Efficient Fault-Tolerant Infrastructure for Cloud Computing

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Outline

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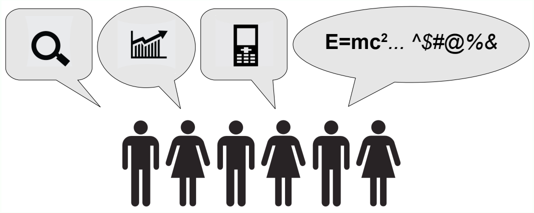
Data-Intensive Computing

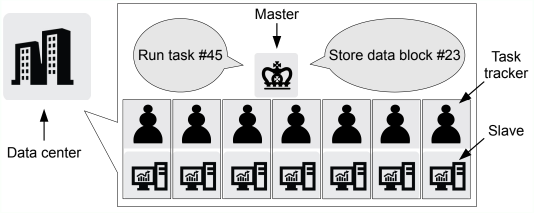


- Data-intensive computing, big data.
  - Store and process large volumes of data (search/transactions/scientific computing/etc...).
  - Scalability, parallelism, and efficiency.
- Solution: Cloud computing!

Cloud Computing

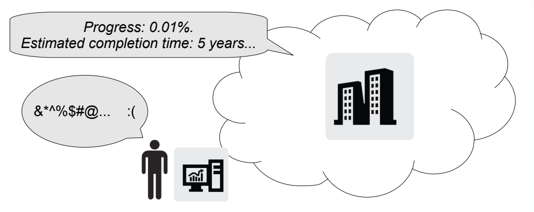
- Envisions shifting data storage and computing power from local servers to data centers.
- Includes both of
  - Applications delivered over the Internet,
  - Supporting hardware & software infrastructure.

Cloud Computing Infrastructure



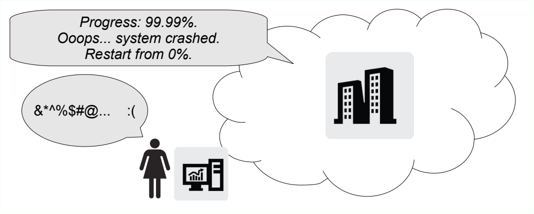
- Google's MapReduce and Apache Hadoop.
- Built on large clusters of commodity machines.
- Storage: Data block replication in distributed file systems.
- Execution: Job partitioning into tasks for parallel execution.

Challenges for Infrastructure Design



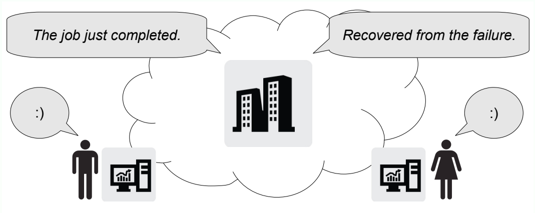
- Efficiency challenge: Load balancing for fast job completion.
- Data locality is critical.
  - The time to load data dominates the actual computation time.
  - Running tasks on servers not holding input data → remotely load through network → communication cost plus disk access.
  - The more data needs to be remotely loaded → the more congested the network becomes → higher remote access cost.

Challenges for Infrastructure Design (cont.)



- Fault tolerance challenge: Inevitable failures due to scale.
- Completions of long-running jobs are affected by failures.
  - Checkpointing & rollback.
  - Literature: Checkpoints are stored in totally reliable storage.
  - Checkpoints may fail & their failure probability can be significant.
  - Faster job completion can sometimes be achieved by using less reliable but cheaper checkpoints.

Our Contributions



- Efficiency: Assignment of tasks to servers for minimizing job completion time.
- Fault tolerance: Checkpoint placements that consider possible checkpoint failure.

Outline

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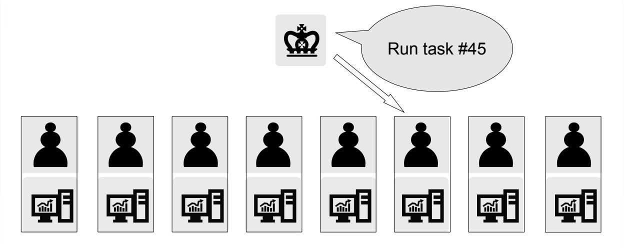
Quick Overview of Hadoop Infrastructure

- Storage: Hadoop distributed file system (HDFS).
  - Files are split into large equal-sized blocks.
  - Each data block is replicated on multiple servers.
- Job execution: Break jobs down into small independent tasks.
  - Each task processes a single input data block.
  - Tasks are assigned to and run on servers.
  - Tasks assigned to the same server run sequentially.
  - Tasks assigned to different servers run in parallel.
  - Job completes when all tasks finish. Thus, the job completion time is the completion time of the task that finishes last.

Scheduling Tasks to Servers

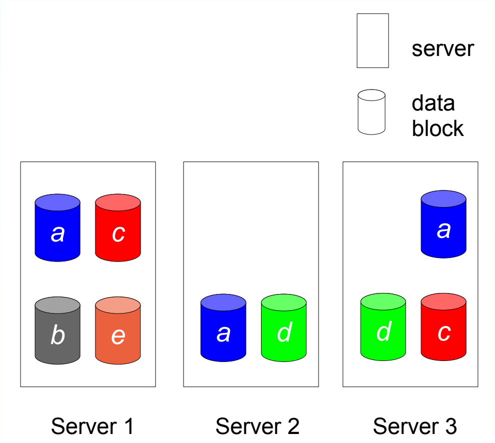


- Challenge: How to assign tasks to servers to simultaneously balance load and minimize cost of remote data access?

Components of the Model

- A set of tasks and a set of data blocks. Each data block has a corresponding task to be performed on it.
- A set of servers that store data blocks and perform tasks.
- A data placement that maps a data block to the set of servers on which replicas of that block are stored.
- A task assignment that specifies the server on which a task will run.
- A cost function that determines the run time of a task on a server to which it is assigned.

Data Placement on Servers

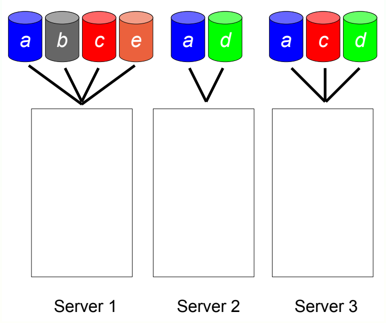


Every data block occurs on at least one server.

Block	#Replicas
a	3
b	1
c	2
d	2
e	1

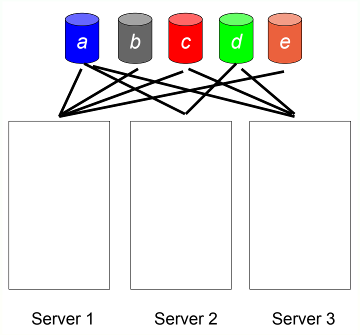
Data Placement Graph

Slide data blocks up out of the server rectangles and connect them by lines. This leads to a bipartite graph where each replica has an edge to its corresponding server.

