Quadratic Voting vs. Proportional Approval Voting

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Introduction

Quadratic voting (QV) is a form of voting that was partly inspired by the work of Hylland and Zeckhauser[1], but formalized by Weyl[2] in 2013. Though it is a fairly new voting mechanism, formalizing and applying QV to various settings has been an extensively pursued research area. A particularly well-studied application of this mechanism is in corporate governance. Using QV as a method for shareholder voting was introduced and popularized in a paper by Posner and Weyl[3]. Inspired by the range of fields QV can be applied to, through this project we apply the mechanism to a common real world setting of deciding on cities to visit during a group vacation. This analysis can be extended in general to scenarios where a group of individuals are trying to choose a set of alternatives among the choices they have.

Proportional approval voting (PAV) is an extension of approval voting which selects multiple winners. The idea was suggested by Simmons (2001)[4]. Similar to the committee selection problem of picking multiple representatives, we are interested in choosing places instead of persons which can represent more individuals. In the project, we compare the differences between two mechanisms. Efficiency and fairness are the common measurements of a mechanism. While QV does better in terms of efficiency and PAV aims for fairness, we analyze their overall performance under different criteria.

Mechanism - Quadratic Voting

Suppose there are n voters and m alternatives to vote on. In the following discussion, voters are represented as $V_i, i = 1 \dots n$, and alternatives are represented as $A_j, j = 1 \dots m$. Each voter casts a non-negative number of votes on every alternative. For each alternative, the voter can purchase any number of votes by paying a cost equal to the square of the number of votes purchased. For example, V_1 can purchase 1 vote for A_1 at a cost of \$1, 3 votes for A_2 at a cost of \$9, and so on. The alternative(s) with the most votes is(are) chosen as winner(s).

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[1] Aanund Hylland and Richard Zeckhauser, *A Mechanism for Selecting Public Goods when Preferences must be Elicited* (70, Kennedy School of Government Discussion Paper D, 1979)

[2] E. Glen Weyl, *Quadratic Vote Buying* (Univeristy of Chicago Working Paper, Apr 1, 2013), online at http://ssrn.com/abstract=2003531 (visited May 1, 2016)

[3] Eric Posner, E. Glen Wyl, *Quadratic Voting as Efficient Corporate Governance* (81 University of Chicago Law Review 251, 2014)

[4] Laslier, J., & Sanver, M. R. (2010). Handbook on approval voting. New York: Springer. pp. 114 – 115.

By allowing the cost to vary as a quadratic function, this mechanism allows for true intensity of preferences to be expressed. To understand this better, consider the following example. Suppose there is a community polling, and votes from residents are used to decide whether a swimming pool or a tennis court is a better use of a newly acquired space. If an individual values a tennis court in the neighborhood at $1000, and has no value for a swimming pool, how should she vote? How many votes should she purchase for each alternative? Assuming that there is fierce competition, and every vote sways the decision in favor of the individual by 1%, benefit from an additional vote = 1% of $1000 = $10. If she buys 1 votes, it costs her $1, but she gains a benefit of $10. If she buys 2 votes, it costs her $4, her marginal cost is $3 and she gains an additional benefit of $10. If she buys 3 votes, it costs her $9, a marginal cost of $7 but she still gains an additional benefit of $10. If she is a rational person (which we assume she is), she would buy as many votes as she can until her marginal cost is less than or exactly offset by her benefit from the vote. That is, she would buy 5 votes at the cost of $25, with a marginal cost of $9 (16-25), but buying 6 votes would cost her $36 and a marginal cost of $11 which exceeds her additional benefit from the vote. Thus, she would settle at buying 5 votes for the tennis court and none for the swimming pool since she has no value for it, and we are making a simplification assumption that negative votes are not allowed.

	Total Cost	Marginal Cost	Additional Benefit
1 Vote	$1	$1	$10
2 Votes	$4	$(4-1) =$3	$10
3 Votes	$9	$(9-4) = $5	$10
4 Votes	$16	$(16-9) = $7	$10
5 Votes	$25	$(25-16) = $9	$10
6 Votes	$36	$(36-25) = $11	$10

Table 1: Marginal Cost and Additional Benefit

To calculate the optimal number of votes for an individual to buy[3], let

v = number of votes

v^2 = cost of votes

$10v$ = value gained from votes.

Then, marginal cost = $\frac{d}{dv}(v^2) = 2v$ and marginal benefit = $\frac{d}{dv}(10v) = 10$.

Setting these equal to each other, $2v = 10$, and we get $v = 5$ which is the same as what we had before.

