Fig.7.1 illustrates this process. Here AB and CD are the two curves whose equations are $y_1 = f_1(x)$ and $y_2 = f_2(x)$. PQ is a vertical strip of width dx.

Then the inner rectangle integral means that the integration is along one edge of the strip PQ from P to Q (x remaining constant), while the outer rectangle integral corresponds to the sliding of the edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

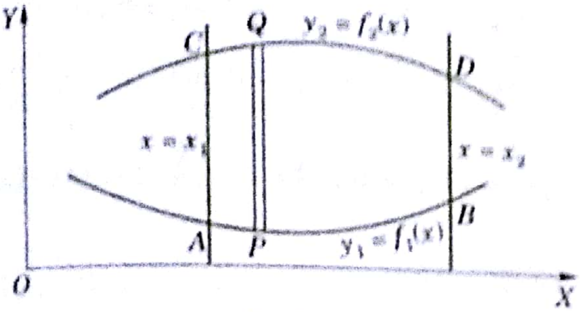

Fig. 7.1

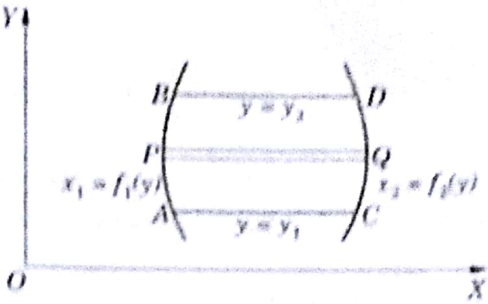

Fig. 7.2

(ii) *When x_1, x_2 are functions of y and y_1, y_2 are constants*, $f(x, y)$ is first integrated w.r.t. x keeping y fixed, within the limits x_1, x_2 and the resulting expression is integrated w.r.t. y between the limits y_1, y_2, *i.e.*,

$$I_2 = \boxed{\int_{y_1}^{y_2} \boxed{\int_{x_1}^{x_2} f(x, y)\, dx}\, dy}$$ which is geometrically illustrated by Fig. 7.2.

Here AB and CD are the curves $x_1 = f_1(y)$ and $x_2 = f_2(y)$. PQ is a horizontal strip of width dy.

Then inner rectangle indicates that the integration is along one edge of this strip from P to Q while the outer rectangle corresponds to the sliding of this edge from AC to BD.

Thus the whole region of integration is the area $ABDC$.

(iii) *When both pairs of limits are constants*, the region of integration is the rectangle $ABDC$ (Fig. 7.3).

In I_1, we integrate along the vertical strip PQ and then slide it from AC to BD.

In I_2, we integrate along the horizontal strip $P'Q'$ and then slide it from AB to CD.

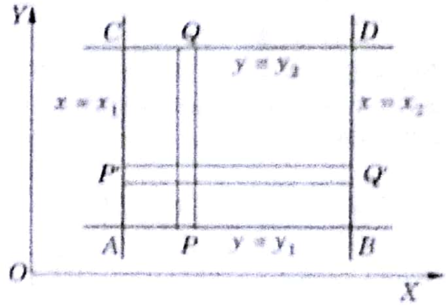

Fig. 7.3

Here obviously $I_1 = I_2$.

Thus *for constant limits, it hardly matters whether we first integrate w.r.t. x and then w.r.t. y or vice versa.*

Example 7.1. *Evaluate* $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dx dy$.

Solution. $$I = \int_0^5 dx \int_0^{x^2} (x^3 + xy^2)\, dy = \int_0^5 \left[x^3 y + x \cdot \frac{y^3}{3} \right]_0^{x^2} dx = \int_0^5 \left[x^3 \cdot x^2 + x \cdot \frac{x^6}{3} \right] dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3} \right) dx = \left| \frac{x^6}{6} + \frac{x^8}{24} \right|_0^5 = 5^6 \left[\frac{1}{6} + \frac{5^2}{24} \right] = 18880.2 \text{ nearly.}$$

Example 7.2. *Evaluate* $\iint_A xy\, dx\, dy$, *where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.* (V.T.U., 2016)

Solution. The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $L(2a, a)$. Figure 7.4 shows the domain A which is the area OML.