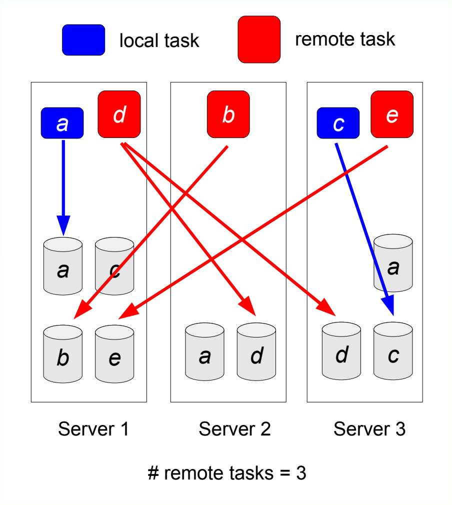


Data Placement Graph (cont.)

Merge the replica nodes with the same block label together, giving the bipartite graph called the data placement graph G_ρ.



Assigning Tasks to Servers



- A task is local if it is on the same server as its corresponding data block. Its cost is w_{loc}.
- A task is remote if it is on a different server from its corresponding data block. Its cost is w_{rem}.
- We assume $w_{\mathrm{rem}} \geq w_{\mathrm{loc}}$.

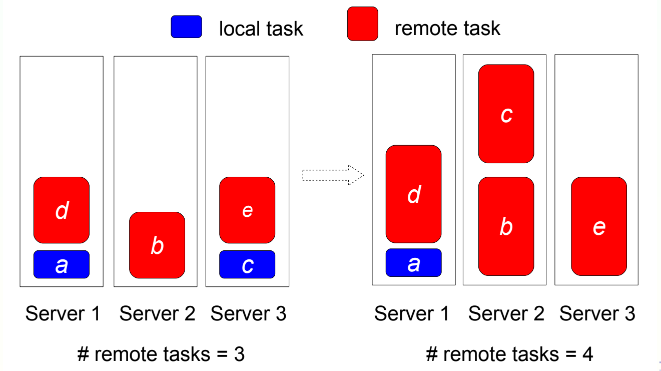
Remote Cost

- Network becomes more congested with more remote tasks.
- Network congestion increases the cost for remote data access.
- To reflect the increased communication cost, w_{rem} is a monotone increasing function of the number of remote tasks.

Remote Cost (cont.)

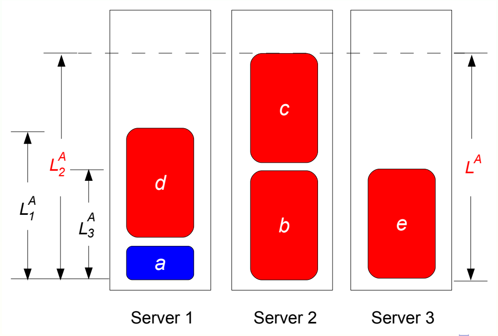
Reassigning task *c* from server 3 to server 2 gives a worse result.

- The cost of task *c* becomes remote cost.
- Remote costs for all remote tasks increase due to the increased number of remote tasks.



Server Load and Maximum Load

- The load L_s^A of a server s under assignment A is the sum of costs introduced by all tasks that A assigns to s.
- The maximum load L^A of assignment A is the maximum server load under A. In the following example, $L^A = L_2^A$.



Outline

HTA Problems

Definition

The *HTA Optimization Problem* is, given a Hadoop system, to find a task assignment that minimizes the maximum load.

Definition

The *HTA Decision Problem* is, given a Hadoop system and a server capacity k, to decide whether there exists a task assignment with maximum load $\leq k$.

NP-Completeness

By reduction from 3SAT, we show that

Theorem

The HTA decision problem is NP-complete.

Actually the HTA problem remains hard even if the server capacity is 3, the cost function only has 2 values $\{1, 3\}$ in its range, and each data block has at most 2 replicas.

Approximation Algorithms

We present and analyze two approximation algorithms.

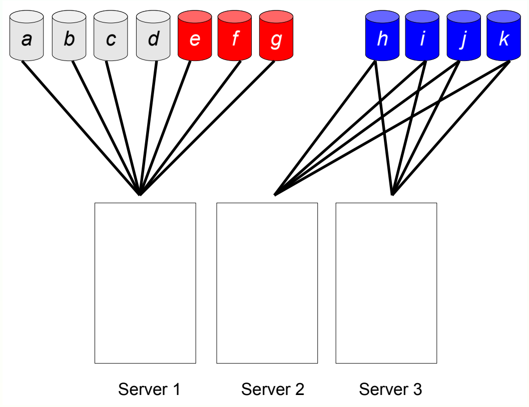
- "Round Robin" algorithm computes assignments that deviate from the optimal by a factor $(w_{\mathrm{rem}}/w_{\mathrm{loc}})$.
- The flow-based algorithm "Max-Cover-Bal-Assign" computes assignments that are optimal to within an additive gap w_{rem}.

Outline of the Max-Cover-Bal-Assign Algorithm

Given a threshold τ, the algorithm runs in two phases:

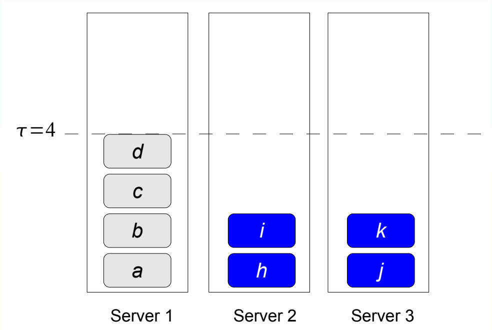
- Max-cover uses network flow to maximize the number of local tasks such that no single server is assigned more than τ local tasks.
- Bal-assign finishes the assignment by greedily assigning each unassigned task to a server with minimal load.

Example Data Placement



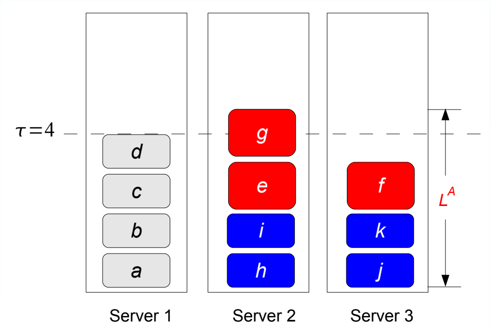
Max-cover with Threshold $\tau = 4$

Max-cover assigns each server the maximum number of local tasks, subject to the threshold $\tau = 4$.