Why should the cost be quadratic? Why not linear or cubic or any other polynomial function? As Posner and Weyl say[3], “If we assume that the marginal benefit of a vote increases at a linear rate, then the marginal cost of the vote must also increase at a linear rate; this is only possible if the total cost of votes increases by the square of the number of votes.”

Mechanism – Proportional Approval Voting

Proportional approval voting (PAV) is an extension of Approval voting (AV). Suppose there are n voters and m candidates. For every voter V_i, she casts a vote A_{ij} to each candidate C_j, where $i \in [1, \mathrm{n}]$, $\mathrm{j} \in [1, \mathrm{m}]$ and A_{ij} is a binary variable with 1 representing approval or 0 otherwise. Under AV, the winners are the candidates that receive most approvals from voters. That is, for every candidate C_j, we calculate the total amount of votes he receives, i.e., $\sum_{i \in [1,\mathrm{n}]} A_{ij}$, and pick the ones with the maximum number of votes. Under PAV, the calculation is similar but the difference lies in the assumption of diminishing marginal returns—the more you get, the less additional happiness you will receive. For example, if there are k' winners that the voter V_i approves of, her utility is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k'}$. Our goal is to maximize the sum of voters' utilities. Assume we want to select k winners. Suppose $a(i) = \sum_{\mathrm{j} \in [1,k']} \frac{1}{j}$, where k' is the number of winners that the voter V_i approves of. The winners are the ones with highest PAV-score, with $\sum_{i \in [1,n]} a(i)$. As an additional note, it has been proved recently that computing PAV is NP-hard[5]. The goal of PAV is to seek fairness among voters by choosing a candidate which an additional voter approves, instead of adding more value to voters who have some of their preferred candidates already chosen.

Survey Setup

For this project, we set up an online survey that polled students from our class to decide on cities to visit during a European vacation we were taking together. The six alternatives they got to vote on were London, Paris, Prague, Berlin, Zurich and Venice.

For quadratic voting, each student was given a budget constraint of a 100 coins which they could use to purchase any number of votes for each alternative (including fractional values)[6]. Staying within the budget constraint, students got to spend a certain number of coins on each city, the square root of which was the number of votes they casted respectively.

For proportional approval voting, voters either approve or disapprove of a city by casting their votes[7]. There are two cases which we consider, which are independent of each other.

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[5] Aziz, H.; Gaspers, S.; Gudmundsson, J.; Mackenzie, S.; Mattei, N.; and Walsh, T. 2014. *Computational aspects of multi-winner approval voting*. In MPREF-14. (as cited in Aziz, H.; Brill, M.; Conitzer, V.; Elkind, E.; Freeman, R.; and Walsh, T. 2014. *Justified representation in approval-based committee voting*)

[6] Refer to "Quadratic Voting Online Survey" in Appendix

[7] Refer to "Proportional Approval Voting Online Survey" in Appendix

Case 1: Travelling to 3 cities[8]

Using mixed integer programming, we select the top 3 cities with the highest PAV value

Case 2: Finding the best route for the trip[9]

Since time is money, we don't want to spend too much time on travelling instead of enjoying what a city has to offer. Therefore, in addition to PAV, we now have an additional time budget of total travelling hours for the whole group. The distance between two cities imposes an additional travelling cost, which is captured by measuring the non-stop flight hours travelling between two cities. The reason why we take planes but not trains is because saving transportation time is our main concern in the trip. Similar as mixed integer program in PAV, the program picks the maximum number of cities under the constraint, which satisfies most voters with the highest total utilities. Moreover, the program selects the best route starting and ending at the same city so that we can all catch the plane from and back to RDU.

The travelling salesman problem (TSP) is similar to the routing problem in our program. Given a list of cities and a matrix of the distance between two cities, TSP tries to find the shortest route which visits each city once and returns to the same city where it starts from. In our program, we modify TSP and incorporate the idea into PAV to maximize the sum of total utilities given the time constraint. Programming details can be found in Appendix, but the idea of beginning and ending at the same city is to prevent suboptimal tour. Also, we make sure that each city is visited at most once, with the constraint that the city is marked as visited when exit. When we enter a city, we must exit the city at some point as a traveler. With above additional constraints, we are able to find the best route of our summer vacation to Europe.

Survey Results

Quadratic Voting

After gathering votes from students, we used a simple linear program[10] to pick cities we would visit. The cities with the highest cumulative number of votes from all students were picked. The results were as follows:

1 city: Paris

2 cities: London and Paris

3 cities: London, Paris and Prague

4 cities: London, Paris, Prague and Venice.

PAV Case 1

Given the feedback from 19 voters and picking three cities to visit, the utility obtained is 29

[8] Refer to "Mixed integer program of PAV" in Appendix.

