite sum

$$F(t) = \Sigma(a_n \sin n\omega t + b_n \cos n\omega t) \tag{17.21}$$

where $\omega = 2\pi/T = 2\pi f$. The function has been decomposed into harmonic components whose frequencies are integer multiples of f, the frequency of the function. The Fourier coefficients a_n and b_n indicate the amplitude of the nth harmonic function. The process of determining these coefficients is called **Fourier analysis.**

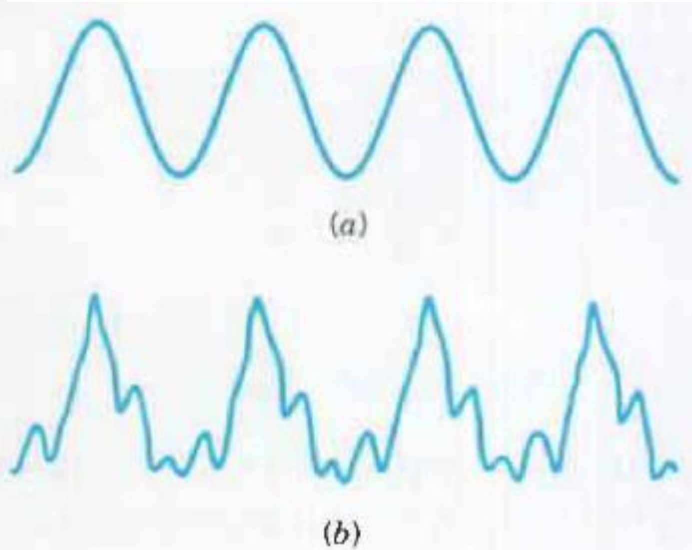


FIGURE 17.15 (a) A tuning fork produces a sinusoidal variation in pressure. (b) A hypothetical pressure fluctuation produced by a musical instrument.

Consider the periodic function depicted in Fig. 17.16. It is a square function for which $F(t) = +A$ between $t = 0$ and $t = T/2$, and $F(t) = -A$ between $t = T/2$ and $t = T$. The period of the function is T. (If the function were periodic in space, we might prefer to label the period by L or λ.) It can be shown that

$$F(t) = \frac{4A}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots\right)$$

In this particular case, only the sine functions are involved; furthermore, only the odd harmonics are present. The first three harmonic terms, each with its appropriate amplitude, are drawn in Fig. 17.17. When they are superimposed as in Fig. 17.18*a* we see that just three terms lead to a fairly good representation of $F(t)$. Figure 17.18*b* shows the effect of including ten terms.

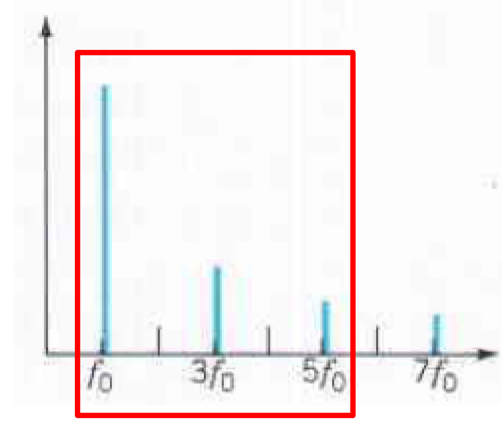
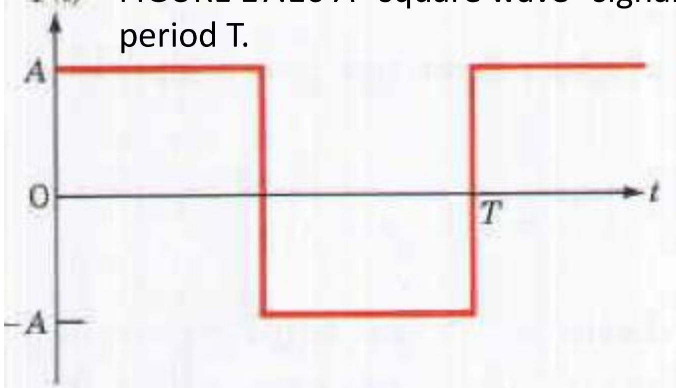
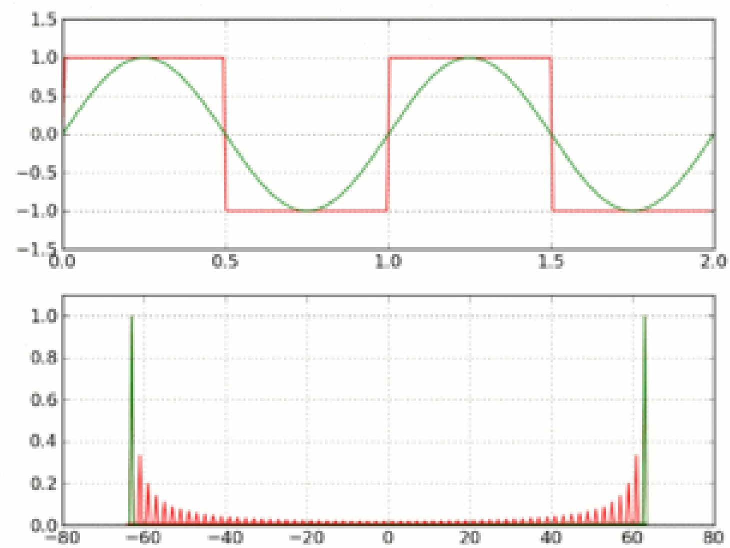


FIGURE 17.16 A "square wave" signal with a period T.



