A brief introduction of the constitutive model

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1. Back ground
2. General components
3. MCC model
4. UH model
5. CSUH model
6. Tips in FEM subroutine implementation
7. FEM calculation examples

Brief introduction of the constitutive model

1. Non-linear models
   - Duncan-Chang model [DC70][1]
   - A series extended models based on Duncan-Chang model
   - ...
2. Elastoplastic models
   - Models for metal
   - Original Cam-Clay model (OCC) [RSW58][2]
   - Modified Cam-Clay model (MCC) [KB70] [3]
   - Unified hardening model based MCC (UH)
   - Critical state model based on the unified hardening model (CSUH)
   - ...

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[1] Duncan JM, Chang CY (1970) Nonlinear analysis of stress and strain in soils.
[2] Roscoe KH, Schofield AN, Wroth CP (1958) On the yielding of soils.
[3] K.H. Roscoe, Burland JB (1970) On the generalized stress-strain behavior of “wet” clay.

General components summary

- Yield function
- Hardening function
- flow rule (plastic energy function)

Yield function

There is a range of yield functions mainly including quantities derived from the stress tensor $\boldsymbol{\sigma}$:

- Mohr-Coulomb (frictional materials)

$$\begin{cases} \tau_m - \sigma_m \sin\phi \leq 0 \\ \sigma_m = \dfrac{\sigma_1 + \sigma_3}{2}, \tau_m = \dfrac{\sigma_1 - \sigma_3}{2} \end{cases} \tag{1}$$

- Mises stress (metal/cohesive materials)

$$\sigma_v = \sqrt{3J_2} = \sqrt{\frac{3}{2} s_{ij} s_{ij}} \tag{2}$$

- Original Cam Clay model (OCC)

$$f(q, p, p_c) = q + Mp \ln\left(\frac{p}{p_c}\right) \tag{3}$$

- Modified Cam Clay model (MCC)

$$f(q, p, p_c) = q^2 + M^2 p(p - p_c) \tag{4}$$

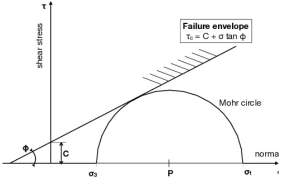


Figure 1: Mohr-Coulomb circle

Acorrding to the Mohr-Coulomb circle, we have:

$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin\phi}{1 - \sin\phi} \tag{5}$$

In the model of MCC, the critical stress ratio $M = \frac{q}{p}$ can be represented as:

$$\begin{cases} M_{compression} = \dfrac{\sigma_1 - \sigma_3}{(\sigma_1 + 2\sigma_3)/3} = \dfrac{6\sin\phi}{3 - \sin\phi} \\ M_{extension} = \dfrac{\sigma_1 - \sigma_3}{(2\sigma_1 + \sigma_3)/3} = \dfrac{6\sin\phi}{3 + \sin\phi} \end{cases} \tag{6}$$

SMP failure criterion:

$$\begin{cases} \dfrac{I_1 I_2}{I_3} \leq \text{const.} \\ \text{const.} = \dfrac{(\sigma_1 + 2\sigma_3)\left(2\sigma_1\sigma_3 + \sigma_3^2\right)}{\sigma_1\sigma_3^2} \\ \qquad = \dfrac{9 - \sin^2\phi}{1 - \sin^2\phi} \\ q_c = \dfrac{2I_1}{3\sqrt{(I_1 I_2 - I_3)/(I_1 I_2 - 9I_3)} - 1} \end{cases} \tag{7}$$

Modified Lade failure criterion:

$$\begin{cases} \dfrac{I_1^3}{I_3} - 27 + \eta \leq 0 \\ \eta = \dfrac{4\tan^2\phi(9 - 7\sin\phi)}{1 - \sin\phi} \end{cases} \tag{8}$$

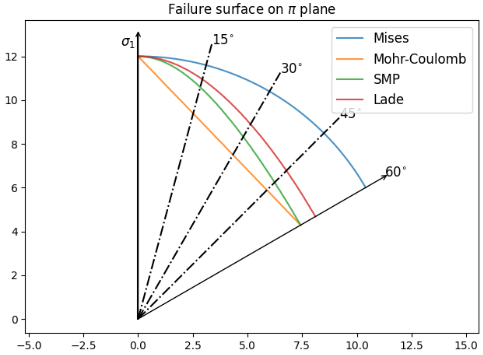


Figure 2: Yield surfaces under different criterion on the π plane

The Mises can be transformed to SMP through the paper[4].

