Time is Money: Combining Feller's Limit Theorem and Peters' Time Resolution to Reframe the St Petersburg Paradox

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Abstract

The St. Petersburg game is famously known for having an infinite expected payout despite the intuition that the fair price for playing the game should be finite. We explore the problem of determining a fair entry fee for the St. Petersburg game by investigating two approaches. The first approach is William Feller's limit theorem, which modifies the law of large numbers. Feller estimates that the fair entry fee for a single game is a function ($\log_2 \sqrt{N}$) of the number of games played, N. The second approach, proposed by Ole Peters, is to instead examine the time average of the game, rather than the standard expected value. We use an algorithm to approximate Peters' solution, and determine that his fair entry fee is also logarithmic, although it is a function of wealth rather than time. We assess how well these two approaches perform on simulations over large datasets, and also provide conditions for when the two approaches recommend the same entry fee. Namely, this occurs when the number of games to be played, N, is equal to a quarter of the player's initial wealth, w. This result enables us to reframe the fair entry fee as a function of both N and w, and we conjecture that the two aforementioned approaches always give upper and lower bounds for the true fair entry fee.

1 Introduction

1.1 Description of Problem

The St. Petersburg Paradox is a fascinating problem in probability that has drawn attention from mathematicians for hundreds of years. In the early 18th century, Nicolaus Bernoulli remarked that the game has an infinite expected value, as shown in Definition 1.1.1. This suggests that the entry fee to play the St. Petersburg game cannot be finite, which clashes with common sense intuition. Since then, many mathematical approaches have been employed to resolve the St. Petersburg Paradox by characterizing a finite entry fee. We consider two prominent, but vastly dissimilar resolutions of the paradox. In Section 2, we reconstruct Feller's asymptotic analysis of the sample mean and in Section 3, we provide Peters' time resolution of the paradox. In Section 4, we compare the effectiveness of both solutions with respect to simulations and also analyze the conditions that ensure the equivalence of the two solutions. Finally, in Section 5, we use these results to build a multivariable framework for the St. Petersburg Paradox, using Feller's and Peters' estimates to provide upper and lower bounds for the fair price of a game.

Definition 1.1.1: A St. Petersburg game unfolds by flipping a fair coin until it lands on tails, and the payout is 2 raised to the number of heads. We denote j as the number of flips in the game. It follows that the number of consecutive heads in the game is $j-1$ and that the payout is \$$2^{j-1}$. As convention, let the distribution of the payout of a game be given by the random variable X.

$\mathbb{E}(X) = \sum_{j=1}^{\infty} \frac{1}{2^j} 2^{j-1} = \sum_{j=1}^{\infty} \frac{1}{2} = \infty$ This suggests that no fair finite entry fee exists.

Definition 1.1.2: Let $\{X_n\}_{n=1}^{N}$ be a sequence of N St. Petersburg games, such that X_n is the payout of the nth St. Petersburg game. We define $S_N = \sum_{n=1}^{N} X_n$ as the cumulative payout of the first N games. It follows that $\frac{S_N}{N}$ is the average payout, also generally known as the *sample mean*, for the first N games.

Definition 1.1.3: The sample mean $\frac{S_N}{N}$ converges in probability to a real number μ if $\forall \epsilon > 0$, $\lim_{n\to\infty} P(|\frac{S_N}{N} - \mu| > \epsilon) = 0$. Informally, the event that the sample mean is not arbitrarily close to μ has a probability that tends to 0 as N grows. We denote this as $\frac{S_N}{N} \xrightarrow{p} \mu$. In particular, convergence in probability is guaranteed by the Law of Large Numbers if the X_n's are identically distributed and have finite variance (which is not the case for St. Petersburg)

Example 1.1.4: Let $D_1, D_2, ..., D_N$ denote the results of N dice rolls. Then the sample mean $\frac{S_N}{N}$ is the sum of the dots on the dice divided by N. Since the dice rolls are identically distributed (with distribution D, which places probability $\frac{1}{6}$ on each of 1, 2, 3, 4, 5, and 6), and since the variance of any given roll is finite, the Weak Law of Large Numbers (WLLN) tells us that $\frac{S_N}{N} \xrightarrow{p} \mu$, where $\mu = \mathbb{E}(D) = 3.5$

2 Feller's Limit Theorem

Since the fair entry fee of a game X is typically given by the expected payout $\mathbb{E}(X)$, and since $\mathbb{E}(X) = \infty$ for the St. Petersburg game, we would like to make sense of a different notion of "average" that is finite. A natural candidate for this average is the sample mean ($\frac{S_N}{N}$) and Feller precisely characterizes the asymptotic rate of growth of $\frac{S_N}{N}$. Namely, if $\frac{S_N}{N}$ eventually grows at the rate of a function $r(N)$, then it is reasonable to conclude that the fair entry price for a single game, given that N games will be played, is $r(N)$. The main result of this section is that $r(N) = \frac{\log_2 N}{2} = \log_2 \sqrt{N}$

It would be useful to apply the law of large numbers to prove this, but as remarked in Definition 1.1.3, this is not directly possible since each St. Petersburg game X_n has infinite variance. Nonetheless, we will show that it is possible to approximate these games with random variables $\overline{X_n}$ with finite variance. Also, note that the crucial last step of the proof of the WLLN is the invokation of Chebyshev's inequality, which uses the fact that $\frac{Var(S_N)}{N^2} \to 0$ as $N \to \infty$. Equivalently, $\text{Var}(S_N)$ is $o(N^2)$. This is what allows us to conclude that $\frac{S_N}{N}$ converges in probability to $\mathbb{E}(X)$. But if $\text{Var}(S_N)$ is $o(g(N))^2$ where $g(N)$ is not necessarily N, then this would provide a similar, albeit weaker, result about convergence in probability. This thought process is central to Theorem 2.1, which effectively applies Chebyshev's inequality on the approximating sequence $\overline{X_n}$. In this sense, it can be thought of as a modified WLLN that extends to random variables X_n with infinite variance, provided that they behave well enough.