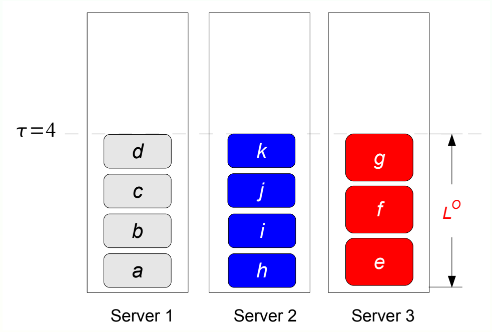
Bal-assign

Bal-assign finishes the task assignment by greedily assigning each unassigned task to a server with minimal load.



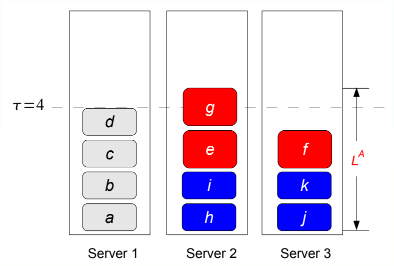
Optimal Assignment O

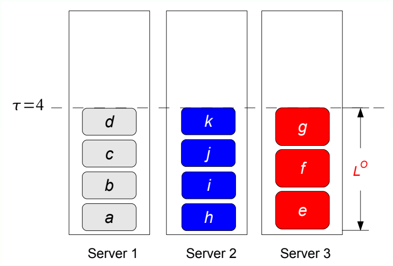
However, optimal assignment can do better...



OPT vs Flow-Based

But the difference is smaller than w_{rem}.





The Approximation Gap

By trying all possible thresholds τ and picking the best assignment, the flow-based algorithm performs surprisingly well.

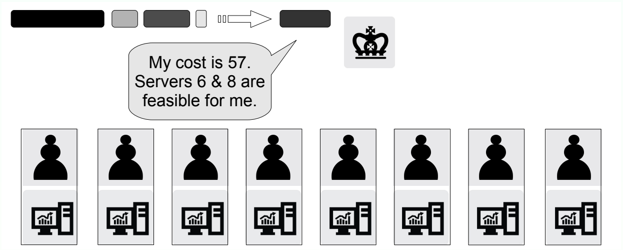
Theorem

Let n be the number of servers. For $n \geq 2$, the flow-based algorithm computes an assignment A such that

$$L^A \leq L^O + \left(1 - \frac{1}{n-1}\right) w^O_{\mathrm{rem}}$$

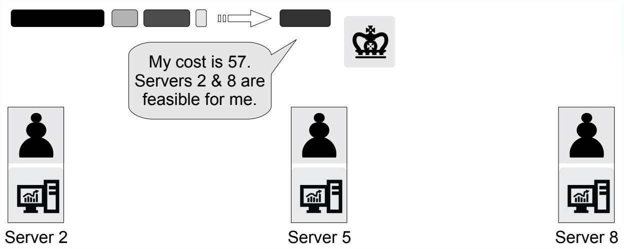
Outline

Online Scheduling with Restricted Assignment



- Distinctions from the HTA problem:
  - Tasks arrive sequentially in an online fashion.
  - Tasks must be assigned to their feasible (local) servers.
  - Tasks may have different costs (but do not depend on which server the tasks are assigned to).
- Goal: Assign each task to a feasible server upon its arrival to minimize the maximum load.

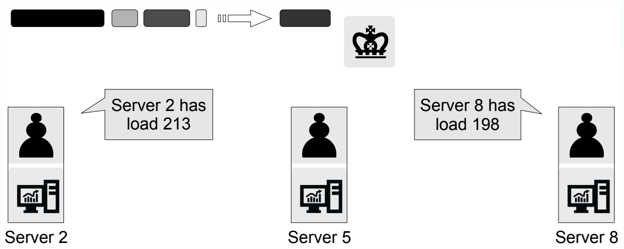
The GREEDY Algorithm



The GREEDY algorithm:

- Chooses a feasible server with minimum load.
- Breaks ties arbitrarily.

The GREEDY Algorithm (cont.)



The GREEDY algorithm:

- Chooses a feasible server with minimum load.
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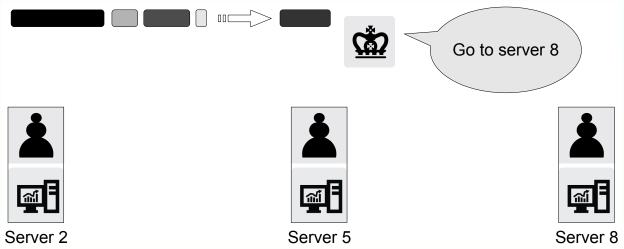
The GREEDY Algorithm (cont.)



The GREEDY algorithm:

- Chooses a feasible server with minimum load.
- Breaks ties arbitrarily.

Performance Measure and Our Approach

- Assume an optimal offline algorithm OPT knows the task sequence in advance and has unbounded computational power.
- The competitive ratio is the maximum load ratio between GREEDY and OPT under all possible inputs.

Theorem (Upper bound on competitive ratio of GREEDY)

For any problem instance σ, let n be the number of servers. Let A_σ and O_σ be computed by GREEDY and OPT, respectively.

$$\frac{L^{A_\sigma}}{L^{O_\sigma}} \leq \log n + 1 - \delta(\sigma),$$

where $0 \leq \delta(\sigma) \leq \log n$. $\delta(\sigma) = 0$ if and only if O_σ perfectly balances load over all servers and L^{A_σ} is a multiple of L^{O_σ}.

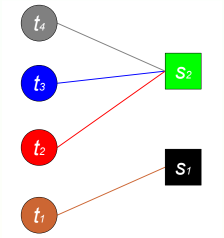
Mathematical Structures for Competitive Analysis

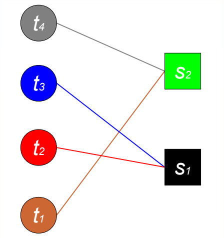
The proof of the competitive ratio upper bound makes use of two novel structures.

- Witness graphs.
- Token Production Systems (TPSs).

Witness Graphs

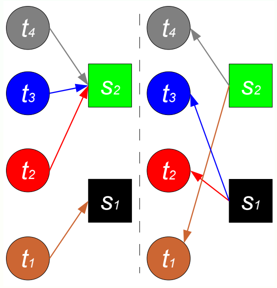
- Witness graphs are ordered weighted multidigraphs.
- Each witness graph embeds two assignments and allow them to be compared in the same framework.

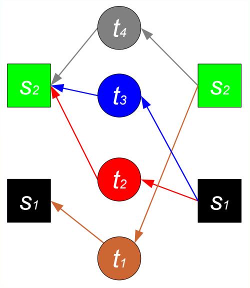




Witness Graph from Assignments

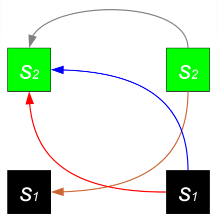
- Step 1: Add directions on edges.
- Step 2: Merge task vertices.