Animation of the additive synthesis of a square wave with an increasing number of harmonics



Period function

- Any function satisfies $f(x) = f(x+T)$
  where T is a constant and is called the *period* of the function.

- Example 1: $\sin nx = \sin n(x+T) = \sin nx \cos nT + \cos nx \sin nT$

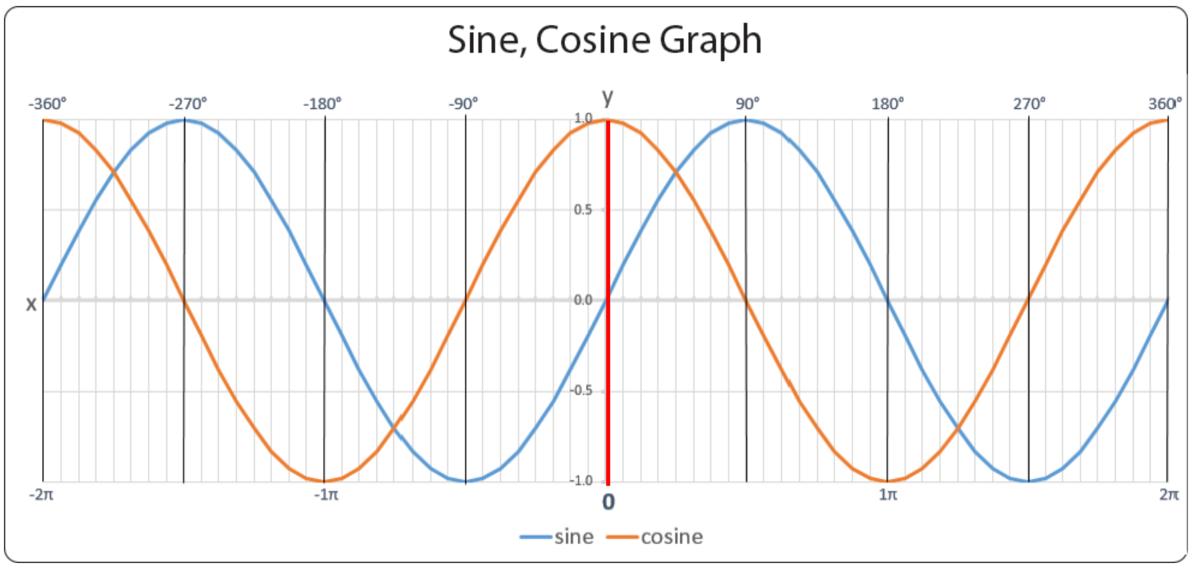
$$\Rightarrow \begin{Bmatrix} \cos nT = 1 \\ \sin nT = 0 \end{Bmatrix} \Rightarrow nT = 2 = n2\pi \Rightarrow T = 2\pi$$

- Example 2: $f(x) = \cos\frac{x}{3} + \cos\frac{x}{4} = \cos\frac{1}{3}(x+T) + \cos\frac{1}{4}(x+T)$

$$\Rightarrow \begin{cases} \frac{T}{3} = 2m\pi \\ \frac{T}{4} = 2m\pi \end{cases} \Rightarrow \begin{cases} T = 6m\pi \\ T = 8m\pi \end{cases} \Rightarrow T = 24\pi$$

Parity of sine and cosine function

- Any function satisfies $f(x) = f(-x)$ is an even function.
- Any function satisfies $f(x) = -f(-x)$ is an odd function.



$$\sin(x) = -\sin(-x) \quad \Rightarrow \textit{sine function is an odd function}$$

$$\cos(x) = \cos(-x) \quad \Rightarrow \textit{cosine function is an even function}$$

Fourier Series

- Fourier series are named after Joseph Fourier (1768-1830), who made important contributions to the study of trigonometric series, in connection with the solution of the heat equation.
- A Fourier series may be defined as an expansion of a function in a series of sines and cosines such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \text{ or } f(x) = a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

- The function *f* satisfies the following conditions:

  (1) f(x) is a periodic function;

  (2) f(x) has only a finite number of finite discontinuities;

  (3) f(x) has only a finite number of extrem values, maxima and minima in the interval [0,2π].

