Thermal and Electrical Waves Experiment

Second Year Laboratory 2022 - 2023

Blackett Laboratory, Physics Department

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Overview of the experiment

Welcome to the 2nd year lab 'Waves' experiment! Over the next 8 sessions you will investigate and learn about the **propagation of waves through different media** and how the propagation of such **waves depends on frequency**.

The experiment is divided into 3 parts:

- In part 1 you will perform **Fourier analysis of a square wave** to understand how it can be deconstructed into harmonics with different frequencies. This will be very helpful in parts 2 and 3. You will also **perform numerical integration** of a mock dataset in preparation for part 2.

- In part 2 (**'Thermal Waves'**) you will investigate the propagation of **temperature oscillations through a solid** on a timescale of minutes. These are the waves that Fourier studied when he developed his famous harmonic analysis technique. Thus, an important aspect of the data analysis is the familiarisation you will gain with the important and generally applicable Fourier-Bessel techniques. In addition, you will gain useful experience of data analysis as you correlate your temperature wave data.

- In part 3 (**'Electrical Waves'**) you will investigate the propagation of **voltage oscillations along a lumped transmission line** on timescales of microseconds. You will gain hands on practical experience of measuring parameters of the transmission line such as characteristic impedance, phase and group velocities, and the effects of impedance matching and mismatching.

Despite the very different timescales involved for the thermal and electrical waves studied, you will find that the distortions in the waveforms which arise due to the frequency dependent attenuations and wave speeds are equivalent and measurable.

Some additional remarks

This script contains all the basic information you will need for the whole experiment. It also contains **additional information in the 'Section 4 - Appendix'** at the end which can be useful if you want to 'dig in' more into the physics behind the experiment.

The experiment has been planned as 8 x 3 hour teaching sessions, however you will need to spend time outside the lab hours to prepare for each session, perform further data analysis, make sure your lab book is up to date, etc.

What is expected of you as a student?

- **That you will go through the script <u>before each lab session</u>**. Each lab session has defined goals (linked to tasks throughout the script), thus spending time reading the script for the first time during the sessions is not a very efficient use of your time.

- **No one expects you to know all the physics and maths presented in the script**. This is a teaching lab and thus we expect that you will learn new things!

- That you will make annotations, write down questions, look up things that you do not fully understand (or that you have never heard of). Where? **In your lab book**.

- That you will talk to your demonstrators. However **always try thinking about a possible answer first**.

What is expected of your demonstrators?

- They will **teach and guide you throughout each session**. They will be inquisitive of your work throughout the experiment. Your demonstrators **are not 'invigilators', they are here to teach you**.

- They will help answering your questions however, most of the times, **they will guide you towards being able to the answer the question yourselves**. If they do not know the answer, they will get back to you with more information.

Part 1: Fourier analysis of a square wave

1.1 Aims

- To understand how a function can be decomposed into a series of sinusoidal functions with different frequencies.
- To perform numerical integration.

1.2 Fourier series

A periodic function in time $T(t)$ with period τ can be expressed as a Fourier series, i.e. an infinite sum of sine and cosine functions:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n}{\tau}\,t\right) + \, b_n \sin\left(\frac{2\pi n}{\tau}\,t\right)\right] . \tag{1.1}$$

where a_n and b_n are constant coefficients (amplitudes) given by the expressions:

$$a_n = \frac{2}{\tau}\int_0^{\tau} T(t)\cos\left(\frac{2\pi n}{\tau}\,t\right)\,dt \tag{1.2}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} T(t)\sin\left(\frac{2\pi n}{\tau}\,t\right)\,dt \tag{1.3}$$

Alternatively, Eq. (1.1) can be re-written in ‘amplitude-phase’ form as:

$$T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\beta_n \sin\left(\frac{2\pi n}{\tau}\,t\, - \,\Delta\phi_n\right) . \tag{1.4}$$

where β_n and $\Delta\phi_n$ are the amplitude and phase lag respectively:

$$\beta_n = \sqrt{{a_n}^2 + {b_n}^2} \tag{1.5}$$

$$\Delta\phi_n = -\arctan\,(a_n/b_n) \tag{1.6}$$

Each component of these series labelled by *n* represents a ‘harmonic’ mode, a sinusoidal function with angular frequency $\omega_n = \frac{2\pi n}{\tau}$.

NOTE: The value of $\Delta\varphi$ in Eq. (1.6) depends on the sign of the argument a_n/b_n. You will use this equation in the Thermal Waves experiment.

✔ **Task 1.1:** Prove the amplitude-phase form of the Fourier series (Eq. (1.4)).

1.3 Fourier analysis of a square wave

You will perform Fourier analysis of a square function by representing it as a Fourier series, i.e. a sum of sinusoidal functions with different frequencies.

You will be using square functions/pulses and looking at these components throughout the Thermal Waves and Electrical Waves parts of this experiment!

Task 1.2: Fourier series

Assume that your square function has a period of τ = 240 with an amplitude between T = [0, 100]. Units are not strictly needed for this exercise, but you can choose to use [s] and [˚C] if you want.

a) Calculate (with 'pen a paper'!) the amplitudes a_n and b_n of the Fourier series up to n = 3 and make a table of your results **for future reference**.

b) Create a dataset that represents a square wave and plot it using e.g. Excel, Python, Google Sheets, etc. Make sure you choose an appropriate number of time divisions (timestep) in your plot.

c) Use your calculated coefficients to plot the Fourier series of the square function using Eq. (1.1) and add it to your plot (see Fig. 1.1). The n = 1 component is known as the 'fundamental frequency' - why?

d) Is your Fourier analysis able to reproduce the square wave? Explain.

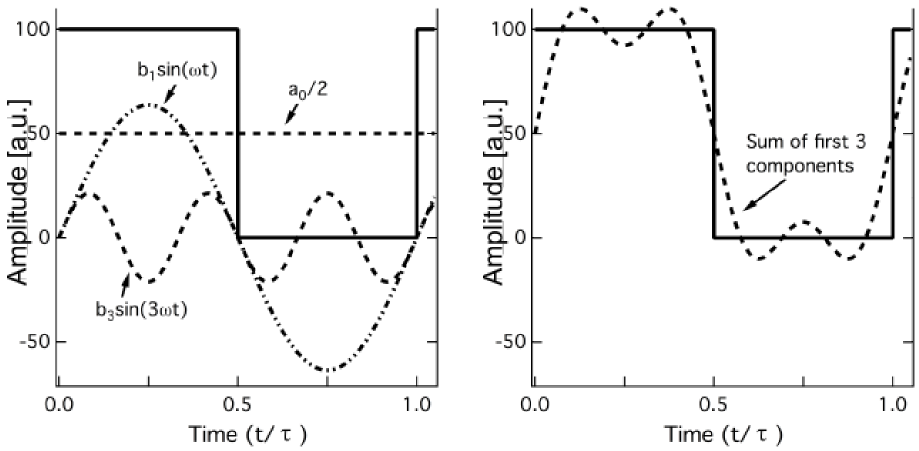


***Fig. 1.1:** Fourier components (harmonics) of a square wave up to n = 3.*

Visit the ImpVis (Imperial Visualisations) website and check out the 'Fourier series' visualisation to see a similar plot to Fig. 1.1 and much more:

https://impvis.co.uk/launch/fourier-series/

1.4 Numerical integration of a test dataset

In this part you will integrate a test function numerically, for instance using the 'rectangle rule' or the 'trapezoid rule'. Most importantly, you will need to graphically (and clearly) show your methods and answers.

Task 1.3: Numerical integration test

Download datasets (as *.txt* files) from Blackboard for either a **semi-circle** or a **sine function**. Each dataset has a 'low resolution' and 'high resolution' versions (see Fig. 1.2). Then:

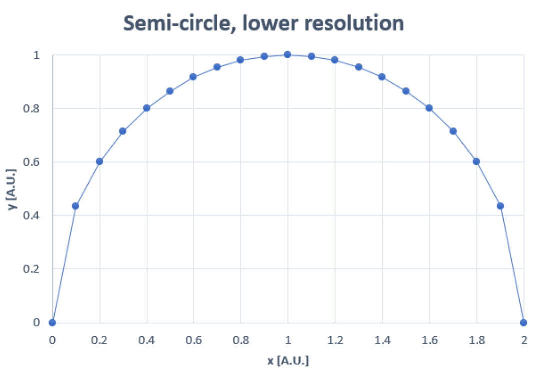
a) Make a sketch of how to implement numerical integration graphically using 'rectangles' or 'trapezoids' for either of your chosen functions.

b) Apply either of your numerical integration methods to your chosen function with lower and higher resolution and show your numerical results. How different are these results to the expected 'analytical' answer?

If you choose to use Python, you can use the following Numpy routine:

```
x1, y1 = numpy.loadtxt(“Task1.3_Semicircle_low.txt”, unpack=True, skiprows=1)
```

If you choose to use a spreadsheet like Excel, you can easily import the file.



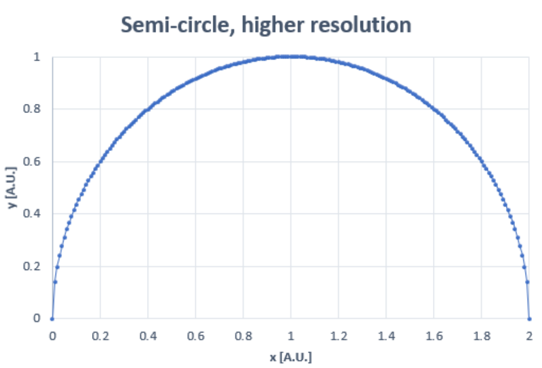


Fig. 1.2: *One of the test functions for numerical integration: a semi-circle with radius 1 and a resolution of 0.1 and 0.01 (in arbitrary units).*

Part 2: Thermal Waves experiment

2.1 Introduction and aims of the experiment

This experiment is an investigation of the propagation, by thermal conduction, of a periodic temperature “wave” through a solid. The propagation is analysed in terms of the Fourier components of an input square temperature wave and, as you will study in the Electrical Waves experiment, the propagation of these components is frequency dependent. The frequencies of the temperature wave are about 0.01 Hz instead of the ~kHz of the voltage waves in the Electrical Waves experiment.

The aims of this experiment include familiarisation with:

- Quantitative ideas of thermal conduction.
- Quantitative use of Fourier and Bessel functions.
- Experience in data analysis.

The overall objective of this experiment is to obtain a best estimate of the thermal diffusivity of PTFE and its associated uncertainty.

2.2 Experiment overview

The thermal diffusivity of a material D depends on its thermal conductivity, specific heat and density. We will assume that the thermal diffusivity is constant over the duration of the experiment.

To measure D, a regular temperature fluctuation is established at one point in a solid and the variation of temperature is measured at another point. The solid in this case is a cylinder of PTFE (poly-tetra-fluoro-ethylene). A regular temperature wave is launched at the outer surface of the cylinder by dipping it alternately in boiling and icy water. Equal times (of the order of minutes) are spent in the hot and cold sources. Thus, a “square” temperature wave going from ~0°C to ~100°C is established at the outer surface of the cylinder. The period of this wave corresponds to the sum of the times spent in the hot and cold baths over one full cycle.

The cylinder takes some time to heat and cool as the wave of temperature variation propagates radially inwards. A concentric central hole in the cylinder contains a small electronic thermometer which has a low heat capacity, thus responding rapidly to the temperature on the walls of the central hole. This thermometer is connected to a computer so that the inner wall temperature can be measured during the cyclic temperature changes.