[4] Matsuoka H, Yao YP, Sun D (1999) The Cam-clay models revised by the SMP criterion.

Yield surfaces in 3D space

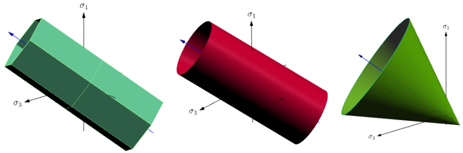


Figure 3: (a)Tresca surface; (b)Von-mises surface; (c)Drucker-Prager surface

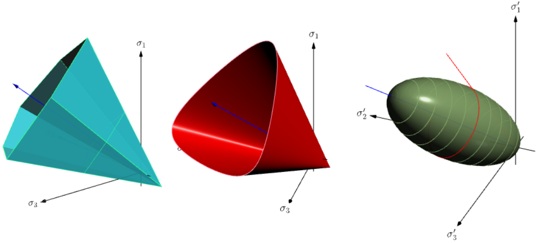


Figure 4: (a)Mohr-Coulomb surface; (b)SMP surface; (c)MCC surface

Hardening model

There are many hardening models, as the plastic strain or the plastic energy used as the internel variables.

- Exponential hardening model (generally used for metals)

$$H = A + B(\epsilon_0 + \epsilon_s^p)^n \tag{9}$$

- Maximum of the mean pressure (MCC and OCC) models, and a series extended model based on the MCC model

$$\begin{cases} H = \frac{\epsilon_v^p}{c_p} \\ c_p = \frac{\lambda - \kappa}{1 + e_0} \end{cases} \tag{10}$$

Flow rules

The fLow rules decide the direction of the plastic strain in the plastic return mapping calculation, which is used to make sure the stress still on the yield function once loading into plastic stage.

The flow rules can be generally summarised as:

- associated
- non-associated

Actually, in the most widely used elastoplastic models, the associated flow rule is employed.

The yield function $f(\sigma_{ij}, H)$ decides the stress stress while the plastic energy function $g(\sigma_{ij})$ governs the direction of plastic deformation through:

$$\mathrm{d}\epsilon_{ij}^{p} = \mathrm{d}\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{11}$$

MCC model

In the MCC model, at first, we need to use the consolidation line to describe the link between void ratio and the consolidation pressure (mean stress p (kPa)).

The consolidation line:

$$v = v_\lambda - \lambda \ln \frac{p}{p_1} \tag{12}$$

And the swelling line:

$$v = v_\kappa - \kappa \ln \frac{p}{p_1} \tag{13}$$

According to this line we can calculate the elastic bulk modulus:

$$K = \frac{\mathrm{d}p}{\mathrm{d}\epsilon_v} = \frac{p(1+e)}{\kappa} \tag{14}$$

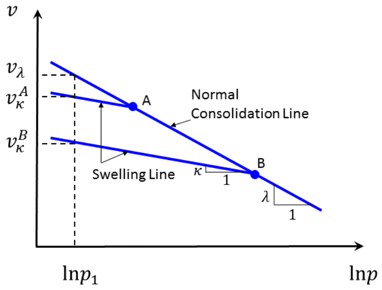


Figure 5: Normal consolidation line

MCC model

The yield function of MCC; because of the associated flow rule, the plastic energy function equals to yield function:

$$f(q,p,p_c) = g(q,p) = q^2 + M^2 p(p-p_c) \quad (15)$$

Then we get the direction of the plastic deformation once the material is loaded into plastic stage:

$$\mathrm{d}\epsilon_{ij}^{p} = \mathrm{d}\lambda \frac{\partial g}{\partial \sigma_{ij}} \quad (16)$$

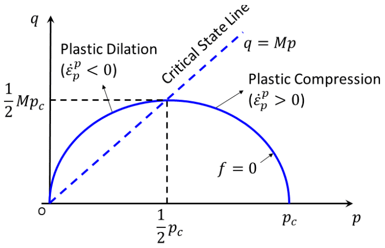


Figure 6: Yield function of the MCC model

MCC model: plastic return mapping

Then comes to the return mapping caculation. According to the consistent condition:

$$\mathrm{d}f = \frac{\partial f}{\partial \sigma_{ij}}\mathrm{d}\sigma_{ij} + \frac{\partial f}{\partial H}\frac{\partial H}{\partial \epsilon_{ij}^{p}}\mathrm{d}\epsilon_{ij}^{p} = 0 \tag{17}$$