Orthogonal relations

$$\int_0^{2\pi} \sin mx \sin nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 0, & m = 0, \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx dx = \begin{cases} \pi\delta_{m,n}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases}$$

$$\int_0^{2\pi} \sin mx \cos nx dx = 0$$ for all integer m and n.

Fourier coefficients

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\text{multipling } \cos mx, \text{ and then integral from } 0 \text{ to } 2\pi$$

$$\int_0^{2\pi} \cos(mx) f(x) dx = \frac{a_0}{2}\int_0^{2\pi} \cos(mx) dx + \sum_{n=1}^{\infty}(a_n \int_0^{2\pi} \cos(nx)\cos(mx) dx + b_n \int_0^{2\pi} \sin(nx)\cos(mx) dx)$$

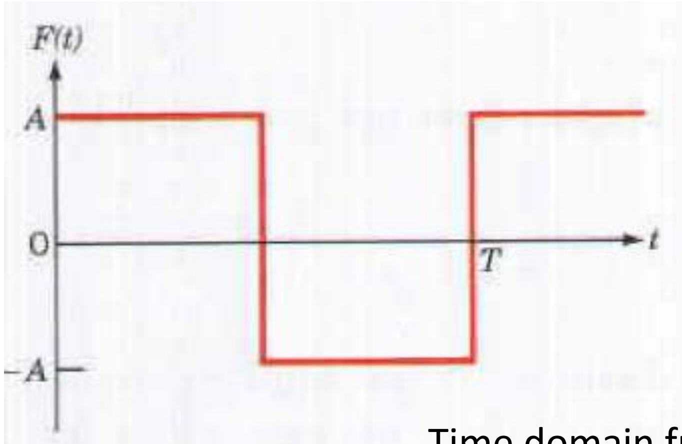
Similarly

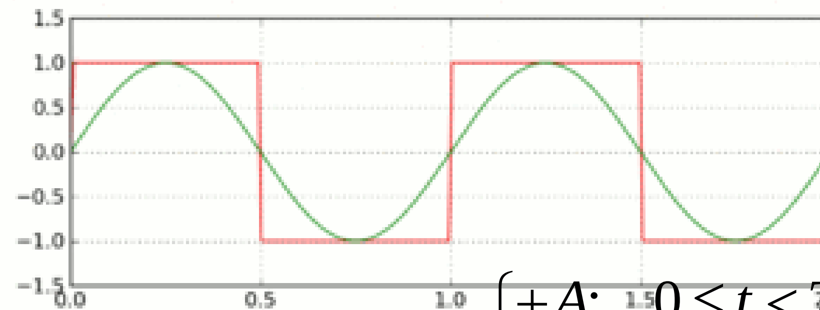
$$\int_0^{2\pi} \sin(mx) f(x) dx = \frac{a_0}{2}\int_0^{2\pi} \sin(mx) dx + \sum_{n=1}^{\infty}(a_n \int_0^{2\pi} \cos(nx)\sin(mx) dx + b_n \int_0^{2\pi} \sin(nx)\sin(mx) dx)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx, \quad n = 0,1,2$$