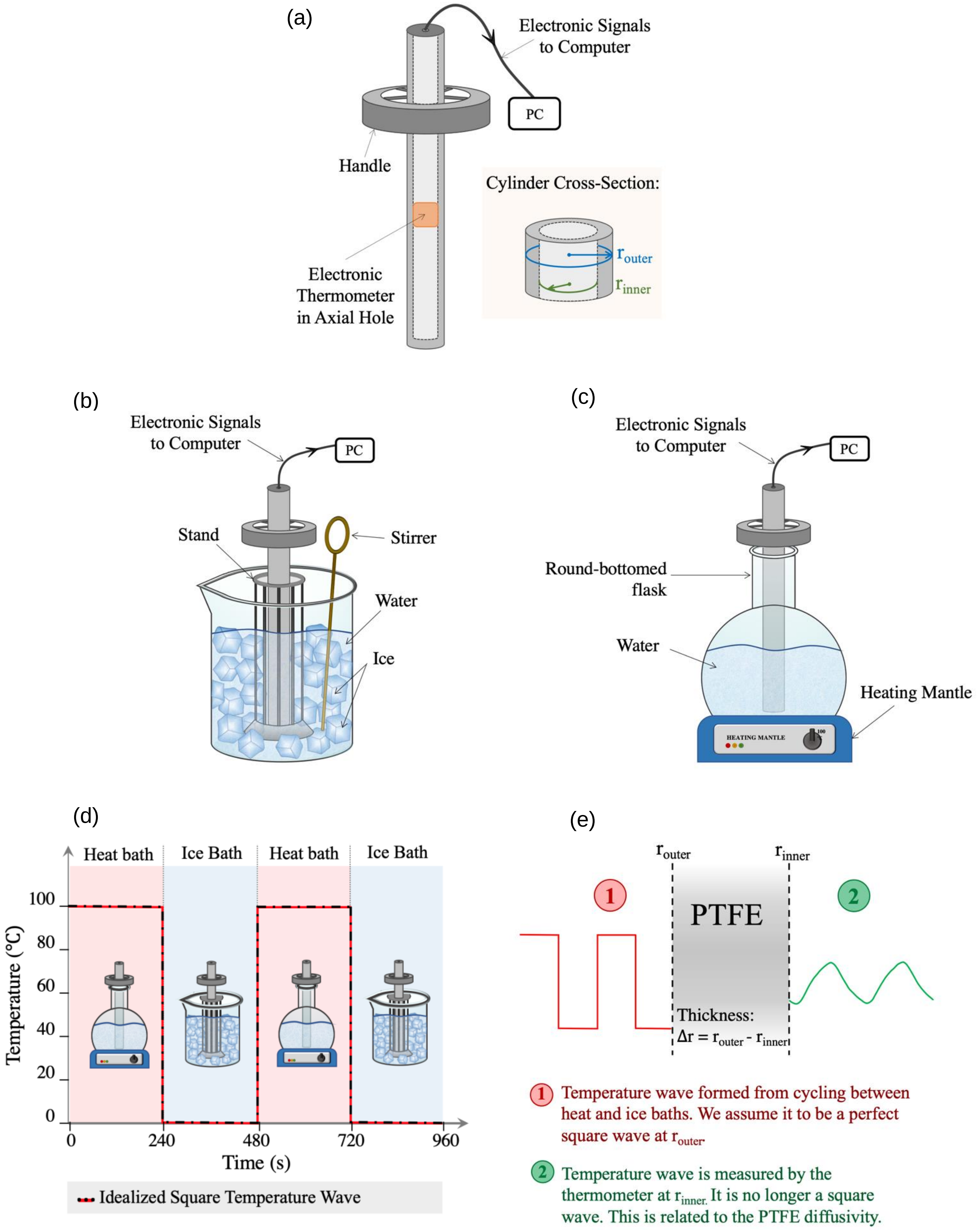


Fig. 2.1: Overview of the Thermal Waves experimental setup. *(a) The PTFE cylinder (not to scale). (b) Illustration of the PTFE cylinder in the ice bath (0˚C). (c) Illustration of the PTFE cylinder in the hot bath (100˚C). (d) Representation of the idealized square temperature wave from the cycling between heat and ice baths. (d) Diagram to illustrate difference between the induced wave and measured waves. Further explanation on this is given in section 2.4.2*

2.3 Experimental details

A sketch of the PTFE cylinder is shown in Fig. 2.1(a). The inner radius of the PTFE cylinder is given as r_{inner} = 2.50 $\pm$ 0.05 mm. **You can measure the outer diameter of the cylinder r_{outer} directly from the experimental apparatus on display in 2nd year lab.**

Some other important experimental details:

- To ensure good thermal contact of the electronic thermometer with the walls of the axial hole, it is permanently fixed inside.
- **The acquisition of data in the computer commences at a cold to hot transition and is set to stop automatically after four full periods**. The duration of the period is chosen by the student. The temperature is recorded every 0.1 s.
- To ensure that the heat flow is radial and that edge effects can be neglected (as is assumed in the analysis), the thermometer is fixed halfway through the height of the cylinder.

Note that there are a number of issues that may affect the ideal behaviour of the temperature bath, particularly the cold one:

- When the hot cylinder is introduced into the cold bath, the water in its vicinity is heated by the hot cylinder. If there is no effective flow of water, then the water temperature around the cylinder rises above 0 ^{0}C.
- Thus, the average temperature in the central hole, which would be expected to be ~50 ˚C, can be few ˚C higher. To reduce this effect a stirrer is used in the ice-water beaker to remove this layer of warmer water while thermal waves are being recorded, and hence make sure the outer surface of the cylinder is close to 0 ^{0}C.

In the hot flask, the boiling water (lower viscosity) is thoroughly mixed by rising bubbles, so no corresponding layer of cooler water builds up round the cylinder.

2.3.1 Note about temperature transients and acquisition of data

As the PTFE cylinder is cycled between hot and cold baths, we expect the average temperature at the centre of the cylinder to be ≈50˚C (or a few ˚C higher).

However, this average temperature is only reached after a few cycles – if the cylinder was at room temperature immediately prior to initiation of the temperature wave at the outside surface, the heat needed to warm (or cool) it to about 50˚C will take some minutes to be conducted to the centre of the cylinder.

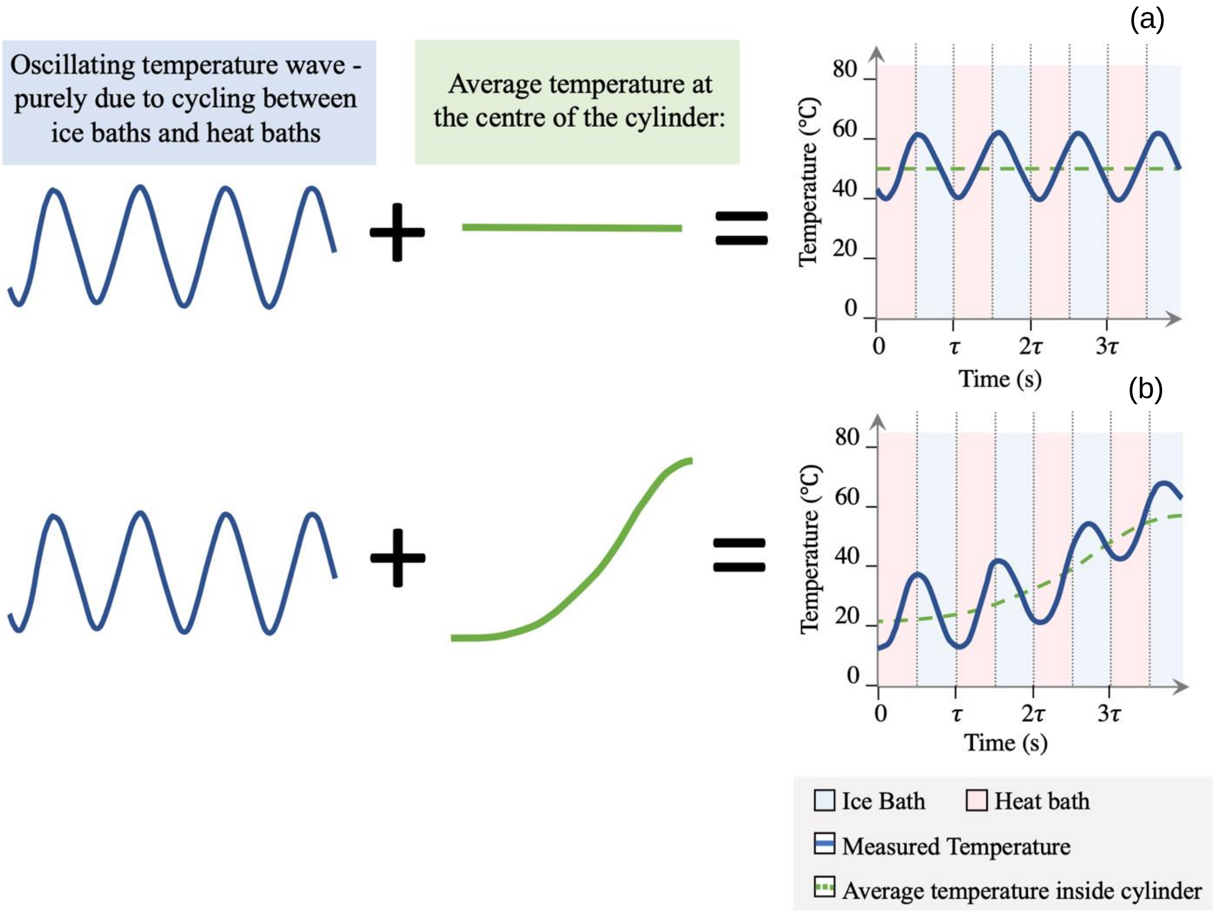


***Fig. 2.2: Measured wave as a combination of average temperature and oscillation due to cycling. (a)** Data if average at centre of the cylinder is ≈50˚C. (b) Data if cylinder was initially at room temperature ≈22˚C. In case (b), it takes a few cycles for the temperature inside the cylinder to rise from ≈22˚C to ≈50˚C. As shown above, this distorts the measured wave.*

This will give rise to initial temperature transients which partially obscure the temperature wave measured at the centre of the cylinder, as illustrated in Fig. 2.2. One way of avoiding this transient is by cycling the cylinder between the heat and ice baths for 5 or 6 cycles before starting to acquire the inner temperature data. *Be careful when choosing your datasets – a strong transient could lead you to large uncertainties in your thermal diffusivity values!*

2.4 Plane slab model

The flow of heat in a medium is described by the heat equation, which for a plane slab (i.e. in 1-dimension) is given by:

$$\frac{\partial T(x,t)}{\partial t} = D\frac{\partial^2 T(x,t)}{\partial x^2}\,, \tag{2.1}$$

where $T(x,t)$ is the temperature in the medium as a function of space and time and D is the thermal diffusivity of the medium.

Eq. (2.1) is a **diffusion equation**, which is a **linear differential equation** like Schrodinger's equation and the wave equation in the Electrical Wave experiment.

A solution for $T(x,t)$ decays in space and oscillates in time with angular frequency ω:

$$T(x,t) = Ce^{-\sqrt{\omega/2D}\,x}\,sin(\sqrt{\omega/2D}\,x - \omega t)\,, \tag{2.2}$$

where C is a constant dependent on initial and boundary conditions.

For a derivation of this equation and its solution, please refer to the Appendix.

