Expectation of the Largest Betting Size in Labouchère System

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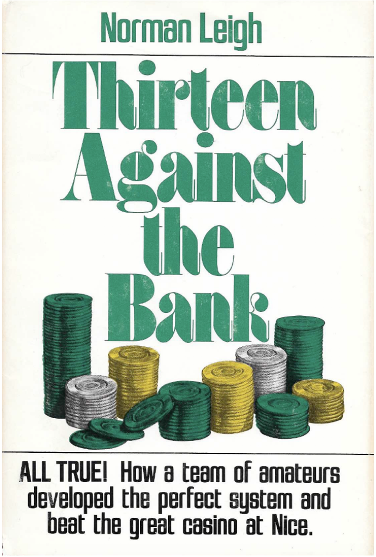
Thirteen Against the Bank

- In 1966 a team of 13 Englishmen and women set out to beat a French casino at roulette using a betting system known as the reverse Labouchère system.
- They won 800,000 francs ($160,000) in 8 days and were banned from further play.
- Their story is documented in a 1976 nonfiction book by Norman Leigh.



Description of Labouchère system

- Assumes one wants to win \$10 at the end of the game. Labouchère system gives a strategy to determine the betting size in each bet.
- To initialize, the bettor writes a list of positive integers on his scoresheet (e.g., 1; 2; 3; 4).
- The next bet size is always the sum of the first and last terms on the current list. (Exception if only one term.)
- After each trial, the list is updated:
  - After a win, the first and last terms are cancelled.
  - After a loss, the amount just lost is appended to the list as a new last term.
- Betting is stopped once the list is empty.
- Then you win \$10 :)

Illustration of the Labouchère system

Table: An illustration of the Labouchère system. Here the initial list is $(1, 2, 3, 4)$.

trial	bet	result	list	cumulative profit
			1, 2, 3, 4	
1	5	Win	2, 3	5
2	5	Lost	2, 3, 5	0
3	7	Lost	2, 3, 5, 7	-7
4	9	Lost	2, 3, 5, 7, 9	-16
5	11	Win	3, 5, 7	-5
6	10	Lost	3, 5, 7, 10	-15
7	13	Win	5, 7	-2
8	12	Win	$\varnothing$	10

History of Henry Labouchère

- Member of British Parliament, journalist, editor/publisher of Truth.
- Infamous for the Labouchère Amendment of 1885, outlawing "gross indecency" - led to Oscar Wilde's incarceration for two years; Alan Turing also convicted under the law. (Repealed in 1967.)
- Popularized the system, but credited it to Condorcet (1743-1794).

Figure: Labouchère

Notations

Let:

- l_0 be the length of initial list, $L_0 \in \mathbb{R}^{l_0}$ be the initial list. (e.g., in previous example, $l_0 = 4$, $L_0 = (1, 2, 3, 4)$)
- $l_n \in \mathbb{Z}$ be the length of the list after n-th betting,
- $L_n \in (\mathbb{R}^+)^{l_n}$ be the list after n-th betting,
- $\xi_1, \xi_2, \cdots$ be i.i.d. random variables with

  $$\mathbb{P}(\xi_1 = 1) = p \qquad \text{and} \qquad \mathbb{P}(\xi_1 = -1) = q$$

  where p denotes the probability of winning in each betting,
- B_n be the betting size at n-th betting, S_n be the sum of the list after n-th betting, T_n be the bettor's cumulative profit after n-th betting,
- N be the first time such that $L_N = \emptyset$ or equivalently $l_N = 0$,
- $B^\star$ be the largest betting size, i.e., $B^\star \doteq \max_{1 \le i \le N} B_i$.

An asymmetric random walk on $\mathbb{Z}$

- The length of the Labouchère bettors list l_n is a random walk on the set of nonnegative integers that takes two steps to the left with probability p and one step to the right with probability q. The initial state is the length of the initial list l_0.
- Easy analysis shows when $p \geq \frac{1}{3}$, then absorption at 0 occurs eventually with probability 1.
- Let N be the first absorption time, then distribution of N can be derived using an extension of ballot theorem.

Lemma (Ethier, 2008)

$$\mathbb{P}_{l_0}(N \geq n) \sim D_{l_0}(n) n^{-\frac{3}{2}} \rho^{\frac{n}{3}},$$

where l_0 is the length of the initial list, $D_{l_0}(n)$ is a constant only depending on l_0 and $n(\text{mod } 3)$,and$\rho \doteq \frac{27}{4} p(1-p)^2 \leq 1$.

Total amount of betting $B_1 + \cdots + B_N$

Why this betting strategy still fails? Martingale analysis shows the total amount of betting has infinite expectation.

Theorem (Grimmett & Stirzaker, 2001, One Thousand Exercises in Probability, 12.9.15)

When $\frac{1}{3} \le p \le \frac{1}{2}$, $\mathbb{E}(B_1 + B_2 + \cdots B_N) = \infty$.

Proof

If not, since $T_n = T_{n-1} + \xi_n B_n$ forms a supermartingale. Since $T_{n\wedge N} \le B_1 + B_2 + \cdots B_N$ and $T_{n\wedge N} \to S_0$ a.s., by dominated convergence theorem, $\mathbb{E}\, T_{n\wedge N} \to S_0$, however $\mathbb{E}\, T_{n\wedge N} \le \mathbb{E}\, T_0 = 0$, contradiction.

The largest betting size

Recall $B^\star = \max_{1 \le i \le N} B_i$, a natural question is, does $B^\star$ have infinite expectation?

- The list evolves in a complicated history dependent manner, it is hard to analyze it using combinatoric tools directly.
- When $p = \frac{1}{2}$, martingale maximal inequality will not give useful bounds.

