Prop: If P irreducible + aperiodic
then there exists $t>0$ s.t. $P^t(x,y)>0$ for every (x,y)

Pf:
We start with a fact that if
$\{a_i\}_{i\in I}$ satisfies $gcd=1$ then there is
large enough N s.t. every $n\geq N$ can be represented
by $n=\sum_{i=1}^{k} n_i a_i$ $n_i\geq 0$

Pf of fact:
First suppose $|I|=2$, Bezout's identity says
$1=\lambda_1 a_1+\lambda_2 a_2$ $\lambda_i\in\mathbb{Z}$
then
$m= ca_1+da_2$ for $c,d\in\mathbb{Z}$
$=(c+\lambda a_2)a_1+(d-\lambda a_1)a_2$ $\forall\, a$.
we can chose λ s.t
$m=\tilde{c}a_1+\tilde{d}a_2$ with $0\leq\tilde{c}<a_2$
if $m>a_1a_2$ then we know $\tilde{d}\geq 0$ $\square$.

For general case, we can proof by induction.

After knowing this fact, since aperiodic

we can find N_x for every x, s.t.

$\mathbb{P}^t(x,x) > 0$ if $t \geq N_x$

then $\mathbb{P}^{t'}(x,x) > 0$ $\forall x$ if $t' > \max_{x} N_x$

then $\mathbb{P}^{\tilde{t}}(x,y) > 0$ if $\tilde{t} \geq 2\max_{x} N_x + \max_{x,y} N_{x,y}$

as $\mathbb{P}^{\tilde{t}}(x,y) \geq \mathbb{P}^{\frac{\tilde{t}-t(x,y)}{2}}(x,x)\, \mathbb{P}^{t(x,y)}(x,y)\, \mathbb{P}^{\tilde{t}-t(x,y)/2}(y,y) > 0.$

Finally, we define (Markovian) coupling of a M.C. ~~stochastic~~

Def.

Given a M.C. $\mathbb{P}$, we define a (Markovian) coupling of $\mathbb{P}$ (with itself) is a M.C. $\{(X_t, Y_t)\}$ on $\mathcal{X}\times\mathcal{X}$ which satisfies

$\mathbb{P}(X_{t+1} = x' \mid X_t = x \quad Y_t = y) = \mathbb{P}(x, x')$

$\mathbb{P}(Y_{t+1} = y' \mid X_t = x \quad Y_t = y) = \mathbb{P}(y, y')$

[Equivalently, Markovian coupling can be viewed as a family of couplings between $\mathbb{P}(x,\cdot)$ and $\mathbb{P}(y,\cdot)$]

In particular, for any fixed x, if $X_t = Y_t = x$, ~~we often choose~~

we often choose $X_{t+1} = Y_{t+1} \sim \mathbb{P}(x,\cdot)$

Therefore, if two coupled chains meet, they meet forever

Thm: Suppose we have a coupling $\bar{P}$ that is faithful meaning that if $X_s = Y_s$ for some s, then $X_t = Y_t$ for $t \geq s$.

Define $\tau_{couple} = \min\{t : X_s = Y_s \text{ for all } s \geq t\}$

then $\| P^t(x,\cdot) - P^t(y,\cdot) \| \leq P_{x,y}(\tau_{couple} > t)$

Pf: $\mathbb{P}_{x,y}(X_t \neq Y_t) \geq \| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV}$ for any coupling

For our coupling $\mathbb{P}_{x,y}(X_t \neq Y_t) = \mathbb{P}_{x,y}(\tau_{couple} > t)$

Moreover, if π is stationary distribution, then

$$\| P^t(x,\cdot) - \pi \|_{TV} \leq \mathbb{P}_{x,\pi}(\tau_{couple} > t)$$

$$\leq \max_y P_{x,y}(\tau_{couple} > t).$$

Pf: Let $X_0 = x$, $Y_0 \sim \pi$ and apply our coupling $\bar{P}$

Then $\mathbb{P}_{x,\pi}(X_t \neq Y_t) = \mathbb{P}_{x,\pi}(\tau_{couple} > t)$

$$\geq \| P^t(x,\cdot) - \pi \|_{TV}$$

Meanwhile $\mathbb{P}_{x,\pi}(\tau_{couple} > t) = \sum_y P_{x,y}(\tau_{couple} > t)\, \pi(y)$

$$\leq \max_y \mathbb{P}_{x,y}(\tau_{couple} > t)$$

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This tells us if we can bound

$\max_{x,y} \mathbb{P}(T_{couple} > t)$, we can bound the convergence speed.

