

OPTION PRICING WITH NON-CONSTANT VOLATILITY

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND APPLIED MATHEMATICS,

FACULTY OF SCIENCE

AT THE UNIVERSITY OF CAPE TOWN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Shih-Hsun Lin

March 2002

Supervised by

Peter W. Ouwehand

Abstract

For the past three decades, researchers have developed models to price options with non-constant asset price volatility. These models can be divided into deterministic volatility models and stochastic volatility models. Deterministic volatility models assume that volatility is determined by some variables observable in the market. Stochastic volatility models suggest that volatility follows a stochastic process, whose parameters are not directly observable in the market. However, most of these authors have compared the results of their models with the classical Black-Scholes model [6], which assumes that volatility is constant.

This dissertation investigates whether there is any model that can completely describe the market. Therefore, instead of comparing the results of the models with that of the Black-Scholes model, we have compared them with the market. For the purpose of this research, the S&P 500 Index option prices extracted from market are used. We investigate and compare for models: the GARCH(1,1) model, the Constant Elasticity of Variance model, the Hull and White model, and the Heston model. The former two belong to deterministic volatility models and the latter two are stochastic volatility models.

We conclude that none of the models under consideration can fully describe the market prices. Moreover, no model dominates the others by producing better results for all options.

Acknowledgement

I am most grateful to my supervisor, Dr Peter Ouwehand, for his guidance and helpful suggestions through the course of this research. I also thank my family for their support and patience throughout my period of study. Lastly, I would like to thank Fernando Durrell for his inspiration and input, which have been very helpful.

Contents

Abstract	i
Acknowledgement	ii
Contents	iii
1 Introduction	1
2 Background	3
2.1 Basics	3
2.1.1 Brownian Motion	3
2.1.2 Asset Price Dynamics	4
2.1.3 Estimating Volatility from Historical Data	5
2.2 The Cause of Volatility	6
2.3 Options	7
2.4 The Black-Scholes Model	8
2.4.1 The Black-Scholes PDE	8
2.4.2 The Black-Scholes Option Pricing Formulas	11
2.4.3 Effects of Dividend Yields	11
2.5 Risk-Neutral Valuation	11
2.6 The Feynman-Kac Formula	12
3 Problem Definition	14
4 Deterministic Volatility Models	18
4.1 Volatility as a Deterministic Function of Asset Price and Time	18
4.2 The Autoregressive Conditional Heteroscedasticity (ARCH) Model	19
4.3 The Exponentially Weighted Moving Average (EWMA) Model	21

4.4	The Geralised ARCH Model	22
4.4.1	GARCH(1,1)	23
4.4.2	Using GARCH(1,1) to Forecast Future Volatility	23
4.5	The Constant Elasticity of Variance (CEV) model.....	24
4.5.1	The Absolute Diffusion Process.....	26
4.5.2	The Square Root Diffusion Process	27
4.5.3	The Non-Central Chi-Square Distribution	27
5	Stochastic Volatility Models	29
5.1	The Two-Factor PDE.....	29
5.2	Stochastic Volatility Models	35
5.2.1	Power Series Method of Hull and White.....	36
5.2.2	Fourier Inversion Method of Stein and Stein	39
5.2.3	Heston Model with Closed-Form Solutions	41
6	Data and Methodology	47
6.1	Data	47
6.1.1	Background on the S&P 500 Stock Index	47
6.1.2	The S&P 500 Index Option	47
6.1.3	Data Screening Procedure	48
6.2	Methodology.....	48
6.2.1	Parameter Estimation.....	48
6.2.2	Estimating Parameters in GARCH(1,1) Model	49
6.2.3	Interpolating Option Prices in the CEV Model	50
6.2.4	Interest Rate and Dividend Yields.....	51
6.2.5	Measuring Model Performances.....	51
7	Results	53
7.1	Data Screening Result	53
7.2	Historical Volatility.....	53
7.3	Option Categories.....	53
7.4	Model Performances.....	54
7.4.1	GARCH(1,1) Model	54
7.4.2	The Constant Elasticity of Variance Model.....	57
7.4.3	Hull and White Model	59
7.4.4	Heston Model	60
7.5	Choosing between Models	61