The theorem was originally proposed be Feller [2] in 1945, and it is described in more explicit detail in Durrett's "Probability: Theory and Examples" (Theorem 2.2.11) [1]. However, the theorem is formalized with triangular arrays of random variables, i.e. families with double indices $(X_{n,k})$. This is unnecessary for the application of the theorem to the St. Petersburg problem, where each game X_n follows the same distribution X. For simplicity, we have worked out the specification of the theorem with single indices. We begin by stating some definitions and elementary results in probability before stating the theorem along with its proof.

Definition 2.0.1: We denote $\overline{X_n} = X_n \mathbb{1}_{X_n \leq g(N)}$ as a truncated version of X_n such that $\overline{X_n}$ behaves identically to X_n for all values less than or equal to the value specified by $g(N)$, and such that $\overline{X_n} = 0$ whenever $X_n > g(N)$.

Remark 2.0.2: The truncated sequence will be the approximating sequence for $\{X_n\}_{n=1}^N$. Since each X_n is distributed identically according to X, it follows that each $\overline{X_n}$ is distributed according to $\overline{X}$.

Definition 2.0.3: We denote $\overline{S_N} = \sum_{n=1}^N \overline{X_n}$.

Result 2.0.4: Expectation is linear: $\mathbb{E}(X_1 + cX_2) = \mathbb{E}(X_1) + c\mathbb{E}(X_2)$.

Result 2.0.5: $\text{Var}(X)=\mathbb{E}(X^2) - (\mathbb{E}(X))^2$. Thus, $\text{Var}(X) \leq \mathbb{E}(X^2)$

Result 2.0.6: a) $\text{Var}(cX)=c^2\text{Var}(X)$

b) $\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2)$ if X_1 and X_2 are independent.

Result 2.0.7 (Chebyshev's Inequality): $\text{P}(|X - \mathbb{E}(X)| > \epsilon) \leq \frac{Var(X)}{\epsilon^2}$

2.1 Feller's Limit Theorem:

Let $X_1, X_2, ... \geq 0$ be a family of independent identically distributed random variables with distribution X. If $g(N)$ satisfies:

- $\lim\limits_{N\to\infty} \sum_{n=1}^N P(X_n \neq \overline{X_n}) = 0$ (Recall $\overline{X_n} = X_n \mathbb{1}_{X_n \leq g(N)}$)

- $\lim\limits_{N\to\infty} \frac{N\mathbb{E}(\overline{X}^2)}{g(N)^2} = 0$

Then, $\frac{S_N}{g(N)} \xrightarrow{p} \frac{N\mathbb{E}(\overline{X})}{g(N)}$ i.e. $\lim_{N\to\infty} P(|\frac{S_N}{g(N)} - \frac{N\mathbb{E}(\overline{X})}{g(N)}| > \epsilon) = 0\ \forall \epsilon > 0$.

The first condition requires that the X_n's are asymptotically identical to their truncated version $\overline{X_n}$ with probability 1, which implies that they are a very close approximation indeed. The second condition implies that $\text{Var}(\overline{S_N})$ is $o(g(N)^2)$ from Result 2.05 and 2.06b.

Proof: Let $\epsilon > 0$.

Then, $P(|\frac{S_N}{g(N)} - \frac{N\mathbb{E}(\overline{X})}{g(N)}| > \epsilon) \leq P(S_N \neq \overline{S_n}) + P(|\frac{\overline{S_N}}{g(N)} - \frac{N\mathbb{E}(\overline{X})}{g(N)}| > \epsilon)$.

Let $N \to \infty$, then the first term goes to 0 since $P(S_N \neq \overline{S_N}) \leq P(\bigcup_{n=1}^{N} X_n \neq \overline{X_n}) \leq \sum_{n=1}^{n} P(X_n \neq \overline{X_n})$ by countable subadditivity, and the limit as $N \to \infty$ is 0 by hypothesis (first condition).

Similarly, the second term goes to 0 from condition 2:

Remark that $\mathbb{E}(\frac{\overline{S_N}}{g(N)}) = \frac{N\mathbb{E}(\overline{X})}{g(N)}$ by linearity of $\mathbb{E}$

Thus, $\text{P}(|\frac{\overline{S_N}}{g(N)} - \frac{N\mathbb{E}(\overline{X})}{g(N)}| > \epsilon) \leq \frac{\text{Var}(\frac{\overline{S_N}}{g(N)})}{\epsilon^2}$ by Chebyshev's inequality

However, $\text{Var}(\frac{\overline{S_N}}{g(N)}) = \frac{1}{g(N)^2}\text{Var}(\overline{S_N}) = \frac{N\text{Var}(\overline{X})}{g(N)^2} \leq \frac{N\mathbb{E}(\overline{X}^2)}{g(N)^2} \to 0$ as $N \to \infty$ by hypothesis

The first two equalities follow from 2.0.6a and 2.0.6b, and the inequality follows from 2.05.
Hence, $\frac{S_N}{g(N)} \xrightarrow{p} \frac{N\mathbb{E}(\overline{X})}{g(N)}$ since $\epsilon > 0$ is arbitrary. □

2.2 Application of Theorem 2.1 to St. Petersburg:

It is not completely obvious how applying Theorem 2.1 will lead to the desired result. We will see that the analysis of an appropriate choice of an approximating sequence will yield the desired result $r(N) = \frac{\log_2 N}{2} = \log_2 \sqrt{N}$. The key is to pick the smallest possible function $g(N)$ that satisfies both conditions in Theorem 2.1 so that we can precisely interpret the asymptotic probability.

