Lab00-Proof

CS214-Algorithm and Complexity, Xiaofeng Gao, Spring 2021.

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1. Prove that for any integer $n > 2$, there is a prime p satisfying $n < p < n!$. (Hint: consider a prime factor p of $n! - 1$ and prove by contradiction)

   Proof.
   Case 1: p is a prime factor of $n! - 1$, then p is a prime factor of $n!$. The statement is obviously true.

   Case 2: $n! - 1$ has no prime factor, then $n! - 1$ is a prime number and $n - 1 \geq 1$, such that $p = n! - 1$ is a prime factor of $n!$. □

2. Use the minimal counterexample principle to prove that for any integer $n \geq 7$, there exists integers $i_n \geq 0$ and $j_n \geq 0$, such that $n = i_n \times 2 + j_n \times 3$.

   Proof. We proof the statement is true for $n \geq 7$ by induction.

   Basis step:
   $n = 7$: $\exists i_n = 2, j_n = 1, \quad st.n = i_n \times 2 + j_n \times 3$
   $n = 8$: $\exists i_n = 1, j_n = 2, \quad st.n = i_n \times 2 + j_n \times 3$
   $n = 9$: $\exists i_n = 0, j_n = 3, \quad st.n = i_n \times 2 + j_n \times 3$
   $n = 10$: $\exists i_n = 2, j_n = 2, \quad st.n = i_n \times 2 + j_n \times 3$
   Induction Hypothesis: Assume that exists $n > 10$, there any intergers $i_n \geq 0$ and $j_n \geq 0$, such that $n \neq i_n \times 2 + j_n \times 3$

   Proof of Induction Step: Define a new variable m,

$$m = \begin{cases} n - 2 & i_n \neq 0 \\ n - 3 & i_n = 0 \end{cases} \tag{1}$$

   Because $n \geq 10$, therefore $m \geq 7$ and $m < n$ and

$$m \neq \begin{cases} (i_n - 1) \times 2 + j_n \times 3 & i_n \neq 0 \\ i_n \times 2 + (j_n - 1) \times 3 & i_n = 0 \end{cases} \tag{2}$$

   $i_n - 1$ and $j_n - 1$ can take arbitrary values in $\mathbb{N}$. □

3. Suppose the function f be defined on the natural numbers recursively as follows: $f(0) = 0$, $f(1) = 1$, and $f(n) = 5f(n-1) - 6f(n-2)$, for $n \geq 2$. Use the strong principle of mathematical induction to prove that for all $n \in N$, $f(n) = 3^n - 2^n$.

   Proof. We proof $f(n) = 3^n - 2^n$ is true for $n \geq 2$ by induction.

   Basis step: For $n = 0, \quad f(n) = 3^0 - 2^0 = 0$.

   Induction Hypothesis: Let $n > 0$. Assume that for any integer $k \in [0, n]$ and $k > 0$, such that $f(k) = 3^k - 2^k$.

Proof of Induction Step: Now let us prove that $f(n+1) = 3^{n+1} - 2^{n+1}$ is true.

$$\begin{aligned} f(n+1) &= 5f(n) - 6f(n-1) && (3)\\ &= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) && (4)\\ &= 9 \times 3^{n-1} - 4 \times 2^{n-1} && (5)\\ &= 3^{n+1} - 2^{n+1} && (6) \end{aligned}$$

□

4. An n-team basketball tournament consists of some set of $n \geq 2$ teams. Team p beats team q iff q does not beat p, for all teams $p \neq q$. A sequence of distinct teams p_1, p_2,..., p_k, such that team p_i beats team p_{i+1} for $1 \leq i < k$ is called a ranking of these teams. If also team p_k beats team p_1, the ranking is called a *k-cycle*.

   Prove by mathematical induction that in every tournament, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Proof. We prove the statement is true by induction.
   Basis step: n=2: Suppose that team team p and team q are in the tournament. Because team p beats team q iff q does not beat p, the winner is the champion.

   Induction Hypothesis: Assume that for any tournament having n teams, either there is a "champion" team that beats every other team, or there is a 3-cycle.

   Proof of Induction Step: For any tournament having $n+1$ teams denoted as $p_1, p_2 \ldots, p_n, p_{n+1}$ there are two cases:

   Case 1: There is a team denoted as p_n that beats all other $n-1$ teams in $\{p_1, p_2 \ldots, p_n\}$. If p_n beats p_{n+1}, then p_n is the champion. If p_{n+1} beats p_n, there are two cases:

   Case 1.1: p_{n+1} beats all of the teams in $\{p_1, p_2 \ldots, p_{n-1}\}$, then p_{n+1} is the champion.

   Case 1.2: Exists p_i in $\{p_1, p_2 \ldots, p_{n-1}\}$, such that p_i beats p_{n+1}. Then $p_n, p_{n+1}, and \quad p_i$ consists of a 3-cycle.

   Case 2: There is a 3-cycle in $\{p_1, p_2 \ldots, p_n\}$. Such 3-cycle will be maintained after the participation of team p_{n+1} . □

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