8 Conclusion	64
A Program Listing	66
A.1 Black-Scholes Program: BS.m	66
A.2 CEV Program: f.m	66
A.3 CEV Program: CEV.m	67
A.4 CEV Program: Q1.m	67
A.5 CEV Program: Q3.m	68
A.6 Hull and White Program: HW.m	68
A.7 Heston Program: Heston.m	69
A.8 Heston Program: P1_integrand.m	70
A.9 Heston Program: P2_integrand.m	71
A.10 Heston Program: f1.m	71
A.11 Heston Program: f2.m	71
A.12 Heston Program: A1.m	72
A.13 Heston Program: A2.m	72
A.14 Heston Program: B1.m	72
A.15 Heston Program: B2.m	73
Bibliography	74

Chapter 1

Introduction

One problem that investors and traders face in a financial market is to obtain the correct value of an option. Black and Scholes [6] first developed their option pricing formula for valuing European options. They assumed that volatility, which measures the dispersion of an asset price about its mean, is constant.

However the Black-Scholes implied volatilities appear to vary systematically with respect to strike prices and time to maturities [1]. For example, if the Black-Scholes formula is used to determine the implied volatility for near-to-the-money options, the longer the time to maturity, the lower the implied volatility [24].

Many researchers, such as Cox and Ross [13] and Hull and White [24], have worked on solving the problem of option pricing with non-constant volatility. However, most of these authors have compared the results of their models with that of the Black-Scholes model. We argue that it is more important to see how well the model can capture the variations of the market than to see how well the model is comparing to the Black-Scholes model.

In this research, we will consider four models, which will be simulated using Matlab [26]. We will specifically look at the relative errors produced by the model prices with respect to the market prices. By comparing the relative errors, we expect to find the best model that can fully describe the market. The model sensitivities to changes in their parameter values will also be considered to examine whether the model is robust.

This dissertation is organised as follow: In Chapter 2, some of the basic techniques and results relevant in this research are reviewed briefly. The problems considered in this research are described in Chapter 3. In Chapter 4 and 5 respectively, the relevant deterministic and

stochastic volatility models are presented. The data and methodology used in this research are described in Chapter 6. In Chapter 7, the results of simulations are described and discussed. Summary and conclusions are given in Chapter 8.

A diskette is provided with this dissertation. In the diskette, there are programs written for this dissertation as well as the results of simulations.

Chapter 2

Background

In this chapter we review some of the fundamental ideas and techniques that will later be used or quoted in this research. A good source of these is Hull [23].

2.1 Basics

2.1.1 Brownian Motion

A Brownian motion with drift is a stochastic process $(X_t : t \geq 0)$ such that

1. $X_{t+s} - X_t \sim N(\mu t, \sigma^2 t)$. Here μ and σ^2 are constants. μ is the *drift rate*, and σ^2 is the *variance rate*. Also $X_0 = 0$.
2. For $0 \leq t_1 < t_2 < \dots < t_n$, the variables $(X_{t_2} - X_{t_1}), (X_{t_3} - X_{t_2}), \dots, (X_{t_n} - X_{t_{n-1}})$ are *independent*.
3. Sample paths are continuous a.s.

Here $X \sim N(\alpha, \beta^2)$ means that the random variable X is normally distributed with mean α and variance β^2 .

A *Standard Brownian Motion* (or a *Wiener process*) W_t is a Brownian motion with drift having $\mu = 0$ and $\sigma^2 = 1$. Thus $W_t \sim N(0, t)$. If W_t is a standard Brownian motion, then $X_t = \mu t + \sigma W_t$ is a Brownian motion with drift rate μ and variance rate σ^2 . We can write X_t in differential form as

$$dX_t = \mu dt + \sigma dW_t \quad (2.1)$$

Therefore the change in X_t is due to a deterministic term μdt and a random term σdW_t .