Claim 2.2.1: $2^{log_2 N + k(N)}$ is a valid choice of $g(N)$ as long as $\lim_{N\to\infty} k(N) = \infty$
For simplicity, let $m(N) = \log_2 N + k(N)$, so that $g(N) = 2^{m(N)}$, and also remark that we may choose $k(N)$ so that $2^{m(N)}$ is always integer-valued. This enables us to disregard floors and ceilings later on.

Proof: We check the first condition:

$$\lim_{N\to\infty} \sum_{n=1}^{N} P(X_n \neq \overline{X_n}) = 0, \text{ where } \overline{X_n} = X_n \mathbb{1}_{X_n \leq 2^{m(N)}}$$

Since we have defined our $g(N)$ as a power of 2, $\overline{X_n}$ agrees with X_n for games with up to (and including) $m(N) + 1$ flips, and is 0 for games that require more flips.
Thus, $\sum_{n=1}^{N} P(X_n \neq \overline{X_n}) = NP(X \neq \overline{X})$ since all N games have identical distribution.

$$= N \sum_{n=m(N)+2}^{\infty} \frac{1}{2^n} = N(\frac{1}{2^{m(N)+2}} \frac{1}{1 - \frac{1}{2}}) \text{ by geometric series}$$

$$= N\frac{1}{2^{m(N)+1}} = \frac{N}{2}\frac{1}{N2^{k(N)}} = \frac{1}{2^{k(N)+1}} \to 0 \text{ as } N \to \infty \text{ since } \lim_{N\to\infty} k(N) = \infty$$

Now we check the second condition: $\lim_{N\to\infty} \frac{N\mathbb{E}(\overline{X}^2)}{g(N)^2} = 0$

$$\mathbb{E}(\overline{X}^2) = \sum_{n=1}^{m(N)+1} \frac{1}{2^n}(2^{n-1})^2 \text{ since } \overline{X}^2 = \overline{X} = 0 \text{ when flips } (\overline{X}) > m(N)+1$$

$$\mathbb{E}(\overline{X}^2) = \sum_{n=1}^{m(N)+1} 2^{n-2} = \frac{1}{2}[2^{m(N)+1} - 1] = \frac{1}{2}[N2^{k(N)+1} - 1]$$

$$\text{So } \frac{N\mathbb{E}(\overline{X}^2)}{g(N)^2} \le \frac{N^2 2^{k(N)+1}}{(N2^{k(N)})^2} = \frac{N^2 2^{k(N)+1}}{N^2 2^{2k(N)}} = \frac{1}{2^{k(N)-1}} \to 0 \text{ given our choice of } k(N)$$

Thus, we see that as long as $\lim_{N\to\infty} k(N) = \infty$ (no matter how slowly it grows), both conditions are satisfied by $g(N) = 2^{m(N)}$ □

We are now ready to prove the primary objective of this section.

Claim 2.2.2: $\frac{S_N}{N}$ grows asymptotically at the rate of $r(N) = \frac{\log_2 N}{2}$

Proof: Since $g(N) = 2^{m(N)}$ satisfies the conditions of the theorem, we compute

$$N\mathbb{E}(\overline{X}) = N \sum_{n=1}^{m(N)+1} \frac{1}{2^n} 2^{n-1} = N \sum_{n=1}^{m(N)+1} \frac{1}{2} = \frac{N(\log_2 N + k(N) + 1)}{2}$$

$$\text{Then, by Theorem 2.1: } \forall \epsilon_0 > 0, \lim_{N\to\infty} P(|\frac{S_N}{g(N)} - \frac{\frac{N}{2}(\log_2 N + k(N) + 1)}{g(N)}| > \epsilon_0) = 0$$

Since $k(N)$ can be any function that tends to ∞, we may let $k(N) = O(\log_2(\log_2 N))$

$$\lim_{N\to\infty} P(|\frac{S_N}{N\log_2 N} - \frac{\frac{N}{2}(\log_2 N + O(\log(\log N)))}{N\log_2 N}| > \epsilon_0) = 0$$

$$\lim_{N\to\infty} P(|\frac{S_N}{N\log_2 N} - \frac{1}{2}| > \epsilon_0) = 0 \text{ since } \log(\log N) \text{ is } o(N\log_2 N)^1$$

Multiplying the argument by 2, and setting $\epsilon = \frac{\epsilon_0}{2}$ we see that

$$\lim_{N\to\infty} P(|\frac{\frac{S_N}{N}}{\frac{\log_2 N}{2}} - 1| > \epsilon) = 0$$

Since $\epsilon > 0$ is arbitrary, we conclude that with probability 1, the average payout $\frac{S_N}{N}$ eventually grows proportionally to $\frac{\log_2(N)}{2}$. Given this, we may conclude that a fair entry price is $\frac{\log_2(N)}{2}$ dollars per game given that N games are to be played. To play N games, the player pays \$$N\log_2(\sqrt{N})$. □

Remark 2.2.3: Claim 2.2.2 is a weaker claim than convergence in probability.

Continuing from 2.2.2, we multiply the argument by $\frac{\log_2 N}{2}$, $\lim_{N\to\infty} P(|\frac{S_N}{N} - \frac{\log_2 N}{2}| > \epsilon \log_2 N) = 0$
So the direct probability convergence carries a logarithmic error, but the asymptotic proportional growth does not.

[1] We delve deeper into this step in the the next section

3 Peters' Time Resolution

Again, knowing that the expected value of a game, $\mathbb{E}(X)$ is infinite, we will consider a third kind of average. We consider a player who plays the St. Petersburg game infinitely many times and average their fluctuation in wealth. This is called the time average and for the St Petersburg problem, it is not equal to the expectation.