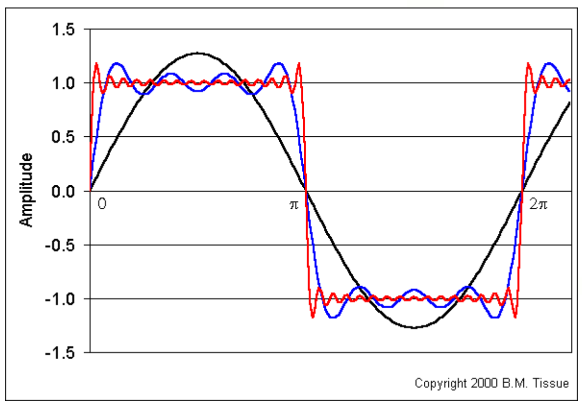
Synthesis of a square wave



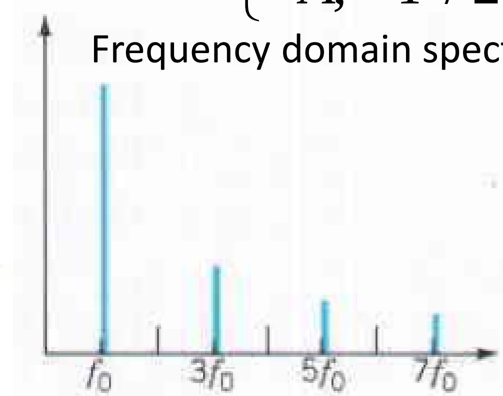


$$f(t)=\begin{cases}+A; & 0\le t<T/2\\ -A; & T/2\le t<T\end{cases}$$

Time domain function



Frequency domain spectrum

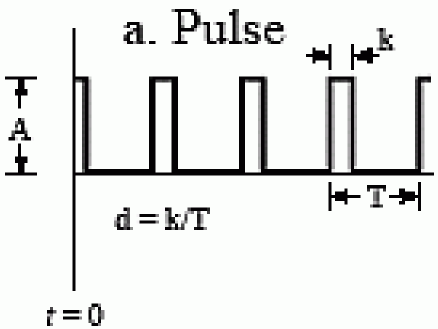


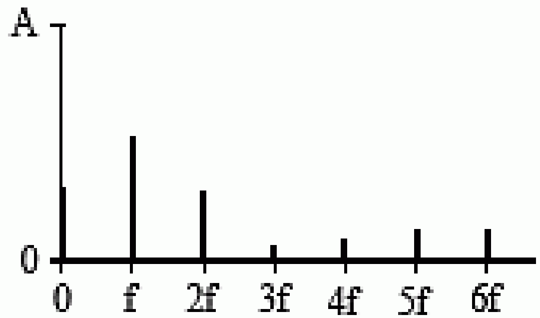
$$f(t)=a_o+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right)$$

$$f(t)=\frac{4A}{\pi}\left(\sin\omega t+\frac{1}{3}\sin 3\omega t+\frac{1}{5}\sin 5\omega t+\frac{1}{7}\sin 7\omega t+\cdots\right);\quad \omega=2\pi f_o$$

Time Domain

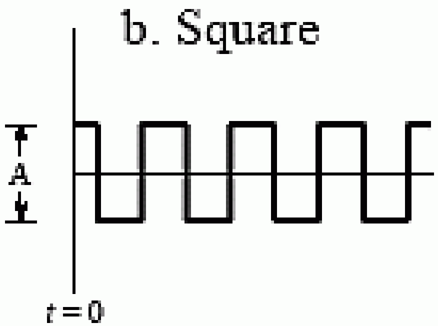
Frequency Domain

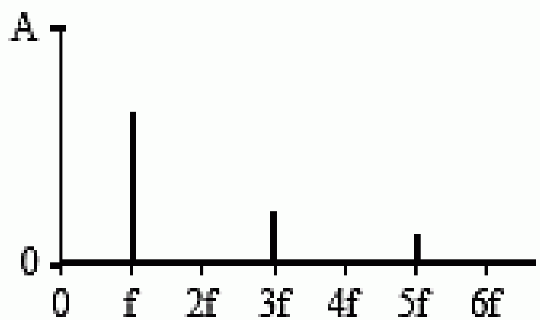




$$a_0 = A\,d$$
$$a_n = \frac{2A}{n\pi}\sin(n\pi d)$$
$$b_n = 0$$

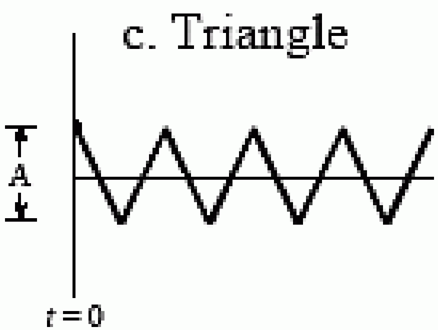
($d = 0.27$ in this example)

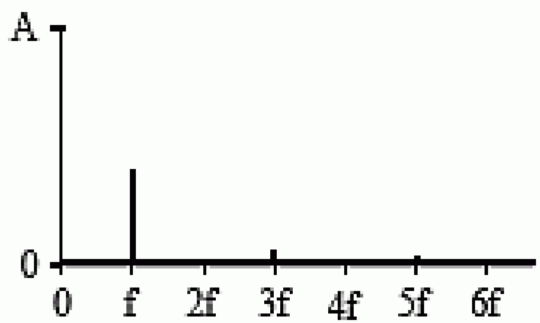




$$a_0 = 0$$
$$a_n = \frac{2A}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$
$$b_n = 0$$

(all even harmonics are zero)