Task 2.1:

a) From Eq. (2.1), what are the units of thermal diffusivity?
b) Look up examples of typical materials and their thermal diffusivities. Remember to record your sources of information.
c) Prove that Eq. (2.2) is a solution to the heat equation

In the plane slab model, the cylinder is crudely treated as a 1-dimensional slab of thickness $\Delta r = r_{outer} - r_{inner}$ with the x-axis pointing radially inwards. We define $x = 0$ at the outer boundary of the cylinder, which implies $x = \Delta r$ at the inner boundary.

Task 2.2: Consider a sinusoidal temperature of angular frequency ω and phase lag 0 starting from r_{outer} and propagating inwards through the cylinder.

a) Evaluate $T(x,t)$ at the inner and outer boundaries of the cylinder. Which one has greater amplitude? Is the thermal wave at r_{inner} leading or lagging the one at r_{outer}?

b) The amplitude transmission factor is defined as

$$\gamma = \frac{Amplitude|_{r_{inner}}}{Amplitude|_{r_{outer}}},$$

and the phase lag is defined as

$$\Delta\phi = \ Phase|_{r_{inner}} - \ Phase|_{r_{outer}} .$$

Show that in the case considered here $\gamma = e^{-\sqrt{\omega/2D}\,\Delta r}$ and $\Delta\phi = \sqrt{\omega/2D}\,\Delta r$. Comment on their angular frequency dependence.

c) Hence show that the thermal diffusivity is given by:

$$D = \frac{\omega\Delta r^2}{2\,\ln(\gamma)^2} = \frac{\omega\Delta r^2}{2\,\Delta\phi^2}$$

This suggests that if either the attenuation or phase lag are measured for a known value of Δr and ω, then D can be found. This is the principle of the analysis.

2.4.1 About the input square temperature wave at r_{outer}

In this experiment, the input temperature variation is a square wave, as opposed to a pure sine wave, like in the example above. So how can we adapt the approach taken in Task 2.2 to our investigations? From Part 1, we know that a periodic function like our square wave can be written as a superposition of sines in a Fourier series. Due to orthogonality between Fourier components and the linear nature of the diffusion equation, we can treat each harmonic mode independently as a sine wave of frequency ω_n, propagating through the PTFE.

2.4.2 About the measured temperature wave r_{inner}

We can interpret the digitally recorded temperature wave at r_{inner} as a superposition of all the square wave sinusoidal modes after they have travelled through the PTFE cylinder.

In Task 2.2, we found that a sine wave propagating through a medium undergoes amplitude attenuation and acquires a phase lag. Both these effects depend on the wave's angular frequency, ω_n. When treating a square wave as a superposition of sinusoidal harmonics, we can say that each mode n will also be attenuated and phase-shifted according to ω_n. Because each mode is affected differently, the transmitted harmonics no longer add up to a square wave in the measured dataset.

If we perform Fourier analysis on the recorded thermal wave, we can recover the individual harmonics. Since ω_n does not change when the wave traverses a medium, each mode of the square wave (launched from the outer boundary) can be related to a mode in the signal which is measured at the inner boundary. As a result, each sinusoidal mode n can be treated independently in the same fashion as in Task 2.2.

2.4.3 Thermal diffusivity calculations

Generalizing the expressions from Task 2.2, the thermal diffusivity for the n^{th} mode from its **transmission factor** $\boldsymbol{\gamma_n}$ is given by

$$D_{TF} \;=\; \frac{\omega_n \, \Delta r^2}{2(\ln(\gamma_n))^2}\,, \tag{2.3}$$

and the thermal diffusivity for the n^{th} mode from its **phase lag** $\boldsymbol{\Delta\varphi_n}$ is given by

$$D_{PL} \;=\; \frac{\omega_n \, \Delta r^2}{2\,\Delta\varphi_n{}^2}\,. \tag{2.4}$$

In both equations $\Delta r = r_{outer} - r_{inner}$ (i.e. the "thickness" of the PTFE cylinder) and ω_n is the frequency of the n^{th} mode.

Note that this **1- dimensional model** can give only an approximate answer, since a semi-infinite slab is not an ideal approximation to a cylinder. A more accurate solution can be obtained by using a cylindrical model.

2.4.4 Frequency dependence of amplitude attenuation and phase lag

We have already seen that phase lag and amplitude attenuation are effects that increase with an increasing angular frequency of the sinusoidal Fourier modes. Since ω_n is given by $\omega_n = \frac{2\pi n}{\tau}$, where τ is the period of the square temperature wave, the duration of the τ directly affects the shape of the measured waveform at r_{inner}.

- **Higher frequency harmonics: The fundamental mode approximation**

If we work at a square wave of short period and a frequency that is high enough, then nearly all the frequency components except the fundamental will be attenuated to "negligible" values. In this case, the temperature wave registered by the thermometer will be almost purely **sinusoidal**, since it will only contain the transmitted fundamental square wave mode.

- **Lower frequency harmonics**

If we make measurements with long square wave periods (i.e. low frequency), then we observe that the shape of the temperature wave at r_{inner} is **no longer purely sinusoida**l. That is because harmonic modes other than the fundamental have non-negligible amplitudes when they reach the centre of the cylinder, and hence also contribute to the measured waveform.

In conclusion, while it is acceptable to use the fundamental mode approximation for short periods, Fourier methods provide a much better approach when dealing with long periods, when multiple harmonics are present. The next task (2.4) will aim to improve your understanding of these concepts.

2.5 Data analysis

*You will be analysing **real data** obtained with the experimental apparatus on display in 2nd Year lab!*

You can download real datasets of temperature as a function of time for different periods at the 2nd year lab Thermal & Electrical Waves Microsoft Teams channel:

File name	Square Wave Period τ [min]
thermal_1min_a.txt	1
thermal_1min_b.txt	1
thermal_2min_a.txt	2
thermal_2min_b.txt	2
thermal_4min_a.txt	4
thermal_4min_b.txt	4
thermal_6min.txt	6
thermal_8min.txt	8
thermal_16min.txt (*)	16 (*)

Note: the time data is given in deciseconds (10^{-1}s)

(*) Contrarily to all the other datasets, the τ = 16 min data was not recorded for 4 periods. Thus, if you want to analyse this dataset, you will have to think about what is the best approach to take here!

Task 2.3: First ‘back of the envelope’ estimate of D using the Fundamental mode approximation:

a) Focus specifically on the 4 min data. On the same plot, superimpose the expected ideal square wave at the outer wall of the PTFE cylinder. Can you see the amplitude attenuation and the phase difference between the two waveforms?

b) Assume that the fundamental frequency is the only harmonic present in the temperature measured inside the PTFE and calculate:
   I. The Transmission Factor for the fundamental frequency
   II. The Phase Lag for the fundamental frequency
   III. Show clearly in your plot these two measured quantities
   IV. Using Eqs. (2.3) and (2.4), estimate D (with error)

NOTE: To calculate $\Delta\varphi$ in Eq. (1.6) use the “2-argument arctangent” function (*atan2*, available in Excel and Python) as otherwise you might get unphysical phase lags.

c) Look up the expected value of D for PTFE (remember to make a note of your sources) and compare with your first estimate for D. Are they compatible?

Task 2.4: Physical meaning and key features of all datasets

a) Plot all datasets on separate axes. Do not forget to label the axes and the period represented on each figure.
b) Explain why none of the measured temperature waves are square waves.
c) Identify the datasets which are affected by strong temperature transients and discard them from the analysis.
d) For the remaining datasets, decide which ones look approximately sinusoidal. In each case, explain your reasoning. Can you identify any pattern for increasing or decreasing period?
e) In which cases could the fundamental mode approximation be sufficient? In which cases would we benefit the most from performing the Fourier analysis of the measured wave?

Task 2.5: More 'back of the envelope' analysis – comparing different values for D

a) Now that you have done 4 min, you will look at the datasets for τ = 1 min and τ = 8 min.
   I. Repeat the 'back of the envelope' calculation for these two additional datasets and obtain their respective values of D (with error).

b) Time permitting, repeat the analysis for τ = 2 min and τ = 6 min.

c) Plot your values of D as a function of period and discuss your plot. Superimpose the expected value of D as a constant.

d) Looking at the measured wave + square wave plot for each investigated period, can you identify a pattern in the overall amplitude attenuation? Is this what you expected?

Recall: to obtain the Fourier series of your dataset you need to calculate the constant coefficients a_n and b_n. The numerical integration methods discussed in Task 1.3 might be useful in this process!

Task 2.6: Fourier analysis of the 4 min data – Obtaining a better estimate for D using the 1-D plane slab model

a) Perform Fourier analysis of the τ = 4 min dataset to obtain an amplitude-phase Fourier Series of the data, truncated at n = 3. Write down the amplitudes and phase lags for n = 1 (fundamental), 2 and 3.

b) Calculate values of D for each harmonic, using both the Transmission Factor Diffusivity, D_{TF}, and Phase Lag Diffusivity, D_{PL} .

c) Compare your diffusivity value from Fourier analysis to your diffusivity value from the 'back of the envelope' analysis and comment on it.

Task 2.7: Fourier analysis of the remaining datasets – comparing different values for D

a) Repeat the Fourier analysis procedure in Task 2.7 for other datasets and obtain the respective values of D for different harmonics. Feel free to extend the truncation to $n > 3$ if and when you judge necessary.

b) Plot the values of D_{TF} and D_{PL} from Fourier analysis for all the datasets on the same plot. Can you decide on a unique value for D? Compare this to your diffusivity from the 'back of the envelope' analysis and comment on it.

2.5. Cylindrical model – Bessel functions

2.5.1. Introduction and theory

What follows is an introduction to the maths of heat conduction in axisymmetric polar coordinates, which might not be as straightforward as the 1-D plane slab model presented before. Even if you cannot follow the maths in detail, it is useful to get a 'flavour' of the maths/physics involved in solving the problem. In the end, values for D will be obtained using a program that works with Bessel functions.

A more accurate modelling of heat conduction in the PTFE must consider its actual geometrical shape, **i.e. not a plane slab but a cylinder instead**.

In this case, the heat/diffusion equation to solve involves the radial coordinate r:

$$\frac{\partial T(r,t)}{\partial t} = D\frac{\partial^2 T(r,t)}{\partial r^2} + D\frac{1}{r}\frac{\partial T(r,t)}{\partial r} \,. \tag{2.5}$$

The solution is then given by

$$T(r,t) = constant \cdot \Re e\left[J_0\left(\sqrt{\frac{i\omega}{D}}r\right)\right], \tag{2.6}$$

where ω is the angular frequency of the wave, and J_0 is a Bessel function of first kind. Because J_0 is complex, $T(r,t)$ can be written in amplitude-phase form as

$$T(r,t) = constant \cdot \Re e\left[M_0\left(\sqrt{\frac{i\omega}{D}}r\right) e^{i\left(\omega t + \Phi_0\left(\sqrt{\frac{i\omega}{D}}r\right)\right)}\right], \tag{2.6}$$

where M_0 and Φ_0 are the modulus and phase factor of J_0, respectively. Both have complex argument $\sqrt{i\omega/D}\,r$.

For a detailed derivation of the heat equation for a cylinder and its solution involving Bessel functions, please refer to the Appendix.