Let us consider the St. Petersburg paradox, with our player beginning with some initial wealth, w. After the first game, his new wealth, w_1, will be his initial wealth minus the cost to play, plus the payout of the game. In general:

$$w_i = w_{i-1} - c + X_i, \qquad \text{where } \begin{aligned} w_i &: \text{ wealth after } i^{th} \text{ game,} \\ c &: \text{ entry fee per game,} \\ X_i &: \text{ payout from } i^{th} \text{ game} \end{aligned}$$

Definition 3.1: Let r_i be the proportional change in the player's wealth after playing game i.

$$r_i = \frac{w_{i-1} - c + X_i}{w_{i-1}} = \frac{w_i}{w_{i-1}}$$

From this, we can express the player's wealth after a finite number of games, N, as: $w(N) = w \prod_{i=1}^{N} r_i$

Definition 3.2: As done by Ole Peters[4] in his paper on the time resolution of the St. Petersburg paradox, we introduce the notion of exponential growth rate. Let $g = \log_2(r) \implies 2^{g_i} = r_i = \frac{w_i}{w_{i-1}}$

With these new definitions, our expected value, similarly to before, is:

$$\mathbb{E}(r_w) = \sum_{j=1}^{\infty} p_j r_j = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{w - c + X_j}{w}$$

Using definition 3.2, we write: $\mathbb{E}(g_w) = \log_2 \left(\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{w - c + 2^{j-1}}{w} \right)$

Theorem 3.3: The time-average exponential growth rate in the St Petersburg game is:

$$\overline{g} = \sum_{j=1}^{\infty} \frac{1}{2^j} \left[\log_2(w - c + 2^{j-1}) - \log_2(w) \right]$$

Proof: By taking the ratio of growth of wealth between all games played, we can obtain an average wealth change over N games played. The N^{th} root of the total fractional change defines the finite-time average, $\overline{r_N}$.

$$\overline{r_N} = \left(\prod_{i=1}^{N} r_i \right)^{\frac{1}{N}} = (r_1 r_2 r_3 ... r_N)^{\frac{1}{N}} = \left((\frac{w_1}{w})(\frac{w_2}{w_1})(\frac{w_3}{w_2})...(\frac{w_N}{w_{N-1}}) \right)^{\frac{1}{N}} = (\frac{w_N}{w})^{\frac{1}{N}}$$

Similarly to definition 3.2, we'd like to express this in terms of j, the number of coin tosses. Let k_j be the number of games in the sequence of N games where j flips occurred, and $j_N^{\max}$ be the largest number of heads observed in a game in the sequence.

$$\implies \overline{r_N} = \left(\prod_{j=1}^{j_N^{\max}} r_j^{k_j} \right)^{\frac{1}{N}},$$

As the number of games N grows to infinity, the ratio $\frac{k_j}{N}$ approaches the probability of getting j flips, resulting in the time-average growth factor $\overline{r}$, and not a stochastic variable $(\overline{r_N})$. Remark also that $j_N^{\max} \to \infty$ as $N \to \infty$

$$\overline{r} \;=\; \lim_{N\to\infty} \overline{r_N} \;=\; \lim_{N\to\infty} \prod_{j=1}^{j_N^{\max}} r_j^{\frac{k_j}{N}} \;=\; \prod_{j=1}^{\infty} r_j^{p_j}$$

Again we use definition 3.2 and obtain:

$$\overline{g} \;=\; \log_2\left(\prod_{j=1}^{\infty} r_j^{p_j}\right) \;=\; \sum_{j=1}^{\infty} p_j \log_2 r_j = \sum_{j=1}^{\infty} \frac{1}{2^j}\left[\log_2(w-c+2^{j-1}) - \log_2(w)\right] \quad \square$$

To obtain a fair price as a function of c, we would want to solve for some c such that $\overline{r}$ to equal 1. From our exponential growth rate relation, $\overline{g} = 0$

$$\implies \sum_{j=1}^{\infty} \frac{1}{2^j}\left[\log_2(w-c+2^{j-1}) - \log_2(w)\right] = 0$$

This was the conclusion reached by Peters, and will be used as the basis for our simulations. We'd like to solve for the cost per game for some fixed initial wealth. That is, we solve for c such that $\sum_{j=1}^{\infty} \frac{1}{2^j}\left[\log_2(w-c+2^{j-1}) - \log_2(w)\right] = 0$. Given that we cannot solve this directly, we will write an algorithm to approximate the solution.

4 Numerical Analysis

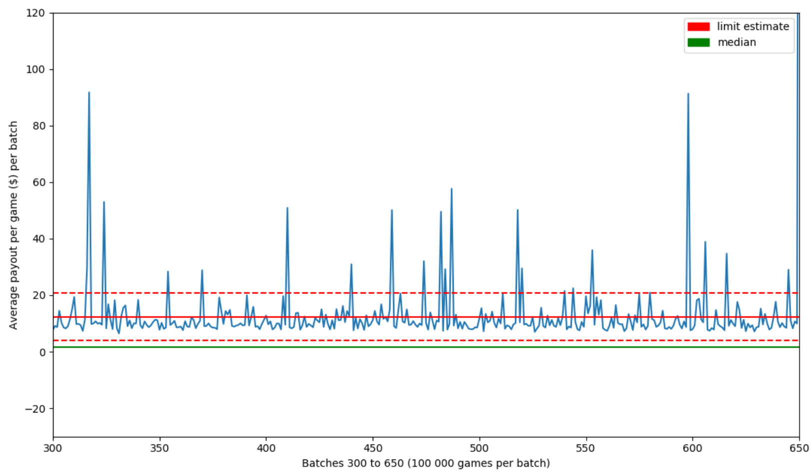
4.1 Testing Theorem 2.1 with Simulations

Section 2 informed us that the asymptotic growth of the sample mean $\frac{S_N}{N}$ is equivalent to $\log_2(\sqrt{N})$. From Remark 2.2.3, and using the definition of a limit, we know that for any $\epsilon > 0$ and any $\delta > 0$, there is some N such that $P(|\frac{S_M}{M} - \frac{\log_2 M}{2}| > \epsilon \log_2 M) \leq \delta$, $\forall M \geq N$. To clarify, ϵ serves as a threshold to demarcate outliers that cannot be adequately approximated by $\frac{\log_2 N}{2}$ from non-outliers. δ represents an upper bound for the probability of getting an outlier as specified by ϵ. Then, fixing any ϵ and δ, there is some N large enough that satisfies the inequality. To get a more precise reading into this behaviour, we use simulations with graphs.