Integrating first over a vertical strip PQ, *i.e.*, w.r.t. y from $P(y = 0)$ to $Q(y = x^2/4a)$ on the parabola and then w.r.t. x from $x = 0$ to $x = 2a$, we have

$$\iint_A xy\, dx\, dy = \int_0^{2a} dx \int_0^{x^2/4a} xy\, dy = \int_0^{2a} x \left[\frac{y^2}{2} \right]_0^{x^2/4a} dx$$

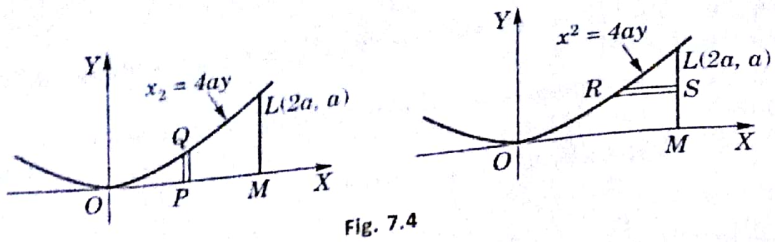

Fig. 7.4

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx = \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a} = \frac{a^4}{3}.$$

Otherwise integrating first over a horizontal strip RS, *i.e.*, w.r.t. x from, R ($x = 2\sqrt{ay}$) on the parabola to S ($x = 2a$) and then w.r.t. y from $y = 0$ to $y = a$, we get

$$\iint_A xy\, dx\, dy = \int_0^a dx \int_{2\sqrt{(ay)}}^{2a} xy\, dx = \int_0^a y\left[\frac{x^2}{2}\right]_{2\sqrt{(ay)}}^{2a} dy$$

$$= 2a\int_0^a (ay - y^2)\, dy = 2a\left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \frac{a^4}{3}.$$

Example 7.3. *Evaluate* $\iint_R x^2\, dx\, dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at A (8, 2) while the line $y = x$ intersects this hyperbola at B (4, 4). Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

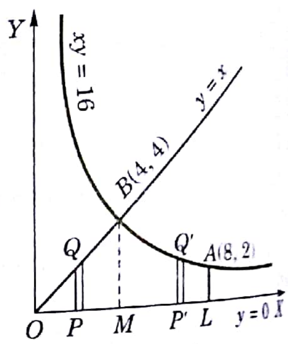

Fig. 7.5

$$\therefore \iint_R x^2\, dxdy = \int_{x \text{ at } O}^{x \text{ at } M}\int_{y \text{ at } P}^{y \text{ at } Q} x^2\, dxdy + \int_{x \text{ at } M}^{x \text{ at } L}\int_{y \text{ at } P'}^{y \text{ at } Q'} x^2\, dxdy$$

$$= \int_0^4\int_0^x x^2\, dxdy + \int_4^8\int_0^{16/x} x^2\, dxdy$$

$$= \int_0^4 x^2\, dx\left|y\right|_0^x + \int_4^8 x^2\, dx\left|y\right|_0^{16/x}$$

$$= \int_0^4 x^3\, dx + \int_4^8 16x\, dx = \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7.2 CHANGE OF ORDER OF INTEGRATION

In a double integral with variable limits, the change of order of integration changes the limit of integration. While doing so, sometimes it is required to split up the region of integration and the given integral is expressed as the sum of a number of double integrals with changed limits. To fix up the new limits, it is always advisable to draw a rough sketch of the region of integration.

The change of order of integration quite often facilitates the evaluation of a double integral. The following examples will make these ideas clear.

Example 7.4. *By changing the order of integration of* $\int_0^\infty\int_0^\infty e^{-xy}\sin px\, dxdy$, *show that*

$$\int_0^\infty \frac{\sin px}{x}\, dx = \frac{\pi}{2}.$$

Solution. $\int_0^\infty\int_0^\infty e^{-xy}\sin px\, dxdy = \int_0^\infty\left(\int_0^\infty e^{-xy}\sin px\, dx\right) dy$

$$= \int_0^\infty \left| \frac{-e^{-xy}}{p^2 + y^2}(p\cos px + y \sin px) \right|_0^\infty dy$$

$$= \int_0^\infty \frac{p}{p^2+y^2}\,dy = \left| \tan^{-1}\left(\frac{y}{p}\right)\right|_0^\infty = \frac{\pi}{2} \qquad ...(i)$$

On changing the order of integration, we have

$$\int_0^\infty \int_0^\infty e^{-xy} \sin px\, dx\, dy = \int_0^\infty \sin px \left\{\int_0^\infty e^{-xy}\,dy\right\} dx$$

$$= \int_0^\infty \sin px \left| \frac{e^{-xy}}{-x} \right|_0^\infty dx = \int_0^\infty \frac{\sin px}{x}\,dx \qquad ...(ii)$$

Thus from (*i*) and (*ii*), we have $\int_0^\infty \frac{\sin px}{x}\,dx = \frac{\pi}{2}$.

