1. $$\varepsilon \frac{d^2\Phi}{dx^2} + \frac{d\varepsilon}{dx}\frac{d\Phi}{dx} = 0$$

$$\varepsilon\left(\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta x^2}\right) + \frac{d\varepsilon}{dx}\left(\frac{\Phi_{i+1} - \Phi_{i-1}}{2\Delta x}\right) = 0$$

$$\Phi_{i+1}\underbrace{\left(\frac{\varepsilon}{\Delta x^2} + \frac{1}{2\Delta x}\frac{d\varepsilon}{dx}\right)}_{\alpha} - \Phi_i\underbrace{\left(\frac{2\varepsilon}{\Delta x^2}\right)}_{\beta} + \Phi_{i-1}\underbrace{\left(\frac{\varepsilon}{\Delta x^2} - \frac{1}{2\Delta x}\frac{d\varepsilon}{dx}\right)}_{\gamma} = 0$$

3. decouple using Jordan decomposition

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -1000 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ +1 & +1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1000 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} -y_1 - 2y_2 \\ y_1 + y_2 \end{bmatrix} = \begin{bmatrix} -1000 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -y_1 - 2y_2 \\ y_1 + y_2 \end{bmatrix}$$

let $v = -y_1 - 2y_2$, $w = y_1 + y_2$

$$\dot{v} = -1000\,v$$

$$\dot{w} = -w$$

$V = -Y_1 - 2Y_2$

$W = Y_1 + Y_2$

$Y_1 = W - Y_2.$

$V = -W + Y_2 - 2Y_2 = -W - Y_2$

$Y_2 = -W - V$

2c. $x'' = w y' (1 + \frac{1}{2} y)$

$y'' = w x' (1 + \frac{1}{2} y)$

decouple by changing variables:

$y = \alpha_1$ $\quad\quad$ $x = \beta_1$

$y' = \alpha_2$ $\quad\quad$ $x' = \beta_2$

$\alpha_1' = y' = \alpha_2$ $\quad\quad$ $\beta_1' = x' = \beta_2$

$\underline{\alpha_2' = w \beta_2 (1 + \frac{1}{2} \alpha_1)}$ $\quad\quad$ $\underline{\beta_2' = w \alpha_2 (1 + \frac{1}{2} \alpha_1)}$

ICs:

$\alpha_1(0) = 0$ $\quad\quad$ $\alpha_2(0) = 1000$

$\beta_1(0) = 0$ $\quad\quad$ $\beta_2(0) = 1000$