Thm (Convergence thm). If $\mathbb{P}$ irreducible + aperiodic then there is $r \in (0,1)$ s.t.

$$\max_x \| \mathbb{P}^t(x,\cdot) - \pi \| \leq (1-r)^t$$

Pf. we know there is N s.t. $P^N(x,y) > \varepsilon$ $\forall x,y$, wlog we assume $N=1$, therefore $\mathbb{P}(x,y) > \varepsilon$ $\forall x,y$

For any (x,y), we maximally couple $\mathbb{P}(x,\cdot)$ and $\mathbb{P}(y,\cdot)$ so if $X \sim P(x,\cdot)$, $Y \sim \mathbb{P}(y,\cdot)$ and they are maximally coup

$$\mathbb{P}(X \neq Y) = \| P(x,\cdot) - P(y,\cdot) \|_{TV}$$

$$= 1 - \sum_z \min\{ P(x,z) ; P(y,z) \}$$

$$\leq 1 - |\mathcal{X}| \varepsilon$$

Therefore fix any $(x \neq y)$. if $(X_0, Y_0) = (x,y)$

$$\mathbb{P}(X_1 \neq Y_1) \leq (1 - |\mathcal{X}|\varepsilon)$$

$$\mathbb{P}(X_2 \neq Y_2) \leq (1 - |\mathcal{X}|\varepsilon)^2 \quad \cdots$$

$$\mathbb{P}(X_t \neq Y_t) \leq (1 - |\mathcal{X}|\varepsilon)^t$$

$\square$

Mixing time:

Given M.C. $\mathbb{P}$, the ε-mixing time is defined as

$t_{mix}(\varepsilon) := \inf_t \{t \mid d(t) \le \varepsilon\}$ where

$t_{mix} := t_{mix}(1/4)$.

$d(t) = \sup \max_x \| \mathbb{P}^t(x,\cdot) - \pi \|_{TV}$ $\quad \bar{d}(t) = \max_{x,y} \| \mathbb{P}^t(x,\cdot) - \mathbb{P}^t(y,\cdot) \|_{TV}$

Fact ① $d(t) \le \bar{d}(t) \le 2d(t)$

② $\bar{d}(t+s) \le \bar{d}(t)\,\bar{d}(s)$

③ $t_{mix}(\varepsilon) \le \lceil \log_2 1/\varepsilon \rceil \, t_{mix}$

Pf ③. $d(t_{mix}) \le 1/4 \Rightarrow \bar{d}(t_{mix}) \le 1/2$

$\bar{d}(2t_{mix}) \le (1/2)^2 \quad \cdots \quad \bar{d}(t_{mix}\cdot k) \le (1/2)^k$

set $k = \lceil \log_2 \frac{1}{\varepsilon} \rceil$ we know $\bar{d}(\lceil \log_2 \frac{1}{\varepsilon} \rceil t_{mix}) \le \varepsilon$

$\Rightarrow$ $d(\lceil \log_2 1/\varepsilon \rceil t_{mix}) \le \varepsilon$.

Example ① Ehrenfest urn (Random walk on hypercube).

Given $X_t \in \{0,1\}^n$, each time pick an index $\in \{1,2,\cdots n\}$

flip this index w.p. 1/2, do nothing w.p. 1/2.

Therefore $\mathbb{P}(x,y) = \begin{cases} 1/2 & y = x \\ \frac{1}{2n} & \sum_i |y_i - x_i| = 1 \\ 0 & \text{otherwise.} \end{cases}$

$\pi = \mathrm{Unif}\{\{0,1\}^n\}$.

Claim: $t_{mix} = O(n\log n)$

Pf: We couple two random walk by picking the same index and force them to be the same.

Let $T = \inf\{t \mid \text{all indexes has been selected}\}$.

Then $\mathbb{E}_{x,y}(T_{couple}) \le \mathbb{E}_{x,y}(T) = n\left(\frac{1}{n} + \frac{1}{n-1} + \cdots +\right.$

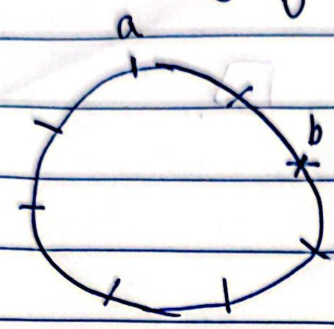
$\le n(\log n + 1)$

$\mathbb{P}(T_{couple} > 4\,\mathbb{E}_{x,y}(T_{couple})) \le \frac{1}{4}$

$\Rightarrow t_{mix} \le 4n(\log n + 1)$.

② Lazy random walk on cycle

$$P(x,y) = \begin{cases} 1/4 & y-x = \pm 1 \pmod n \\ 1/2 & y = x \\ 0 & \text{otherwise} \end{cases}$$



Fix any two points a, b, we couple in this way

$(a,b) \to (a, b+1)$

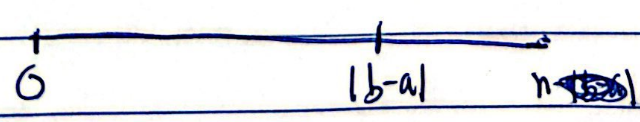
$(a,b) \to (a, b-1)$

$(a,b) \to (a-1, b)$

$(a,b) \to (a+1, b)$

[only allow one chain moves]

If we look at $|X_t - Y_t|$ it is a simple r.w. on cycle beginning at $|b-a|$. Calculating the hitting time at 0 is the same as the hitting time of a random walk on $\mathbb{Z}$ at 0 or $n-|b-a|$



$$E_{x,y}(\tau_{meet}) = (n-|x-y|)|x-y|$$

$$\leq \frac{n^2}{4}$$

$$\Rightarrow t_{mix} = O(n^2).$$

$\square$.