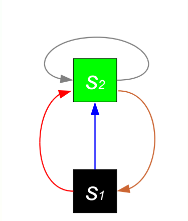


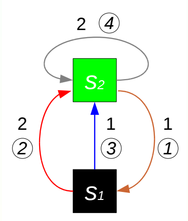


Witness Graph from Assignments (cont.)

- Step 3: Eliminate task vertices.
- Step 4: Merge server vertices.
- Step 5: Add edge weights and ordering.

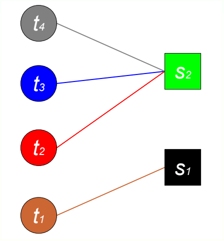


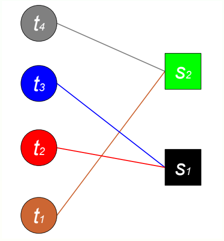


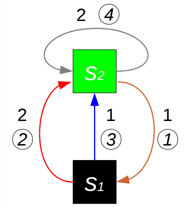


Witness Graph from Assignments (cont.)

- Each server is represented by a vertex, and each task is represented by an edge.
- Server load under Assignment 1 equals incoming edge weight.
- Server load under Assignment 2 equals outgoing edge weight.
- The total task load equals the total edge weight.

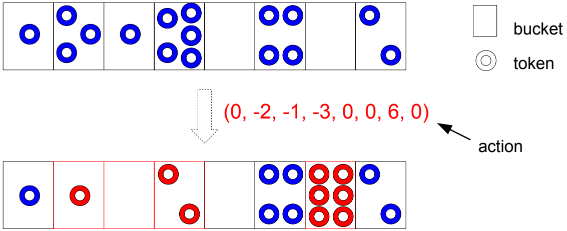






Token Production Systems

- A token production system consists of a list of buckets and some number of tokens.
- A set of vectors called actions rule the production and consumption of tokens in buckets. Each action has a cost.
- A sequence of actions form an activity and its cost is the sum of action costs.



Token Production Systems (cont.)

- A witness graph can be embedded into a token production system.
- Each vertex is represented by an action.
- The maximum weight of incoming edges over all vertices equals the number of buckets.
- The total edge weight equals the cost of an activity.

Outline

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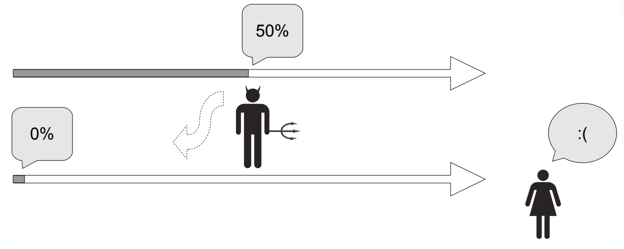
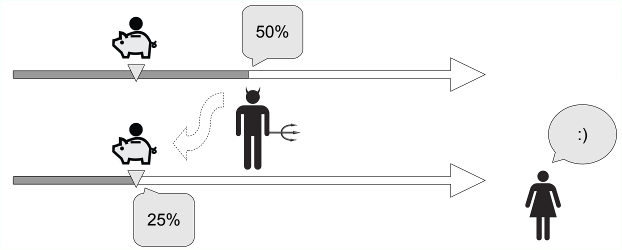
Long-Running Jobs Affected By Failures



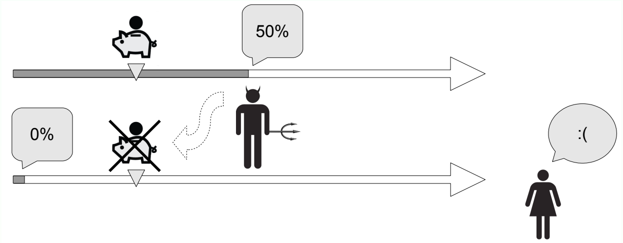
- Hardware failures are the norm, not the exception, in large systems.
- The chance of job completion without any failures becomes exponentially small with increasing job processing time.
- Assume a job execution fails. After faulty parts are fixed or replaced, jobs must be restarted and lost data must be recomputed, incurring substantial overhead.

Checkpoints Avoid Restart Overhead



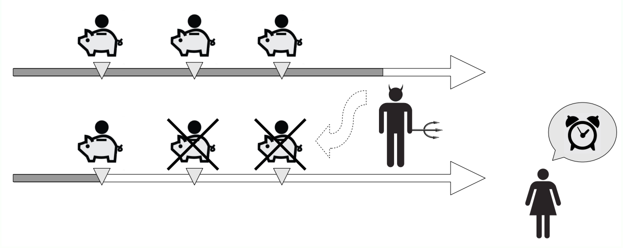
- Checkpoint: Save the job state from time to time.
- Rollback: When a failure is detected, a rollback operation restores the job state to one previously saved in a checkpoint, and recomputation begins from there.
- This avoids always restarting the job from the beginning.

However, Checkpoints Might Also Fail



- Checkpoints may fail and their failure probability can be significant.
- Faster job completion can sometimes be achieved by using less reliable but cheaper checkpoints.

Impact of Checkpoint Failures

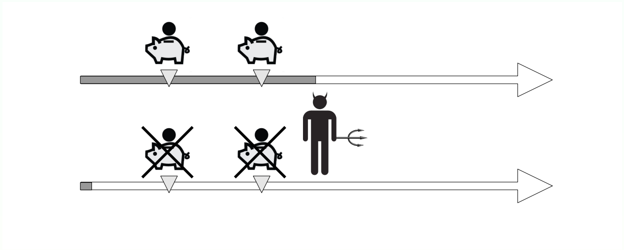


- Challenges:
  - How to evaluate the impact of checkpoint failures on job completion time?
  - How to place checkpoints to minimize the expected job completion time.

Job Execution with Unreliable Checkpoints

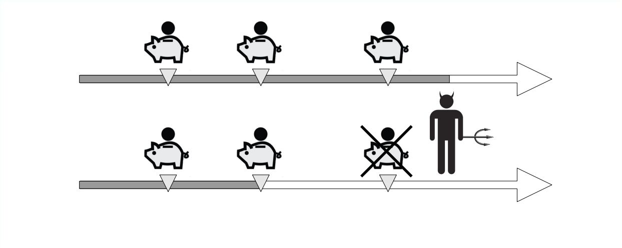
- A set of tasks $t_1, t_2, \cdots, t_n$ to be executed sequentially.
- Each run of t_i succeeds with positive probability p_i.
- Upon success, store the output of t_i into checkpoint c_{i+1}.
- When a run of t_i fails, recovery from checkpoint c_i is attempted. With probability q_i, the recovery succeeds and the job is resumed from there. If the access to c_i fails, access to the previous checkpoint c_{i-1} is attempted and repeat.
- Restarting the job from the beginning is always possible.