Definition 4.1.1 A batch is a sequence of N games. Thus, each batch has a corresponding sample mean $\frac{S_N}{N}$. We primarily consider batches of size $1000, 10000, 100000$, and 1000000.

This allows us to determine the percentage of "good batches" from outliers, as specified by ϵ. As N increases, we expect the percentage of outliers to decrease, since δ is smaller. Let us now examine the data.

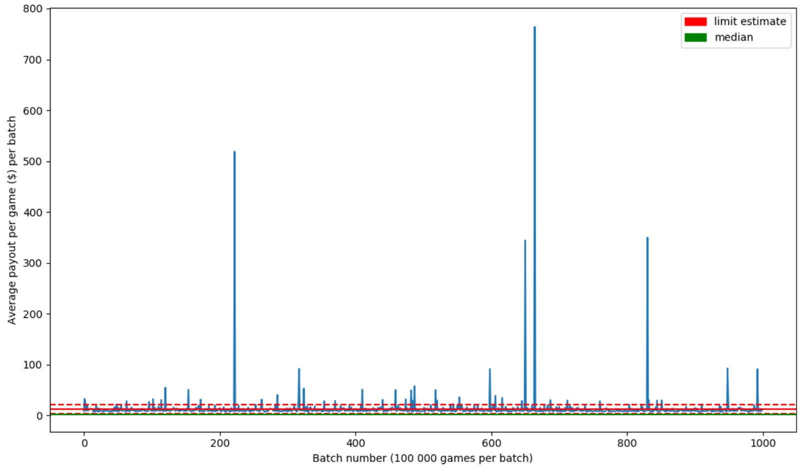
Figure 4.1.2: A partial plot graphing average payout per game for each batch ($N = 100000, \epsilon = 0.5$)



The solid red line represents the estimate given by $\log_2(\sqrt{100000}) \approx 8.34$ and the dotted red lines demarcate the outliers specified by $\epsilon = 0.5$. The vast majority of batches fall within the ϵ-band. Over the full plot, which considers 1000 batches, the percentage of outliers is about 11%. The data is therefore consistent with Theorem 2.2.1, and it suggests that $N = 100000$ is sufficient for the case when $\epsilon = 0.5$ and $\delta = 0.11$. We have also plotted the median solution of 1.7 dollars per game, which was proposed by Okabe et al.[3]. The graph shows that this solution completely fails when a large number of games are played, and that Feller's solution is essentially as good of an estimate as possible with respect to capturing the most number of batches in the acceptable range. As expected, the percentage of outliers decreases when we raise N. For example, when $N = 1000000$, ceteris paribus, the proportion of outliers drops to 0.05.

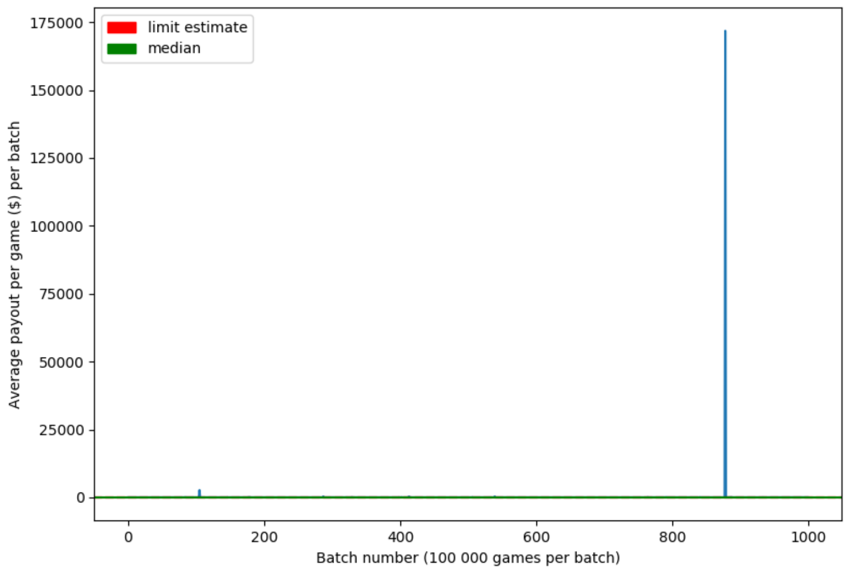
However, although the data is concentrated near the estimate, the inherent randomness of the game manifests in large spikes. Even when the batches contain a large number of games (100 000), the sample mean of each batch can vary wildly. Thus, when the estimate is off, it can be way off, and is typically a large underestimate. The following graph (4.1.3) is the full plot from which Figure 4.1.2 was taken from.

Figure 4.1.3: Average payout per game for 1000 batches ($N = 100000, \epsilon = 0.5$)



Then, repeating the simulation produced an exceptional batch with average payout $\approx \$17000$, which corresponds to a batch revenue in the vicinity of \$170 000 000. This spike makes the graph virtually unreadable, and also illustrates why a casino would never run the St. Petersburg game. Despite this, the data ended up reflecting a comparable probability of outliers (about 13%) to the previous simulation.