2.1.2 Asset Price Dynamics

Let S denote the price of a financial asset (e.g. a stock). Return of such a asset is defined as the change in asset price divided by the original asset price, i.e.

$$\frac{\Delta S}{S}.$$

Suppose that there is no risk (uncertainty) and the expected rate of return of S is μ . This means that over an amount of time Δt the return of S is expected to be $\mu\Delta t$. Thus

$$\frac{\Delta S}{S} = \mu\Delta t$$

In the limit as $\Delta t \rightarrow 0$, we get the ordinary differential equation (ODE)

$$\frac{dS}{S} = \mu dt \quad (2.2)$$

with solution

$$S(t) = S(0)e^{\mu t}.$$

In practice, uncertainties always do exist. Thus a more practical model for asset price process can be obtained if we add a random (non-deterministic) term to Equation (2.2)

$$\frac{dS}{S} = \mu dt + \sigma dW_t$$

or

$$dS = \mu S dt + \sigma S dW_t \quad (2.3)$$

where W_t is a standard Brownian motion. μ is the drift rate. σ is the *volatility*.

Equation (2.3) is called a *Geometric Brownian Motion*, and is the model most widely used to describe asset price process. It complies with the fundamental idea that the expected return and random effect are independent of the asset price.

By Ito's lemma,

$$d(\ln S) = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$$

Thus $(\ln S)_t$ is a Brownian motion with drift, and therefore

$$\ln S_T - \ln S_t \sim N\left[\left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right]$$

Thus

$$\ln \frac{S_T}{S_t} \sim N\left[\left(\mu - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right] \quad (2.4)$$

In other words, $\ln(S_T / S_t)$ is normally distributed with mean $(\mu - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$.

2.1.3 Estimating Volatility from Historical Data

The simplest way of estimating volatility is by looking backward. People extract information from the past, and use these to predict the future. This method is adequate to the extent that the future is like the past i.e. to the extent that history repeats itself.

Constant Volatility

Assuming no intermediate cash flows such as dividends, let

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

where S_i is the closing pricing of the asset at the end of the i th time-interval (usually every day, week, or month). Here $i = 0, 1, \dots, n$ for n observations in the data. Then $S_i = S_{i-1}\exp(u_i)$ and thus u_i is the continuously compounded asset return in the i th interval. The standard deviation of return, s , is therefore given by

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (u_i - \bar{u})^2}$$

where \bar{u} is the mean of the u_i 's.

From Equation (2.4), the standard deviation of the u_i , s is $\sigma\sqrt{T-t}$. Therefore σ can be estimated using $s/\sqrt{\tau}$ where s is the standard deviation of the u_i 's and $\tau = T - t$. This σ is referred to as the historical volatility. The standard error of this estimate is $\sigma/\sqrt{2n}$. For the purpose of this research, we shall look at the daily volatility i.e. u_i is the return for the i th day. Therefore $\tau = 1 \text{ day} = 1/252 \text{ years}$, considering there are 252 trading days in a year. Thus

$$\sigma = \frac{s}{\sqrt{1/252}} = s\sqrt{252}$$

In other words, the annualised volatility is estimated to be $\sqrt{252}$ times the standard deviation of the returns. This is the volatility we use as input for the Black-Scholes formula.

***m*-Windowed Moving Average Volatility**

It may not be appropriate to use all available historical asset prices when estimating historical volatility even if more data generally leads to more accuracy. The reason is that some data may be too old to be relevant when it comes to forecasting the future volatility level. An alternative way is to use an *m*-windowed moving average method. This method involves choosing an appropriate value for *m* to decide how long back we should consider when estimating volatility.

