CSCI 2270: Data Structures

Lecture 17: Binary Search Trees

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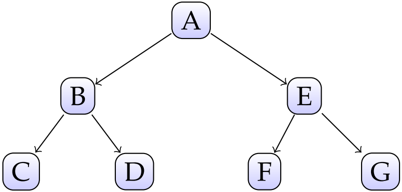

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Recursive (Inductive) Definition

"In order to understand recursion you must first understand recursion."

Recursive (Inductive) Definition

Definition (Recursion Definition)

- Defining an object using recursive definition.
- Sometimes it is difficult to define a function or object explicitly; it is easier to define the function or object in terms of the function or object itself. This is called a recursive definition or recursion.

Recursive Definition

Recursive Definition

Recursive Definition: Examples

1. Set of natural numbers $\mathbb{N}$:
   - $0 \in \mathbb{N}$ (base case)
   - if $n \in \mathbb{N}$ then $(n+1) \in \mathbb{N}$. (general case)
2. Factorial $n! = n \cdot (n-1) \cdot \cdots \cdot 1$:

$$n! = \begin{cases} 1 & \text{if } n = 1 \\ n \cdot (n-1)! & \text{otherwise} \end{cases}$$

3. Fibonacci numbers $Fib(n)$:

$$Fib(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ Fib(n-1) + Fib(n-2) & \text{otherwise.} \end{cases}$$

Recursively Defined Structures

1. Strings.
   - ε is the empty string containing no symbols.
   - if w is a string and x is a character, then wx is a string.
2. Well-formed parenthetical expression over $\{(, \{, [,), \},]\}$
   - empty string is a well-formed parenthetical expression
   - If E is well-formed then so is (E) and $\{E\}$ and $[E]$.

Recursively Defined Structures

1. A Linked List L is:
   - either null,
   - or a node with a "key" and a next pointer pointing to a linked list L', i.e. $Node(key : a, next : L')$.
2. A Binary Search Tree T is
   - either null;
   - or a node with a "key" and a left and a right pointer pointing to a binary search trees T_ℓ and T_r, i.e., $Node(key : a, left : T_\ell, right : T_r)$.

Recursively-Definition and Induction

When a sequence is defined recursively, mathematical induction can be used to prove properties about it.

Theorem

For every $n \in \mathbb{N}$ the property $P(n)$ defined as

$$1 + 2 + 3 + \ldots + n = n(n+1)/2$$

holds.

Proof.

- (base case): Verify that $P(n)$ is true for $n = 0$
- (inductive step): Assuming that for an arbitrary n property $P(n)$ holds, show that it holds for $P(n+1)$.

From mathematical induction, $P(n)$ then holds for all $n \in \mathbb{N}$. □

Recursively-Definition and Structural Induction

When a structure is defined recursively, *structural induction* can be used to prove properties about it.

Theorem

For every linked-list L we have that

$$size(L) = \begin{cases} 0 & L = null \\ 1 + size(L') & L = Node(key : a, next : L'). \end{cases}$$

Proof.

- (base case): Verify that the property holds for the base case $L = null$.
- (inductive step): Assuming that property holds for all sub-structures used in the inductive definitions, show that the property will hold for the defined structure.

From structural induction, the property then holds for all structures. □

Recursively-Definition and Structural Induction

When a structure is defined recursively, *structural induction* can be used to prove properties about it.

Theorem

For every linked-list L, its reverse list is equal to:

$$reverse(L) = \begin{cases} null & L = null \\ reverse(L') : a & L = Node(key : a, next : L') \end{cases}$$

Proof.

- (base case): Verify that the property holds for the base case $L = null$.
- (inductive step): Assuming that property holds for all sub-structures used in the inductive definitions, show that the property will hold for the defined structure.

From structural induction, the property then holds for all structures. $\square$

Recursively-Definition and Structural Induction

When a structure is defined recursively, *structural induction* can be used to prove properties about it.

Theorem

For every binary search tree T we have that:

$$size(T) = \begin{cases} 0 & T = null \\ 1 + size(T_\ell) + size(T_r) & T = Node(key : a, left : T_\ell, right : T_r) \end{cases}$$

Proof.

- (base case): Verify that the property holds for the base case $L = null$.
- (inductive step): Assuming that property holds for all sub-structures used in the inductive definitions, show that the property will hold for the defined structure.

From structural induction, the property then holds for all structures. □

Recursively-Definition and Structural Induction

When a structure is defined recursively, *structural induction* can be used to prove properties about it.

Theorem

For every binary search tree T and a value k we have that $search(T, k)$ equals

$$\begin{cases} false & T = null \\ (k == a) || search(T_\ell, a) || search(T_r, a) & T = Node(key: a, left: T_\ell, right: T_r) \end{cases}$$

Proof.

- (base case): Verify that the property holds for the base case $L = null$.
- (inductive step): Assuming that property holds for all sub-structures used in the inductive definitions, show that the property will hold for the defined structure.

From structural induction, the property then holds for all structures. □

Exercise

Implement the following using recursion.

1. Search for an element in a linked list.
2. Compute the size of a linked list.
3. Search for an element in a binary search tree.
4. Compute the number of elements in a binary search tree.
5. Find the minimum element in a BST (recursive).
6. Find the maximum element in a BST (recursive).
7. Print all of the elements in the tree.
   - In-order
   - Pre-order
   - Post-order