Example 7.5. *Change the order of integration in the integral*

$$I = \int_{-a}^{a} \int_0^{\sqrt{(a^2-y^2)}} f(x, y)\, dx\, dy.$$

Solution. Here the elementary strip is parallel to x-axis (such as PQ) and extends from $x = 0$ to $x = \sqrt{(a^2 - y^2)}$ (*i.e.*, to the circle $x^2 + y^2 = a^2$) and this strip slides from $y = -a$ to $y = a$. This shaded semi-circular area is, therefore, the region of integration (Fig. 7.6).

On changing the order of integration, we first integrate w.r.t. y along a vertical strip RS which extends from R $[y = -\sqrt{(a^2 - y^2)}]$ to S $[y = \sqrt{(a^2 - y^2)}]$. To cover the given region, we then integrate w.r.t. x from $x = 0$ to $x = a$.

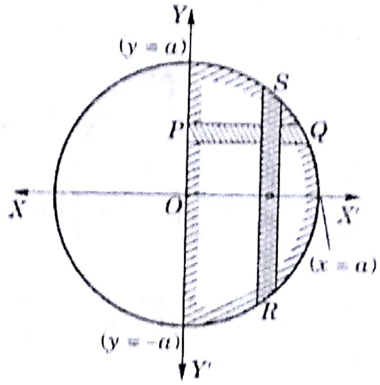

Fig. 7.6

Thus $$I = \int_0^a dx \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)\, dy$$

or $$= \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} f(x, y)\, dydx.$$

Example 7.6. *Evaluate* $\int_0^1 \int_{e^x}^{e} dydx/\log y$ *by changing the order of integration.*

Solution. Here the integration is first w.r.t. y from P on $y = e^x$ to Q on the line $y = e$. Then the integration is w.r.t. x from $x = 0$ to $x = 1$, giving the shaded region ABC (Fig. 7.7).

On changing the order of integration, we first integrate w.r.t. x from R on $x = 0$ to S on $x = \log y$ and then w.r.t. y from $y = 1$ to $y = e$.

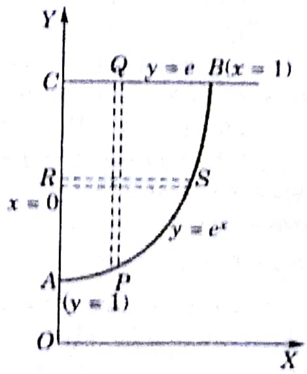

Fig. 7.7

Thus $$\int_0^1 \int_{e^x}^{e} \frac{dydx}{\log y} = \int_1^e \int_0^{\log y} \frac{dxdy}{\log y}$$

$$= \int_1^e \frac{dy}{\log y} \left| x \right|_0^{\log y} = \int_1^e dy = \left| y \right|_1^e = e - 1.$$

Example 7.7. *Change the order of integration in* $I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dydx$ *and hence evaluate.*

(Andhra., 2016 ; Delhi., 2016 ; Nagpur, 2009 ; P.T.U., 2009 S)

Solution. Here integration is first w.r.t. y and P on the parabola $x^2 = 4ay$ to Q on the parabola $y^2 = 4ax$ and then w.r.t. x from $x = 0$ to $x = 4a$ giving the shaded region of integration (Fig. 7.8).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = 4a$

$$\therefore \quad I = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx dy = \int_0^{4a} dy \left| x \right|_{y^2/4a}^{2\sqrt{ay}} = \int_0^{4a} (2\sqrt{ay} - y^2/4a)\, dy$$

$$= \left| 2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{12a} \right|_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}.$$

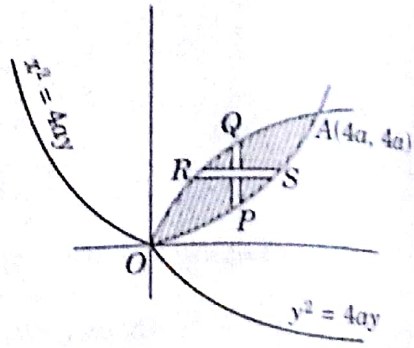

Fig. 7.8

Example 7.8. *Change the order of integration and hence evaluate*

$$I = \int_0^a \int_{\sqrt{ax}}^a \frac{y^2\, dx dy}{\sqrt{(y^4 - a^2x^2)}}$$

(C.S.V.T.U., 2011)

Solution. Here integration is first w.r.t. y from P on the parabola $y^2 = ax$ to Q on the line $y = a$, then w.r.t. x from $x = 0$ to $x = a$, giving the shaded region OAB of integration (Fig. 7.9).