[9] Refer to "Mixed integer program of PAV under additional time and distance constraints" in Appendix.

[10] Refer to "Linear Program for Quadratic Voting" in Appendix

and the optimal choice is London, Paris and Venice. If all voters had approved London, Paris and Venice, we would have achieved a maximum utility of 34.83. Comparing this maximum utility with what we achieved in our survey, our outcome is good with a vast majority of the voters approving at least some cities of the three chosen destinations. Except for a voter who is against any city, all voters appear to have at least one city of the approvals chosen. The result shows that PAV is generally fair among voters.

PAV Case 2

Given the same feedback from 19 voters, which we collect once for both cases, we find the optimal tour for different time budgets. Starting from $\text{time} = 1$(hour), we relax the budget by 1 hour every time until $\text{time} = 10$. As we can see from the result[11], the optimal decision for 1 hour travelling budget is to stay in Paris for the total trip because travelling to and forth any pair of cities would exceed 1 hour. As time budget increases, the total utility for voters increases as more cities are added to the tour and the cities chosen satisfy more voters. When the time budget is 8 hours or more, all 6 cities are able to be included in the vacation and the maximum total utilities are reached among voters. The only difference between budget of 8 hours or more is the order of cities visited.

Comparison Analysis I: Total Efficiency

For our analysis, we interpret the number of votes an individual cast for an alternative as the amount of utility they receive if that city was selected to be a part of the vacation. We measure efficiency as maximizing the utility among all voters using QV and PAV.

Since voting with PAV only involved approving or disapproving an alternative, we make an assumption of consistency and use utilities from QV. Suppose in PAV, a voter approves London, Paris and Berlin, then the assumption of consistency implies that he has positive votes on these alternatives under QV. The intuition is that voters will only cast votes, either approval or numerical votes, for alternatives which would result in a positive utility if they were selected. Among alternatives that give voters zero utility according to QV, the voters are indifferent between approving or disapproving them in PAV. Moreover, voters would not cast votes or approve alternatives that give them negative utilities because if these are chosen, they would make the voters worse off. In our model, we would not be able to distinguish between indifference and negative utilities. Nevertheless, we capture the intensity of preferences over the alternatives and use this for our analysis.

There are two points regarding inconsistencies in our survey data that we would like to clarify. The first inconsistency is that, there are voters that approve a city under PAV, but have zero votes cast under QV. And the second is that, there are voters that disapprove a city but cast

[11] Refer to “PAV Case 2 Results” in Appendix

a positive number of votes under QV. Our interpretation of these anomalies are discussed briefly below and all votes were included in our analysis.

Inconsistencies in cities that receive zero votes in QV, but approved by the same voter in PAV can be seen as the voter being conservative. Since PAV does not have a cost, he approves all cities that he wants to go to and also the ones he is indifferent about. However, in QV the voter might not want to spend his limited resources by voting for ones he is indifferent about, and instead use his budget to represent his preferences among his preferred choices.

Cities that are disapproved in PAV, but receive a positive number of votes in QV can be seen as a way for the voter trying to avoid a different alternative that gives him a negative utility. For example, suppose a voter definitely wants to go to London, but does not want to visit either Berlin or Venice. However, among Berlin and Venice he really dislikes Venice. Under PAV he would approve London and disapprove Berlin and Venice. Under QV, he would have most of his votes on London, but a small amount on Berlin to avoid going to Venice. So, though Berlin was disapproved in PAV, it received a positive vote in QV.

Consider QV and the case of PAV where we ignore distance. Then, picking top three places to visit using the votes we gathered results in London, Paris and Prague under QV and London, Paris and Venice under PAV. These cities provide a combined utility of 150.75 for all voters under QV and utility of 134.81 under PAV[12].

If we pick only one city to visit, both mechanisms select Paris. This results in a utility of 63.04 for all voters under QV and utility of 60.8 under PAV. If we pick two cities to visit, the results are London and Paris under QV and Paris and Venice under PAV. The cities generate a total utility of 107.39 under QV and 96.86 under PAV. Finally, if we pick four cities to visit, we select London, Paris, Prague and Venice under QV for a utility of 189.05 and London, Paris, Berlin and Venice for a utility of 161.09.