Therefore s_n^2 , the variance rate on day *n* based on the most recent *m* observations is given by

$$s_n^2 = \frac{1}{m-1} \sum_{i=1}^m (u_{n-i} - \bar{u})^2 \quad (2.5)$$

where

$$\bar{u} = \frac{1}{m} \sum_{i=1}^m u_{n-i}$$

m is usually chosen so that 30, 60, 90, or 180 trading days in the past are considered. When pricing options, the relevant volatility is not what has occurred in that past, but what is expected to happen in the future. However, historical volatility is useful when volatility is considered to be auto-correlated through time.

2.2 The Cause of Volatility

There has been some dispute about what causes volatility. Some authors claim that the volatility of an asset price is caused solely by the random arrival of new information about future returns from a stock, whereas others claim that it is caused mostly by trading.

The proponents of the first view base their ideas on the *Efficient Market Hypothesis*. In a efficient market, investors' expectations about future returns are (correctly) based upon all

information available, and share price reflects all the information currently available. Thus new information instantaneously adjusts the share price. This can be summarised, in mathematical terms, by saying that $(S_t : t \geq 0)$ is a *Markov process*:

$$E(S_{t_n} | S_{t_1}, \dots, S_{t_{n-1}}) = E(S_{t_n} | S_{t_{n-1}}).$$

The empirical research conducted by Fama [17] tested this argument. Using stock closing prices over a long period of time, he calculated:

1. The variance of stock returns between the closes of two consecutive trading days.
2. The variance of stock returns between the closes on Fridays and on Mondays.

If trading and non-trading days are equivalent, the second case should have variance three times greater than the first case. Fama [17] found that it was only 22% greater.

This result suggests that volatility is greater during trading hours than non-trading hours. People may argue that most new information on stock arrives during trading hours. However, studies of futures prices on agricultural commodities, which depend mostly on the weather, have shown similar results; that is, they are more volatile during trading hours [23]. Therefore a reasonable conclusion is that volatility is caused mostly by trading.

2.3 Options

Hull [23] defines a *derivative* (or derivative security) as

...a financial instrument whose value depends on the value of other, more basic underlying variable.

There are many different types of derivatives such as *forwards*, *futures*, *swaps* ... etc. *Option* is one of the derivative securities. Options on stocks were first traded in 1973 and are now traded in many exchanges throughout the world. The underlying assets include stocks, stock indices, foreign currencies, debt instruments, commodities, and futures contract.

There are two basic types of options. A *call option* gives the holder the right, but not obligation, to buy the underlying asset for a predetermined price (the *exercise price* or *strike price*) at a predetermined date (the *expiry date* or *maturity*). A *put option* gives the holder the right to sell the underlying asset for some strike price at the maturity date.

There are two styles of options. *European* options can only be exercised at maturity, whereas *American* options can be exercised any time prior to the maturity date.

By arbitrage arguments, it can be shown that European call option and put option must satisfy the put-call parity

$$C - P = S - PV(K) \quad (2.6)$$

where C and P represent prices of European call and put respectively, S is the price of the underlying asset, and $PV(K)$ is the present value of the strike price K .

2.4 The Black-Scholes Model

2.4.1 The Black-Scholes PDE

In 1973, Black and Scholes [6] published their model on option pricing, which later becomes the famous Black-Scholes model. The assumptions of their model are as follows:

1. The underlying asset price follows a geometric Brownian motion described in Equation (2.3), with constant drift rate μ and constant volatility σ .
2. Short selling is permitted.
3. There are no transaction costs or taxes.
4. There are no dividends paid during the life of the option.
5. There are no arbitrage opportunities.
6. Assets are perfectly divisible.
7. Security trading is continuous.
8. The risk-free interest rate remains constant until option maturity.

Let $V(S, t)$ be the value of a derivative (e.g. an option), which depend on S , the price of the underlying asset, and time t . Consider a portfolio Π , containing:

$$\begin{aligned} -1: & \quad \text{derivatives} \\ +\Delta: & \quad \text{assets} \end{aligned}$$

The value of Π is then

$$\Pi = -V + \Delta S$$

and the change of portfolio $d\Pi$ is:

$$d\Pi = -dV + \Delta dS$$

following the derivation of Black and Scholes [6]¹.