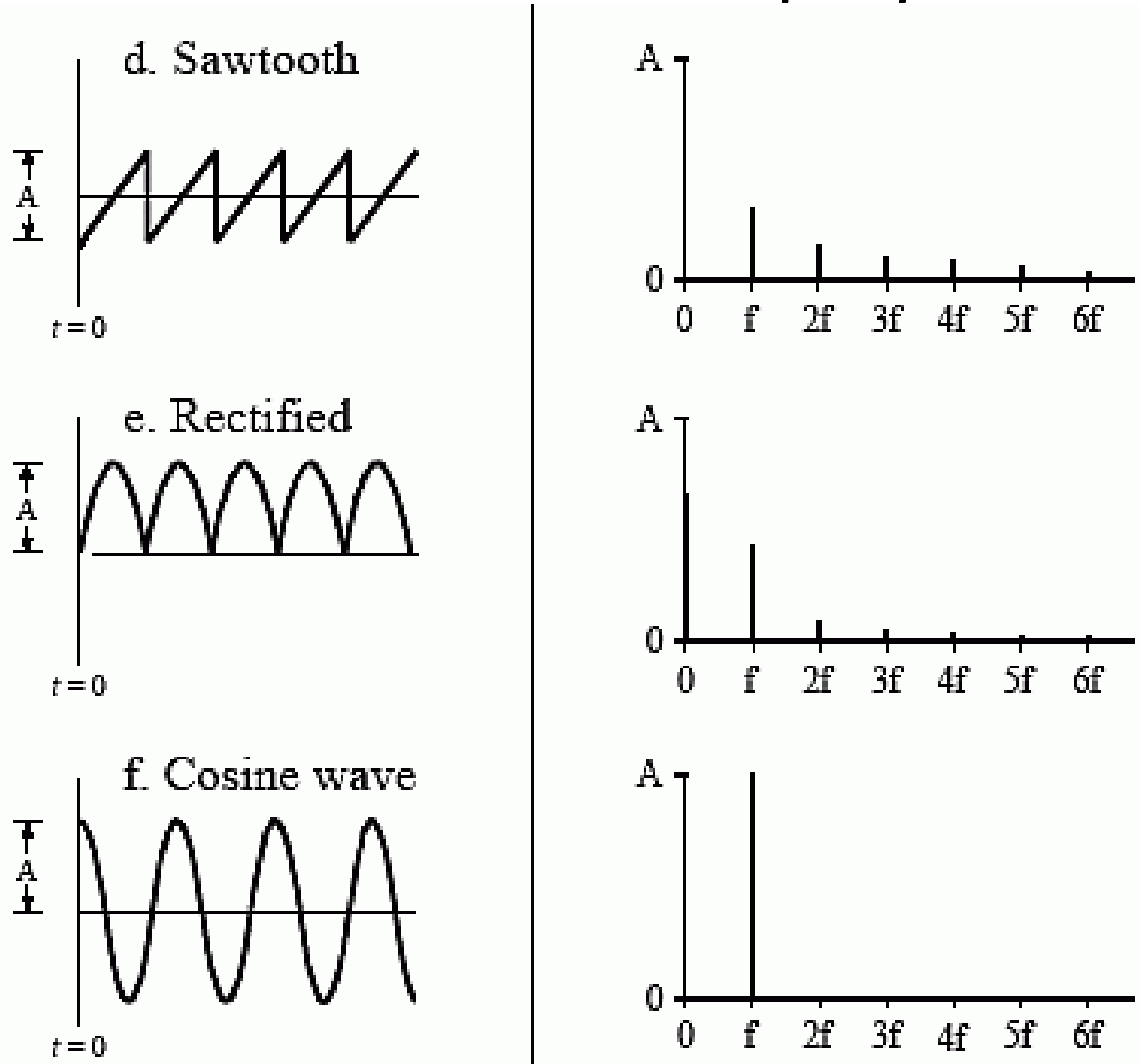




$$a_0 = 0$$
$$a_n = \frac{4A}{(n\pi)^2}$$
$$b_n = 0$$

(all even harmonics are zero)

Time domain | **Frequency domain**



d. Sawtooth:

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{A}{n\pi}$$

e. Rectified:

$$a_0 = 2A/\pi$$
$$a_n = \frac{-4A}{\pi(4n^2-1)}$$
$$b_n = 0$$

f. Cosine wave:

$$a_1 = A$$

(all other coefficients are zero)

Fourier transform

- 類比信號轉換至頻率空間
- 以$e^{-i\omega x}$為正交基底的振幅表示
  - 比較: (x,y,z)為正交基底,座標位置表示向量大小
  - 轉換將函數轉換至頻率域
  - 逆轉換由頻率域轉換回值域

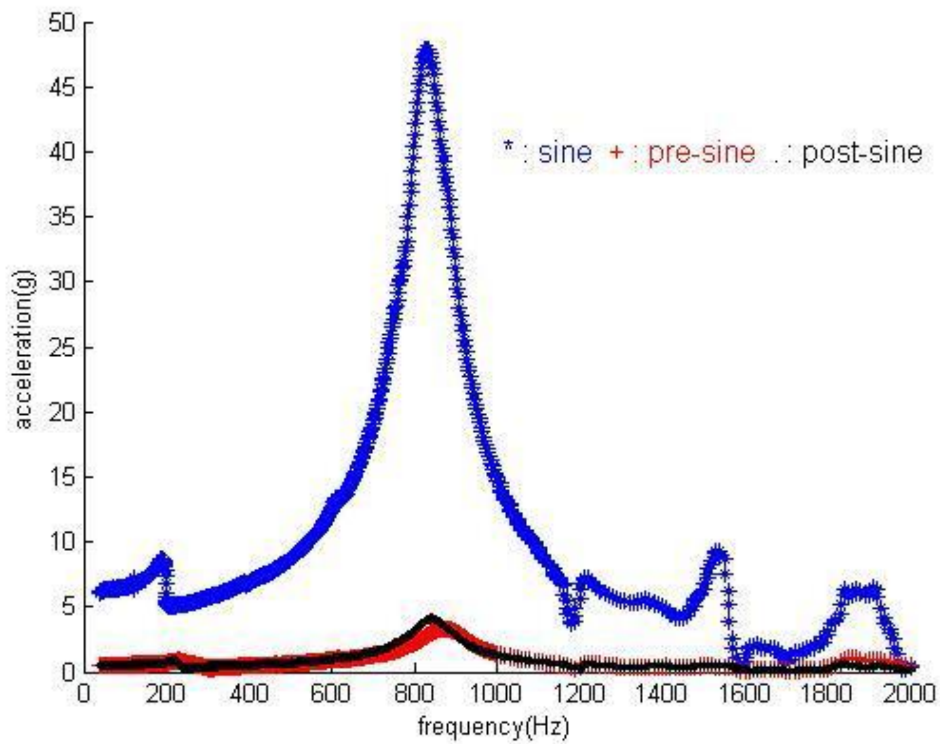
$$\hat{f}(\omega) = \int_{\mathbf{R}^n} f(x) e^{-i\omega\cdot x}\, dx.$$