# of cities	utility in QV	utility in PAV	difference	marginal loss*
1	63.04	60.80	2.24	3.55%
2	107.39	96.86	10.53	9.81%
3	150.75	134.81	15.94	10.57%
4	189.05	161.09	27.96	14.79%

Table 2: Efficiency Comparison as a Measure of Total Utility Difference and Marginal Utility Difference

**Marginal loss measures the percentage of efficiency loss by using PAV instead of QV. The calculation takes the absolute value of (utility in PAV – utility in QV)/ (utility in QV).*

As the results show, the utilities calculated under PAV are lower than the utilities under QV. They are consistent with the objectives that QV aims at maximizing the efficiency while PAV focuses on fairness among voters. As we can see, though PAV results in lower utilities, they are not far from their counterparts in QV. This indicates that the efficiency achieved using in PAV is

[12] Refer to "Measuring an individual's utility under PAV using QV utilities" in Appendix

not too disappointing. Another thing to notice is that in general, as we select more cities, the combined utilities differ more between the two mechanisms and the marginal difference becomes larger. This indicates that the efficiency in PAV decreases more as more cities are selected.

Comparison Analysis II: Individual Efficiency

As a second measure of comparison we look at individual efficiency. Individual efficiency is calculated as the difference between the utility a voter would have got if his top cities were selected and the actual utility realized based on the voting mechanism. We want this value to be as small as possible. For every voter, we calculate the utility he gets from every possible combination of three cities, and the maximum of these values is "Utility_maximum" in our table. Note that cities that the voter doesn't approve under PAV generate 0 value. Calculating the difference between the maximum attainable, and the utility realized by picking our top three choices, we see that the forgone utility for QV is 44.28 and for PAV is 59.67.

	QV	PAV
Utility_maximum	195.03	194.48
Utility_3 picked cities in QV	150.75	134.81
Utility voters give up	44.28	59.67
% of Utility Loss*	22.70	30.68

Table 3: Efficiency Comparison of How Much an Individual Gives Up

** % of Utility Loss= |utilities for 3 picked cities in QV – maximum utilities|/ (maximum utilities).*

As the result indicates, voters generally give up more utilities under PAV than QV, and efficiency loss is great under PAV. In addition, the marginal efficiency loss, which is measured in percentage value, is larger under PAV. Although it is less efficient to use PAV in individual sense, QV is not better by much because voters give up their best choices of 3 cities in order to travel as a group.

Comparison Analysis III: Fairness[13]

As another measure of comparison, we use fairness. Here, fairness is defined as minimizing the number of voters that do not have at least one of their approved cities included in the vacation. Different from the measurement of efficiency, where utilities for PAV were calculated using QV votes, here we interpret positive votes under QV to be approvals and zero votes are disapprovals. Running the PAV program on the interpreted QV votes, the utilities for all voters is a combined 33. In comparison to the 29 under PAV, this result seems counter-intuitive. Though PAV

[13] Refer to "Measuring an individual's utility under QV using PAV utilities" in Appendix

is geared toward maximizing fairness, it seems that the fairness is higher under QV. This is because there are 6 voters who approve none of the cities under QV, the utility maximization can focus only on 13 voters who have some approved cities. Also, among the 13 voters who have at least one approved city, there are 5 voters who approve of all cities and 7 voters who approve at least 3 cities. Therefore, it is easier to find 3 cities which satisfy more voters because of high approval rate among less voters who really care.

If we take a closer look at how individuals are satisfied, i.e., how many approved cities they actually get, it is obvious that PAV is fairer than QV.

# of approved cities one gets	# of voters under QV	# of voters under PAV
3	8 (42%)	9 (47%)
2	4 (21%)	7 (37%)
1	1 (5%)	2 (11%)
0	6 (32%)	1 (5%)