Since V is a function of S and t , then by Ito's lemma

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2$$

from Equation (2.3), we get $(dS)^2 = \sigma^2 S^2 dt$. Then

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (\mu S dt + \sigma S dW_t) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (\sigma^2 S^2 dt) \\ &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW_t \end{aligned}$$

and thus

$$\begin{aligned} d\Pi &= -dV + \Delta dS \\ &= -\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - \sigma S \frac{\partial V}{\partial S} dW_t + \Delta (\mu S dt + \sigma S dW_t) \\ &= -\left(\frac{\partial V}{\partial t} + \mu S \left(\frac{\partial V}{\partial S} - \Delta \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - \left(\frac{\partial V}{\partial S} - \Delta \right) \sigma S dW_t \end{aligned}$$

If we let

$$\Delta = \frac{\partial V}{\partial S}$$

we can eliminate randomness by setting the dW_t parts to zero. Thus

¹ According to Ito's lemma, we should have

$$d\Pi = -dV + \Delta(dS) + S(d\Delta) + (d\Delta)(dS).$$

The statement that

$$d\Pi = -dV + \Delta dS$$

is erroneous. Some authors attempt to justify this by claiming that Δ is “instantaneously constant” or by claiming that the portfolio $(-1, +\Delta)$ is self-financing. Carr [10] pointed out that both of these claims are false. One possible fix would be to enforce the self-financing condition by varying also the option holding, not just the stock holding.

$$\begin{aligned}
d\Pi &= -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \\
&= r\Pi dt \\
&= r(-V + \Delta S_1)dt \\
&= (-rV + r\Delta S)dt
\end{aligned}$$

since Π is now riskless.

We therefore have

$$-rV + r\Delta S = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)$$

which becomes

$$rV - rS \frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Therefore we have

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \quad (2.7)$$

Equation (2.7) is the well-known Black-Scholes partial differential equation (B-S PDE). It is a linear, 2nd order, parabolic diffusion equation. Since the Black-Scholes PDE was derived without specifying which type of derivative V is, it can be applied to many financial instruments such as stocks ($V = S$), and money market account ($V = V_0 e^{rt}$).

The price of a particular type of derivative can be obtained by solving the Black-Scholes PDE with the correct auxiliary conditions, which in this case are the *final conditions*. For example, a European call with strike price K and expiry T have final condition as its payoff function

$$C(S, T) = \max\{S - K, 0\} = (S - K)^+.$$

And in the case of a European put with the same strike price and maturity,

$$P(S, T) = \max\{K - S, 0\} = (K - S)^+.$$

2.4.2 The Black-Scholes Option Pricing Formulas

The Black and Scholes prices for European call options on a non-dividend-paying stock is

$$C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \quad (2.8)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$$

S is the current stock price, K is the strike price, r is the continuously compounded risk-free interest rate, σ is the volatility of stock price, t is the current time, and T is the maturity time of the option. The price of a European put can then be calculated using the put-call parity in Equation (2.6).

2.4.3 Effects of Dividend Yields

Suppose the underlying asset pays a continuously compounded dividend yield at rate q . The price for European call options is then

$$c(S, t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \quad (2.9)$$

where

$$d_1 = \frac{\ln(S/K) + (r - q + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

and the put-call parity becomes

$$C(S, t) - P(S, t) = Se^{-q(T-t)} - Ke^{-r(T-t)} \quad (2.10)$$

2.5 Risk-Neutral Valuation

Risk-neutral valuation was motivated by the observation that the Black-Scholes option pricing formula does not depend on any variables that reflect investors' risk preference. All variables that appear in the formula are independent of risk preferences. Thus option prices are the same irrespective of what forms the investors' risk preference are. Therefore the assumption of risk neutrality can be made.

In the risk-neutral world, investors prefer more wealth to less wealth with concerning about risks. Therefore, in equilibrium, all assets must have the same rate of return, and that is the risk-free interest rate. This means that all assets have drift rate $\mu = r$ in Equation (2.3).