On changing the order of integration, we first integrate w.r.t. x from R to S, then w.r.t. y from $y = 0$ to $y = a$.

$$\therefore \quad I = \int_0^a \int_0^{y^2/a} \frac{y^2\, dy}{\sqrt{(y^4 - a^2x^2)}}\, dx = \frac{1}{a}\int_0^a \int_0^{y^2/a} y^2\, dy \frac{dx}{\sqrt{[(y^2/a)^2 - x^2]}}\, dx$$

$$= \frac{1}{a}\int_0^a y^2 dy \left| \sin^{-1}\left(\frac{xa}{y^2}\right) \right|_0^{y^2/a} = \frac{1}{a}\int_0^a y^2 dy\, [\sin^{-1}(1) - \sin^{-1}(0)]$$

$$= \frac{\pi}{2a}\int_0^a y^2 dy = \frac{\pi}{2a}\left| \frac{y^3}{3} \right|_0^a = \frac{\pi a^2}{6}.$$

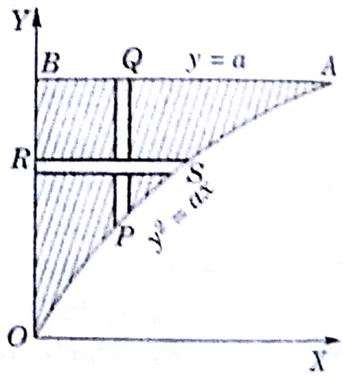

Fig. 7.9

Example 7.9. *Change the order of integration in* $I = \int_0^1 \int_{x^2}^{2-x} xy\, dx dy$ *and hence evaluate the same.*

(Anna, 2016 ; V.T.U., 2016 ; P.T.U., 2013)

Solution. Here the integration is first w.r.t. y along a vertical strip PQ which extends from P on the parabola $y = x^2$ to Q on the line $y = 2 - x$. Such a strip slides from $x = 0$ to $x = 1$, giving the region of integration as the curvilinear triangle OAB (shaded) in Fig. 7.10.

On changing the order of integration, we first integrate w.r.t. x along a horizontal strip $P'Q'$ and that requires the splitting up of the region OAB into two parts by the line AC ($y = 1$), *i.e.*, the curvilinear triangle OAC and the triangle ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = \sqrt{y}$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

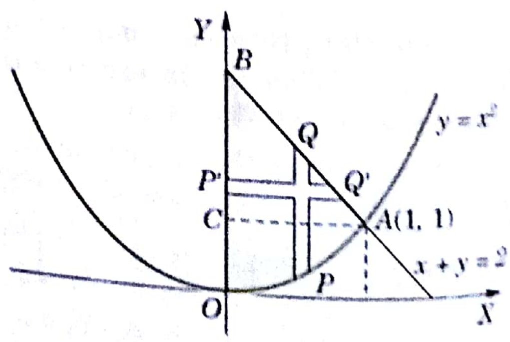

Fig. 7.10

$$I_1 = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx$$

For the region ABC, the limits of integration for x are from $x = 0$ to $x = 2 - y$ and those for y are from $y = 1$ to $y = 2$. So the contribution to I from the region ABC is

$$I_2 = \int_1^2 dy \int_0^{2-y} xy\, dx.$$

Hence, on reversing the order of integration,

$$I = \int_0^1 dy \int_0^{\sqrt{y}} xy\, dx + \int_1^2 dy \int_0^{2-y} xy\, dx$$

$$= \int_0^1 dy \left| \frac{x^2}{2} \cdot y \right|_0^{\sqrt{y}} + \int_1^2 dy \left| \frac{x^2}{2} \cdot y \right|_0^{2-y} = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\, dy = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}.$$

Example 7.10. *Change the order of integration in* $I = \int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx\, dy$ *and hence evaluate it.*

(*Andhra., 2014 ; J.N.T.U., 2005*)

Solution. Here the integration is first w.r.t. y along PQ which extends from P on the line $y = x$ to Q on the circle $y = \sqrt{(2-x^2)}$. Then PQ slides from $y = 0$ to $y = 1$, giving the region of integration OAB as in Fig. 7.11.

On changing the order of integration, we first integrate w.r.t. x from P' to Q' and that requires splitting the region OAB into two parts OAC and ABC.

For the region OAC, the limits of integration for x are from $x = 0$ to $x = 1$ and those for y are from $y = 0$ to $y = 1$. So the contribution to I from the region OAC is

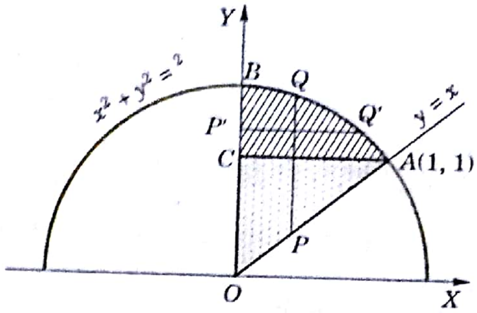

Fig. 7.11

$$I_1 = \int_0^1 dy \int_0^y \frac{x}{\sqrt{(x^2+y^2)}}\, dx.$$

For the region ABC, the limits of integration for x are 0 to $\sqrt{(2-y^2)}$ and these for y are from 1 to $\sqrt{2}$. So the contribution to I from the region ABC is

$$I_2 = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{(2-y^2)}} \frac{x}{\sqrt{(x^2+y^2)}}\, dx$$