2.5.3. Bessel transmission factor and phase lag calculations

Previously, we have defined the transmission factor as the ratio between the plane slab temperature amplitudes evaluated at the inner and outer radius of the PTFE cylinder. Following the same logic as before, but now with an improved mathematical model, we can write the transmission factor for the n^{th} mode in terms of the Bessel temperature amplitudes as

$$\gamma_n = \frac{M_0(\sqrt{\frac{i\omega_n}{D}}r_{inner})}{M_0(\sqrt{\frac{i\omega_n}{D}}r_{outer})} \,. \tag{2.7}$$

Similarly, the phase lag can be written in terms of Bessel phase factors as

$$\Delta\phi_n = \Phi_0\left(\sqrt{\frac{\omega_n}{D}}\, r_{inner}\right) - \Phi_0\left(\sqrt{\frac{\omega_n}{D}}\, r_{outer}\right). \tag{2.8}$$

Since the physical meanings of the transmission and phase lag factors are the same as the ones adopted throughout the experiment, (2.7) and (2.8) are directly comparable to γ_n and $\Delta\phi_n$ as calculated from the square wave Fourier modes and the Fourier decomposed data at r_{inner}.

Because D is contained within the argument of the Bessel functions, we should be able to evaluate transmission diffusivity D_{TF} by solving for the parameter $\alpha_n = \sqrt{\omega_n/D}\ r_{inner}$ in the following equation:

$$\gamma_{n\,(calculated)} - \frac{M_0(\sqrt{i}\,\alpha_n)}{M_0\left(\sqrt{i}\,\frac{r_{outer}}{r_{inner}}\alpha_n\right)} = 0\,, \tag{2.9}$$

Similarly, for phase lag diffusivity D_{PL} , we have

$$\Delta\phi_{n\,(calculated)} - \left[\Phi_0(\sqrt{i}\,\alpha_n) - \Phi_0\left(\sqrt{i}\,\frac{r_{outer}}{r_{inner}}\,\alpha_n\right)\right] = 0\,. \tag{2.10}$$

Equations (2.9) and (2.10) can be solved using the program BESSEL, which relies on an iterative process and is available in both Excel and Python versions. It handles the Bessel functions and uses root-finding algorithms to obtain the best value for α_n. You simply have to provide your transmission and phase lag data, their associated periods and run the program as instructed. Once BESSEL has calculated α_n, it will return the corresponding value for D, given by

$$D = \frac{{r_{inner}}^2\,\omega_n}{{\alpha_n}^2}\,, \tag{2.11}$$

from the definition of α_n. Depending on the data you enter, it will return you the transmission diffusivities D_{TF}, the phase lag diffusivities D_{PL}, or both.

As in the plane slab analysis, remember to treat each sinusoidal, square wave mode n as independently propagating by using Fourier methods.

In summary, the same set of results for γ_n and $\Delta\phi_n$ can be analysed using two different models to estimate diffusivity: one is a 1-D plane slab approximation, and the other is a more realistic cylindrical model which uses Bessel functions. They are two independent interpretations of the same problem and can give different values for D and different systematic errors.

Task 2.8: Bessel analysis of thermal data

You have access to the program BESSEL in either Excel or Python formats - both do exactly the same thing. Indeed, this is a ‘black box’, however it is one that you can look into if you want to!

a) Choose one temperature dataset (e.g. τ = 8 min).
   I. Input the Fourier transmission factor, phase lag, and associated period for all calculated harmonic modes into the program BESSEL to output values for D_{TF} and D_{PL}.
   II. Plot your values of D_{TF} and D_{PL} for this dataset on the same plot.

b) Repeat (a) for all the other datasets.

c) Plot the values of D_{TF} and D_{PL} from Fourier analysis for all the datasets on the same plot. Can you decide on a unique value for D? Compare this to your diffusivity from the plane slab Fourier analysis and comment on it.

Make sure to include the associated errors in your values of D!

Part 3: Electrical Waves experiment

3.1 Introduction and aims of the experiment

You will study the propagation of electrical signals, specifically **square pulses** and **sine waves**, in **a dispersive medium**. Think about the most common medium to transport such signals, a cable, a continuous pair of conductors arranged in a cylindrical symmetry. We will **model such a cable as a 'lumped transmission line'**, a series of discrete inductors and capacitors.

The aims of this experiment include familiarisation with:

- A lumped transmission line circuit and its main parameters, both theoretically and experimentally.
- The reflection of waves at interfaces between media of different impedances.
- The concept of dispersion, phase and group velocity.

These phenomena, studied here in the context of electrical signals in a transmission line, have wider application in many fields of physics, for instance the propagation of information in fibre optics, antennas, etc.

3.2 Theory

You will work with a lumped transmission line circuit consisting of 40 sections as illustrated roughly in Fig. 3.1. "Lumped" means that the *L* and *C* occur as individual inductors and capacitors, in contrast to a "continuous" line, such as a coaxial cable where the *L* and *C* are distributed along the inner and outer conductors.

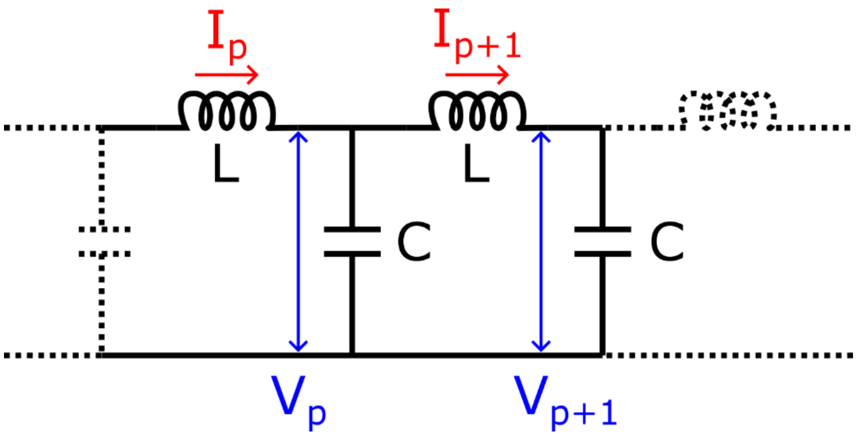


Fig. 3.1: *Circuit diagram of a single section of the lumped transmission line circuit.*

Under certain conditions, we find that the voltage *V(x,t)* is described by the wave equation:

$$\frac{\partial^2 V(x,t)}{\partial t^2} = \frac{1}{LC}\frac{\partial^2 V(x,t)}{\partial x^2} \tag{3.1}$$

where *x* is a spatial coordinate along the sections of the transmission line.

As this is a wave equation, we can distinguish the term *1 / LC* as being related to the speed of propagation of the wave, the phase velocity, measured in sections/sec:

$$V_{phase} \quad = \quad 1/\sqrt{LC} \tag{3.2}$$

A solution for *V(x,t)* is oscillatory in space and time with a frequency ω and wavenumber *k*:

$$V(x,t) = V_0 \exp\big(i(kx - \omega t)\big) \tag{3.3}$$

where V_0 is a constant dependent on the initial conditions.

> *For a detailed derivation of the Wave equation in a lumped transmission line, please refer to the Appendix. What assumptions underly this derivation? Can you repeat the derivation without making those assumptions? How does this affect the circuit's behaviour?*

3.2.1 Characteristic impedance and cut-off frequency

To derive the characteristic impedance of the line Z_0, we assume an infinitely long version of the circuit in Fig. 3.1 with inductors and capacitors having impedances Z_L and Z_C respectively. Thus, it is possible to derive the following expression (as shown in the 'Feynman Lectures', Vol. 2, 22-6):

$$Z_0 \quad = \quad \sqrt{Z_L Z_C (1/(1 + Z_L/4Z_C))} \tag{3.4}$$

and replacing $Z_L = i\omega L$ and $Z_C = 1 / i\omega C$ this results in:

$$Z_0(\omega) \quad = \quad \sqrt{(L/C)(1/(1 - \omega^2 LC/4))} \tag{3.5}$$

In the limit of $\omega \ll \omega_c = \sqrt{4/LC}$, the characteristic impedance is constant:

$$Z_0 \quad \approx \quad \sqrt{L/C} \tag{3.6}$$

where ω_c is known as the 'cut-off frequency'.

3.2.2 Reflections at changes of impedance

If a signal propagating in a medium A of impedance Z_a meets an interface with a medium B where the impedance changes to Z_b the signal will be only partially transmitted from A to B with the rest being reflected back into A.

If the signal is a voltage incident from A with amplitude V_i then a signal with amplitude V_r is reflected from the interface back into A. The voltage reflection coefficient R can be found to be

$$R = \frac{V_r}{V_i} = \frac{(Z_b - Z_a)}{(Z_b + Z_a)} \quad . \tag{3.7}$$

R is plotted as a function of Z_b / Z_a in Fig. 3.2 and some key values are tabulated below.

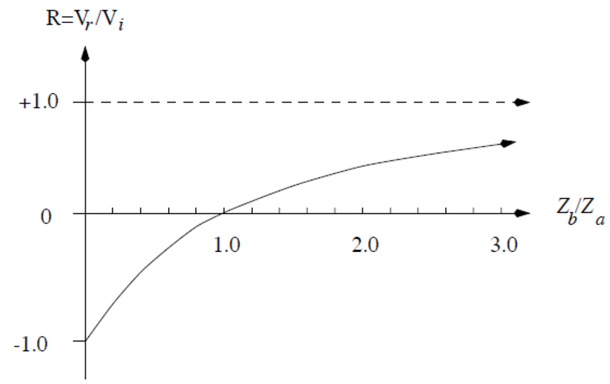


Fig. 3.2. *The voltage reflection coefficient V_r / V_i as a function of Z_b / Z_a.*

So if	$Z_a = Z_b$	$R = 0$	i.e. there is no reflection
if	$Z_b = 0$	$R = -1$	i.e. complete reflection with inversion
if	$Z_b = \infty$	$R = +1$	i.e. complete reflection without inversion

Task 3.1: Transmission line theory

a) Demonstrate that Eq. (3.3) is a solution to the wave equation and, in doing so, derive the dispersion relation $\omega(k)$.

b) Research about what are transmission lines and some known examples/applications. **Remember to keep track of your sources**.

c) Look up typical values of L, C, Z_0 and V_{phase} for a radiofrequency cable, for instance the coaxial BNC cables in your experimental kit. These are typically 'RG-59/U' or similar.

3.3 Experimental setup

The equipment you will need for this part of the experiment is composed of:

- 1 x lumped transmission line circuit made of 40 sections
  - Note that each circuit is labelled – Keep this consistent throughout the experiment!
- 1 x oscilloscope: Rohde & Schwarz model RTB2004, 100 MHz bandwidth, with built-in signal generator (and MANY other features, e.g. touch screen)
- 2 x BNC probes
- A digital multimeter
- Coaxial BNC cables
- A USB stick to download dataset and images from the oscilloscope
  - Remember to not take the USB stick with you!

Details of the 40-section lumped transmission line circuit are shown in Fig. 3.3. Note that, in the actual circuit, the 40 sections are arranged in a "C" shape starting from the BNC input on the top-right part of the circuit and ending with the switch and variable resistor on the bottom-right. There are 40 inductors and 43 capacitors, each with the same nominal value of L = 330 μH and C = 15 nF.

