EP 501 Homework 4: Differentiation, Integration, and Multidimensional functions

November 7, 2019

Instructions:

- Submit all source code and publish Matlab results in .pdf form via Canvas. Please zip all contents of your solution into single file and then submit in a single zip file.
- Discussing the assignment with others is fine, but you must not copy anyone's code.
- I must be able to run your code and produce all results by executing a single top-level Matlab script, e.g. `assignment1.m` or similar.
- You may use any of the example codes from our course repository: `https://github.com/mattzett/EP501/`.
- Do not copy verbatim any other codes (i.e. any source codes other than from our course repository). You may use other examples as a reference but you must write you own programs (except for those I give you).

Purpose of this assignment:

- Deal with numerical differentiation to solve complex problems.
- Develop good coding and documentation practices, such that your programs are easily understood by others.
- Exercise good judgement in numerical problem setup.
- Demonstrate higher reasoning in synthesize a problem and devising a basic algorithm to solve it.

1. Vector derivatives and multidimensional plotting:

   (a) Plot the two components of the vector magnetic field defined by the piecewise function:

$$\mathbf{B}(x,y) = \begin{cases} \frac{\mu_0 I}{2\pi a^2}\sqrt{x^2+y^2}\left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right) & \left(\sqrt{x^2+y^2} < a\right) \\ \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}}\left(-\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_x + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_y\right) & \left(\sqrt{x^2+y^2} \geq a\right) \end{cases} \tag{1}$$

   Use an image plot (`imagesc`, `pcolor`) for each component. Make sure you add a colorbar and axis labels to your plot. You will need to define a range and resolution in x and y, and create a meshgrid from that. Be sure to use a resolution fine enough to resolve important variations in this function.

   (b) Make a quiver plot of the magnetic field; add labels, etc.

   (c) Write a function to compute the curl of a vector field, i.e. $\nabla \times \mathbf{U}$. Compute the three components of the numerical curl of this field and plot using `imagesc`, or `pcolor`

   (d) Compute the curl analytically and plot this alongside your numerical approximation. Demonstrate that they are suitably similar.

   (e) Compute and plot the scalar field:

$$\Phi(x,y) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\left(1+\frac{\alpha\sqrt{x^2+y^2}}{2}\right) \tag{2}$$

   Make sure you avoid having $(x,y) = (0,0)$ in your grid (the function is singular at this point). This is probably best done by implementing a "regulator" that enforces some minimum value for x and y. Be sure to use a resolution fine enough to resolve important variations in this function.

   (e) Write a function to compute the Laplacian of a scalar field, i.e. $\nabla^2\Phi$. Compute the numerical Laplacian of the function:

2. Integration in multiple dimensions.

   (a) Numerically compute the electrostatic energy per unit length, defined by the integral:

$$W_E = -\iint \left(\epsilon_0 \nabla^2 \Phi\right) \Phi dy dx \tag{3}$$

   using a method that you have coded yourself.

   (b) Compute and plot the parametric path

$$\mathbf{r}(\phi) = \cos\phi\ \hat{\mathbf{e}}_x + \sin\phi\ \hat{\mathbf{e}}_y \qquad (0 \leq \phi \leq 2\pi) \tag{4}$$

   in the x, y plane on the same axis as your magnetic field components (plot the path in each). You will need to define a grid in ϕ to do this.

   (c) Plot the two components of the magnetic field at the x, y points along $\mathbf{r}$ and visually compare against your image plots of the magnetic field.

   (d) Numerically compute the tangent vector to the path $\mathbf{r}$ by performing the derivative: $d\mathbf{r}/d\phi$. Compare your numerical results against the analytical derivative (e.g. plot the two) and adjust your grid in ϕ such that you get visually acceptable results.

   (e) Numerically compute the magnetic field integrated around the path $\mathbf{r}$, i.e.:

$$\int_{\mathbf{r}} \mathbf{B} \cdot d\boldsymbol{\ell} \tag{5}$$

   where the differential path length is given by:

$$d\boldsymbol{\ell} = \frac{d\mathbf{r}}{d\phi} d\phi \tag{6}$$