Illustration of a Job Execution



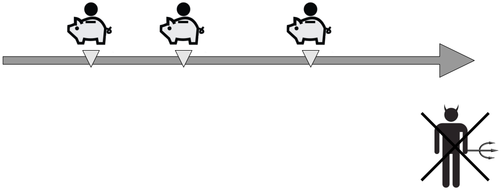
- When a task fails, accesses to checkpoints are attempted.
- When both checkpoints fail, the job is restarted from the beginning.

Illustration of a Job Execution (cont.)



- When another task fails, accesses to checkpoints are attempted.
- In this example, the access to the most recent checkpoint fails, but the second most recent one succeeds. Recovery resumes from there.

Illustration of a Job Execution (cont.)



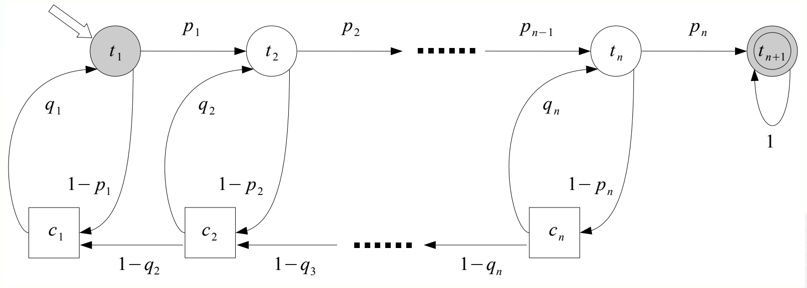
- Finally the job successfully completes.

Two Placement Models

- The discrete (modular program) model:
  - Task lengths are pre-determined.
  - Checkpoints can only be placed at task boundaries.
  - Task success probability may be independent of task length.
  - A task failure is detected at the end of the task run.
- The continuous model:
  - Checkpoints can be placed at arbitrary points in the job.
  - Task lengths are determined by distance between adjacent checkpoints.
  - Task failures follow the Poisson distribution. Task success probability depends on task length and failure rate.
  - A task failure is detected immediately after its occurrence.

Job Execution as a Markov Chain

- For each $1 \leq i \leq n$, state t_i represents the start of task t_i, and state c_i represents the attempted access to checkpoint c_i.
- For convenience, state t_{n+1} represents the job completion.
- A job execution is a random walk from state t_1 to state t_{n+1}.



Outline

Number of Successful Task Runs

- Let S_i be the number of successful runs of task t_i during a job execution.
- The sample space consists of all possible job executions.
- Then the expectation of S_i satisfies the recurrence relation

$$\begin{aligned} \mathbf{E}[S_n] &= 1 \\ \mathbf{E}[S_{i-1}] &= \left(\frac{1-q_i}{p_i}\right)\mathbf{E}[S_i] + q_i, \quad \text{for all } i \neq 1 \end{aligned}$$

Number of Successful Task Runs (cont.)

Extreme values of q:

$$\mathbf{E}[S_{i-1}] = \left(\frac{1-q_i}{p_i}\right)\mathbf{E}[S_i] + q_i$$

- If $q_i = 0$, $\mathbf{E}[S_{i-1}] = \mathbf{E}[S_i]/p_i$ (exponential growth). This is the case without any checkpoints.
- If $q_i = 1$, $\mathbf{E}[S_{i-1}] = 1$ (constant). This is the case with totally reliable checkpoints.

Growth Rate of Expected Completion Time

- Consider the uniform case where $p_i = p$ and $q_i = q$ for all i.
- $p + q$ is critical in determining the growth rate of expected job completion time as a function of job length n.
- Let $r = (1 - q)/p$.

$$T = \begin{cases} \frac{q}{p(1-r)} n + \Theta(1) = \Theta(n) & \text{if } p + q > 1, \\ \frac{q}{2p} n^2 + \frac{2-q}{2p} n = \Theta(n^2) & \text{if } p + q = 1, \\ \frac{1}{p(r-1)} r^n + \frac{q}{p(1-r)} n + \Theta(1) = \Theta\left(r^n\right) & \text{if } p + q < 1. \end{cases}$$

- System designers should configure checkpoints such that $p + q > 1$.

Checkpoint Retention Policy

- Sometimes the storage for checkpoints is limited. Assume at most k checkpoints can be stored at the same time.
- Consider the uniform case where $p_i = p$ and $q_i = q$ for all i such that $p + q > 1$.
- A simple retention policy keeps only the most recent k checkpoints and discards older ones.
- When $k = \Omega(\log n)$, the expected job completion time is $T = \Theta(n)$. The penalty is only a constant factor.

First Order Approximation to Optimum Interval

- In many systems, it is reasonable to make checkpoints over regular intervals, e.g., once an hour. One might want to decide the optimum checkpoint interval to minimize expected job completion time.
- Let q be checkpoint reliability, g be the cost for generating a checkpoint, and M be mean time between failure.
- When $g \ll M$, a first order approximation to the optimum checkpoint interval is

$$w = \sqrt{\left(\frac{q}{2-q}\right) 2gM}$$

- This expression generalizes the result $w = \sqrt{2gM}$ in the literature for totally reliable checkpoints.

Two Symmetry Properties

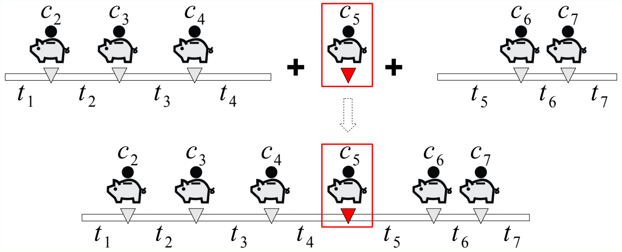
- Faster expected job completion can be achieved with the freedom to place checkpoints over non-equidistant points.
- Represent task lengths by a vector $\mathbf{w} = (w_1, \cdots, w_n)$. Let $\mathbf{w}^R = (w_n, \cdots, w_1)$ be its reversal.
- Two symmetry properties:

$$\begin{aligned} T(\mathbf{w}) &= T(\mathbf{w}^R) \\ T(\mathbf{w}) &\geq T\left(\frac{\mathbf{w} + \mathbf{w}^R}{2}\right) \end{aligned}$$

- Both symmetry properties are proved by induction and multiple applications of a weighted sum theorem.

Outline

Concatenation of Jobs



Concatenating two jobs with a checkpoint results in a larger job.

Concatenation of Jobs (cont.)

Theorem (Weighted Sum)

Let $T(q)$ be the expected completion time of the concatenation of two jobs with a checkpoint of reliability q. $T(q)$ satisfies

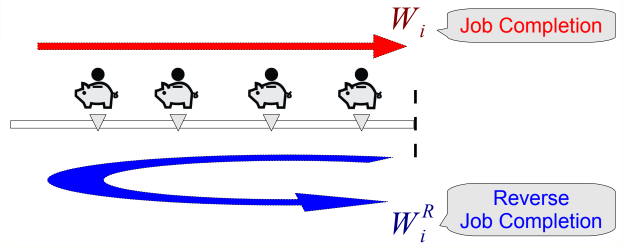
$$T(q) = qT(1) + (1-q)T(0)$$

In other words, $T(q)$ is the weighted sum of the two extreme cases where checkpoints are either totally reliable or totally unreliable.