Hence $$I = \int_0^1 \left| (x^2+y^2)^{1/2} \right|_0^y dy + \int_1^{\sqrt{2}} \left| (x^2+y^2)^{1/2} \right|_0^{\sqrt{(2-y^2)}} dy$$

$$= \int_0^1 (\sqrt{2}-1)\, y\, dy + \int_1^{\sqrt{2}} \sqrt{(2-y)}\, dy = \frac{1}{2}(\sqrt{2}-1) + \sqrt{2}\sqrt{(2-1)} - \frac{1}{2} = 1 - 1/\sqrt{2}.$$

7.3 DOUBLE INTEGRALS IN POLAR COORDINATES

To evaluate $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$, we first integrate w.r.t. r between limits $r = r_1$ and $r = r_2$ keeping θ fixed and the resulting expression is integrated w.r.t. θ from θ_1 to θ_2. In this integral, r_1, r_2 are functions of θ and θ_1, θ_2 are constants.

Figure 7.12 illustrates the process geometrically.

Here AB and CD are the curves $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$. PQ is a wedge of angular thickness $\delta\theta$.

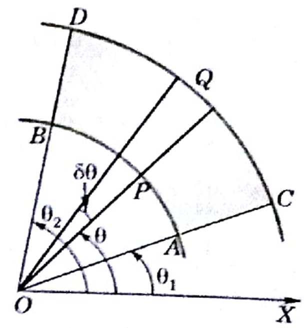

Fig. 7.12

Then $\int_{r_1}^{r_2} f(r, \theta)\, dr$ indicates that the integration is along PQ from P to Q while the integration w.r.t. θ corresponds to the turning of PQ from AC to BD.

Thus the whole region of integration is the area $ACDB$. The order of integration may be changed with appropriate changes in the limits.

Example 7.11. *Evaluate* $\iint r \sin\theta \, dr \, d\theta$ *over the cardioid* $r = a(1 - \cos\theta)$ *above the initial line.*

(*Andhra., 2016 ; Kerala, 2005*)

Solution. To integrate first w.r.t. r, the limits are from 0 $(r = 0)$ to P $[r = a(1 - \cos\theta)]$ and to cover the region of integration R, θ varies from 0 to π (Fig. 7.13).

$$\therefore \quad \iint_R r \sin\theta \, dr d\theta = \int_0^\pi \sin\theta \left[\int_0^{r = a(1-\cos\theta)} r dr\right] d\theta$$

$$= \int_0^\pi \sin\theta \, d\theta \left|\frac{r^2}{2}\right|_0^{a(1-\cos\theta)} = \frac{a^2}{2}\int_0^\pi (1-\cos\theta)^2 \, . \sin\theta \, d\theta$$

$$= \frac{a^2}{2}\left|\frac{(1-\cos\theta)^3}{3}\right|_0^\pi = \frac{a^2}{2} \cdot \frac{8}{3} = \frac{4a^2}{3}.$$

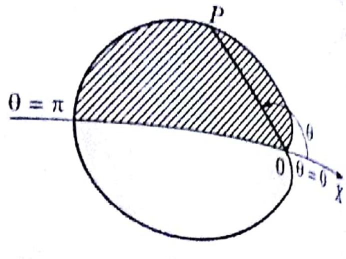

Fig. 7.13

Example 7.12. *Calculate* $\iint r^3 \, dr \, d\theta$ *over the area included between the circles* $r = 2\sin\theta$ *and* $r = 4\sin\theta$

Solution. Given circles $r = 2\sin\theta$ ...(*i*)

and $r = 4\sin\theta$ ...(*ii*)

are shown in Fig. 7.14. The shaded area between these circles is the region of integration.