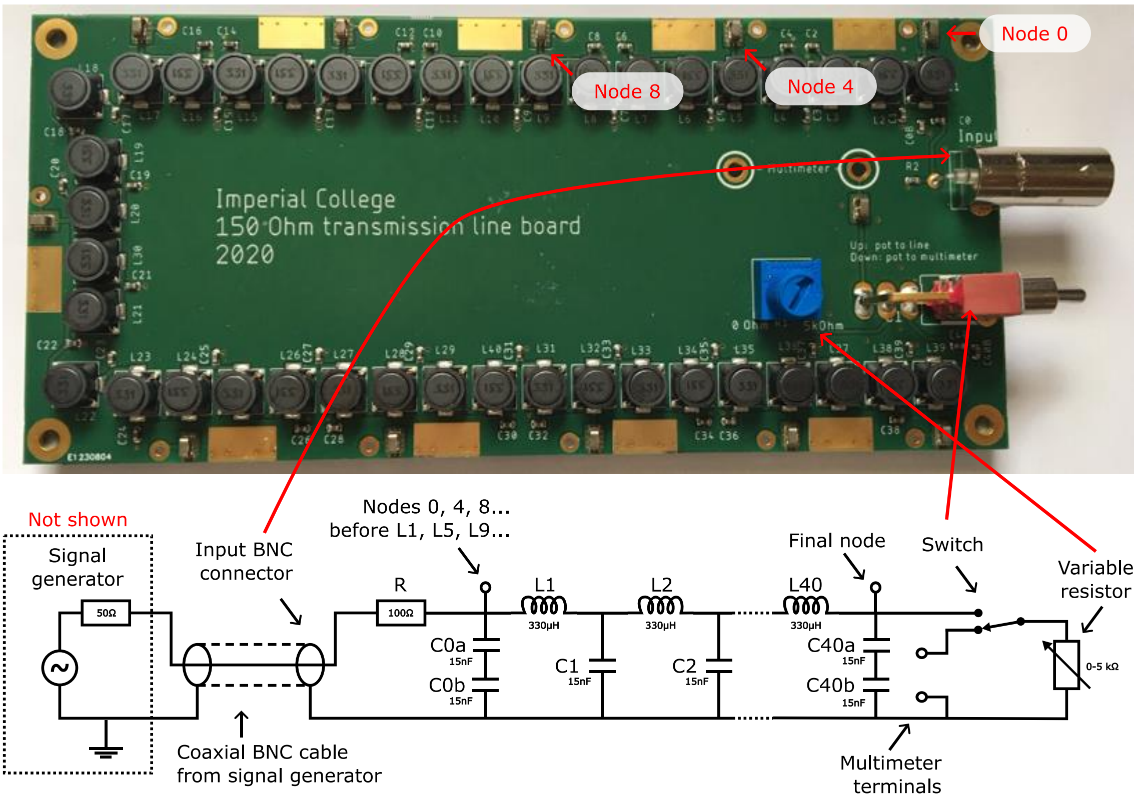


Fig. 3.3: *Details of the lumped transmission line circuit. (top) Photo of the circuit (bottom) Equivalent circuit. Each capacitor and inductor in the circuit have the same value of L and C.*

3.4 Measurement 1: Square pulses

3.4.1 Initial investigations

Your first task is to produce a series of square pulses and send them through the transmission line to investigate the behaviour. Note that square pulses are not exactly the same as square waves (like those in the Thermal Waves experiment). Although square pulses are still periodic, they will be set up such that only a single pulse propagates through the transmission line circuit.

Your oscilloscope is a Rohde & Schwarz model RTB2004 with a built-in signal generator (Fig. 3.4). The output signal comes from the 'Aux Out' BNC connector on the bottom left of the oscilloscope. There are 4 input BNC channels ('Ch1' to 'Ch4) on the bottom right.

About the oscilloscope: *Besides the actual transmission line circuit, your oscilloscope is the most important piece of equipment in this experiment. It will allow you to send and measure signals in the line with high accuracy. Think of the oscilloscope as a multimeter combined with a computer. Do not be discouraged by the daunting number of buttons and options displayed either in the panel or on screen. You might find that many times a specific option can be found in several different ways!*

Fig. 3.4: *Front view of the Rohde & Schwarz RTB2004 oscilloscope. The labels show (1) 'Aux Out' connector from the built-in signal generator, (2)-(3) input BNC channels 'Ch1' and 'Ch2', (4) 'Universal rotary knob', useful to select values when it is blue, (5) 'Gen' button to quickly access to the signal generator options, (6) Use the 'Autoset' and 'Preset' buttons at your own risk! (7) 'Apps' quick access button. Note that knobs such as (4) have push functions which can be quite useful to set up your signals quickly.*

A recap of oscilloscopes: Oscilloscopes are multi-purpose devices used to measure voltages and plot how they change over time. They are ubiquitous in research physics laboratories because they allow you to accurately plot and analyse almost any quantity that you can measure.

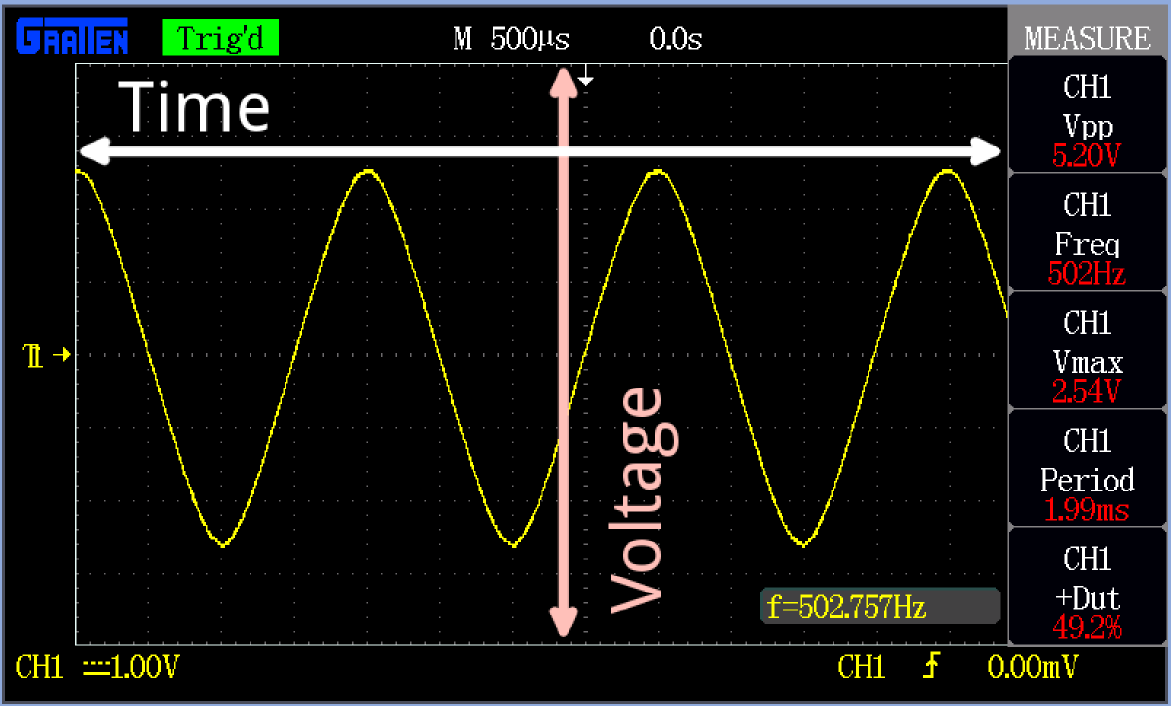


Figure reproduced from https://learn.sparkfun.com/tutorials/how-to-use-an-oscilloscope

Normally, an oscilloscope plots a measured voltage on the Y axis against time on the X axis. You control the scale of both the X and the Y axes, allowing you to zoom into interesting feature of the signal you are measuring. You also control when an oscilloscope starts measuring by setting its trigger level. For a recap of these concepts, see https://learn.sparkfun.com/tutorials/how-to-use-an-oscilloscope/all.

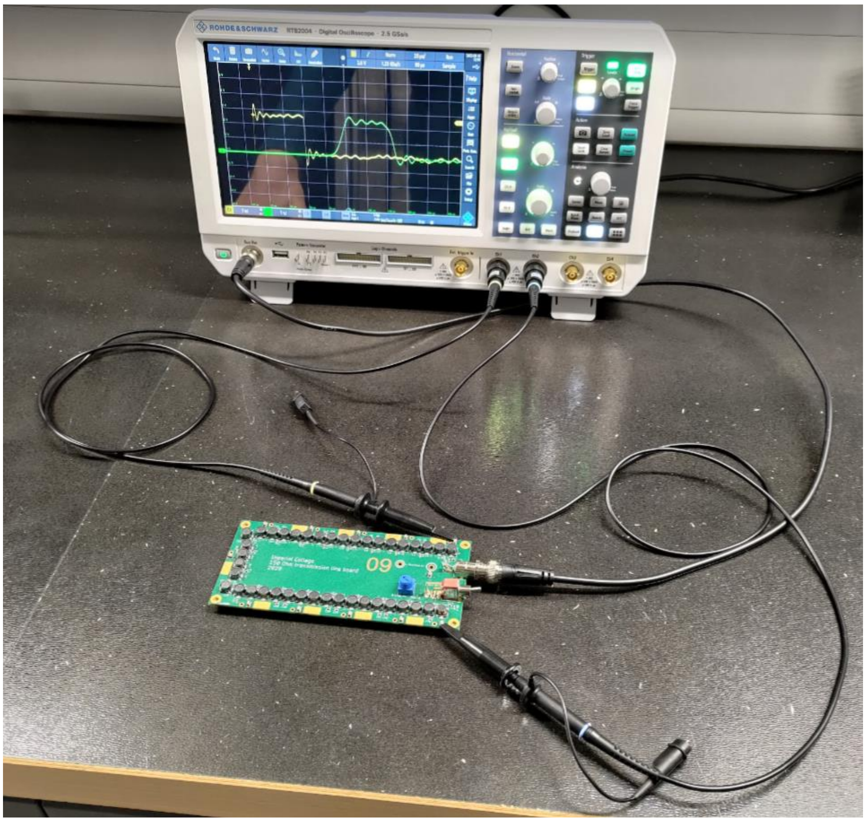


Fig. 3.5: *View of the connections needed between the oscilloscope and transmission line, and the signals you will measure. Note that the probes can conveniently hook into the hoops on the circuit board: here they are shown hooked into nodes 0 and 40.*

Task 3.2: Setting up a square pulse using the oscilloscope's 'Function Generator'

a) Use the "Pulse" function of the Function Generator (see (5) in Fig. 3.4) to produce a 50 μs pulse ranging from 0 V to 5 V and repeating at 2 kHz. Ensure that the "Load" option of the signal generator is set to "High-Z".

Hint: You will need to use the duty cycle setting to achieve this.

b) Use a BNC coaxial cable to deliver this pulse to the "Input" of the transmission line and use a probe on channel 1 of the oscilloscope to monitor the pulse at the first test-point of the circuit. Adjust the oscilloscope to clearly visualise the pulse.

You can save oscilloscope screenshots and data traces to the USB flash drive provided by inserting it in the front USB slot.

- *To save a screenshot (.PNG by default), press the*  *button*
- *To save data traces, press* 

and go to "Waveforms" to save voltage as a function of time as, e.g. .CSV or .TXT

Remember: Do not take the USB stick home with you!

Task 3.3: Sending square pulses through the transmission line

a) Have a careful look at your transmission line circuit and make sure you can distinguish the main components following the schematic diagram in Fig. 3.3.

b) Check that the switch at the end of the line is 'down' (i.e. open circuit, variable resistor disconnected from the circuit). Plug a 2nd probe into CH2 and hook the probe into a series of different points along the line. Keep a record of your measurements and explain the time delays and behaviour of the pulses observed.

   What could be causing the pulse to lose its ideal square shape?

3.4.2 Measure the characteristic impedance of the line

Now move the switch 'up' to connect the variable resistor (potentiometer, 0 to 5 kΩ) at the end of the circuit and look at the behaviour of the square pulse as you vary the resistance.