Proof of the Weighted Sum Theorem

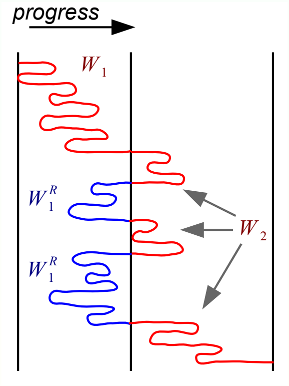
For job i, define

- W_i: expected completion time (left in right out).
- W_i^R: expected reverse completion time (right in right out).
- p_i: success probability.



Proof of the Weighted Sum Theorem (cont.)

$$T(q) = W_1 + W_2 + (1-q)\left(\frac{1}{p_2} - 1\right) W_1^R$$



- $W_1 + W_2$: Both jobs must complete.
- $(1-q)(1/p_2 - 1)$: On average, job 2 fails for $(1/p_2 - 1)$ times. Each time job 2 fails, access to the connecting checkpoint is attempted. Each access fails with probability $(1-q)$.
- W_1^R: Each time when the connecting checkpoint fails, job 1 must reverse complete.

Proof of the Weighted Sum Theorem (cont.)

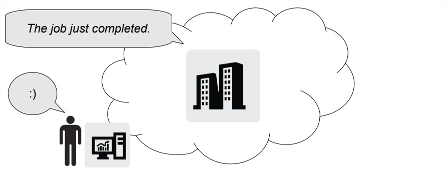
$$T(q) = W_1 + W_2 + (1-q)\left(\frac{1}{p_2} - 1\right) W_1^R$$

Plugging $q = 1$ and $q = 0$ into above equation gives

$$\begin{aligned} & qT(1) + (1-q)T(0) \\ = \quad & q\,(W_1 + W_2) + (1-q)\left(W_1 + W_2\left(\frac{1}{p_2} - 1\right) W_1^R\right) \\ = \quad & W_1 + W_2 + (1-q)\left(\frac{1}{p_2} - 1\right) W_1^R \\ = \quad & T(q) \end{aligned}$$

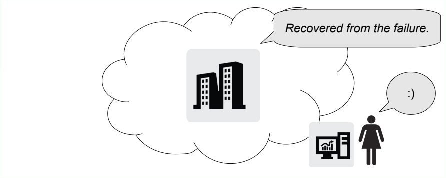
Outline

Efficiency



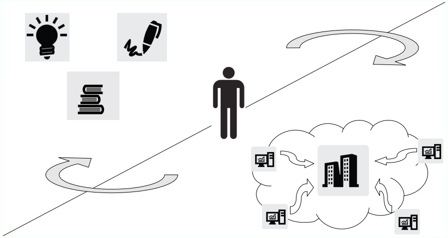
- A model for studying the problem of assigning tasks to servers so as to minimize job completion time.
- $\mathcal{NP}$-completeness result for HTA problem.
- Good approximation algorithms for task assignment.
- New competitive ratio bound for online GREEDY.

Fault Tolerance



- A model that includes checkpoint reliability in evaluating job completion time.
- $p + q$ determines the growth rate of expected completion time.
- Logarithmic checkpoint retention with constant factor penalty.
- First order approximation to optimum checkpoint interval.
- Two symmetry properties for non-equidistant placements.

Sidenotes on Non-Theoretical Work

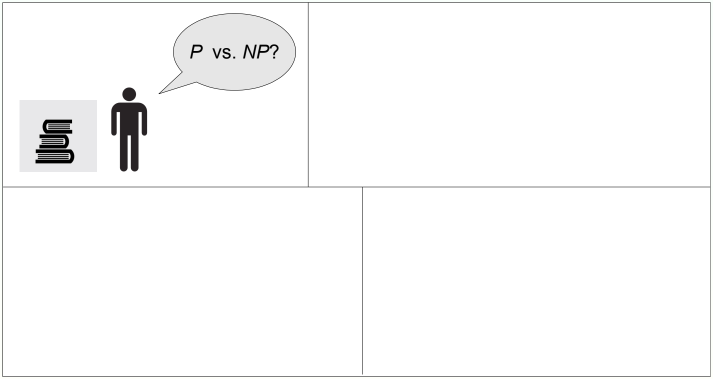


- Google: Chunk Allocation Algorithms for Colossus (the 2nd gen. of Google File System). *Do No Evil. No publication!*
- ORACLE®: Oracle In-Database Hadoop: When MapReduce Meets RDBMS. *In SIGMOD 2012 & a feature in Oracle Database 12c*.

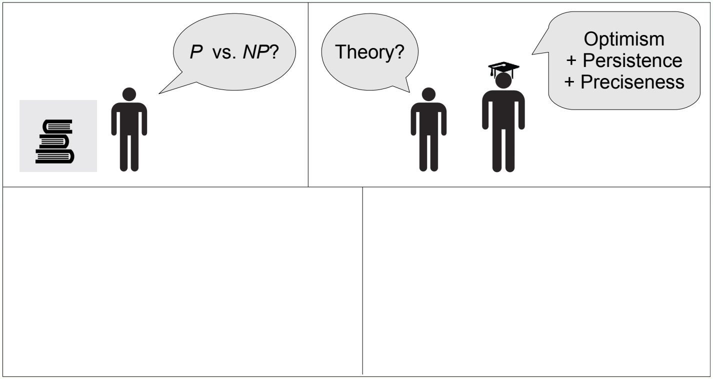
Before Finishing My Talk...

Acknowledgements

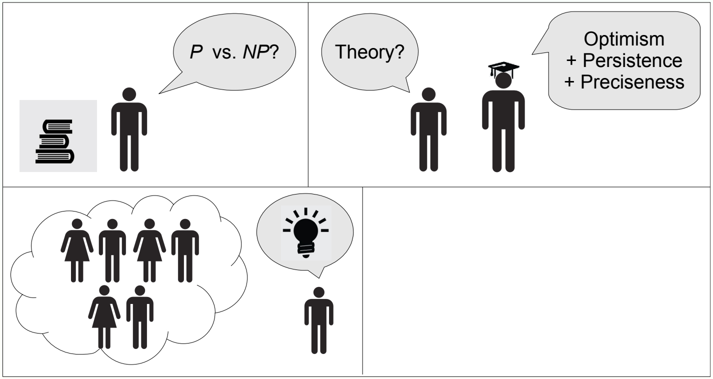
Hopeless Little X.