Table 4: Fairness Comparison under Two Mechanisms

** The percentage number inside () represents the fraction of voters who are in the category.*

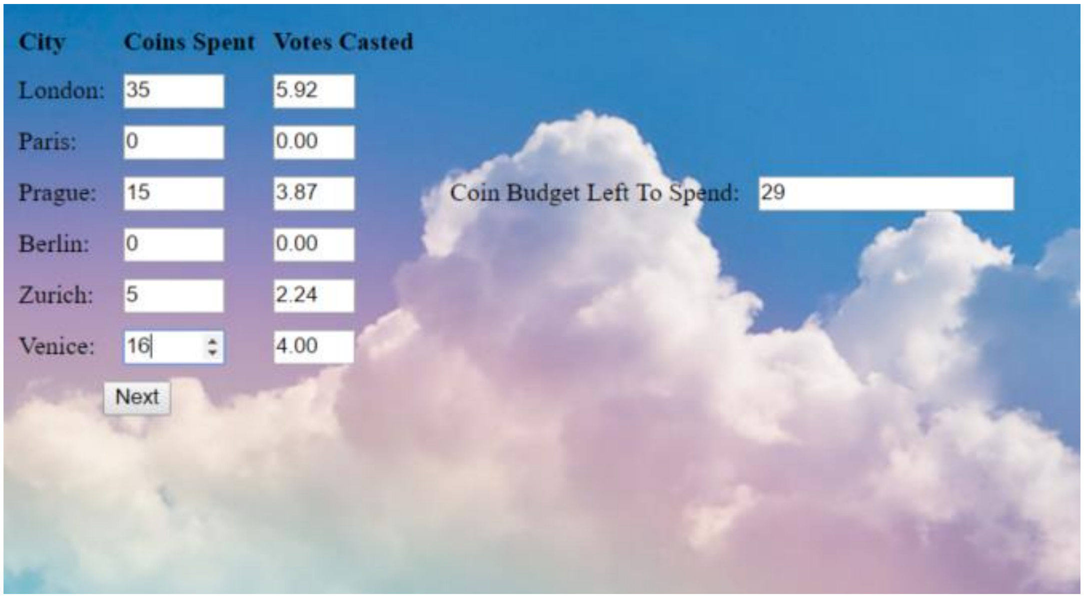
From the above table, over 80% of voters get at least 2 cities in approval under PAV compared to approximately 60% of voters under QV. Also, there is only 5% of voters who get none of the approved cities under PAV while the percentage is almost one third under QV. As a result, even though the maximum total utility is higher under QV, PAV is a better measurement of fairness.

Conclusion

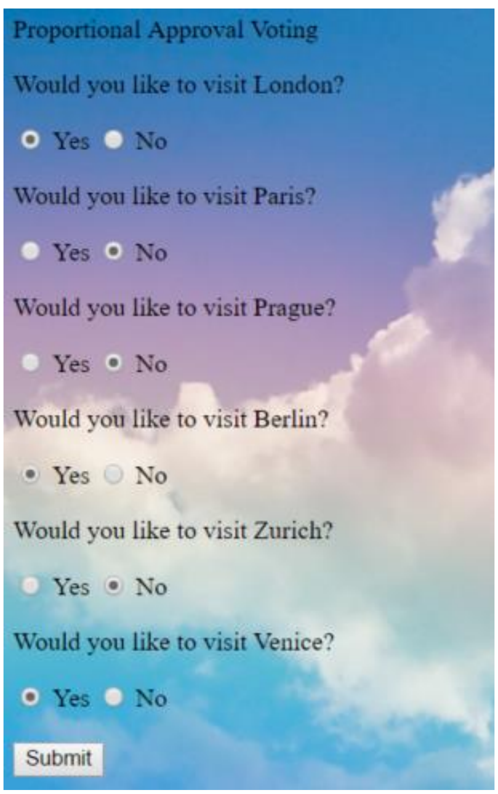
As we have demonstrated through this paper, both Quadratic Voting and Proportional Approval voting have their own merits and disadvantages. QV allows voters to express their true intensity of preferences and has a higher efficiency when compared to PAV. PAV is an easy to understand, straightforward voting mechanism and maximizes fairness among voters.

APPENDIX

Quadratic Voting Online Survey



Proportional Approval Voting Online Survey



Linear Program for Quadratic Voting

```
set CITIES; /* set of cities that the voters were choosing from */
var selected {j in CITIES}, binary; /*whether city j was picked or not*/
param num_voters;
param votes {i in 1..no_voters, j in CITIES};

/* Objective: pick top 3 cities based on number of votes*/
maximize utility: sum {i in 1..no_voters, j in CITIES} selected[j]*votes[i, j];
s.t. time_constraint: sum {j in CITIES} selected[j] <= 3; /*picking 3 out of all options*/

data;
set CITIES: = London Paris Prague Berlin Zurich Venice;
param num_voters: = 19;
param votes: London Paris   Prague  Berlin  Zurich  Venice :=
  A          5.48    5.48   3.16    0       0       5.48
  B          0       0      0       0       0       0
  C          5.48    4.47   4.47    0       5.48    0
  D          2.24    2.45   2.65    2.83    3       3.16
  E          0       0      0       0       0       0
  F          5.1     2.24   3.16    4.9     5.48    2.24
  G          4.9     4      5.48    4       2       3.16
  H          1.73    6      0       0       6       5
  I          4.47    4.47   6.32    3.16    3.16    0
  J          1       3.16   5.48    2.45    1.73    7.07
  K          0       4.47   6.32    4.47    0       4.47
  L          4.47    7.07   0       5.48    0       0
  M          3.16    3.16   6.32    2.24    2.24    5.48
  N          6.32    6.32   0       4.47    0       0
  O          0       0      0       0       0       0
  P          0       0      0       0       0       0
  Q          0       0      0       0       0       0
  R          0       0      0       0       0       0
  S          0       9.75   0       0       0       2.24
end;
```