If we integrate first w.r.t. r, then its limits are from $P(r = 2\sin\theta)$ to $Q(r = 4\sin\theta)$ and to cover the whole region θ varies from 0 to π. Thus the required integral is

$$I = \int_0^\pi d\theta \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr = \int_0^\pi d\theta \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta}$$

$$= 60\int_0^\pi \sin^4\theta \, d\theta = 60 \times 2\int_0^{\pi/2} \sin^4\theta \, d\theta = 120 \times \frac{3 \, . \, 1}{4 \, . \, 2} \cdot \frac{\pi}{2} = 22.5\,\pi.$$

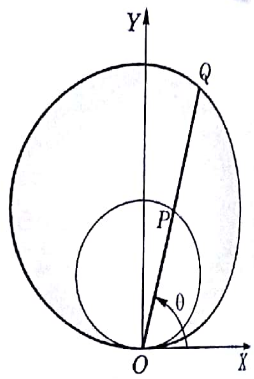

Fig. 7.14

PROBLEMS 7.1

Evaluate the following integrals (1–7) :

1. $\int_1^2 \int_y^{y^2} dxdy$ (*K.T.U., 2016*)

2. $\int_1^2 \int_3^4 (xy + e^y) \, dydx$ .. (*V.T.U., 2013*)

3. $\int_0^1 \int_0^x e^{x/y} \, dxdy$. (*P.T.U., 2005*)

4. $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dydx}{1 + x^2 + y^2}$ (*Rajasthan, 2005*)

5. $\iint xy \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. (*Anna., 2016 ; Rohtak, 2011 S ; Rajasthan, 2006*)

6. $\iint (x + y)^2 dxdy$ over the area bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$. (*U.K.T.U., 2013 ; Kurukshetra, 2009 S*)

7. $\iint xy(x + y) \, dxdy$ over the area between $y = x^2$ and $y = x$. (*V.T.U., 2015*)

Evaluate the following integrals by changing the order of integration (8–15) :

8. $\int_0^a \int_y^a \frac{xdxdy}{x^2 + y^2}$. (*Bhopal, 2008*)

9. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y) \, dxdy$. (*Andhra., 2016 ; V.T.U., 2005*)

10. $\int_0^1 \int_x^{\sqrt{(2-x^2)}} \frac{x\,dydx}{\sqrt{(x^2+y^2)}}$ (Kurukshetra, 2013 ; P.T.U., 2010 ; Marathwada, 2008)

11. $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log(x^2+y^2)\,dxdy\ (a>0)$. (C.S.V.T.U., 2015)

12. $\int_0^1 \int_x^{\sqrt{x}} xy\,dydx$. (V.T.U., 2010)

13. $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} xy\,dx\,dy$ (Anna, 2009)

14. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dydx$. (C.S.V.T.U., 2012 ; P.T.U., 2011 ; Bhopal, 2009)

15. $\int_0^\infty \int_0^x xe^{-x^2/y}\,dydx$. (U.P.T.U., 2013 ; Rohtak, 2012 ; C.S.V.T.U., 2008)

16. Sketch the region of integration of the following integrals and change the order of integrations,

(i) $\int_0^{2a} \int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x,y)\,dxdy$ (J.N.T.U., 2015 ; Rajasthan, 2006) (ii) $\int_a^{ae^{\pi/4}} \int_{2\log(r/a)}^{\pi/2} f(r,\theta)\,r\,drd\theta$.

17. Show that $\iint_R r^2 \sin\theta\,drd\theta = 2a^3/3$, where R is the semi-circle $r = 2a\cos\theta$ above the initial line. (Andhra, 2013)

18. Evaluate $\iint \frac{r\,drd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2 = a^2\cos 2\theta$. (Rohtak, 2006 S ; P.T.U., 2005)

19. Evaluate $\iint r^3\,drd\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

(Anna, 2009 ; Madras, 2006)

7.4 AREA ENCLOSED BY PLANE CURVES

(1) *Cartesian coordinates.*

Consider the area enclosed by the curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = x_1$, $x = x_2$ [Fig. 7.15 (a)].

Divide this area into vertical strips of width δx. If $P(x, y)$, $Q(x + \delta x, y + \delta y)$ be two neighbouring points, then the area of the small rectangle $PQ = \delta x \delta y$.

$\therefore$ area of strip $KL = \underset{\delta y \to 0}{\text{Lt}} \Sigma\, \delta x\, \delta y$.

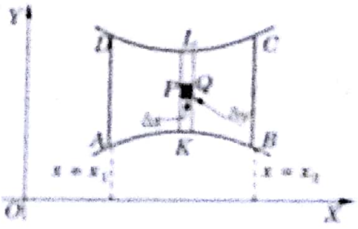

Fig. 7.15(a)

Since for all rectangles in this strip δx is the same and y varies from $y = f_1(x)$ to $y = f_2(x)$.

$$\therefore \text{ area of the strip } KL = \delta x \underset{\delta y \to 0}{\text{Lt}} \sum_{f_1(x)}^{f_2(x)} dy = \delta x \int_{f_1(x)}^{f_2(x)} dy.$$

Now adding up all such strips from $x = x_1$ to $x = x_2$, we get the area $ABCD$

$$= \underset{\delta x \to 0}{\text{Lt}} \sum_{x_1}^{x_2} \delta x \cdot \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dxdy$$

