Question: Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and $x^3(y^3 + z^3) = 2012(xyz + 2)$.

Solution: First note that x divides $2012 \cdot 2 = 2^3 \cdot 503$. If $503 \mid x$ then the right-hand side of the equation is divisible by 503^3, and it follows that $503^2 \mid xyz + 2$. This is false as $503 \mid x$. Hence $x = 2^m$ with $m \in \{0,1,2,3\}$. If $m \geq 2$ then $2^6 \mid 2012(xyz + 2)$. However the highest powers of 2 dividing 2012 and $xyz + 2 = 2^m yz + 2$ are 2^2 and 2^1 respectively. So $x = 1$ or $x = 2$, yielding the two equations

$$y^3 + z^3 = 2012(yz + 2),$$
$$y^3 + z^3 = 503(yz + 1)$$

In both cases …… It follows that $y \equiv -z(mod\ 503)$ as claimed. Therefore $y + z = 503k$ with $k \geq 1$. In view of $y^3 + z^3 = (y + z)((y - z)^2 + yz)$ the two equations take the form

$$k(y - z)^2 + (k - 4)yz = 8 \quad (1)$$
$$k(y - z)^2 + (k - 1)yz = 1 \quad (2)$$

In (1) we have $(k - 4)yz \leq 8$, which implies k ≤ 4 ……
Therefore (1) has no integer solutions.
Equation (2) implies $0 \leq (k - 1)yz \leq 1$, so that $k = 1$ or $k = 2$.
Also $0 \leq k(y - z)^2 \leq 1$, hence $k = 2$ only if $y = z$. However then $y = z = 1$, which is false in view of $y + z \geq 503$.
Therefore $k = 1$ and (2) takes the form $(y - z)^2 = 1$, yielding $z - y = |y - z| = 1$. Combined with $k = 1$ and $y + z = 503k$, this leads to $y = 251, z = 252$.
In summary the triple $(2,251,252)$ is the only solution.

Final answer: $(2,251,252)$

Subfield: Number theory

Answer type: Tuple

Question type: Open-ended