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\omega) e^{i\omega\cdot x}\, d\omega.$$

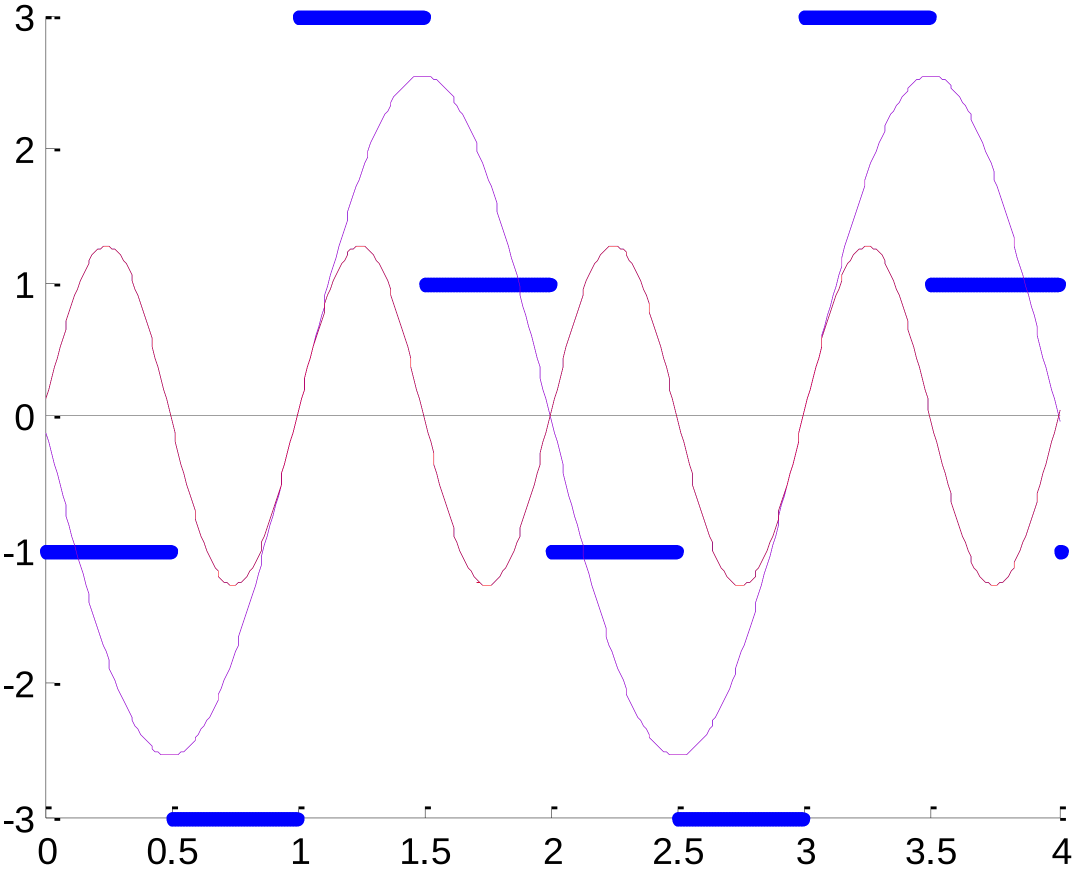
- 實務應用
  - 頻譜分析
  - 振動測試

振動測試

- 特徵頻率分析
- 量測結果為時變的振幅(時域)轉換為頻率域



範例



正弦及餘弦函數的正交特性

- 相乘後一個週期積分

$$\int_{-\pi}^{\pi} \sin(mx)\,\sin(nx)\,dx = \pi\delta_{mn}, \quad m,n \geq 1$$

$$\int_{-\pi}^{\pi} \cos(mx)\,\cos(nx)\,dx = \pi\delta_{mn}, \quad m,n \geq 1,$$

$$\int_{-\pi}^{\pi} \cos(mx)\,\sin(nx)\,dx = 0;$$

傅立葉級數

- 將函數以如下形式表示

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{inx},$$

將複數以三角函數展開

$$s_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} A_n \cdot \sin(\tfrac{2\pi nx}{P} + \phi_n), \quad \text{for integer } N \geq 1.$$

