Delta method and asymptotic variances

Recall that the asymptotic variance of an estimator $\widehat{\theta}$ for a parameter θ is defined as $V(\widehat{\theta})$, if

$$\sqrt{n}(\widehat{\theta} - \theta) \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{N}(0, V(\widehat{\theta})).$$

The arguments that we use to establish asymptotic normality are often the same in our setups, namely the Law of Large Numbers, the Central Limit Theorem, and the Delta Method. First, we review the assumptions and statements of those theorems:

Let $X_1, X_2 \ldots,$ be random variables.

The (Weak) Law of Large Numbers

This says that with

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

we have

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1].$$

providing that $\mathbb{E}[X_i]$ are all finite and that the X_i are i.i.d.

The Central Limit Theorem

The Central Limit Theorem states that, under some assumptions, there is V such that

$$\sqrt{n}(\overline{X}_n - \mathbb{E}[X_1]) \xrightarrow{(\mathrm{D})} \mathcal{N}(0, V).$$

The assumptions are that $\mathbb{E}[|X_i|]$ *and* $\mathsf{Var}(X_i)$ are finite for all i, and that the X_i are i.i.d.

The Delta Method

The Delta Method gives us a way to determine the asymptotic variance of a transformation of a random variable whose asymptotic variance we do know.

Let $\theta \in \mathbb{R}$ be a parameter and $Z_n \in \mathbb{R}$ be a sequence of random variables that satisfies

$$\sqrt{n}(Z_n - \theta) \xrightarrow[n\to\infty]{(\mathrm{d})} \mathcal{N}(0, V)$$

for some $V > 0$.
Given a function $g : \Omega \subseteq \mathbb{R} \to \mathbb{R}$,

$$\sqrt{n}(g(Z_n) - g(\theta)) \xrightarrow[n\to\infty]{\text{(d)}} \mathcal{N}(0, g'(\theta)^2 V).$$

providing $g(\theta)$ is continuously differentiable at θ.

Delta Method applied to a Poisson Statistical Model

Argue that the proposed estimators $\widehat{\lambda}$ and $\widetilde{\lambda}$ below are both consistent and asymptotically normal. Then, give their asymptotic variances $V(\widehat{\lambda})$ and $V(\widetilde{\lambda})$, and decide if one of them is always bigger than the other.

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathsf{Poiss}(\lambda)$, for some $\lambda > 0$. Let $\widehat{\lambda} = \overline{X}_n$ and $\widetilde{\lambda} = -\ln(\overline{Y}_n)$, where $Y_i = \mathbf{1}\{X_i = 0\}, i = 1, \ldots, n$.

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For $\widehat{\lambda}$, by the Law of Large Numbers,

$$\widehat{\lambda} = \overline{X}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[X_1] = \lambda$$

By the Central Limit Theorem,

$$\sqrt{n}(\overline{X}_n - \lambda) \sim \mathcal{N}(0, \mathsf{Var}(X_1)) = \mathcal{N}(0, \lambda)$$

Hence

$$V(\widehat{\lambda}) = \lambda$$

For $\widetilde{\lambda}$, first observe that by the Law of Large Numbers,

$$\overline{Y}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[\overline{Y}_1] = \mathbb{P}(\overline{X}_1 = 1) = e^{-\lambda},$$

So with $g(t) = -\log(t)$,

$$\widetilde{\lambda} = g(\overline{Y}_n) \xrightarrow[n\to\infty]{\mathbf{P}} g(e^{-\lambda}) = \lambda.$$

Hence we see that both $\widehat{\lambda}$ and $\widetilde{\lambda}$ are consistent estimators of λ.
Now, $\overline{Y}_n$ is a Bernoulii random variable with parameter $e^{-\lambda}$, so by the Central Limit Theorem

$$\sqrt{n}(\overline{Y}_n - \mathbb{E}[Y_1]) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, e^{-\lambda}(1 - e^{-\lambda})\right)$$

Since $\widetilde{\lambda} = -\log(\overline{Y}_n) = g(\overline{Y}_n)$ is a function $\overline{Y}_n$, whose variance we know, we can use the Delta method to find the variance of $\widetilde{\lambda}$. To do this we compute

$$g'(t) = -\frac{1}{t}, \quad g'(e^{-\lambda}) = -e^{\lambda}, \quad \left(g'(e^{-\lambda})\right)^2 = e^{2\lambda},$$

which results in

$$\sqrt{n}\left(g(\overline{Y}_n) - g(e^{-\lambda})\right) = \sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow[n\to\infty]{\text{(d)}} \mathcal{N}\left(0, \left(g'(e^{-\lambda})\right)^2 \left(e^{\lambda} - 1\right)\right) = \mathcal{N}(0, e^{\lambda} - 1).$$

The Delta Method applied to an Exponential Statistical Model

As above argue that the proposed estimators $\widehat{\lambda}$ and $\widetilde{\lambda}$ below are both consistent and asymptotically normal. Then, give their asymptotic variances $V(\widehat{\lambda})$ and $V(\widetilde{\lambda})$, and decide if one of them is always bigger than the other.
Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathsf{Exp}(\lambda)$ for some $\lambda > 0$. Let $\widehat{\lambda} = \frac{1}{\overline{X}_n}$ and let $\widetilde{\lambda} = -\log(\overline{Y}_n)$ where $Y_i = \mathbf{1}\{X_i > 1\}, i = 1, \ldots, n,$
For $\widehat{\lambda}$ we have,

$$\overline{X}_n \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1] = \frac{1}{\lambda}$$

Hence by the continuous mapping theorem, with $g(t) = 1/t$ we have that

$$\widehat{\lambda} = \frac{1}{\overline{X}_n} \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\mathbb{E}[X_1]} = \lambda.$$

By the Central Limit theorem,

$$\sqrt{n}(\overline{X}_n - \frac{1}{\lambda}) \sim \mathcal{N}(0, \mathsf{Var}(X_1)) = \mathcal{N}\left(0, \frac{1}{\lambda^2}\right).$$

but what we want is

$$\sqrt{n}(\widehat{\lambda} - \lambda) \sim \mathcal{N}(0, \mathsf{Var}(\widehat{\lambda})) = \mathcal{N}(0, \lambda)$$

To find $\mathsf{Var}(\widehat{\lambda})$ we use the Delta method. Now,

$$g'(t) = \frac{-1}{t^2} \text{ so } \left(g'\left(\frac{1}{\lambda}\right)\right)^2 = \lambda^4.$$

Hence, application of the Delta Method gives us

$$V(\widehat{\lambda}) = \left(g'\left(\frac{1}{\lambda}\right)\right)^2 \cdot \mathsf{Var}(X_1) = \lambda^4 \frac{1}{\lambda^2} = \lambda^2$$

For $\widetilde{\lambda}$, first observe that it is the average of Bernoulli variables, and by the Law of Large Numbers,

$$\overline{Y}_n \xrightarrow[n\to\infty]{\mathbf{P}} \mathbb{E}[Y_1] = \mathbf{P}(X_1 > 1) = \exp(-\lambda),$$

so with $\tilde{g}(t) = -\log(t)$,

$$\tilde{\lambda} = \tilde{g}(\overline{Y}_n) \xrightarrow[n\to\infty]{\mathbf{P}} \tilde{g}(\exp(-\lambda)) = \lambda.$$

which results in

$$\sqrt{n}(\tilde{\lambda} - \lambda) \xrightarrow[n\to\infty]{\text{(d)}} \mathcal{N}(0, \exp(\lambda) - 1).$$