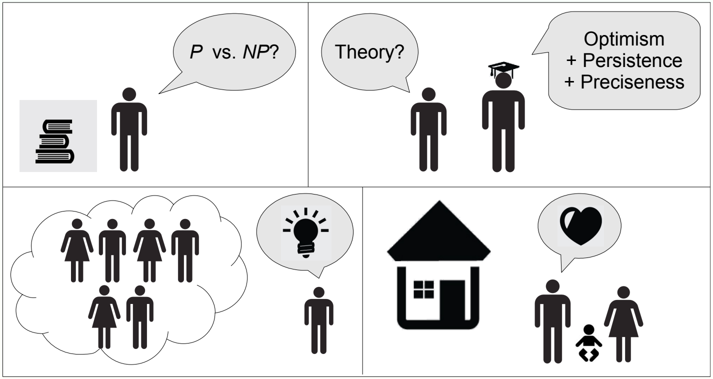
The Wise Mike



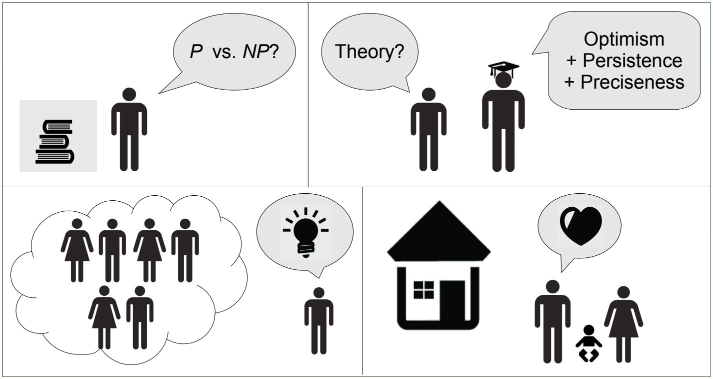
Inspiring Committee & Friends



Supportive Family



The End



Thank You!

Backup Slides

Backup Slides

Outline

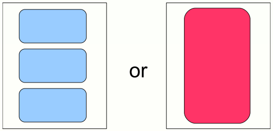
Proof Method

Given a 3CNF formula, we construct an instance of the HTA decision problem such that

- the local cost is 1,
- the remote cost is 3,
- the server capacity is 3.

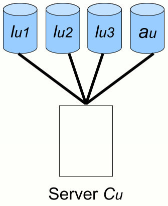
The resulting HTA instance has a task assignment of maximum load at most 3 if and only if the 3CNF formula is satisfiable.

Note that a server can be assigned at most 3 local tasks or 1 remote task.



A Piece of the Structure

- Let $C_u = (l_{u1} \vee l_{u2} \vee l_{u3})$ be a clause in the 3CNF formula and each l_{uv} be a literal.
- Construct a corresponding clause gadget, containing 1 clause server C_u, 3 literal tasks l_{u1}, l_{u2}, l_{u3} and 1 auxiliary task a_u.
- The server can only accept 3 local tasks, and thus 1 of the tasks needs to be assigned elsewhere.



Intuition Behind the Structure

There exists a feasible task assignment such that:

- The auxiliary task is assigned locally to its corresponding clause server.
- All false literal tasks are assigned locally to their corresponding clause servers.

Therefore, some literal task must be assigned elsewhere.
This will imply that the corresponding literal of the 3CNF formula is true.

Please refer to my thesis for more details...

Outline

A Weaker Approximation Bound

Lemma

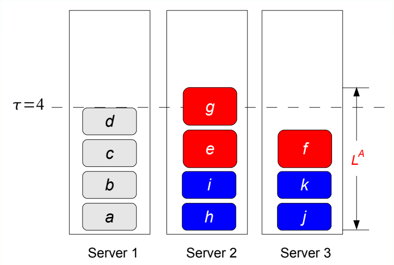
Let n be the number of servers, $L^A \leq L^O + (1 - 1/n)\, w_{\mathrm{rem}}^O$.

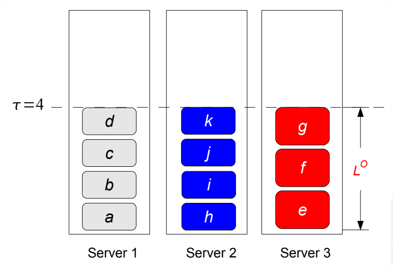
Proof intuition:

- Max-cover guarantees no fewer local tasks than O.
- Bal-assign balances remaining remote tasks.

Proof of the Approximation Bound

- Let K^A and K^O be the total number of local tasks under assignments A and O, respectively.
- Max-cover uses network flow to maximize the number of local tasks with respect to the threshold τ.
- When $\tau^A = \tau^O$, we have $K^A \geq K^O$.





Proof of the Approximation Bound (cont.)

- The number of local tasks $K^A \geq K^O$ implies that A assigns no more remote tasks than O.
- By the monotonicity of w_{rem}, we have $w^A_{\mathrm{rem}} \leq w^O_{\mathrm{rem}}$
- Let H^A and H^O be the total load under assignments A and O, respectively. It is straightforward that $H^A \leq H^O$.

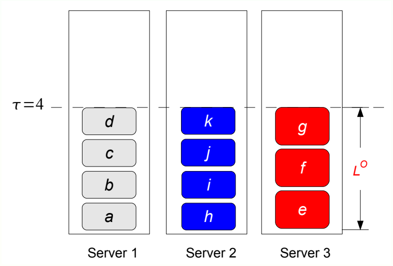
Proof of the Approximation Bound (cont.)

- The maximum load cannot be smaller than the average. Thus,

$$L^O \geq H^O / n$$

- This implies that

$$H^O \leq n \cdot L^O$$



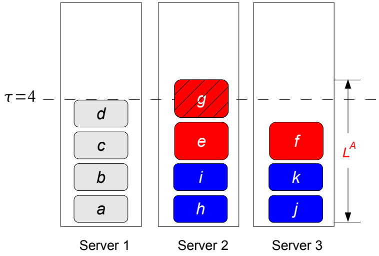
Proof of the Approximation Bound (cont.)

Consider the server with maximum load (server 2) and the last task assigned to it (task g).

- If g is assigned by max-cover, then $L^A = L^O = \tau \cdot w_{\text{loc}}$.
- If g is assigned by the greedy bal-assign, before g is assigned,

$$\forall s \neq 2, L_s^A \geq L^A - w_{\text{rem}}^A$$

$$\rightarrow \quad H^A \geq (n-1) \cdot (L^A - w_{\text{rem}}^A) + L^A$$



Proof of the Approximation Bound (cont.)