The principle of risk-neutral valuation is that the initial value of a financial asset (or a contingent claim) is equal to the expected value of the discounted future cash flow, i.e.

$$V_0 = E_Q[e^{-rT}V_T] \quad (2.11)$$

where Q is the risk-neutral measure associated with the risk-neutral world. In other words, Q is a measure such that, for any asset price S ,

$$E_Q[S_T] = S_0 e^{rT}.$$

So that the rate of return for any asset under risk-neutral measure is r .

For example, applying this principle to valuing a European call we get

$$C(S_0, 0) = e^{-rT} E_Q[(S - K)^+]. \quad (2.12)$$

2.6 The Feynman-Kac Formula

Assume $F : \mathfrak{R} \times \mathfrak{R}^n \rightarrow \mathfrak{R}$ is the solution of

$$\begin{cases} \frac{\partial F(t, x)}{\partial t} + \sum_{i=1}^n \mu_i(t, x) \frac{\partial F(t, x)}{\partial x_i} + \frac{1}{2} \sum_{i,j} c_{ij}(t, x) \frac{\partial^2 F(t, x)}{\partial x_i \partial x_j} + r(t, x) F(t, x) = 0 \\ F(T, x) = \Phi(x) \end{cases}$$

where $x = (x_1, x_2, \dots, x_n)$, c is an $n \times n$ matrix which can be represented as $c = \sigma \sigma^T$ for some $n \times d$ matrix $\sigma(t, x)$.

Then F has representation

$$F(t, x) = E_{t,x} \left[\Phi(x_T) \exp \left\{ \int_t^T r(u, X_u) du \right\} \right] \quad (2.13)$$

where X_s is a solution of the n -dimensional stochastic differential equation

$$\begin{cases} dX_s = \begin{pmatrix} \mu_1(s, X_s) \\ \vdots \\ \mu_n(s, X_s) \end{pmatrix} ds + \sigma(s, X_s) dW_s \\ X_t = x \end{cases}$$

where W_s is a d -dimensional standard Brownian motion.

Equation (2.13) can also be written as

$$F(t, x) = E_Q \left[\Phi(x_T) \exp \left\{ \int_t^T r(u, X_u) du \right\} \middle| X_t = x \right].$$

There are two distinctive approaches to the Mathematics of Finance. One is via PDE's and the other is by risk-neutral valuation. The Feynman-Kac formula is really the link between the two separate routes of Mathematics of Finance. For example we look at the Black-Scholes model. If we go with the PDE approach, we use the arbitrage argument to derive the Black-Scholes PDE, and then we can use the Feynman-Kac formula to solve it and the solution. If we go with the risk-neutral valuation approach, we would first determine the risk-neutral dynamics of asset price (since we know what the rate of return of any asset must be r). In doing so, we have fixed the risk-neutral measure. By doing Feynman-Kac in reverse, we can then get the Black-Scholes PDE.

Chapter 3

Problem Definition

One important assumption that Black and Scholes [6] made in deriving their option pricing formula is that the option's underlying asset price follows a geometric Brownian motion. That is

$$\frac{dS}{S} = \mu dt + \sigma dW_t$$

where S is the asset price, μ and σ are the drift rate and the volatility respectively. W_t is a standard Brownian motion. Volatility is a measure of the dispersion of asset price about its mean level. The Black-Scholes model assumes that this volatility to be constant over the life of the option.

The Black-Scholes option pricing formula depends on six parameters:

- S = current underlying asset price,
- K = strike price,
- r = continuously compounded risk-free interest rate,
- q = continuously compounded dividend yield of the underlying asset,
- σ = volatility of the asset, and
- τ = time to maturity.

All of the above parameters, except volatility, are “observable” from the market. Thus, it is extremely important to obtain a correct volatility estimate.