Task 3.4: Pulse reflection and characteristic impedance Z_0 of the line

a) Connect the potentiometer to the circuit by flicking the switch at the end of the line to the 'up' position. Vary the potentiometer and look at the pulses along the line at a fixed position, e.g. section 20 half-way along the circuit. Explain the change of amplitude and polarities observed and the relation with the value of the variable resistor by referring to the content in Section 3.2.2 ('Reflections at changes of impedance').

b) Vary the potentiometer until the reflected pulse vanishes. Measure the value of resistance in the potentiometer (to do so, make sure you disconnect the potentiometer from the circuit by leaving the switch 'down').

   This is the characteristic impedance Z_0 of the transmission line. Explain.

c) Estimate the uncertainty δZ_0 of your measurement.

d) Compare you calculated value of Z_0 to the value given by Eq. (3.6) using the manufacturer's values for the inductors and capacitors of $L = 330 \pm 20\%$ μH and $C = 0.015 \pm 10\%$ μF. Are these two values compatible?

3.4.3 Measure the speed of propagation of pulses

Task 3.5: Speed of propagation of pulses

During this measurement, ensure that the line is terminated to avoid reflection.

a) Measure the propagation speed of the pulse through the transmission line in units of 'sections/second'. (You may find the Cursor feature useful here). By taking velocity measurements over various intervals the line, estimate the uncertainty of your measurement.

b) Now calculate the phase velocity of the line from Eq. (3.2) and its uncertainty using the manufacturer's values for the inductors and capacitors. In this case your values of L and C should be in units of [H/section] and [F/section].

3.4.4 Pulse distortion

You have seen how the pulse degrades and attenuates as it propagates down the line. The distortion of the pulse can be approximately characterised by the parameters indicated in Fig. 3.6.

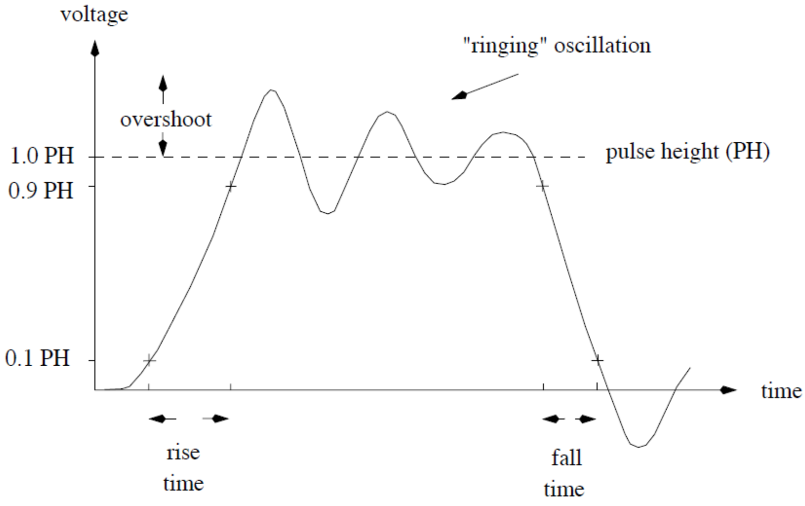


Fig. 3.6: *Schematic of pulse distortion.*

Task 3.6: Pulse distortion

a) Investigate how the rise time changes as the pulse propagates along the transmission line.

b) Decrease the duty cycle from 10% to 1%. What happens to the pulse as it propagates down the line? Why? Comment on why this could be important in practical transmission lines.

3.5 Measurement 2: Sine waves

You will now investigate the frequency response of the transmission line by sending sine waves with variable frequency down the line. This contrasts with the square pulse with, approximately a single 'low' frequency. If the transmission line is dispersive, the speed of the waves going through the line will change as a function of frequency.

Your aim in this part is to measure the dispersion relation of the transmission line, i.e. a graph of ω vs *k*, where ω is the angular frequency and *k* is the wavenumber.

3.5.1 Speed of propagation of sine waves with different frequencies

In general, the phase speed of a wave is defined as:

$$V_{phase} = \lambda f \tag{3.8}$$

where λ is the wavelength, in units of sections in this case, and *f* the frequency in units of Hz (for electromagnetic waves in vacuum, $V_{phase} = c$).

To measure the phase speed for a given frequency, we will look at the relative phase of a sinusoidal wave at two fixed points along the line, and in doing so estimate the wavenumber in terms of the 'distance' between those two points. This is shown schematically in Fig. 3.7.

Setup

Firstly, adjust the variable resistor so there are no reflections of signals from the end of the line back towards the pulse generator, i.e. 'match' the end of the line with its characteristic impedance so it seems to be infinitely long.

As with the square pulses, connect 'Aux Out' to the input of the transmission line using a coaxial BNC cable. On the 'Function generator', chose 'Sine' with a start frequency of '20 Hz', a peak-to-peak amplitude of '4 V' and an offset of '0 V' to obtain a sine wave that oscillates between -2 V and 2 V.

Now connect your first probe (CH1) at the start of the circuit and your second probe (CH2) at the end of your circuit on section 40.

To look at the phase difference of the sine wave at the input and output of the line, we could look at the two signals superimposed visually on the oscilloscope display. However, it is much more accurate to display phase differences by using the 'XY' mode on the oscilloscope, also known as 'Lissajous figures'. Press the 'Apps' button ((7) in Fig. 3.4) and chose the 'XY' view. You should see something similar to Fig. 3.8.

When working in 'XY mode', make sure you adjust the values of vertical position, magnitude, time scale and number of samples as you vary the frequency, as these will influence how the Lissajous figures are displayed.

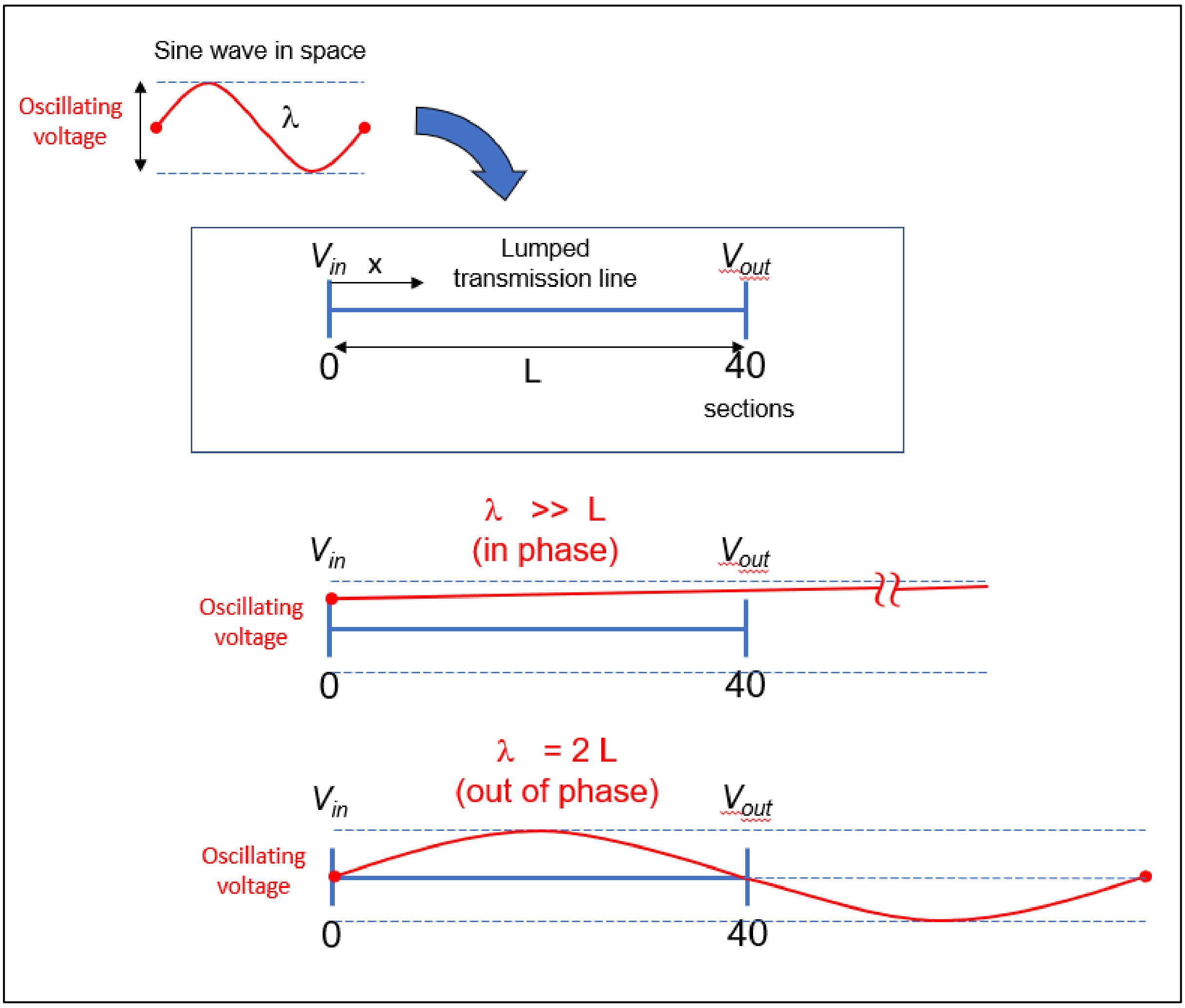


Fig. 3.7: *Schematic diagram showing a wave with a variable wavelength λ sent through a transmission line with length L = 40 sections.*

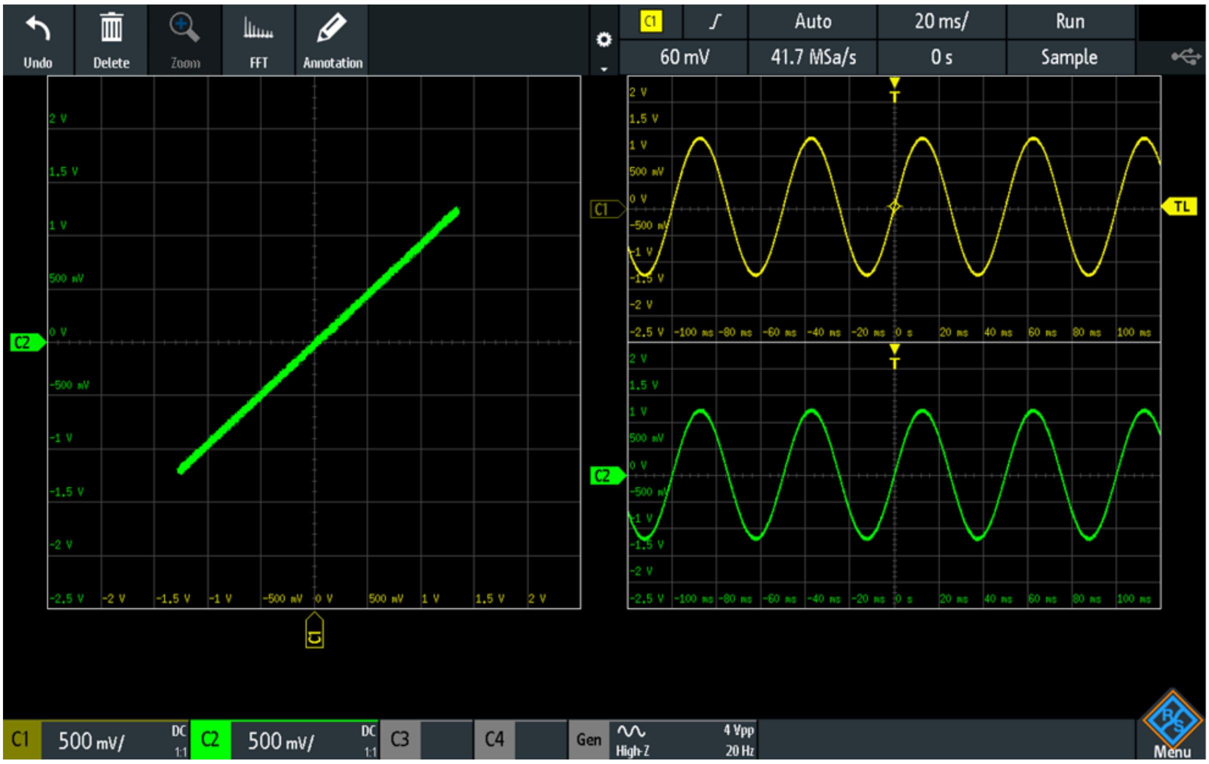


Fig. 3.8: *Oscilloscope screenshot showing the XY mode for sine waves at the input and output of the transmission line.*

Note that this image is for visual purposes only as you probably cannot read the small numbers and labels. This is an example of what you SHOULD NOT DO in your report…

Task 3.7: Sine waves - Measuring phase differences in the line

a) Explain, with the aid of figures, the shape of the Lissajous figures as you increase the frequency.

b) Using the sketch in Fig. 3.7, find a general expression for the wavelength λ_n, as a function of the length of the line and whether the wave is in or out of phase. From this, calculate an expression for the wavenumber k_n.
Hint: From the first 2 examples in the figure, work out the next 'in/out' of phase case and carry on from there.

c) Your measurements:

- Measure all the frequencies at which the wave is in and out of phase by scanning the frequency range starting from ~20 Hz, where the signals are nearly in phase, to the maximum frequency at which signals will propagate down the line.