Substituting Eq. (16) into Eq. (17):

$$\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}^{e}(\mathrm{d}\epsilon_{ij} - \mathrm{d}\lambda\frac{\partial g}{\partial \sigma_{ij}}) + \frac{\partial f}{\partial H}\frac{\partial H}{\partial \epsilon_{ij}^{p}}\mathrm{d}\lambda\frac{\partial g}{\partial \sigma_{ij}} = 0 \tag{18}$$

Then we can derive the plastic scaler $\mathrm{d}\lambda$ as following:

$$\mathrm{d}\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}^{e}\mathrm{d}\epsilon_{kl}}{\frac{\partial f}{\partial \sigma_{ij}}D_{ijkl}^{e}\frac{\partial g}{\partial \sigma_{kl}} - \frac{\partial f}{\partial H}\frac{\partial H}{\partial \epsilon_{ij}^{p}}\frac{\partial g}{\partial \sigma_{ij}}} \tag{19}$$

MCC model: plastic return mapping

With $\mathrm{d}\lambda$ we can calculate the plastic strain:

$$\mathrm{d}\epsilon_{ij}^{p} = \mathrm{d}\lambda \frac{\partial g}{\partial \sigma_{ij}} \tag{20}$$

And the stress:

$$\mathrm{d}\sigma_{ij} = D_{ijkl}^{e}(\mathrm{d}\epsilon_{kl} - \mathrm{d}\epsilon_{kl}^{p}) \tag{21}$$

Substituting Eq. (19) and Eq. (20) into Eq. (21), we get:

$$\mathrm{d}\sigma_{ij} = D_{ijkl}^{e}\left(\mathrm{d}\epsilon_{kl} - \frac{\frac{\partial f}{\partial \sigma_{mn}} D_{mnst}^{e} \mathrm{d}\epsilon_{st}}{\frac{\partial f}{\partial \sigma_{mn}} D_{mnst}^{e} \frac{\partial g}{\partial \sigma_{st}} - \frac{\partial f}{\partial H} \frac{\partial H}{\partial \epsilon_{mn}^{p}} \frac{\partial g}{\partial \sigma_{mn}}} \frac{\partial g}{\partial \sigma_{kl}}\right) \tag{22}$$

MCC model: plastic return mapping

In Eq. (16) after assuming the subscripts in tensor $\mathrm{d}\epsilon_{st} \to \mathrm{d}\epsilon_{kl}$ and $\frac{\partial g}{\partial \sigma_{kl}} \to \frac{\partial g}{\partial \sigma_{st}}$, the elastoplastic matrix can be presented as:

$$\begin{cases} D^{ep}_{ijkl} = D^{e}_{ijkl}\left(1 - \dfrac{\frac{\partial f}{\partial \sigma_{mn}} D^{e}_{mnst} \frac{\partial g}{\partial \sigma_{st}}}{\frac{\partial f}{\partial \sigma_{mn}} D^{e}_{mnst} \frac{\partial g}{\partial \sigma_{st}} - \frac{\partial f}{\partial H} \frac{\partial H}{\partial \epsilon^{p}_{mn}} \frac{\partial g}{\partial \sigma_{mn}}}\right) \\ \mathrm{d}\sigma_{ij} = D^{ep}_{ijkl}\mathrm{d}\epsilon_{kl} \end{cases} \tag{23}$$

Note: because of the assumptions, $D^{ep}_{ijkl}\mathrm{d}\epsilon_{kl} \neq D^{e}_{ijkl}\left(\mathrm{d}\epsilon_{kl} - \mathrm{d}\epsilon^{p}_{kl}\right)$. **According to my experience, the stress increment calculated as Eq. (21) will benefit the nonlinear iteration in the FEM calculation because of it is the accurate value without assumptions.**

MCC model: plastic return mapping

Followed by the plastic return mapping calculation, the internal variable (p_c) in the MCC model should be renewed:

$$\mathrm{d}p_c = \exp\left(\frac{(1+e)\mathrm{d}\epsilon_{ii}^{p}}{\lambda-\kappa}\right)p_c - p_c \tag{24}$$

That's how to calculate the stress while yielding and update the yield surface.

