EP 501 Midterm exam: Chapters 1,3-5

November 12, 2019

Instructions:

- Answer all questions
- You may use the course textbook during this exam
- You may log into your computer and use Matlab.
- You may use *your own* course notes for this exam
- You may use my course notes.
- You may access the course Canvas site during this exam: `https://erau.instructure.com/courses/101937`
- You may visit the course repository: `https://github.com/mattzett/EP501`
- You *may not* use an internet browser to access search capabilities and internet references.

1. Use the Taylor Series method (§5.4 in the textbook) to develop derivatives for a *nonuniform* (unequally spaced) grid consisting of the three points x_{i-1}, x_i, x_{i+1} and function values at those points f_{i-1}, f_i, f_{i+1} defined by:

$$
\begin{aligned}
x_{i-1} &= x_i - \Delta x_b && (1)\\
x_{i+1} &= x_i + \Delta x_f && (2)\\
(\Delta x_b &\neq \Delta x_f) && (3)\\
f_{i-1} &= f(x_i - \Delta x_b) && (4)\\
f_{i+1} &= f(x_i + \Delta x_f) && (5)
\end{aligned}
$$

(a) Develop a *centered* difference approximation for the first derivative with respect to x at the i^{th} grid point, i.e. derive an approximation for:

$$f'(x_i) = \left[\frac{df}{dx}\right]_i \tag{6}$$

$$f(x + \Delta x_f) = f(x_i) + \Delta x_f f^{'}(x_i) + \frac{\Delta x_f^2}{2} f^{''}(x_i) + ...$$

$$f(x - \Delta x_b) = f(x_i) - \Delta x_b f^{'}(x_i) + \frac{\Delta x_b^2}{2} f^{''}(x_i) + ...$$

$$f(x + \Delta x_f) - f(x - \Delta x_b) = (\Delta x_f + \Delta x_b) f^{'}(x_i) + \frac{1}{2}(\Delta x_f^2 - \Delta x_b^2) f^{''}(x_i) + ...$$

$$\boxed{f^{'} = \frac{f(x + \Delta x_f) - f(x - \Delta x_b)}{\Delta x_f + \Delta x_b} = \frac{f_{i+1} - f_{i-1}}{\Delta x_f + \Delta x_b}}$$

(b) Show that the truncation error for your finite difference formula is $\mathcal{O}(\Delta x_f - \Delta x_b)$ Truncated Part of $f^{'}$:

$$\frac{\frac{1}{2}(\Delta x_f^2 - \Delta x_b^2) f^{''}(x_i)}{\Delta x_f + \Delta x_b} = \frac{\frac{1}{2}((\Delta x_f + \Delta x_b)(\Delta x_f - \Delta x_b)) f^{''}(x_i)}{\Delta x_f + \Delta x_b}$$

$$= \frac{1}{2}(\Delta x_f - \Delta x_b) f^{''}(x_i)$$

Therefore, truncation error is $\boxed{\mathcal{O}(\Delta x_f - \Delta x_b)}$

(c) Obtain an approximate second derivative:

$$f''(x_i) = \left[\frac{d^2 f}{dx^2}\right]_i \tag{7}$$

by iteratively applying your first derivative formula derived from part (a), e.g.:

$$\left[\frac{d^2 f}{dx^2}\right]_i \approx \frac{\left[\frac{df}{dx}\right]_{i+1/2} - \left[\frac{df}{dx}\right]_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} \tag{8}$$

$$x_{i+\frac{1}{2}} = x_i + \frac{1}{2}\Delta x_f$$
$$x_{i-\frac{1}{2}} = x_i - \frac{1}{2}\Delta x_b$$
$$x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = \frac{1}{2}(\Delta x_f + \Delta x_b)$$

$$\left[\frac{df}{dx}\right]_{i+1/2} = \frac{f_{i+\frac{3}{2}} - f_{i-\frac{1}{2}}}{\Delta x_f + \Delta x_b}$$

$$\left[\frac{df}{dx}\right]_{i-1/2} = \frac{f_{i+\frac{1}{2}} - f_{i-\frac{3}{2}}}{\Delta x_f + \Delta x_b}$$

$$\left[\frac{d^2 f}{dx^2}\right]_i = \frac{\frac{f_{i+\frac{3}{2}} - f_{i-\frac{1}{2}}}{\Delta x_f + \Delta x_b} - \frac{f_{i+\frac{1}{2}} - f_{i-\frac{3}{2}}}{\Delta x_f + \Delta x_b}}{\Delta x_f + \Delta x_b}$$

$$\boxed{\left[\frac{d^2 f}{dx^2}\right]_i = \frac{2\left[f_{i+\frac{3}{2}} + f_{i-\frac{3}{2}} - f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right]}{(\Delta x_f + \Delta x_b)^2}}$$

2. Derive and explain matrix condition numbers.

   (a) Use a method similar to that presented in §1.6.3.2 to show that the condition number relates variations in the solution vector $\underline{b}$ to variations in the matrix $\underline{\underline{A}}$ via the formula:

$$\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} \leq \mathcal{C}(\underline{\underline{A}}) \frac{\left\|\delta \underline{\underline{A}}\right\|}{\left\|\underline{\underline{A}}\right\|} \tag{9}$$

   where $\mathcal{C}(\underline{\underline{A}})$ is the condition number of the matrix $\underline{\underline{A}}$.