Similarly, dividing the area $A'B'C'D'$ [Fig. 7.15(b)] into horizontal strips of width δy, we get the area $A'B'C'D'$.

$$= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dxdy$$

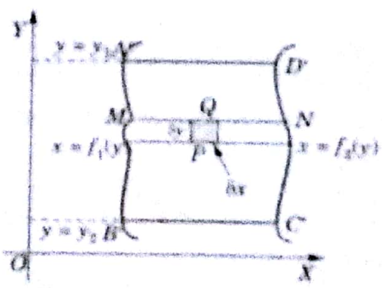

Fig. 7.15 (b)

(2) *Polar coordinates.*

Consider an area A enclosed by a curve whose equation is in polar coordinates.

Let $P(r, \theta)$, $Q(r + \delta r, \theta + \delta\theta)$ be two neighbouring points. Mark circular areas of radii r and $r + \delta r$ meeting OQ in R and OP (produced) in S (Fig. 7.16).

Since arc $PR = r\delta\theta$ and $PS = \delta r$.

$\therefore$ area of the curvilinear rectangle $PRQS$ is approximately $= PR \cdot PS = r\delta\theta \cdot \delta r$.

If the whole area is divided into such curvilinear rectangles, the sum $\Sigma\Sigma r\delta\theta\delta r$ taken for all these rectangles, gives in the limit the area A.

Hence $$A = \underset{\substack{\delta r \to 0 \\ \delta\theta \to 0}}{\text{Lt}} \Sigma\Sigma r\delta\theta\delta r = \iint r\,d\theta dr$$

where the limits are to be so chosen as to cover the entire area.

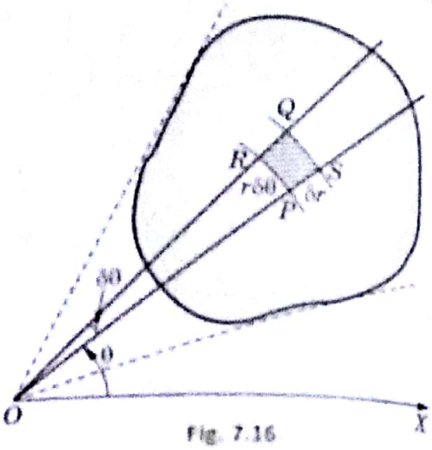

Fig. 7.16

Example 7.13. *Find the area of a plate in the form of a quadrant of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$ (*Anna, 2012*)

Solution. Dividing the area into vertical strips of width δx, y varies from $K(y = 0)$ to $L[y = b\sqrt{(1 - x^2/b^2)}]$ and then x varies from 0 to a (Fig. 7.17).

$\therefore$ required area

$$= \int_0^a dx \int_0^{b\sqrt{(1-x^2/a^2)}} dy = \int_0^a dx\,[y]_0^{b\sqrt{(1-x^2/a^2)}}$$

$$= \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx = \pi ab/4.$$

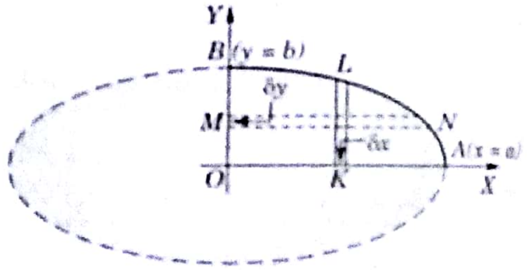

Fig. 7.17

Otherwise, dividing this area into horizontal strips of width δy, x varies from $M(x = 0)$ to $N[x = a\sqrt{(1 - y^2/b^2)}]$ and then y varies from 0 to b.

$\therefore$ required area $$= \int_0^b dy \int_0^{a\sqrt{(1-y^2/b^2)}} dx = \int_0^b dy\,[x]_0^{a\sqrt{(1-y^2/b^2)}}$$

$$= \frac{a}{b}\int_0^b \sqrt{(b^2 - y^2)}\,dy = \pi ab/4.$$

Obs. The change of the order of integration does not in any way affect the value of the area.