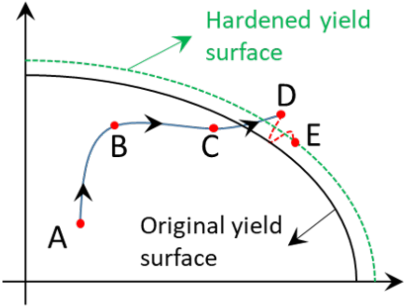


Figure 7: Schematic of the plastic return mapping

UH model

The MCC model, concentrated on the clay, can not reproduce the dilation of the frictional materials due to the non-negtive dilation angle as is shown in Eq. (25).

$$\frac{\mathrm{d}\epsilon_{ii}^{p}}{\mathrm{d}\epsilon_{s}^{p}} = \frac{M^2 - \eta^2}{2\eta} \tag{25}$$

Such as the dense sand which will swell under the conventional drained odometer test. As also controlled by the friction, rockfills will dilate if the initial state is dense enough.

So here introduced Yao's work about the Unified Hardening method to construct the unified hardening rule for both clay and sands[YTZS19].

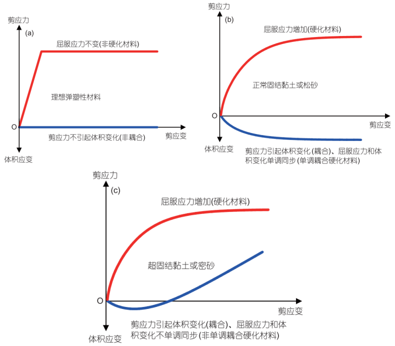


Figure 8: Compression comparison between loose and dense sands

UH model

Compared with the yield function of MCC model Eq. (15), the key modification is on the hardening variables, as is shown in following equation:

$$\begin{cases} f = g = \ln \dfrac{p}{p_0} + \ln \left(1 + \dfrac{q^2}{M^2 p^2}\right) - \dfrac{H}{c_\mathrm{p}} = 0 \\ c_p = \dfrac{1+e}{\lambda - \kappa} \\ H_{MCC} = \varepsilon_\mathrm{v}^\mathrm{p} \\ H_{UH} = \displaystyle\int c(p,q)\,\mathrm{d}\varepsilon_\mathrm{v}^\mathrm{p} = \int \frac{M^4}{M_\mathrm{f}^4} \frac{M_\mathrm{f}^4 - \eta^4}{M^4 - \eta^4}\,\mathrm{d}\varepsilon_\mathrm{v}^\mathrm{p} \end{cases} \tag{26}$$

UH model

In the MCC model we have:

$$\begin{cases} \dfrac{\partial g}{\partial p} = \dfrac{2M^2 p}{M^2 p^2 + q^2} - \dfrac{1}{p} \\ \dfrac{\partial g}{\partial q} = \dfrac{2q}{M^2 p^2 + q^2} \\ \dfrac{\mathrm{d}\epsilon_{ii}^p}{\mathrm{d}\epsilon_s^p} = \dfrac{\partial g}{\partial p} \Big/ \dfrac{\partial g}{\partial p} = \dfrac{M^2 - \eta^2}{2\eta} \\ \mathrm{d}\lambda = c_p \left(\mathrm{d}p + \dfrac{2\eta}{M^2 - \eta^2} \mathrm{d}q \right) \end{cases} \tag{27}$$

After introduced the UH hardening:

$$\begin{cases} \dfrac{\partial g}{\partial p} = \dfrac{2M^2 p}{M^2 p^2 + q^2} - \dfrac{1}{p} \\ \dfrac{\partial g}{\partial q} = \dfrac{2q}{M^2 p^2 + q^2} \\ \mathrm{d}\lambda = \dfrac{c_p}{c(p,q)} \left(\mathrm{d}p + \dfrac{2\eta}{M^2 - \eta^2} \mathrm{d}q \right) \\ c\left(p, q\right) = \dfrac{M^4}{M_{\mathrm{f}}^4} \dfrac{M_{\mathrm{f}}^4 - \eta^4}{M^4 - \eta^4} \end{cases} \tag{28}$$

The dilation angle are the same, while the plastic scaler $\mathrm{d}\lambda$ is changed. **The scaler can be negative once the stress ratio $\eta = \frac{q}{p}$ is larger than the critical stress ratio M, which agrees with the phenomenan that the dilation always happen along with the stress peak.**

CSUH model: highlights

Based on the UH model, some further modifications were carried out to implement the CSUH model [YLL+19]. There are several main points in the CSUH model, compared with the UH model:

- Non-association flow through a newly-introduced variable M_c to control the dilation;
- Internal variable ξ to represent the current stage according to the ACL (Anisotropic compression line);
- A curved line representing the $e - \ln p$ relationship;
- An parameter χ to involved to describe the distance between NCL and CSL.
- Transformed stress space used to consider the medium principal stress coefficient $b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - sigma_3}$