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(\overbrace{a_n}^{A_n \sin(\phi_n)} \cos(\tfrac{2\pi nx}{P}) + \overbrace{b_n}^{A_n \cos(\phi_n)} \sin(\tfrac{2\pi nx}{P}) \right)$$

$$= \sum_{n=-N}^{N} c_n \cdot e^{i\frac{2\pi nx}{P}},$$

$$a_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x) \cdot \cos(\tfrac{2\pi nx}{P})\, dx \qquad b_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x) \cdot \sin(\tfrac{2\pi nx}{P})\, dx$$

由正交特性可知a_n, b_n 僅在同一 $\cos(\frac{2\pi nx}{P})$ $\sin(\frac{2\pi nx}{P})$分量上有值

解題步驟

- 類比訊號取樣成為離散訊號$f(x_i)$
- 同樣的也將sin 及cos 轉為離散訊號$\cos\frac{n\pi x_i}{L}$ 、$s\,in\frac{n\pi x_i}{L}$
- 以正交特性求a_n, b_n

$$a_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x)\cdot\cos(\tfrac{2\pi nx}{P})\ dx \qquad b_n = \frac{2}{P}\int_{x_0}^{x_0+P} s(x)\cdot\sin(\tfrac{2\pi nx}{P})\ dx$$

- 實務上不求積分,矩陣數值相乘求平均(why?)
- 證明:

$$a_n = \frac{2}{P}\sum_{i=1}^{k} f(x_i)\times\cos\frac{n\pi x_i}{L}\times dx = \frac{2}{k\times dx}\sum_{i=1}^{k} f(x_i)\times\cos\frac{n\pi x_i}{L}\times dx$$

$$= \frac{2}{k}\sum_{i=1}^{k} f(x_i)\times\cos\frac{n\pi x_i}{L}$$

HW3

- 傅立葉轉換為利用正弦餘弦函數的正交特性將週期函數展開為一系列傅立葉級數的疊加，函數原本的特性是否對稱以及是否連續會決定級數展開後的品質。作業三將展示此一特性。
- 1.真實世界中為類比訊號，任何時刻都會有值，但在數位世界中則因為取樣而不連續。請以取樣率100HZ畫出1Hz由1開始在[1,-1]間震盪的方波以及三角波，至少需畫出兩個完整的波形。
- 提示:matlab有函式tripuls以及square可用，但也可以自行定義
- 2.請利用三角函數的正交特性，畫出題一中方波1到5階的傅立葉展開結果
  - 提示:有部分的結果會一樣，請思考一下為什麼?
- 3.畫出題一中三角波1到5階的傅立葉級數展開結果
- 4.若有一個波的前半部的波形保持為-1，後半部波形為自-1等斜率升高到1。請畫出這個波的1到5階的傅立葉展開結果。
  - 提示:請觀察展開結果的誤差特性

注意事項

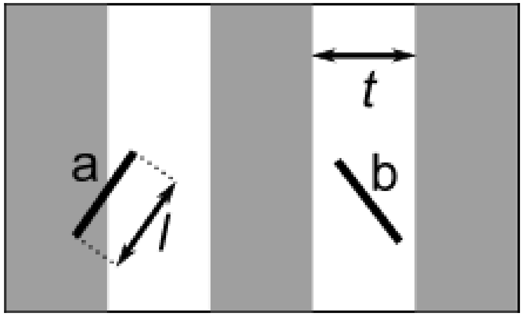
- 請繳交不需更改檔名或是另存新檔便可執行的程式碼。
- 請以取樣的方法求解
  - 不要用matlab內建的函式