Example 7.14. *Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.*

(*V.T.U., 2016 ; Anna, 2013 ; Kerala, 2005*)

Solution. Solving the equations $y^2 = 4ax$ and $x^2 = 4ay$, it is seen that the parabolas intersect at O (0, 0) and A (4a, 4a). As such for the shaded area between these parabolas (Fig. 7.18) x varies from 0 to $4a$ and y varies from P to Q *i.e.*, from $y = x^2/4a$ to $y = 2\sqrt{(ax)}$. Hence the *required area*

$$= \int_0^{4a}\int_{x^2/4a}^{2\sqrt{(ax)}} dydx = \int_0^{4a}(2\sqrt{(ax)} - x^2/4a)\,dx$$

$$= \left| 2\sqrt{a}\cdot\frac{2}{3}x^{3/2} - \frac{1}{4a}\cdot\frac{x^3}{3}\right|_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

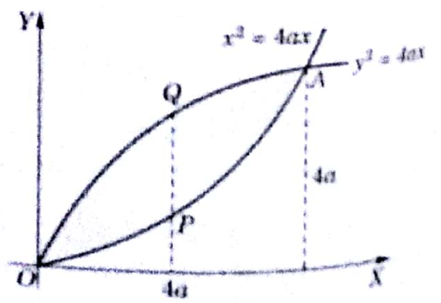

Fig. 7.18

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$\therefore$ required area
$$= 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr d\theta$$
$$= 2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$
$$= a^2\int_0^{\pi/2}(2+\cos^2\theta)\,d\theta = 5\pi a^2/4.$$

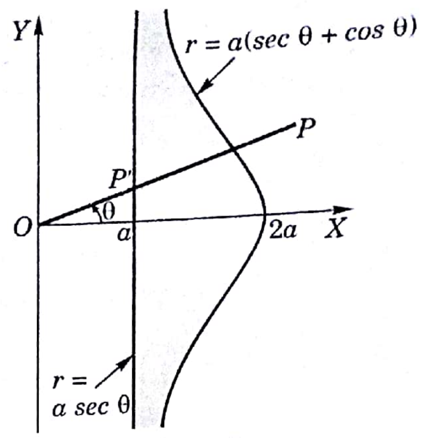

Fig. 7.19

Example 7.16. *Find the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.*

Solution. In Fig. 7.20, $ABODA$ represents the cardioid $r = a(1+\cos\theta)$ and $CBA'DC$ is the circle $r = a$.

Required area (shaded) = 2 (area $ABCA$)
$$= 2\int_0^{\pi/2}\int_{r=OP'}^{r=OP} r\,d\theta\,dr = 2\int_0^{\pi/2}\int_a^{a(1+\cos\theta)}(rdr)\,d\theta$$
$$= 2\int_0^{\pi/2}\left|\frac{r^2}{2}\right|_a^{a(1+\cos\theta)} d\theta = a^2\int_0^{\pi/2}[(1+\cos\theta)^2-1]\,d\theta$$
$$= a^2\int_0^{\pi/2}(\cos^2\theta + 2\cos\theta)\,d\theta = a^2\left(\frac{1}{2}\cdot\frac{\pi}{2}+2\right) = \frac{a^2}{4}(\pi+8).$$

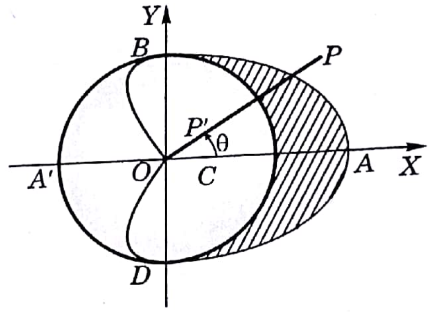

Fig. 7.20

PROBLEMS 7.2

1. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$. *(J.N.T.U., 2015 ; C.S.V.T.U., 2014)*
2. Find the area lying between the parabola $y = x^2$ and the line $x + y - z = 0$. *(Anna, 2009)*
3. By double integration, find the whole area of the curve $a^2x^2 = y^3(2a - y)$.
4. Find, by double integration, the area enclosed by the curves $y = 3x/(x^2+2)$ and $4y = x^2$. *(Andhra., 2013 ; Kurukshetra, 2013)*
5. Find, by double integration, the area of the lemniscate $r^2 = a^2\cos 2\theta$. *(Andhra., 2016)*
6. Find, by double integration, the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$. *(Andhra., 2015 ; Rohtak, 2012 ; P.T.U., 2011 ; Anna 2009)*
7. Find the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1+\cos\theta) = 1$. *(Mumbai, 2007)*
8. Find the area common to the circles $r = a\cos\theta$, $r = a\sin\theta$ by double integration.

7.5 TRIPLE INTEGRALS

Consider a function $f(x, y, z)$ defined at every point of the 3-dimensional finite region V. Divide V into n elementary volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. Let (x_r, y_r, z_r) be any point within the rth sub-division δV_r. Consider the sum
$$\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r .$$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to 0$ is called the *triple integral of $f(x, y, z)$ over the region V* and is denoted by
$$\iiint f(x, y, z)\,dV.$$

For purposes of evaluation, it can also be expressed as the repeated integral
$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\,dxdydz.$$