CSUH model: Non-association

Different from the associated flow rule in the MCC model, a variable M_c is introduced to construct the energy function:

$$g = \ln p + \ln\left(1 + \frac{q^2}{M_c^2 p^2}\right) \tag{29}$$

where, $M_c = M \exp(-m\xi)$, M is the critical stress ratio, ξ is the variable representing the material state, and m is the material constant calibrated through **undrained compression test** regarding:

$$m = -\frac{1}{\xi_c} \ln \frac{M_c^{\xi_c}}{M} \tag{30}$$

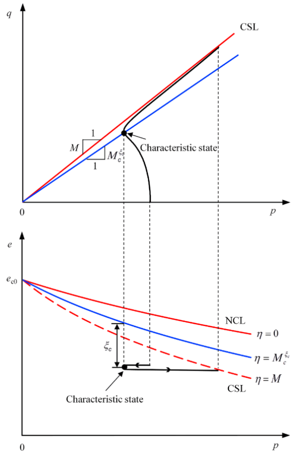


Figure 9: Material constant m calibration

CSUH: internal variable ξ

Variable ξ is used for representing the current stage of the material. According to Fig. (10), we can calculate the e_η wrt the current stress ratio $\eta = q/p$ and the functions of the lines:

$$\begin{cases} e_\eta = N - \lambda \ln p - (\lambda - \kappa) \ln \left(1 + \dfrac{\eta^2}{M^2}\right) \\ \xi = e_\eta - e \end{cases} \quad (31)$$

ξ **is very important in this model.** There are 2 kinds of conditions of the current stages:

- $\xi > 0$: Current material is denser than the corresponding CS (Critical state);
- $\xi < 0$: Current material is looser.

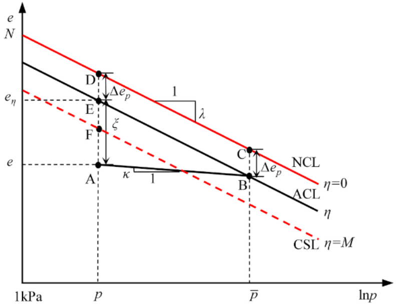


Figure 10: e_η and ξ calculation

CSUH: internal variable ξ

ξ is the only state variable involved to calculate the failure stress ratio M_f and the feature stress ratio M_c.

- M_f representing the failure stress ratio q/p when will the material fails. That's why a peak of stress will emerge in the $\sigma_s - \epsilon_{axial}$ curve;
- M_c mainly influences the direction of the plastic deformation, cuz:

$$\mathrm{d}\epsilon_v/\mathrm{d}\epsilon_s = \frac{M_c^2 - \eta^2}{2\eta}$$

which means that the smaller M_c is, the material is more tend to dilation.

Analogous to the varible ξ in this model, φ is defined in Been and Jefferies model [Jef93][5] to represent the vertical distance between the current state and the CSL.

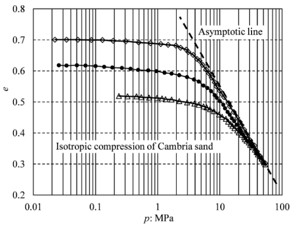
[5] Been and Jefferies, A state parameter for sands. Geotechnique 1985

CSUH: curved line in $e - \ln p$ space

Another thing interesting is that the curve $e - \ln p$ is not a straigt line in the semi-ln (p) space, especially at the lower stress level state. Subsequently, there employed an important modification to improve the model:

$$\begin{cases} \text{MCC model}: & e = N - \lambda \ln p \\ \text{CSUH model}: & e = Z - \lambda \ln\left(\dfrac{p + p_{si}}{1 + p_{si}}\right) \end{cases} \tag{32}$$

where, Z is the void ratio as $p = 1$kPa. **Note: in the this paper, the unit of the p in the $e - \ln p$ space is kPa.** While $p \to \infty$, we have $N - \lambda \ln p = Z - \lambda \ln\left(\frac{p+p_{si}}{1+p_{si}}\right)$. Then $p_{si} = \exp\left(\frac{N-Z}{\lambda}\right) - 1$.