The constant volatility assumption of Black and Scholes [6] was soon challenged. While the formula was originally intended to calculate the price of an option where the volatility is a constant, the reverse can be done. In other words, given an option price, one can use

Black-Scholes formula to find the corresponding volatility input. This value is often referred to as the *implied* (or implicit) volatility, since it is implied by the option price. Under Black-Scholes assumption, implied volatilities from options should be the same regardless of which option is used to compute the volatility. However, this is not the case in practice. Implied volatility appears to be dependent on option maturities as well as strike prices. Figure 3.1 and Figure 3.2 show the dependence of volatility on time and strike price respectively.

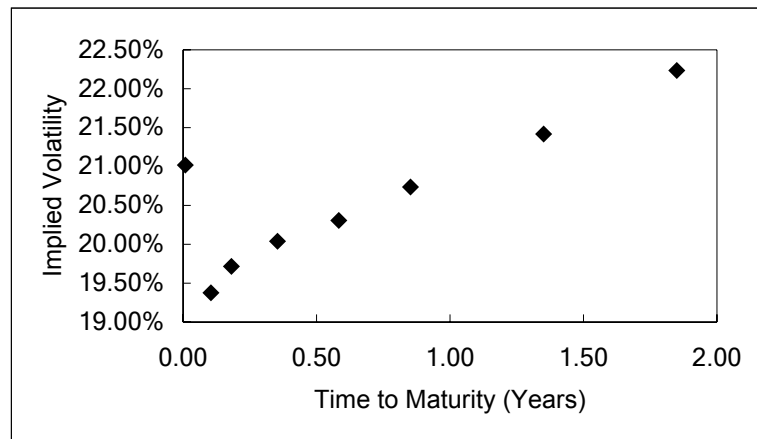


Figure 3.1: Implied volatility vs. Time to Maturity of S&P 500 index options with strike price 1200 points on August 15, 2001.

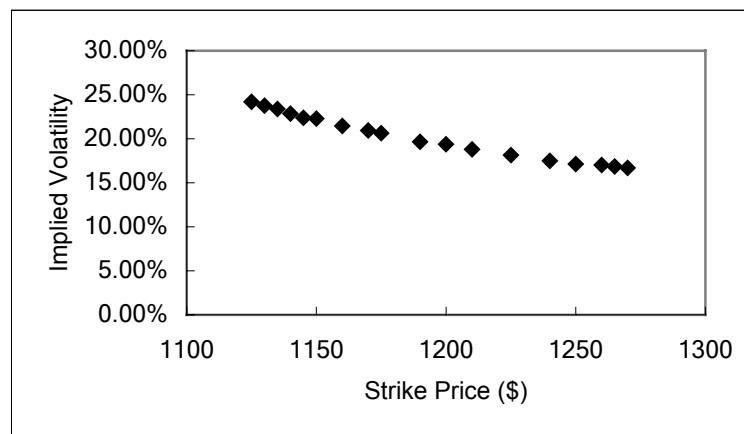


Figure 3.2: The volatility smile implied by the S&P 500 index options on August 15, 2001.

The pattern of the Black-Scholes implied volatilities with respect to strike prices is known as the volatility smile. Figure 3.2 shows such a smile. The downward slope is typical for equity and index options.

If asset prices follow a geometric Brownian motion then returns are normally distributed. However, empirical studies done on the asset returns distribution conducted by pointed out that, in practice, the observed frequency of extreme asset returns is greater than the expected frequency from a normal model². This means that extreme events are far more likely than what would be predicted by a normal distribution. As a rule of thumb³:

1. Every financial market experiences one or more daily prices moves of 4 standard deviations or more per year. However, the normal distribution implies that such a move would occur maybe once every 125 years.
2. Moreover, every year there is usually one market that has a daily move of >10 standard deviations.

Figure 3.3 shows the frequency plot of the S&P 500 Index daily returns from August 15, 1996 to August 14, 2001. It can be seen that it is not perfectly normal as there are extreme values at both the left and right tails.