A useful result for $p = \frac{1}{2}$

Theorem (Han & W., 2018.)

When $p = \frac{1}{2}$, for any initial list L_0, we have $\mathbb{E}((B^\star)^{1+\epsilon}) = \infty$ and $\mathbb{E}((B^\star)^{1-\epsilon}) < \infty$ for any $\epsilon > 0$.

Proof

- $\infty = \mathbb{E}(B_1 + \cdots + B_N) \leq \mathbb{E}(NB^\star) \leq [\mathbb{E}(N^p)]^{\frac{1}{p}}[\mathbb{E}((B^\star)^q)]^{\frac{1}{q}}$ for any $\frac{1}{p} + \frac{1}{q} = 1$, taking $q = 1 + \epsilon$ gives $\mathbb{E}((B^\star)^{1+\epsilon}) = \infty$.
- S_n forms a martingale and $B_n \leq S_{n-1}$ and therefore

$$\mathbb{P}((B^\star)^{1-\epsilon} > \lambda) = \mathbb{P}(B^\star > \lambda^{\frac{1}{1-\epsilon}}) \leq \mathbb{P}(\max_{1 \leq i \leq N} S_i > \lambda^{\frac{1}{1-\epsilon}}) \leq \frac{S_0}{\lambda^{\frac{1}{1-\epsilon}}}$$

The last step uses Doob's maximal inequality, therefore

$$\mathbb{E}((B^\star)^{1-\epsilon}) = \int_0^\infty \mathbb{P}((B^\star)^{1-\epsilon} > \lambda) d\lambda < \infty.$$

Solving the case $p < \frac{1}{2}$ and $p > \frac{1}{2}$

With the above result, we are ready to solve the case where $p < \frac{1}{2}$ and $p > \frac{1}{2}$, the idea is change of measure.

Theorem (Han & W., 2018.)

For any initial list L_0, we have:

- When $p > \frac{1}{2}$, $\mathbb{E}(B^\star) < \infty$,
- When $p > \frac{1}{2}$, $\mathbb{E}(B^\star) = \infty$.

Proof for $p > \frac{1}{2}$

Fix any $p > \frac{1}{2}$, let P be the probability measure over the betting process under winning probability p, and Q be the counterpart under winning probability $\frac{1}{2}$. Note that for any sample path ω with stopping time $N = n$, there must be $\frac{n}{3} + c$ wins and $\frac{n}{3} - c$ losses, where c is a constant depending only on the initial length l_0.

Proof for $p > \frac{1}{2}$

As a result, the likelihood ratio is

$$\frac{dP}{dQ}(\omega) = \frac{p^{\frac{n}{3}+c}(1-p)^{\frac{2n}{3}-c}}{2^{-n}} = \left(\frac{p}{1-p}\right)^c \cdot \left(\frac{p(1-p)^2}{\frac{1}{2}(1-\frac{1}{2})^2}\right)^{\frac{n}{3}} \leq Cr^n$$

where C is a constant only depending on l_0, r is a constant less than 1. As a result,

$$\mathbb{E}_P[B^\star] = \mathbb{E}_Q\left[B^\star \cdot \frac{dP}{dQ}\right] \leq C \cdot \mathbb{E}_Q[r^N B^\star].$$

Since $B^\star \leq 2^N S_0$, therefore

$$\mathbb{E}_Q[r^N B^\star] \leq S_0^\epsilon \cdot \mathbb{E}_Q[(r2^\epsilon)^N \cdot (B^\star)^{1-\epsilon}].$$

Choosing $\epsilon > 0$ small enough such that $r2^\epsilon < 1$ gives us

$$\mathbb{E}_P[B^\star] < \infty.$$

Solving the case $p = \frac{1}{2}$

Theorem (Han & W., 2018.)

When $p = \frac{1}{2}$, for any initial list L_0, we have:

$$\mathbb{E}(B^{\star}) = \infty.$$

Solving the case $p = \frac{1}{2}$

Sketch of Proof:

Assuming that the expectation is finite. Notice that for a fixed target t (say $t = 1$), a fixed initial length d , all the possible initial lists forms a simplex on $\mathbb{R}^d$, i.e.

$$L_0 \in \Delta_d(t) = \{(x_1, \cdots, x_d) | \sum_{i=1}^{d} x_i = S_0, x_i \geq 0, \forall i\}.$$

Each L_0 corresponds to a (finite) expectation of largest betting size

$$\mathbb{E}_{L_0}(B^\star).$$

We define the 'optimal' list with length d be the initial list which attains the minimum of the expectation above, and define

$$f(t, d) \doteq \min_{L_0 \in \Delta_d(t)} \mathbb{E}_{L_0}(B^\star).$$

Solving the case $p = \frac{1}{2}$

Sketch of Proof:

Then we have the following properties of $f(t, d)$

- For any general list system, $f(t, d) = ta_d$, here $a_d = f(1, d)$,
- For any general list system, the sequence a_d is strictly decreasing, i.e.:

$$a_1 > a_2 > a_3 > \cdots$$

- For Labouchère System, there exists $\nu > 1$ such that

$$a_l - a_{l+2} > \nu(a_{l-2} - a_l)$$

Therefore we have $a_d \to -\infty$ as d goes to ∞, contradiction.

References

Yanjun Han, Guanyang Wang (2018)
Expectation of the Largest Betting Size in Labouchère System
arXiv:1807.11729

Stewart Ethier (2008)
Absorption time distribution for an asymmetric random walk
Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz. 31–40

Geoffrey Grimmett, David Stirzaker (2001)
One thousand exercises in probability.
Oxford University Press

Thanks!