Mixed integer program for PAV

```
set CITIES;
set VOTERS;
param num_cities;
param approval{ i in VOTERS, k in CITIES }, binary; /* Does voter_i like city_k */
var is_visited{ k in CITIES }, binary; /* Is city_k visited */
```

```
var get_city{ i in VOTERS, j in 1..num_cities }, binary; /* Does voter_i get at least j cities? */

/* Objective: pick top 3 cities using PAV */
maximize utility: sum{ i in VOTERS, j in 1..3 } 1/j*get_city[i,j];

s.t. num_get_visited{ i in VOTERS }: sum{ j in 1..num_cities } get_city[i,j] <= sum{ k in CITIES }
approval[i,k]*is_visited[k]; /* the cities voter i gets are at most the ones he likes and visited */
s.t. at_least{ i in VOTERS, j in 1..num_cities-1 }: get_city[i,j] >= get_city[i,j+1]; /*If a voter gets 2
cities, then he must get at least 1 city */
s.t. visit_k_cities: sum{ k in CITIES } is_visited[k] <= 3;

data;
set CITIES:= London Paris Prague Berlin Zurich Venice;
set VOTERS:= A B C D E F G H I J K L M N O P Q R S;
param num_cities:= 6;
param approval: London Paris Prague Berlin Zurich Venice :=
 A    1    1    1    1    1    1
 B    0    0    0    0    0    0
 C    1    1    1    0    1    0
 D    0    1    0    1    0    1
 E    1    1    1    1    1    0
 F    1    0    0    1    1    0
 G    1    1    1    1    0    1
 H    1    1    0    0    1    1
 I    1    1    1    1    0    0
 J    0    1    1    1    0    1
 K    1    1    0    0    0    1
 L    0    1    1    0    0    1
 M    1    1    0    1    0    1
 N    1    1    0    0    0    1
 O    0    0    1    1    1    1
 P    1    1    1    0    1    1
 Q    1    1    1    1    1    1
 R    1    1    0    1    1    1
 S    0    1    1    1    0    1;
End;
```

PAV Case 1 Results

```
/* If picking top 1 city */
/* optimal: Paris */
```

```
/* maximum utility: 16 */

/* If picking top 2 cities instead… */
/* optimal: Paris, Venice */
/* maximum utility: 23.5 */

/* If picking top 3 cities instead… */
/* optimal: London, Paris, Venice */
/* maximum utility: 29 */

/* If picking top 4 cities instead… */
/* optimal: London, Paris, Berlin, Venice */
/* maximum utility: 32.91667 */
```

Mixed integer program of PAV under additional time and distance constraints

```
set CITIES;
set VOTERS;
param num_cities;
param approval{ i in VOTERS, k in CITIES }, binary; /* Does voter_i like city_k */
param time; /* time budget for travelling by plane */
param distance{ k in CITIES, m in CITIES }; /* direct flight time from city_k to city_m */
var is_visited{ k in CITIES }, binary; /* Is city_k visited */
var get_city{ i in VOTERS, j in 1..num_cities }, binary; /* Does voter_i get at least j cities */
var connect_city{ k in CITIES, m in CITIES }, binary; /* Is there a connection from city_k to city_m */
var number{ k in CITIES }; /* arbitrary number for city_k*/
var is_start{ k in CITIES },binary; /* start from city_k */

/* Objective: pick a tour that visits as many cities as possible using PAV, under time and distance constraint */
maximize utility: sum{ i in VOTERS, j in 1..num_cities } 1/j*get_city[i,j];

s.t. one_start: sum{ k in CITIES } is_start[k] <= 1; /* start and end at one city */
s.t. exit{ k in CITIES }: sum{ m in CITIES } connect_city[k,m] <= 1; /* exit each city at most once */
s.t. enter{ m in CITIES }: sum{ k in CITIES } connect_city[k,m] <= 1; /* enter each city at most once */
s.t. flow{ k in CITIES }: sum { m in CITIES } connect_city[k,m] = sum { m in CITIES }
```

```
connect_city[m,k]; /* require enter and exit from a city when choose to visit */
s.t. no_subtour{ k in CITIES, m in CITIES }: number[k] - number[m] +
(num_cities)*connect_city[k,m] <= num_cities - 1 + num_cities*is_start[m]; /* prevent
suboptimal tours; start and end at same city */
s.t. num_get_visited{ i in VOTERS }: sum{ j in 1..num_cities } get_city[i,j] <= sum{ k in CITIES }
approval[i,k]*is_visited[k]; /* the cities voter_i gets are at most the ones he likes and visited */
s.t. at_least{ i in VOTERS, j in 1..num_cities-1 }: get_city[i,j] >= get_city[i,j+1]; /*If a voter gets 2
cities, then he must get at least 1 city */
s.t. visit_constraint{ k in CITIES}: is_visited[k] <= sum{m in CITIES} connect_city[k,m]; /* a city
should be marked as visited when exit */
s.t. time_contraint: sum{ k in CITIES, m in CITIES } distance[k,m]*connect_city[k,m] <= time; /*
total travelling time in the tour should meet the time budget */

data;
set CITIES:= London Paris Prague Berlin Zurich Venice;
set VOTERS:= A B C D E F G H I J K L M N O P Q R S;
param num_cities:= 6;
param approval: London Paris Prague Berlin Zurich Venice :=
 A      1   1   1   1   1   1
 B      0   0   0   0   0   0
 C      1   1   1   0   1   0
 D      0   1   0   1   0   1
 E      1   1   1   1   1   0
 F      1   0   0   1   1   0
 G      1   1   1   1   0   1
 H      1   1   0   0   1   1
 I      1   1   1   1   0   0
 J      0   1   1   1   0   1
 K      1   1   0   0   0   1
 L      0   1   1   0   0   1
 M      1   1   0   1   0   1
 N      1   1   0   0   0   1
 O      0   0   1   1   1   1
 P      1   1   1   0   1   1
 Q      1   1   1   1   1   1
 R      1   1   0   1   1   1
 S      0   1   1   1   0   1;
param time:= 10; /* change accordingly */
param distance: London Paris Prague Berlin Zurich Venice:=  /* non-stop flight hours */
 London    0    1.25    1.92    1.83    1.67    2.33
```