Figure 4.1.4: Average payout per game for 1000 batches ($N = 100000, \epsilon = 0.5$)



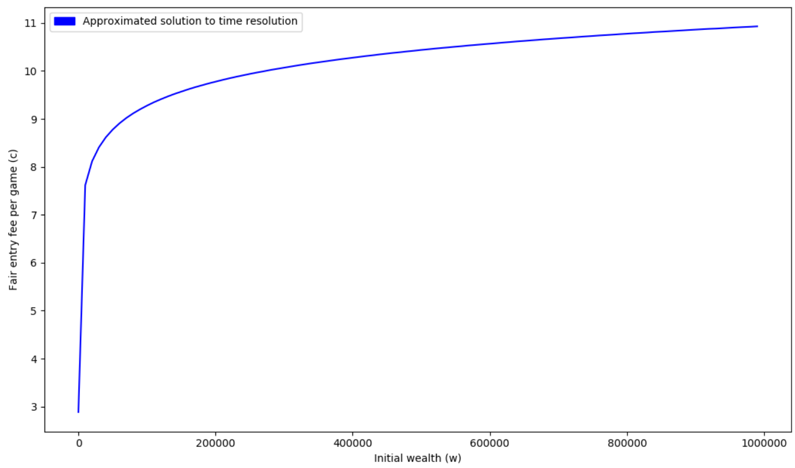
A criticism for Feller's estimate is that it is too low, since the "spiking" outliers deviate far more from the estimate than the "dipping" outliers. For instance, when we consider the average payout of all games across the 1000 batches of 100 000 games, we typically get a number just above 13.6. Meanwhile, $\log_2(100000)/2 \approx 8.34$. In response to this, we must not input the wrong N for our estimate. When we pool together all of the batches, we get $1.0 * 10^8$ games played, so if we wish to approximate the sample mean across the total sequence of games, then Feller's estimate is actually $\log_2(1.0 * 10^8)/2 \approx 13.29$, which is still lower than the experimental sample mean, but not significantly so.

Still, we sought an explanation for why Feller's approach slightly underestimates the sample mean. We return to the footnoted step from Claim 2.2.2. This step assumes that $\frac{O(\log(\log N))}{N \log_2 N}$ is 0. However, to equate this with 0, we would need to implicitly move the limit into the probability argument, and although this term should flatten out to 0 as $N \to \infty$, the effect is that our estimate is just slightly below what it ought to be when we apply it to a fixed N.

4.2 Approximating a Solution to Theorem 3.3 and Testing

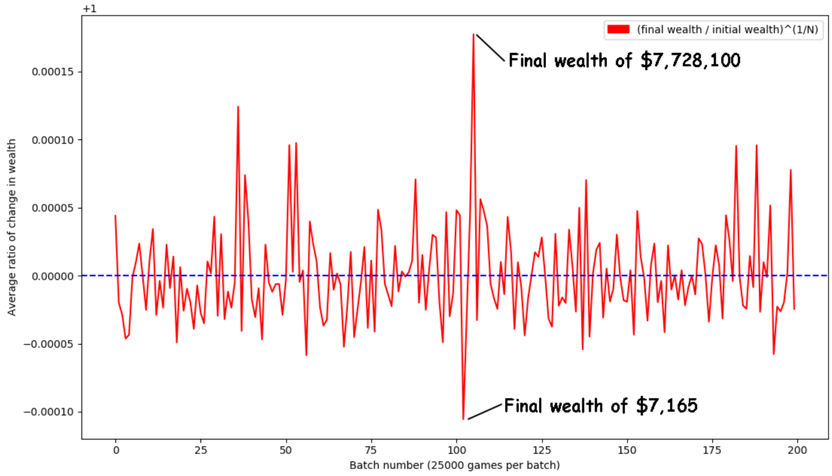
We wrote an algorithm to output the fair cost given initial wealth w. This corresponds to finding c such that the exponential growth rate, given by $\sum_{j=1}^{\infty} \frac{1}{2^j} \left[\log_2(w - c + 2^{j-1}) - \log_2(w)\right]$ is 0. Depending on our choice of c, this infinite sum may be negative or positive, but either way, we notice that the terms in the series decrease in magnitude exponentially due to the $\frac{1}{2^j}$ coefficient. Thus, we can approximate the series with a finite sum, such that we exclude all terms with absolute value less than some specified minuscule number (we chose $1.0 * 10^{-12}$). Once a specified w is fixed, our algorithm works by approximating the series for a preset value c (we chose $c = 5$). It then iteratively approximates the series by adjusting c with either a positive step or a negative step until the series is within $1.0 * 10^{-12}$ of 0. Results from this can be seen plotted in figure 4.2.1, showing cost as a function of initial wealth.

Figure 4.2.1: Cost as a function of initial wealth as calculated by numerical analysis of Theorem 3.3



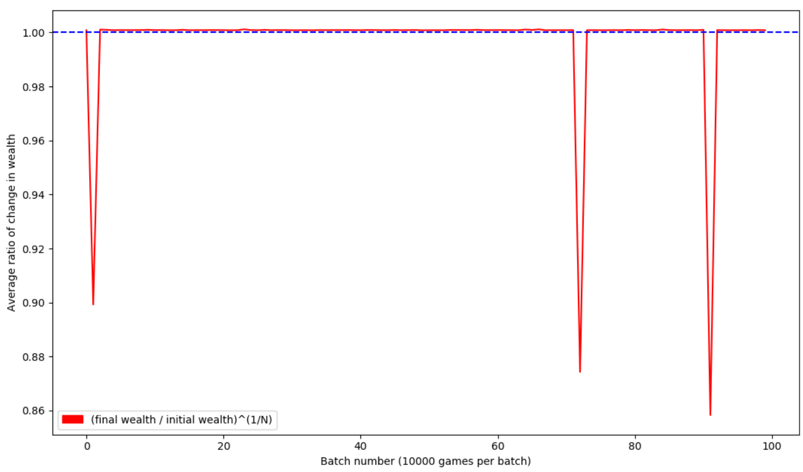
Similarly to section 4.1, we run simulations of multiple batches of N games to test Peter's estimate.