$$\underline{\underline{A}}\underline{x} = \underline{b}$$

$$(\underline{\underline{A}} + \delta \underline{\underline{A}})\underline{x} = (\underline{b} + \delta \underline{b})$$

   Subtract first equation from second equation:

$$\delta \underline{\underline{A}}\underline{x} = \delta \underline{b}$$

   Multiply both sides by $\underline{\underline{A}}^{-1}$:

$$\underline{\underline{A}}^{-1}\underline{x} = \underline{\underline{A}}^{-1}\delta \underline{b}$$

$$\delta \underline{b} = \underline{\underline{AA}}^{-1}\delta \underline{x}$$

$$C(A) = \underline{\underline{AA}}^{-1}$$

   Apply Schwarz inequality and divide both sides by $\|\delta \underline{b}\|$:

$$\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} \leq \mathcal{C}(\underline{\underline{A}}) \frac{\left\|\delta \underline{\underline{A}}\right\| |\underline{x}|}{\|\underline{b}\|}$$

   Multiply right side by $\frac{\left|\left|\underline{\underline{A}}\right|\right|}{\left|\left|\underline{\underline{A}}\right|\right|}$:

$$\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} \leq \mathcal{C}(\underline{\underline{A}}) \frac{\left\|\delta \underline{\underline{A}}\right\| |\underline{x}|}{\|\underline{b}\|} \frac{\left|\left|\underline{\underline{A}}\right|\right|}{\left|\left|\underline{\underline{A}}\right|\right|}$$

   Assume $\left\|\underline{\underline{A}}\right\| |\underline{x}| = \|\underline{b}\|$

$$\boxed{\frac{\|\delta \underline{b}\|}{\|\underline{b}\|} \leq \mathcal{C}(\underline{\underline{A}}) \frac{\left\|\delta \underline{\underline{A}}\right\|}{\left\|\underline{\underline{A}}\right\|}}$$

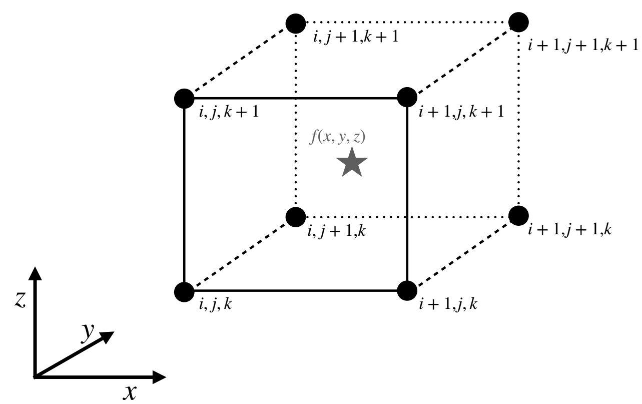
   (b) Provide a brief (several sentence) explanation of the meaning of the condition number $\mathcal{C}(\underline{\underline{A}})$. Include a discussion of what large and small condition numbers mean and why they are undesirable.

   The condition number of $\underline{\underline{A}}$ is a measure of the sensitivity to small changes in the system. If the number is large, then small changes in $\underline{\underline{A}}$ will result in large changes to $\underline{b}$. If the number is small, the system is resistant to small changes in $\underline{\underline{A}}$.

3. *Trilinear* interpolation involves approximation of an underlying three dimensional function using a polynomial of the form:

$$f(x, y, z) \approx a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 xz + a_8 xyz \tag{10}$$

for the region $x_i \leq x \leq x_{i+1}, y_j \leq x \leq y_{j+1}, z_k \leq x \leq z_{k+1}$. Set up a system of equations that can be solved for the coefficients $\underline{a} \equiv a_\ell$ using the value of the function f at the eight points defining the vertices of this cube-shaped region, shown in the diagram below. Express your system of equations in matrix form.



$$\underline{\underline{M}} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 & y_1z_1 & x_1z_1 & x_1y_1z_1 \\ 1 & x_2 & y_2 & x_2y_2 & y_2z_2 & x_2z_2 & x_2y_2z_2 \\ 1 & x_3 & y_3 & x_3y_3 & y_3z_3 & x_3z_3 & x_3y_3z_3 \\ 1 & x_4 & y_4 & x_4y_4 & y_4z_4 & x_4z_4 & x_4y_4z_4 \\ 1 & x_5 & y_5 & x_5y_5 & y_5z_5 & x_5z_5 & x_5y_5z_5 \\ 1 & x_6 & y_6 & x_6y_6 & y_6z_6 & x_6z_6 & x_6y_6z_6 \\ 1 & x_7 & y_7 & x_7y_7 & y_7z_7 & x_7z_7 & x_7y_7z_7 \\ 1 & x_8 & y_8 & x_8y_8 & y_8z_8 & x_8z_8 & x_8y_8z_8 \end{bmatrix}$$

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}, \underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}$$

$$\underline{\underline{M}}\,\underline{a} = \underline{f}$$

4. Suppose we wish to perform a least squares fit (§4.10) to a set of measurements y_i sampled at independent variable locations x_i using the functional form:

$$y(x) = ax^2 + bx^5 \tag{11}$$

Derive a system of equations can can be solved to determine the coefficients a, b. Express your system in matrix form.

No corrections needed, see attached exam.