Paris	1.25	0	1.67	1.67	1.17	1.58
Prague	1.92	1.67	0	0.92	1.25	1.67
Berlin	1.83	1.67	0.92	0	1.50	1.67
Zurich	1.67	1.17	1.25	1.50	0	1.08
Venice	2.33	1.58	1.67	1.67	1.08	0

end;

PAV Case 2 Results

/* time = 1 */
/* optimal: Paris */
/* maximum utility: 16 */
/* optimal tour: (staying in) Paris */

/* time = 2 */
/* optimal: Prague, Berlin */
/* maximum utility: 19 */
/* optimal tour: Prague -> Berlin -> Prague */

/* time = 3 */
/* optimal: London, Paris*/
/* maximum utility: 23 */
/* optimal tour: London -> Paris -> London */

/* time = 4 */
/* optimal: Paris, Zurich, Venice */
/* maximum utility: 27.66667 */
/* optimal tour: Zurich -> Paris -> Venice -> Zurich */

/* time = 5 */
/* optimal: Prague, Belin, Zurich, Venice */
/* maximum utility: 29.75 */
/* optimal tour: Berlin -> Prague -> Zurich -> Venice -> Berlin */

/* time = 6 */
/* optimal: Paris, Prague, Berlin, Venice */
/* maximum utility: 32.08333 */
/* optimal tour: Prague -> Berline -> Paris -> Venice -> Prague */

/* time = 7 */

/* optimal: London, Paris, Berlin, Zurich, Venice */

/* maximum utility: 35.26667 */

/* optimal tour: Berlin -> London -> Paris -> Zurich -> Venice -> Berlin */

/* time = 8 */

/* optimal: London, Paris, Prague, Berlin, Zurich, Venice */

/* maximum utility: 37.78333 */

/* optimal tour: Zurich -> Venice -> Paris -> London -> Berlin -> Prague -> Zurich */

/* time = 9 */

/* optimal: London, Paris, Prague, Berlin, Zurich, Venice */

/* maximum utility: 37.78333 */

/* optimal tour: Zurich -> Prague -> Berlin -> Paris -> London -> Venice -> Zurich */

/* time = 10 */

/* optimal: London, Paris, Prague, Berlin, Zurich, Venice */

/* maximum utility: 37.78333 */

/* optimal tour: Zurich -> Prague -> Berlin -> Paris -> London -> Venice -> Zurich */