Figure 4.2.2: Simulation of playing with fixed initial wealth, $w = 100000 \implies c = 9.30$



Here in figure 4.2.2, the scale of the graph indicates that every batch's time average r_N is extremely close to 1, which validates Peters' result. Also, we observe a player with initial wealth of \$100 000, paying the suggested \$9.30 per game by Peters, will win approximately 50% of the time. This result is promising, however when a similar simulation is run with a smaller initial wealth, we see that Peters' suggested price results in the player almost always winning. Figure 4.2.3 plots a player with an initial wealth of \$10.

Figure 4.2.3: Simulation of playing with fixed initial wealth, $w = 10 \implies c = 2.86$



For this random trial of 100 batches, only 3 sequences of 10000 games resulted in the player losing money. This implies that Theorem 3.3 is not fair for a small initial wealth. This will be revisited in section 4.3.

4.3 Conditions for Equivalent Pricing and Discussion

Figure 4.2.1 gave us a graph of the cost given by the time resolution approach over wealth. To compare these costs with the costs given by Feller's approach, the natural strategy was to plot $c = \log_2(\sqrt{w})$ on the same graph. Amazingly, we saw the curves were nearly identical, noticing only an offset of exactly 1. Wow! The plot was then adjusted to show the curve $c = \log_2(\sqrt{w}) + 1$, alongside our numerical solution to Theorem 3.3, as seen in figure 4.3.2.

Figure 4.3.1: Cost vs. Initial Wealth and Cost vs. Games Played

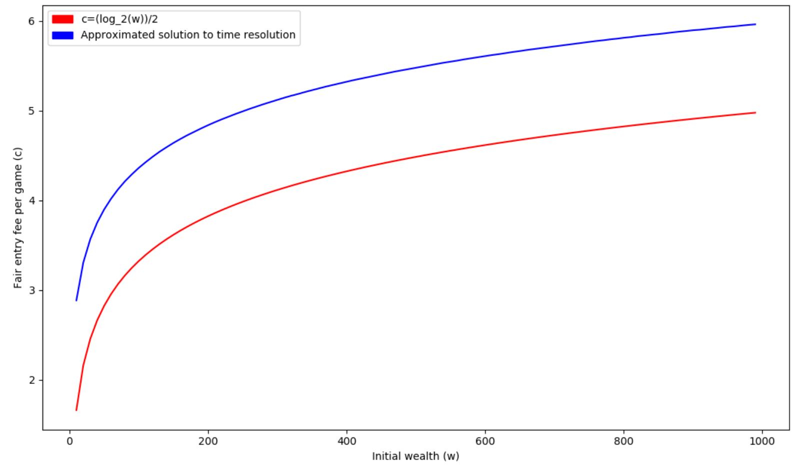
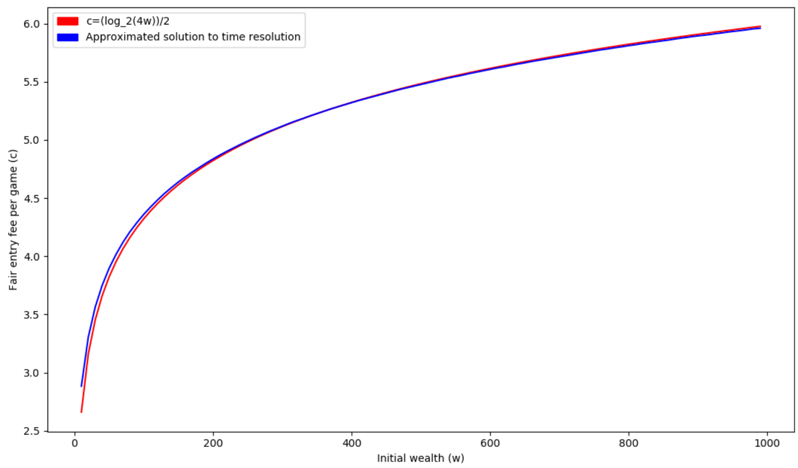


Figure 4.3.2: Cost vs. Initial Wealth and Cost vs. Adjusted Games Played



In figure 4.3.2, we observe that there is a small difference between our approximated solution and this perfect logarithmic curve. We believe this is simply because the algorithm must terminate and cannot continue to add infinitely many terms. Due to the fact that there is no closed form for the cost specified in Theorem 3.3, we were unable further mathematically investigate this claim. However, this sets us up nicely for determining the precise mathematical conditions for when the two approaches produce the same fair entry fee.

5 A Multivariable Framework for the St. Petersburg Paradox

Result 5.1: Feller's fair price equals Peters' fair price when $w = \frac{N}{4}$

The graphs in the previous section show that it is reasonable to represent Peters' fair price solution as $\frac{\log_2(w)}{2} + 1$. This enables us to easily solve for the precise conditions for when the two approaches predict identical fair entry fees. We set $\frac{\log_2(w)}{2} + 1 = \frac{\log_2(N)}{2} \implies \log_2(w) + 2 = \log_2(N) \implies 4w = N$ as desired. $\square$

Therefore, if N games are to be played by a player with initial wealth $w = \frac{N}{4}$, then both approaches agree that the exact solution for the fair entry fee is $\frac{\log_2(N)}{2}$. Ultimately, Feller's estimate of the fair price depends only on the number of games to be played, and Peters' estimate only takes in initial wealth as input. Thus, both approaches produce fair entry fees that are robust under simulations, but each neglects a variable. We conclude this paper by considering what the fair price would be in the case of a player who decides to play N games, and who has w dollars at the start.

Definition 5.2: To formalize the St. Petersburg Paradox such that both wealth and the number of games played are considered, a player with wealth w who wants to play N games will only be able to continue playing the game if they are able to pay the cost of entry. This intuitive definition aligns with realistic financial constraints.