Combining the four inequalities completes the proof.

$$\begin{aligned}
w_{\mathrm{rem}}^{A} &\leq w_{\mathrm{rem}}^{O} \\
H^{A} &\leq H^{O} \\
H^{O} &\leq n \cdot L^{O} \\
H^{A} &\geq (n-1)\cdot(L^{A} - w_{\mathrm{rem}}^{A}) + L^{A} \\
&\rightarrow L^{A} \leq L^{O} + (1 - 1/n)\, w_{\mathrm{rem}}^{O}
\end{aligned}$$

Outline

Successful Task Runs vs. Checkpoint Failures

We prove this theorem by counting arguments.

- $\mathbf{E}[S_n] = 1$ as the job completes once task t_n succeeds.
- For $2 \leq i \leq n + 1$, consider task t_{i-1}.
  - t_{i-1} succeeds at least once.
  - Each time when checkpoint c_i fails, task t_{i-1} needs to be successfully re-executed to restore its output.
  - Let D_i be the number of times that checkpoint c_i fails.

$$\mathbf{E}[S_{i-1}] = 1 + \mathbf{E}[D_i]$$

Checkpoint Failures vs. Failed Task Runs

Consider checkpoint c_i.

- Each time when task t_i fails or checkpoint c_{i+1} fails, an attempted access to checkpoint c_i is made.
- Each attempted access fails with probability $(1 - q_i)$.
- Let F_i be the number of times task t_i fails.

$$\mathbf{E}[D_i] = (1 - q_i)(\mathbf{E}[F_i] + \mathbf{E}[D_{i+1}])$$

Failed Task Runs vs. Successful Task Runs

Task runs are repeated Bernoulli trials, and thus

$$\mathbf{E}[F_i] = \left(\frac{1}{p_i} - 1\right) \mathbf{E}[S_i]$$

Recurrence of Successful Task Runs

Combining the three equalities gives

$$\begin{aligned}
\mathbf{E}[S_{i-1}] &= 1 + \mathbf{E}[D_i] \\
&= 1 + (1 - q_i)(\mathbf{E}[F_i] + \mathbf{E}[D_{i+1}]) \\
&= 1 + (1 - q_i)\left(\left(\frac{1}{p_i} - 1\right)\mathbf{E}[S_i] + \mathbf{E}[S_i] - 1\right) \\
&= \frac{1 - q_i}{p_i}\,\mathbf{E}[S_i] + q_i
\end{aligned}$$

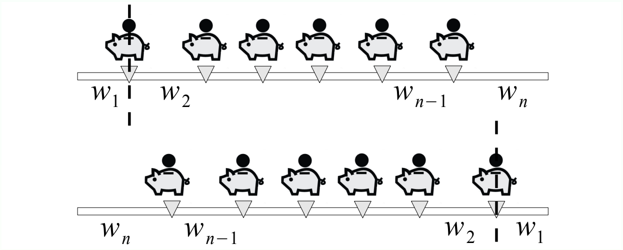
Outline

Proof of the First Symmetry

Inductive proof:

- The base case with $\ell = 1$ task is trivially true.
- Assume the claim holds for $\ell = n - 1$ tasks.
- Now consider the inductive step with $\ell = n$ tasks.

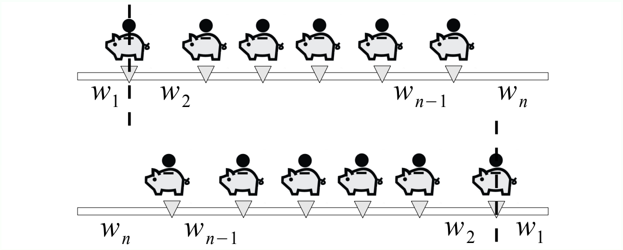
Proof of the First Symmetry (cont.)



- Apply the weighted sum theorem to the checkpoint after the first task in $(w_1, \cdots, w_n)$,

$$\begin{aligned} T_n(w_1, \cdots, w_n) &= q\left[T_1(w_1) + T_{n-1}(w_2, \cdots, w_n)\right] \\ &\quad + (1-q)\, T_{n-1}(w_1 + w_2, w_3, \cdots, w_n) \end{aligned}$$

Proof of the First Symmetry (cont.)



- Apply the weighted sum theorem to the checkpoint before the last task in $(w_n, \cdots, w_1)$,

$$
\begin{aligned}
T_n(w_n, \cdots, w_1) &= q\left[T_{n-1}(w_n, \cdots, w_2) + T_1(w_1)\right] \\
&\quad +(1-q)\,T_{n-1}(w_n, \cdots, w_3, w_2 + w_1)
\end{aligned}
$$

Proof of the First Symmetry (cont.)

By induction hypothesis,

$$\begin{aligned} T_{n-1}(w_2, \cdots, w_n) &= T_{n-1}(w_n, \cdots, w_2) \\ T_{n-1}(w_1 + w_2, w_3, \cdots, w_n) &= T_{n-1}(w_n, \cdots, w_3, w_2 + w_1) \end{aligned}$$

Proof of the First Symmetry (cont.)

Combining the four equalities completes the proof.

$$\begin{aligned} T_n(w_1, \cdots, w_n) &= q\,[T_1(w_1) + T_{n-1}(w_2, \cdots, w_n)] \\ &\quad +(1-q)\,T_{n-1}(w_1 + w_2, w_3, \cdots, w_n) \\ &= q\,[T_1(w_1) + T_{n-1}(w_n, \cdots, w_2)] \\ &\quad +(1-q)\,T_{n-1}(w_n, \cdots, w_3, w_2 + w_1) \\ &= T_n(w_n, \cdots, w_1) \end{aligned}$$

Outline

Construction of Helper Functions

Consider length vector $\mathbf{w} = (w_1, \cdots, w_n)$, where n is the vector length. For any $x, y \geq 0$, define a family of functions f_n.

- $f_0(\mathbf{w}, x, y) = 0$.
- For $n \geq 1$,

$$\begin{aligned} f_n(\mathbf{w}, x, y) &= T_n(x + w_1, w_2, \cdots, w_n) \\ &\quad + T_n(w_1, \cdots, w_{n-1}, w_n + y) \end{aligned}$$

Property of Helper Functions

Lemma

Let $\mathbf{w} = (w_1, \cdots, w_n)$ *be a length vector and* $\mathbf{w}^R = (w_n, \cdots, w_1)$ *be its reversal. For any* $n \geq 0$ *and any* $x, y \geq 0$*, we have*

$$f_n(\mathbf{w}, x, y) \geq f_n\left(\frac{\mathbf{w}+\mathbf{w}^R}{2}, \frac{x+y}{2}, \frac{x+y}{2}\right)$$

This Lemma is proved by induction on n and multiple applications of the weighed sum theorem. Then it follows immediately that

$$T(\mathbf{w}) = \frac{1}{2} f(\mathbf{w}, 0, 0) \geq \frac{1}{2} f\left(\frac{\mathbf{w}+\mathbf{w}^R}{2}, 0, 0\right) = T\left(\frac{\mathbf{w}+\mathbf{w}^R}{2}\right)$$