- As you scan in frequency, measure the voltage amplitudes at the input and output of the line and calculate their 'amplitude ratio', i.e. V_{out} / V_{in} as a function of frequency.

Remember to record your uncertainties!

You should now have a table with frequency, whether the wave is in/out phase, V_{in}, V_{out} and amplitude ratio. Now:

Task 3.8: Sine waves - Frequency response of the transmission line

a) From your measurements, plot the dispersion relation of the line ω vs k. Explain the main features of this plot

b) Work out an expression for the group velocity $V_{group} = \Delta\omega / \Delta k$.

c) Plot V_{phase} and V_{group} vs frequency in the same graph. How is V_{phase} related to your measurements with square pulses? How is V_{phase} and V_{group} related to the dispersion relation?

d) Plot amplitude ratio vs frequency and measure the cut off frequency of your line (with its uncertainty). Compare your estimate with the theoretical value derived from Eq. (3.5).

e) How are your findings in this task connected to your measurements in Task 3.6 ('Pulse distortion')?

You can also look at the frequency response of the lumped transmission line by performing Fast Fourier Transform (FFT) tool in your oscilloscope. Try having a go at this even if it is purely treated as a black box!

The Lissajous plot is only one method to measure the amplitude and phase of sine waves transmitted through the circuit – other methods are possible! If you have time, you could try sending a periodic signal through the transmission line, and analysing the amplitude ratios and phase differences of the Fourier coefficients at the beginning and the end of the line. To excite a sufficiently broad range of frequency components at the input of the line, giving consistent signals up to the cutoff frequency, I recommend either using short pulses (e.g. 5 us, 2 kHz), or using a square wave ("Rectangle" function) and discarding the tiny even-harmonic frequency components from the analysis. Be careful assigning the Fourier component phase lags!

4. Appendix: Derivations and other useful extra information

4.1 Thermal Waves – Additional information

4.1.1 Heat equation in 1-dimension - Plane slab model

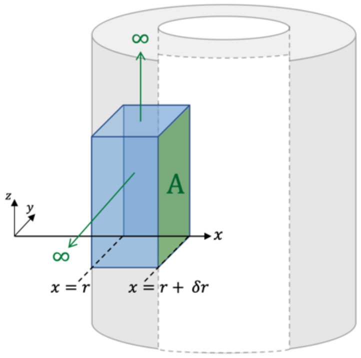


Fig. 4.2: *Infinite plane slab model. The slab in finite in x (thickness = δr) and infinite in y and z. Cross-sectional area of slab = A. Heat propagation occurs along x.*

In a slab geometry as shown in Fig. 4.2, the temperature gradient only exists in the x direction, meaning that the heat conduction occurs in one dimension only. The rate of heat conduction per unit area is given by Fourier's Law[1] as the product of the thermal conductivity κ and the temperature gradient $\partial T(x,t)/\partial x$ in the direction of conduction. Consider the slab volume element from Fig. 4.2. It has a mass given by $m = \rho \cdot A \cdot \delta r$, where ρ is the material's density. The net rate of heat flowing into and out of the volume element is given by

$$\frac{\partial Q}{\partial t} = \left(-A \cdot \kappa \cdot \frac{\partial T(x,t)}{\partial x}\bigg|_{x=r}\right) - \left(-A \cdot \kappa \cdot \frac{\partial T(x,t)}{\partial x}\bigg|_{x=r+\delta r}\right) \tag{4.1}$$

where $T(x,t)$ represents the temperature. Using the chain rule, we can rewrite

$$\frac{\partial Q}{\partial t} = m\left(\frac{1}{m}\frac{\partial Q}{\partial T}\right)\frac{\partial T(x,t)}{\partial t} = m\, c_V \frac{\partial T(x,t)}{\partial t}\ , \tag{4.2}$$

where c_V is the constant volume heat capacity of the material. Substituting our new expression for $\partial Q/\partial t$ into Eq. (4.1), re-arranging the terms and dividing both sides by $m = \rho \cdot A \cdot \delta r$ gives

$$\frac{\partial T(x,t)}{\partial t} = \frac{\kappa}{\rho \cdot c_V}\left(\frac{\frac{\partial T(x,t)}{\partial x}\Big|_{r+\delta r} - \frac{\partial T(x,t)}{\partial x}\Big|_{r}}{\delta x}\right) = D\,\frac{\partial^2 T(x,t)}{\partial x^2}\ , \tag{4.3}$$

since diffusivity $D = \frac{\kappa}{\rho \cdot c_V}$ and the expression in brackets is the definition of a derivative in x. This is the 1-D diffusion equation we have been using in the Thermal waves experiment.

[1] More about Fourier's Law, see e.g. https://en.wikipedia.org/wiki/Heat_flux and references therein.

Assuming a separable solution of the form $T(x,t) = X(x)\,\theta(t)$ and restricting the oscillatory solution to the time domain, we obtain, with $-i\omega$ as separation constant:

- In the time domain (oscillator solution): $\theta(t) = e^{-i\omega t}$
- In the space domain: $X(x) = Ce^{-\sqrt{\frac{\omega}{2D}}x} e^{i\sqrt{\frac{\omega}{2D}}x}$, where C is a constant.

Therefore, the full solution is given by

$$T(x,t) = Ce^{-\sqrt{\frac{\omega}{2D}}x} e^{i\left(\sqrt{\frac{\omega}{2D}}x-\omega t\right)}. \tag{4.4}$$

4.1.2 Heat equation in axisymmetric polar coordinates - Cylindrical model

This section deals specifically with cylindrical geometry, which is a better model of the physical system under consideration.

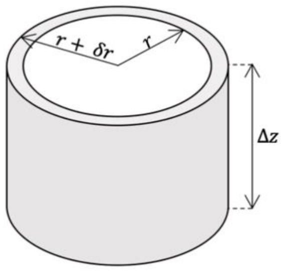


Fig. 4.2: *Volume element in the cylindrical model*

In cylindrical geometry, we consider a volume element whose shape is a cylindrical shell with inner radius r , outer radius $r + \delta r$, and height Δz , as shown in Fig. 4.2. If the cylinder is made of a material of density ρ, then the mass of the volume element is $m = \rho \cdot 2\pi r \Delta z\, \delta r$. The temperature gradient can only exist radially due to symmetry. From Fourier's Law of heat conduction, the heat flowing radially into the element through the inner surface per unit time is given by

$$Q_{in} = -(2\pi r \Delta z)\cdot \kappa \left.\frac{\partial T(r,t)}{\partial r}\right|_{r}, \tag{4.5}$$

where $T(r,t)$ is the temperature and κ is the conductivity of the material. Similarly, the heat flowing out of the volume element through the outer surface per unit time is

$$Q_{out} = -(2\pi(r+\delta r)\Delta z)\cdot \kappa \left.\frac{\partial T(r,t)}{\partial r}\right|_{r+\delta r}. \tag{4.6}$$

From this, we can calculate the rate of heat transfer, $\frac{\partial Q}{\partial t} = Q_{in} - Q_{out}$. Using the chain rule, we can rewrite $\partial Q/\partial t$ in terms of the rate of temperature rise in the volume element $\partial T/\partial t$:

$$\frac{\partial Q}{\partial t} = Q_{in} - Q_{out} = m\left(\frac{1}{m}\frac{\partial Q}{\partial T}\right)\frac{\partial T}{\partial t} = m\, c_V \frac{\partial T}{\partial t}\ , \tag{4.7}$$

where c_V is the constant volume heat capacity of the material. Re-arranging the terms and dividing both sides by $m = \rho \cdot 2\pi r \Delta z\, \delta r$ gives

$$\frac{\partial T(r,t)}{\partial t} = \frac{\kappa}{\rho \cdot c_V}\left(\frac{\left.\frac{\partial T(r,t)}{\partial r}\right|_{r+\delta r} - \left.\frac{\partial T(r,t)}{\partial r}\right|_{r}}{\delta r}\right) + \frac{\kappa}{\rho \cdot c_V}\frac{1}{r}\ \frac{\partial T(r,t)}{\partial r}\ . \tag{4.8}$$

Since the diffusivity $D = \frac{\kappa}{\rho \cdot c_V}$ and the expression in brackets is the definition of a derivative in r, we can rewrite Eq. 4.8 to get the standard diffusion equation that we have been working with for Bessel analysis:

$$\frac{\partial T(r,t)}{\partial t} = D\frac{\partial^2 T(r,t)}{\partial r^2}\ + D\frac{1}{r}\ \frac{\partial T(r,t)}{\partial r}\ . \tag{4.9}$$

Assuming a separable solution of the form $T(r,t) = R(r)\,\theta(t)$ and restricting the oscillatory solution to the time domain, we obtain, with $-i\omega$ as separation constant:

- In the time domain (oscillator solution): $\theta(t) = e^{-i\omega t}$
- In the space domain: $\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \frac{i\omega}{D}R(r) = 0$

Substituting $\zeta = \sqrt{\frac{i\omega}{D}} \cdot r$, the differential equation for the space domain is reduced to the standard Bessel equation (of order 0):

$$\frac{\partial^2 R(\zeta)}{\partial \zeta^2}\ + \frac{1}{\zeta}\frac{\partial R(\zeta)}{\partial \zeta} + R(\zeta) = 0\,. \tag{4.10}$$

Bessel equations frequently occur in physical problems involving a cylindrical geometry. The general solutions are of the form:

$$R(\zeta) = c_1 J_0(\zeta) + c_2 Y_0(\zeta), \tag{4.11}$$

where $J_0(\zeta)$ and $Y_0(\zeta)$ are first and second kind Bessel functions of order zero, and c_1 and c_2 are constants. Evaluation of these functions on normal calculators is not possible, but you can generally find them available in high level programming languages, such as MATLAB and Python.

The Bessel function of second kind diverges as $r \to 0$ ($\zeta \to 0$ in the complex plane along the i direction) and is inconsistent with the fact that there is no source of heat in the centre of the cylinder. Therefore $c_2 = 0$. The boundary condition on the inner radius should be applied to fix the remaining constant, but this is not well defined as we don't know where the thermometer actually is and what the space is filled with. Therefore, we keep $R(\zeta) = c_1 J_0(\zeta)$.