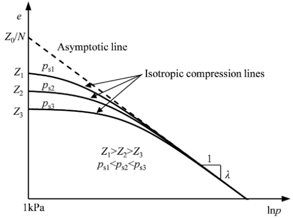


Figure 11: Isotropic compression of Cambria sand

CSUH: parameter χ representing the distance between NCL and CSL

In the original MCC model, according to the yield function $f = \ln\frac{p}{p_0} + \ln\left(1 + \frac{q^2}{M^2p^2}\right) - \frac{\epsilon_v^p}{c_p} = 0$, we can compare the equation in isotropic ($\eta = 0$) and critical state ($\eta = M$):

$$\begin{cases} \text{isotropic:} & \ln\dfrac{p}{p_0} - \dfrac{\epsilon_{v0}^p}{c_p} = 0 \\ \text{cirtical state:} & \ln\dfrac{p}{p_0} + \ln 2 - \dfrac{\epsilon_{v\eta}^p}{c_p} = 0 \end{cases} \tag{33}$$

As is shown in Fig. (12), the distance grows larger as χ increase. We found the vertical distance between the NCL and the CSL is fixed as $\ln 2$ in the $e - \ln p$ curve. Then a parameter χ is added to adjust vertical distance between the two lines, and the modified yield function as:

$$f = \ln\left(\frac{p}{p_0}\right) + \ln\left(\frac{(1+\chi)q^2}{M^2p^2 - \chi q^2} + 1\right) - \frac{\epsilon_v^p}{c_p} = 0 \tag{34}$$

Then the distance is changed to $\ln(2+\chi)$.

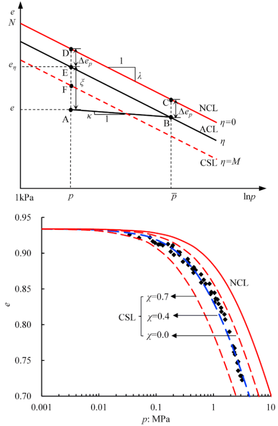


Figure 12: CSL parameter χ

CSUH model: Transformed stress space based on the SMP criterion

To consider the medium stress ratio $b = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$, as is shown in Fig. (2).

Without this consideration, the model can not distinguish the compression from the extension

In the yield function Eq. (34), if we calculate q through the Mises stress equation as $q = \sqrt{3J_2}$, the yield surface will be a circle in the π plane.

Then we calculated the q as q_c Eq. (7):

$$q_c = \frac{2I_1}{3\sqrt{(I_1 I_2 - I_3)\,/\,(I_1 I_2 - 9I_3)} - 1} \tag{35}$$

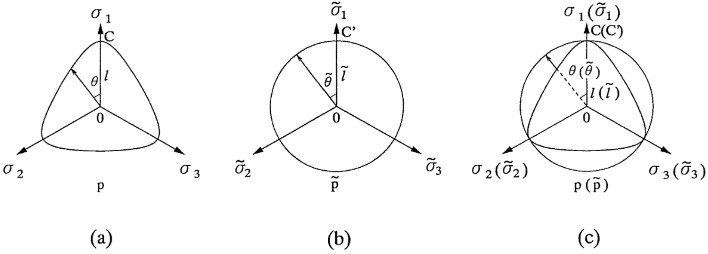


Figure 13: Transformed stress space

Tips in FEM subroutine implementation

1. elasticy and plasticity transformation split
2. Yield value of the last step added to the $\mathrm{d}\lambda$ calculation
3. split the loading step as *Abaqus*
4. **einsum** method in numpy library

```
dsig = np.einsum('ijkl,kl->ij', self.D, deps)
```

5. tolerance of the yield value

Tip 1: elastoplastic transformation split

Bi-section used in the transformation split:

```
def transformSplit(self, deps):
    rmin, rmax, rmid = 0., 1., 0.5
    sig = self.sigma + np.einsum('ijkl, kl->ij', self.D, deps * rmid)
    p = getP(sigma=sig)
    q = getQ(sigma=sig)
    yieldValue = self.yieldFunction(q=q, p=p, H=self.H, px0=self.px0)
    while True:
        if yieldValue > 0.:
            rmax = rmid
        elif yieldValue < -self.yieldTolerance:
            rmin = rmid
        else:
            break
        rmid = .5 * (rmax + rmin)
        sig = self.sigma + np.einsum('ijkl, kl->ij', self.D, deps * rmid)
        p = getP(sigma=sig)
        q = getQ(sigma=sig)
        yieldValue = self.yieldFunction(q=q, p=p, H=self.H, px0=self.px0)
    return rmid, sig, yieldValue
```

Tip 2: yield value in the last return mapping calculation added to current $d\lambda$