參考資料

- Wikipedia
  - 傅立葉轉換
  - 傅立葉級數

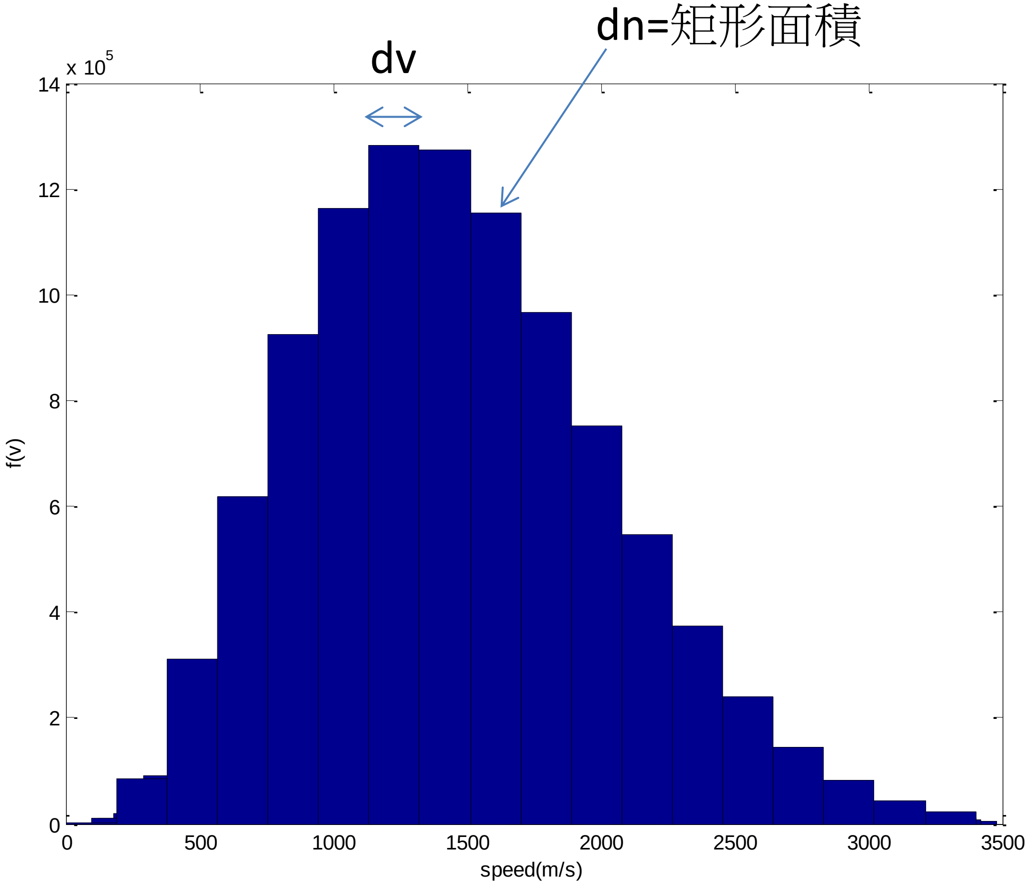
蒙地卡羅法

- 藉由計算機以機率統計處理問題
- 以亂數方式產生隨機的輸入參數
- 經過程式計算每一個輸入參數經過某計算過程後的結果
- 藉由統計方法分析其特徵
- 範例:醉漢走路、社會物理學、布豐針



Maxwell Boltzman distribution

$$
\begin{aligned}
f(v)dv &= 4\pi(2\pi mkT)^{-\frac{3}{2}} e^{\left(-\frac{mv^2}{2kT}\right)} m^3 v^2 dv \\
&= 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{\left(-\frac{mv^2}{2kT}\right)} v^2 dv
\end{aligned}
$$



$$\frac{df(v)}{dv} = 0$$

Typical speeds

- 最可能速率 $\frac{df(v)}{dv} = 0$ $v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}}$
- 平均速率 $\langle v \rangle = \int_0^\infty v\, f(v)\, dv = \sqrt{\frac{8kT}{\pi m}} = \sqrt{\frac{8RT}{\pi M}} = \frac{2}{\sqrt{\pi}} v_p$
- 均方根速率 $\sqrt{\langle v^2 \rangle} = \left(\int_0^\infty v^2\, f(v)\, dv \right)^{1/2} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3}{2}} v_p$

HW4 (1)

- A mole of helium gas is confined in a metal chamber that is keep at temperature T = 400 K.
- (a) Write a code that chooses random velocities satisfying the Maxwell-Boltzmann velocity distribution for 10,000 helium atoms. You may calculate the velocity range from 0 to 3500 m/s and stepped (bin) by 30 m/s. The particle number in each bin is given by inputting these values to the distribution function. And then assume the velocity of the particles in this bin are randomly distribute from v to v+dv. Plot the histogram of your random speed distribution and plot the result along with the analytic Maxwell-Boltzmann distribution function.
- (b) Compute the average, the most probable, the root-mean-square speeds from your data and compare them with theoretical value.

HW4 (2)

- (c) Change the particle number to 10,000, 50,000 and 100,000 and Compute the average, the most probable, the root-mean-square speeds from your data, compare the error of these results.
- (d) Plot the distribution function of helium gas in 400K, 600K and 1000K.
- (e) Plot the distribution function of helium gas and N2 gas.
- Hint: you can use hist in matlab to plot histogram (長條圖).

因為成績結算期限的緣故，本次作業不接受任何補交。請務必在繳交時確認檔案正確可以執行且準時繳交。程式碼請交一份並註解調整各項參數的位置，各項圖形以及計算出的數據貼至文件檔中並且標示清楚題號以及對應的參數，建議轉成PDF以縮小檔案大小。

- 12/11 助教課更改教室 4217
- 12/18 助教課暫停一次