(1) Advanced Macroeconomics

Gustavo Baamonde

May 8, 2021

1 Hodrick and Prescott Filter

We can decompose the time series y_t, where $t = 1, 2, \ldots, T$, in a trend and cycle component. That is:

$$y_t = y_t^p + y_t^C \tag{1}$$

Our objective function is:

$$f = \min_{y_t^p} \sum_{t=1}^{T} \left(y_t - y_t^p\right)^2 + \lambda \sum_{t=2}^{T-1} \left[\left(y_{t+1}^p - y_t^p\right) - \left(y_t^p - y_{t-1}^p\right)\right]^2 \tag{2}$$

Where $\{y_t^p\}_{t=1}^T$ is the trend. Also note that after the minimization, we can calculate the cycle component:

$$\hat{y}_t^C = y_t - \hat{y}_t^p \tag{3}$$

If $\lambda = 0$, the trend and the series are the same. We can show this by solving:

$$\min_{y_t^p} \sum_{t=1}^{T} \left(y_t - y_t^p\right)^2 \tag{4}$$

FOC:

$$\begin{aligned} \frac{df}{dy_t^p} &= 2\left(y_t - y_t^p\right) \\ &= y_t - y_t^p \\ &= 0 \end{aligned} \tag{5}$$

We can do the same thing using matrix notation
The HP filter is obtained by minimizing the objective function

$$\sum_{t=1}^{N} \left(y_t - y_t^p\right)^2 + \lambda \sum_{t=2}^{N} \left(\left(y_t^p - y_{t-1}^p\right) - \left(y_{t-1}^p - y_{t-2}^p\right)\right)^2$$

for y_t^p. It is convenient to express the objective function in matrix form:

$$\left(Y - Y^p\right)' \left(Y - Y^p\right) + \lambda Y^{p'} \nabla^{2'} \nabla^2 Y^p$$

where Y and Y^p are $T \times 1$ vectors of the original data and the trend and ∇^2 denotes the 2nd difference matrix . The solution of this optimisation problem follows from the first order conditions in matrix form:

$$
\begin{aligned}
Y^p &= \left(I + \lambda \nabla^{2'} \nabla^2\right)^{-1} Y \\
Y^C &= Y - Y^p
\end{aligned}
$$

2 Conclusion

- When $\lambda = 0$, the trend and series are the same, that is, there is no cycle.
- When $\lambda \to \infty$, the trend is linear
- If the assumptions postulated by Hodrick and Prescott are valid (Proposition 1 - WHY YOU SHOULD NEVER USE THE HODRICK-PRESCOTT FILTER (Hamilton)), then the HP filter is a good approximation of reality.
- Hamilton's criticism about the filter is that its assumptions are not always valid, that is, the filter is not using the true data-generating process to decompose the series. Therefore, we can't argue that the result is, in fact, the true decomposition of the series.
- Hamilton proposes another method, using the data itself, but it's not relevant for the problem.
- Despite its problems, the HP filter is still broadly used, specially given its simplicity.