This definition allows us to characterize the fair entry fee for a St. Petersburg game as a multivariable function $c(N, w)$. Result 4.3.3 implies that for all pairs (N, w) such that $N = 4w$,

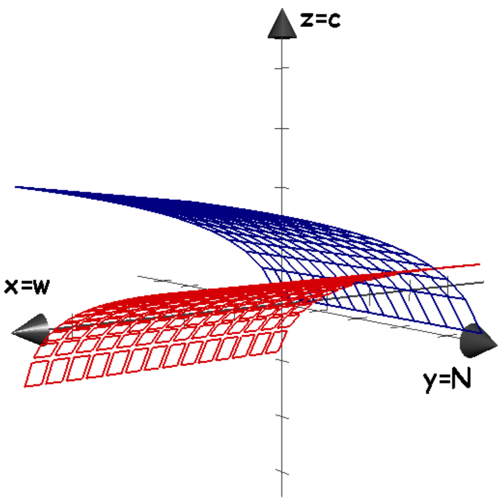
$$c(N, w) = f(N, w) = \frac{\log_2(N)}{2} = p(N, w) = \frac{\log_2(4w)}{2}$$

We denote Feller's solution by $f(N, w)$ and Peters' by $p(N, w)$. However, what can we say about the other cases?

Conjecture 5.3: If $N \geq 4w$, then $p(N, w) \leq c(N, w) \leq f(N, w)$ and if $N \leq 4w$, then $f(N, w) \leq c(N, w) \leq p(N, w)$

Proof: Provided that both approaches are accepted as solutions to the problem, then the conjecture follows from basic multivariable calculus. Letting $x = w$ and $y = N$, Peters' solution is given by $z = \frac{\log_2(4x)}{2}$ and Feller's solution is given by the surface $z = \frac{\log_2(y)}{2}$. $\square$

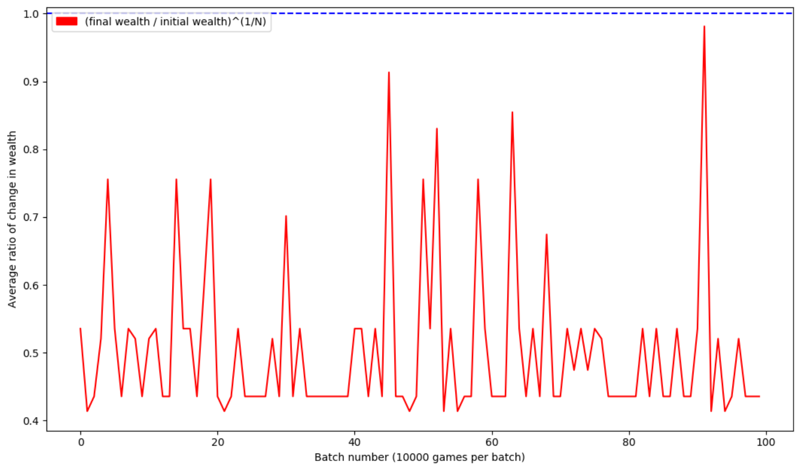
Figure 5.3.1: Feller's surface is in red and Peters' surface is in blue



We label this a conjecture rather than a theorem despite the simple proof because it is somewhat contentious to assert that both approaches are solutions to the problem for all N and w. For instance, Feller's limit law pertains to large N, so his estimate may not be applicable when the number of games played is small. It is possible that the general idea behind this conjecture is true, but that the two estimates may need to be refined.

Nevertheless, this conjecture is generally supported by results from simulations. As a first example, recall Figure 4.2.3, where we observed that Peters' solution ($c = 2.86$) was a notable underestimate for a player with $w = 10$ intending to play 10000 games since all but three batches result in the player finishing with more money than he started with. We may now identify this case as a point in the region $N \geq 4w$, where Peters' solution is in fact a lower bound for the price. Let us now examine what happens when we apply Feller's solution to this case.

Figure 5.3.2: Simulation of playing with fixed initial wealth, $w = 10$, using Feller's cost, $c = 6.64$



If we apply Feller's solution ($c = \frac{\log_2(10000)}{2} \approx 6.64$), the player loses money in every batch. Therefore, the data supports our conjecture: $2.86 \leq c(N, w) \leq 6.64$. This also makes intuitive sense. For small w and large batches, Feller's cost will be too steep since the player may go bankrupt before completing all N games. Similarly, Peters' solution does not consider games played, but since it is proven (by Feller) that the average payout goes up logarithmically as games played increases, Peters' cost is too small for this situation.

We obtain symmetric results when the player has large wealth and when a small number of games are played ($N \leq 4W$). When we let $w = 10000$ and $N = 10$, the player wins about 13% of the time under Peters' price of $\frac{\log_2(4*10000)}{2} \approx 6.64$. Under Feller's price of $\frac{\log_2(10)}{2} \approx 1.66$, the player wins about 88% of the time.

Lastly, we have already seen that when $N = 4w$, the simulations suggest that this is indeed a fair price (4.2.2). Our simulations show that this is particularly true when N and w are large. So the line $N = 4w$ is a special case where we can directly observe a fair price.

In conclusion, we have reconstructed two prominent, yet orthogonal, solutions to the St. Petersburg Paradox in sections 2 and 3. Then in Section 4, we analyzed their performance by using simulations, and in doing so, we coincidentally stumbled upon the realization that both approaches use a near-identical logarithmic estimate about different axes. This chance discovery enabled us to unravel new mathematical insight into the St. Petersburg Paradox. In the last section, we draw upon the previous sections to conclude that the approaches predict the same entry fee along the line $N = 4w$. We could then synthesize the two approaches to construct a more realistic generalized probability model for the St.

Petersburg Paradox that considers N and w. In particular, we conjecture that bounds for the fair entry price at any N and w can be found by using Feller's and Peters' estimates, thus yielding an identifiable fair entry price along the line $N = 4w$. We believe that this kind of probability model is well-suited to the St. Petersburg game and for games of chance in general. It would therefore be beneficial to research the relationship between limit laws and time averages to improve upon our probability model.

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