The Bessel function of first kind $J_0\left(\zeta = \sqrt{i\omega/D}\, r\right)$ can be defined as a series solution:

$$J_0\left(\sqrt{\frac{i\omega}{D}}r\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!^2} \left(\frac{i\omega}{4D}r^2\right)^n \quad . \tag{4.12}$$

The table below shows the first four terms of series:

Term value (U_n)	**Domain**	**Truncated Sum ($J_{0,N}$)**
$U_0 = 1$	$\in \mathbb{R}$	$J_{0,\,0} = 1 + i0$
$U_1 = -i\frac{\omega}{4D}r^2$	$\in \mathbb{I}$	$J_{0,\,1} \ = \ 1 - i\frac{\omega}{4D}r^2$
$U_2 = -\frac{\omega^2}{64D^2}r^4$	$\in \mathbb{R}$	$J_{0,\,2} \ = \ \left(1 - \frac{\omega^2}{64D^2}r^4\right) \ - i \ \frac{\omega}{4D}r^2$
$U_3 = -i\frac{\omega^3}{2304D^3}r^6$	$\in \mathbb{I}$	$J_{0,\,3} \ = \left(1 - \frac{\omega^2}{64D^2}r^4\right) \ - i\left(\frac{\omega}{4D}r^2 + \frac{\omega^3}{2304D^3}r^6\right)$

As exemplified above, for $N > 0$, the truncated sum of the series solution results in a complex number. Hence, we can always write $J_0\left(\sqrt{\frac{i\omega}{D}}r\right)$ as

$$J_0\left(\sqrt{\frac{i\omega}{D}}r\right) = M_0\left(\sqrt{\frac{i\omega}{D}}r\right) \ e^{i\left(\omega t + \Phi_0\left(\sqrt{\frac{i\omega}{D}}r\right)\right)}, \tag{4.13}$$

to extract its amplitude M_0 and phase factor Φ_0. This gives rise to the amplitude-phase form of the temperature $T(r,t)$ given in Eq. 2.6, and the Bessel analysis versions of the amplitude transmission factor and phase lag, given in Eq. 2.7 and 2.8, respectively.

4.1.3 BESSEL solver - Excel and Python versions

There are several different approaches one could take when doing the Bessel analysis of the data. In this section, we will briefly describe the reasoning behind the method adopted in BESSEL.py and BESSEL.xlsm.

Step 1: Construct the truncated J_0 series

Instead of using a Bessel function of first kind from a package, we created our own. This allowed us to tailor it according to our needs for this problem.

From the table above, we can see that J_0 actually consists of two series which grow independently:

- even n : gives purely real terms ($U_{n(even)} \in \mathbb{R}$)
- odd n : gives purely imaginary terms ($U_{n(odd)} \in \mathbb{I}$)

We took this into account when constructing J_0 – we created an iterative process that builds up the even and odd series separately, with the number of iterations determined by the truncation of J_0. In this way, it is easy to then extract the amplitude M_0 and phase factor Φ_0, without ever declaring anything as a complex variable. This is useful because it means that we don't have to restrict ourselves to computational methods that are able to deal with complex numbers - all our values are declared as real!

Step 2: Create functions that represent Eq. 2.9 and 2.10

As explained in section 2.5.3 of the script, we define equations comparing the Fourier calculated transmissions and phase lags to the Bessel ones. In these equations, we look at the amplitudes and phase lags already evaluated at the inner and outer boundaries. To make it easier for us, we define them in terms of a single variable $\alpha = \sqrt{\omega/D}\ r_{inner}$, giving rise to Eq. 2.9 and 2.10 in the script.

In both versions of BESSEL, we create functions equivalent to the LHS of Eq. 2.9 and 2.10:

$$\text{bessel\_trans\_fit} = \gamma_{(calculated)} - \frac{M_0|_{r_{inner}}}{M_0|_{r_{outer}}}, \tag{4.14}$$

$$\text{bessel\_phase\_fit} = \Delta\phi_{(calculated)} - \left(\Phi_0|_{r_{inner}} - \Phi_0|_{r_{outer}}\right). \tag{4.15}$$

Step 3: Use a solver to find the roots of $\text{bessel\_trans\_fit} = 0$ and $\text{bessel\_phase\_fit} = 0$

To find the optimal value of α, we use a root-finding algorithm (solver) to find the roots of Eq. 2.9 and Eq. 2.10.

- BESSEL.py : In Python, we used the secant method to estimate α. It is similar to and can be thought of as an approximation of the Newton-Raphson Method. We chose it because it only requires a function and an initial guess - there is no need to provide an interval or the function's derivative.

  More on the secant method: https://en.wikipedia.org/wiki/Secant_method https://mathworld.wolfram.com/SecantMethod.html

  More on the Python algorithm: https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.newton.html

- BESSEL.xlsm : In Excel, the program is stored in a Macro and is written in VBA (Visual Basic for Applications). The solver used to estimate α was the "Generalized Reduced Gradient (GRG Nonlinear)" solver.

  More on the GRG Nonlinear solver: https://engineerexcel.com/excel-solver-solving-method-choose/

Step 4: From the estimate for α, rearrange to get $D = \ r_{inner}{}^2\ \omega/\alpha^2$.

To understand this in more detail, feel free to look through the actual code for either one of the programs! The Python one in particular has some comment to guide your understanding.

Remember that there are other ways to do Bessel analysis! For instance, you could:

a) Curve fit a function for Bessel transmission or phase lag to your data
b) Create a plot of Bessel transmission factors and phase lags against a range of α. Then, the correct value of α can be obtained from the graph using the measured transmission factor and phase lag as the y-coordinate value.

4.2 Electrical Waves – Additional information

4.2.1 Wave equation in 1-dimension

The waves propagating in a lumped transmission line are voltage / current waves. In the limit that the wavelength (measured in number of sections) is much larger than unity, we can derive the wave equation from some simple assumptions.

Some basic current-voltage relations for inductors and capacitors:

Capacitor: The voltage dropped across a capacitor is the charge stored divided by the capacitance $V\ =\ Q/C$. The current through the capacitor is, of course, the time derivative of the charge $I\ =\ dQ/dt$. Consequently, $I\ =\ C \cdot dV/dt$.

Inductor: The voltage dropped across an inductor is the product of its inductance and the rate of change of current flowing through it $V\ =\ L \cdot dI/dt$.

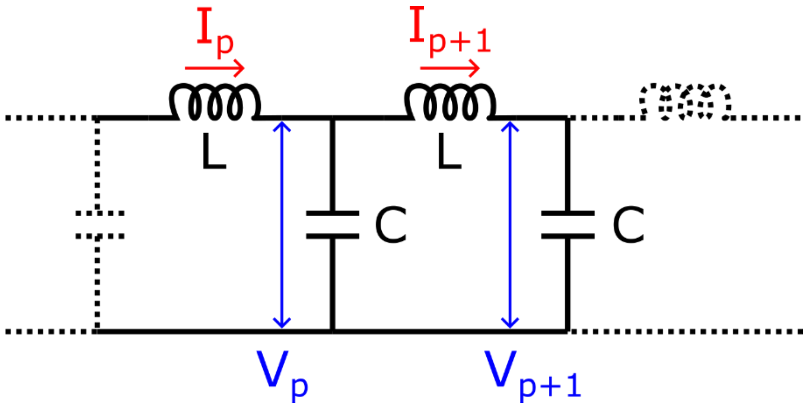


Fig. 4.3: *Sections of lumped transmission line*

Let's consider a section of the lumped transmission line as shown in Fig.4.3. Using the designation of current and voltage in Fig. 4.3, we have the following relations:

- The current flowing through the capacitor at node p+1:
$$I_p - I_{p+1}\ =\ C\ \cdot\ \partial V_p/\partial t$$
- The voltage dropped across the inductor between nodes p and node p+1:
$$V_p\ -\ V_{p+1} = \ L\ \cdot\ \partial I_{p+1}/\partial t$$

- The voltage dropped across the inductor between node p-1 and node p:

$$V_{p-1} - V_p = L \cdot \frac{\partial I_p}{\partial t}$$

Taking the difference between the last two expressions:

$$V_{p+1} + V_{p-1} - 2V_p = L\frac{\partial}{\partial t}\left(I_p - I_{p+1}\right) \ . \tag{4.16}$$

Substituting the 1st expression of difference in current between adjacent nodes into Eq. 4.16 we obtain:

$$V_{p+1} + V_{p-1} - 2V_p = L\frac{\partial}{\partial t}\left(C\frac{\partial V_p}{\partial t}\right) = LC\frac{\partial^2 V_p}{\partial t^2} \ . \tag{4.17}$$

The LHS of the expression can be re-cast as differentials as function of node number:

$$V_{p+1} + V_{p-1} - 2V_p = \frac{\left(\frac{V_{p+1} - V_p}{\Delta p}\right) - \left(\frac{V_p - V_{p-1}}{\Delta p}\right)}{\Delta p} \approx \frac{\partial^2 V_p}{\partial p^2} \ , \tag{4.18}$$

where Δp = 1. This leads to the wave equation:

$$\frac{\partial^2 V}{\partial p^2} = LC \cdot \frac{\partial^2 V}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} \ , \tag{4.19}$$

where v is the phase speed in sections per unit time.

5. Health and safety: Risk assessment

Imperial College London

RISK ASSESSMENT AND STANDARD OPERATING PROCEDURE

1. PERSON(S) CARRYING OUT THIS ASSESSMENT – This assessment has been carried out by the head of experiment.	
Name (Head of Experiment)	Charles Baynham and Richard Hobson
Date	01 October 2022

2. PROJECT DETAILS.						
Project Name	Thermal and Electrical Waves				Experiment Code	W
Brief Description of Project Outline	Wave propagation experiment for Year 2 physics undergraduates					
Location (*)	Campus	South Ken	Building	Blackett	Room	B407

(*) Note face to face teaching location in Blackett is optional for students who can come into campus. However, for students working remotely, the same hazards should be applicable.

3. HAZARD SUMMARY – Think carefully about all aspects of the experiment and what the work could entail. Write down any potential hazards you can think of under each section – this will aid you in the next section. If a hazard does not apply then leave blank.			
Manual Handling		Electrical	X
Mechanical		Hazardous Substances	
Lasers		Noise	
Extreme Temperature		Pressure/Steam	
Trip Hazards	X	Working At Height	
Falling Objects		Accessibility	X
Other			

4. CONTROLS – List the multiple procedures which may be carried out during the experiment along with the controls/ precautions that you will use to minimise any risks. Remember to take into consideration who may be harmed and how – other people such as students, support staff, cleaners etc will be walking past the experimental setup even when you aren't around.	
Brief description of the procedure and the associated hazards	**Controls to reduce the risk as much as possible**
Accessibility	All bags, coats, jumpers, etc. to be placed away from aisles, doors and walkways to have clear evacuation paths.
Electrical	No adjustment of electrical, mechanical or other parts of the experiments. In doubt, please inform a demonstrator or the Head of Experiment.
Trip hazards	No cables are to be left on the floor near the equipment.

5. EMERGENCY ACTIONS – What to do in case of an emergency, for example, chemical spillages, pressure build up in a system, overheating in a system etc. Think ahead about what should be done in the worst case scenario.
All present in room Blackett B407 must be aware of the available escape routes and follow instructions in the event of an evacuation. The nearest fire escape route if following the green sign at the back of the lab, towards Huxley.