Return mapping code based on the CSUH model:

```python
def returnMapping(self,):
        e = elast - (1. + elast) * getVolStrain(deps)
        eta_last = q_last / p_last
        # return mapping
        returnMapping_iter = 0
        while True:
                dg_dsigma = self.get_dg_dsigma(Mc=Mc_last, eta=eta_last,
                                               p=p_last, sigma=sigLast)
                df_dsigma, df_depsvp = self.get_df_dsigma_df_depsp(
                Mf=Mf_last, Mc=Mc_last, sigma=sigLast, p=p_last, q=q_last)
                temp = np.einsum('ij, ijkl, kl->', df_dsigma, Dlast, dg_dsigma) - \
                           df_depsvp * np.trace(dg_dsigma)
                A = (np.einsum('ij, ijkl, kl->', df_dsigma, Dlast, deps) +
                yieldValue_last if yieldValue_last < 1e5 else 0.) / temp
                # A = np.einsum('ij, ijkl, kl->', df_dsigma, Dlast, deps) / temp
                deps_p = A * dg_dsigma
                depsvp = np.trace(deps_p)
                D_ep = Dlast - np.einsum('ijmn, mn, st, stkl->ijkl', Dlast,
                                         dg_dsigma, df_dsigma, Dlast) / temp
                # sig = sigLast+np.einsum('ijkl, kl->ij', D_ep, deps)
                sig = sigLast + np.einsum('ijkl, kl->ij', Dlast, deps - deps_p)
```

Tip 2: yield value in the last return mapping calculation added to current $\mathrm{d}\lambda$

```python
p = getP(sigma=sig)
q = getQ(sigma=sig)
if p < 0.:
    sig = sigLast
    q = q_last
    p = p_last
    deps_p = deps
    depsvp = np.trace(deps_p)
    D_ep = np.zeros_like(self.D)
    xi = xi_last
    H = self.H
    yieldValue = self.yieldFunction(q, p, H, self.px0)
    epsvp = self.epsvp+depsvp
    if self.verboseFlag:
        print('Failed element in the plastic return mapping stage!')
    return sig, q, p, deps_p, D_ep, xi, H, yieldValue, epsvp
# calculate the updated state variables
eta= q / p
xi = self.get_e_eta(eta=eta, p=p) - e
epsvp = self.epsvp + depsvp
Mf_last = 0.5 * (Mf_last + self.getM_f(xi))
```

Tip 2: yield value in the last return mapping calculation added to current $\mathrm{d}\lambda$

```
Mc_last = 0.5 * (Mc_last + self.getM_c(xi))
eta_last = .5 * (eta_last + eta)
# K, G, lam = self.getElasticModulus(p=p_last, e=e)
# Dlast = self.getMaterialMatrix(lam=lam, G=G)
H = self.H + self.get_dH(
Mf=Mf_last,
Mc=Mc_last,
eta=eta_last, depsvp=depsvp)
yieldValue = self.yieldFunction(q=q, p=p, H=H, px0=self.px0)
if np.abs(yieldValue) < self.yieldTolerance:
    if self.verboseFlag:
        print('converged!')
    break
else:
    if self.verboseFlag:
        print('\t\t\t Yield value: %.3e' % yieldValue)
returnMapping_iter += 1
break
```

Tip 3: split the loading step as *Abaqus*

```
while t < loadingStep:
        vel = vel_list[t]
        vel_remain, scaler, remain_du = copy.deepcopy(vel), 1.0, 1.0
        du = Vector(0., Solution(mydomain))
        while True:
                Dbc, Vbc, Nbc = boudaryCondition(loadingPath=loadingPath, nb=nb, fb=fb, vel=vel)
                prob.initialize(f=Nbc, specified_u_mask=Dbc, specified_u_val=Vbc)
                ddu, converge = prob.solve(iter_max=10)
                if converge == False:
                        scaler = 0.5 * scaler
                if scaler < 1. / 2 ** 10:
                        raise ValueError('=' * 80 + '\n \t Can not converge after 4 times split.' +
                        '\n \t please decrease the loading step and retry.')
                vel = 0.5 * vel
                print('\n\t\t Can not converge, original vel_0: %.3e vel_1: %.3e' % (2. * vel, vel))
                else:
                        remain_du = remain_du - scaler
                        du += ddu
                if remain_du <= 0.:
                        break
                t += 1
        disp += du
```

Tip 5: tolerance of the yield value

In the mathematical function, we can easily use 0 to split the elasticity and plasticity.