Measuring an individual's utility under PAV using QV utility

		London	Paris	Prague	Berlin	Zurich	Venice
A	Quadratic	0	9.75	0	0	0	2.24
	Approval	1	1	1	1	1	1
		0	9.75	0	0	0	2.24
B	Quadratic	0	0	0	0	0	0
	Approval	1	1	0	1	1	1
		0	0	0	0	0	0
C	Quadratic	0	0	0	0	0	0
	Approval	1	1	1	0	1	1
		0	0	0	0	0	0
D	Quadratic	0	0	0	0	0	0
	Approval	0	0	1	1	1	1
		0	0	0	0	0	0
E	Quadratic	0	0	0	0	0	0
	Approval	1	1	0	0	0	1
		0	0	0	0	0	0
F	Quadratic	6.32	6.32	0	4.47	0	0
	Approval	1	1	1	1	0	1

		6.32	6.32	0	4.47	0	0
G	Quadratic	3.16	3.16	6.32	2.24	2.24	5.48
	Approval	0	1	1	0	0	1
		0	3.16	6.32	0	0	5.48
H	Quadratic	4.47	7.07	0	5.48	0	0
	Approval	1	1	0	0	0	1
		4.47	7.07	0	0	0	0
I	Quadratic	0	4.47	6.32	4.47	0	4.47
	Approval	0	1	1	1	0	1
		0	4.47	6.32	4.47	0	4.47
J	Quadratic	1	3.16	5.48	2.45	1.73	7.07
	Approval	0	1	1	1	0	1
		0	3.16	5.48	2.45	0	7.07
K	Quadratic	4.47	4.47	6.32	3.16	3.16	0
	Approval	1	1	1	1	0	0
		4.47	4.47	6.32	3.16	0	0
L	Quadratic	1.73	6	0	0	6	5
	Approval	1	1	0	0	1	1
		1.73	6	0	0	6	5
M	Quadratic	4.9	4	5.48	4	2	3.16
	Approval	1	1	1	1	0	1
		4.9	4	5.48	4	0	3.16
N	Quadratic	5.1	2.24	3.16	4.9	5.48	2.24
	Approval	1	0	0	1	1	0
		5.1	0	0	4.9	5.48	0
O	Quadratic	0	0	0	0	0	0
	Approval	1	1	1	1	1	0
		0	0	0	0	0	0
P	Quadratic	2.24	2.45	2.65	2.83	3	3.16
	Approval	0	1	0	1	0	1
		0	2.45	0	2.83	0	3.16
Q	Quadratic	5.48	4.47	4.47	0	5.48	0
	Approval	1	1	1	0	1	0
		5.48	4.47	4.47	0	5.48	0
R	Quadratic	0	0	0	0	0	0
	Approval	0	0	0	0	0	0
		0	0	0	0	0	0
S	Quadratic	5.48	5.48	3.16	0	0	5.48
	Approval	1	1	1	1	1	1
		5.48	5.48	3.16	0	0	5.48
QV Total		44.35	63.04	43.36	34	29.09	38.3
PAV Total		37.95	60.8	37.55	26.28	16.96	36.06

Measuring an individual's utility under QV using PAV utility

		London	Paris	Prague	Berlin	Zurich	Venice
A	Quadratic	0	9.75	0	0	0	2.24
Corresponding PAV		0	1	0	0	0	1
B	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
C	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
D	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
E	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
F	Quadratic	6.32	6.32	0	4.47	0	0
Corresponding PAV		1	1	0	1	0	0
G	Quadratic	3.16	3.16	6.32	2.24	2.24	5.48
Corresponding PAV		1	1	1	1	1	1
H	Quadratic	4.47	7.07	0	5.48	0	0
Corresponding PAV		1	1	0	1	0	0
I	Quadratic	0	4.47	6.32	4.47	0	4.47
Corresponding PAV		0	1	1	1	0	1
J	Quadratic	1	3.16	5.48	2.45	1.73	7.07
Corresponding PAV		1	1	1	1	1	1
K	Quadratic	4.47	4.47	6.32	3.16	3.16	0
Corresponding PAV		1	1	1	1	1	0
L	Quadratic	1.73	6	0	0	6	5
Corresponding PAV		1	1	0	0	1	1
M	Quadratic	4.9	4	5.48	4	2	3.16
Corresponding PAV		1	1	1	1	1	1
N	Quadratic	5.1	2.24	3.16	4.9	5.48	2.24
Corresponding PAV		1	1	1	1	1	1
O	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
P	Quadratic	2.24	2.45	2.65	2.83	3	3.16
Corresponding PAV		1	1	1	1	1	1
Q	Quadratic	5.48	4.47	4.47	0	5.48	0
Corresponding PAV		1	1	1	0	1	0
R	Quadratic	0	0	0	0	0	0
Corresponding PAV		0	0	0	0	0	0
S	Quadratic	5.48	5.48	3.16	0	0	5.48
Corresponding PAV		1	1	1	0	0	1

/* optimal: London, Paris, Berlin */

/* maximum utility: 33 */