While, in numerical calculation, such as the renewed yieldValue f after the return mapping process can not be exactly zero due to the non-linearity since the derivations ($\frac{\partial f}{\partial \sigma_{ij}}$, $\frac{\partial f}{\partial \epsilon_{ij}^{p}}$ and $\frac{\partial g}{\partial \sigma_{ij}}$) used is based on last stage.

So we have to introduce a parameter in the code as the tolerance of the yield value. If the yield value is lower than parameter, the return mapping processure is satisfied.

Von-mises model and co-operation with the $\mathcal{NN}$ model

Here are the simulating results based on the Von-Mises model Eq. (2) and exponential hardening Eq. (9).

(a) Mathematical equations; (b) $\mathcal{NN}$ hardening model involed; (c) both hardening model and the Mises stress calculation are substituted by $\mathcal{NN}$.

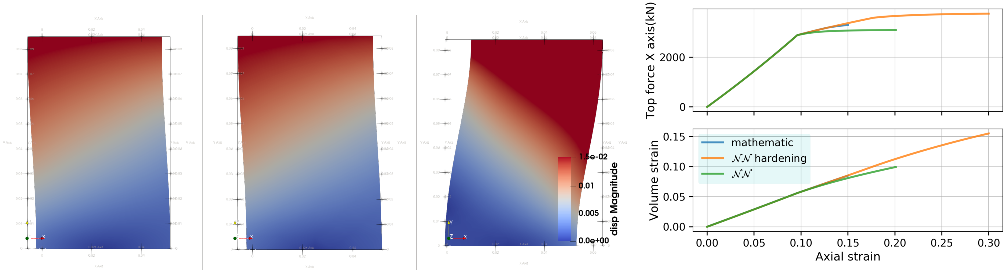


Figure 14: 2D calculation based on the Von-Mises model and exponential hardening rule; Top force and volume strain comparison

MCC model simulation 3D

Here, we completed the MCC model in the FEM calculation. There is no sheaing band in the conventional compression test as is shown in Fig. (15).

In Fig. (16), these are cylic loading simulations under the conventional compression and the isotropic compression. There is an obvious transformation between the elastic and the plastic stages. The $e - \ln p$ curve in Fig. (16b) agrees well with the material constants (λ and κ).

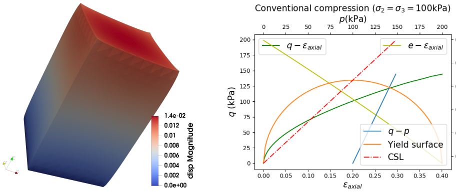


Figure 15: Conventional compression 3D

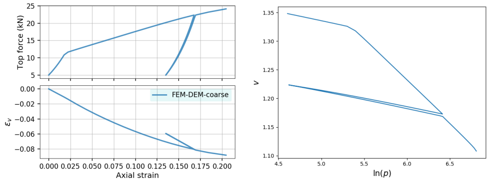


Figure 16: Cylic loading: (a) conventional compression; (b) isotropic compression

CSUH model

The model can describe the dilation, the stress peak and the shearing band (strain localization) of the dense sand (rockfill materials) in the conventioal compression simulations ($e_0 = 0.7$).

Table 1: Parameters of the Toyoura sand

Paramters	Value
M	1.25
λ	0.135
κ	0.04
v	0.3
N	1.973
e_{c0}	0.934
χ	0.4
m	1.8

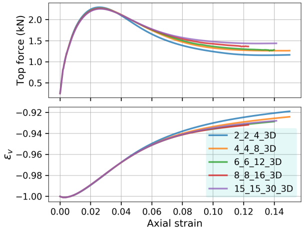


Figure 17: Top force comparison

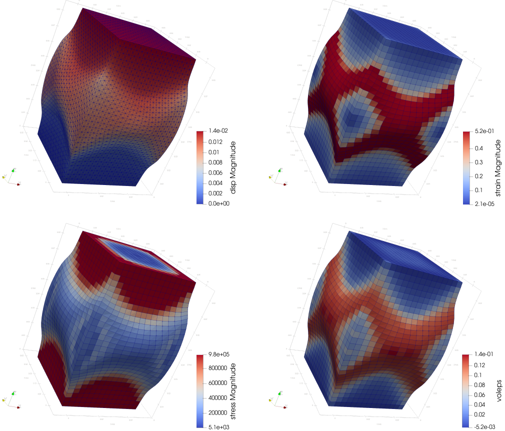


Figure 18: Conventional compression 3D

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Thanks!