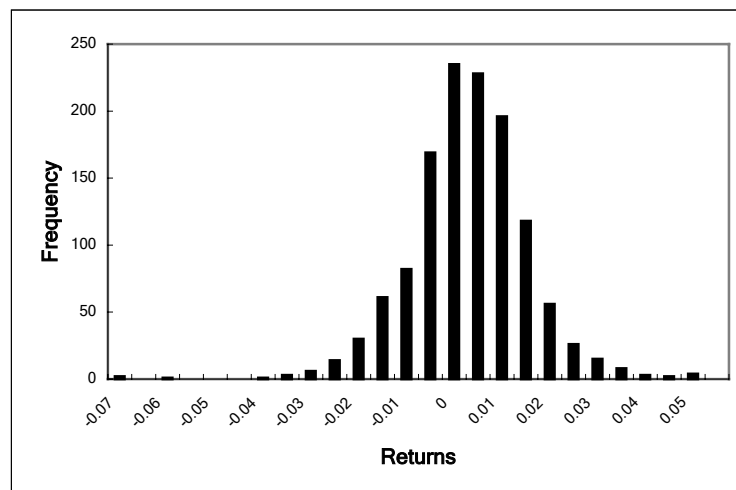


Figure 3.3: Frequency plot of the S&P 500 Index daily returns.

² See Mandelbrot [25] and Fama [17].

³ Source: Financial Risk Manager Handbook 2001-2002: P. Jorion, John Wiles 2001.

Also the left tail is thicker than the right tail. This observation complies with the distribution implied by Figure 3.2. Since high implied volatilities are implied by high option prices, we can say that low strike options are valued higher and high strike options are valued lower than they would be valued using a lognormal distribution. Thus the implied distribution must have a thicker left tail and a thinner right tail.

The existence of volatility smile and the observed thick tails in the empirical distributions of asset returns are proofs that the constant volatility assumption in Black-Scholes model is violated.

Because volatility plays a central role in option pricing, precise modelling of volatility is important. Therefore a number of researchers have suggested alternative models for option pricing, which can incorporate non-constant volatility. These can be divided into two types. The first type suggests that volatility satisfy some deterministic relationships with known variables, such as asset price or returns. Models of this genre are known as the *deterministic volatility* models and include the *Autoregressive Conditional Heteroscedasticity* (ARCH) family [7, 16, 29] and the *Constant Elasticity of Variance* (CEV) models [13, 34]. The other type, which argues that volatility follows a different random process from that of the asset price, and cannot be directly observed. Works of this genre include Hull and White [24], Scott [35], Wiggins [37], Melino and Turnbull [27] (for foreign currency options), Stein and Stein [36], Heston [18], and Ball and Roma [4]. These are in general called the *stochastic volatility* models.

In this research, two deterministic and two stochastic volatility models are selected for investigation. The two deterministic volatility models are the GARCH(1,1) model [7] and the CEV model [13, 34]. The two stochastic volatility models are the Hull and White model [24] and the Heston model [18]. These models are implemented in order to find the model that holds the best solution to the non-constant volatility problem.

Chapter 4

Deterministic Volatility Models

The simplest relaxation of the constant volatility assumption is to allow future volatility being completely determined by its past. In other words, future volatility can be perfectly predicted from its history and possibly some other observable information. Consider the variance of asset returns σ_{t+1}^2 being describe by the following equation:

$$\sigma_{t+1}^2 = \theta + \kappa \sigma_t^2$$

The future volatility can be completely determined by a constant and a constant proportion of last period's volatility. This is a simple example of deterministic volatility.

4.1 Volatility as a Deterministic Function of Asset Price and Time

The Black-Scholes implied volatility varies systematically with respect to option's strike price and time to maturity. Dumas, Fleming, and Whaley [14] had done empirical tests on deterministic volatility models using samples of S&P 500 index options. They specified 4 different volatility functions intending to capture variation with asset price and time. The details can be found in [14]. Their major findings are:

1. Deterministic volatility models always do better than the constant volatility (Black-Scholes) model, because of the flexibility of the volatility function's specification.
2. However, when the fitted volatility function is used to value options one week later, the model's prediction does not appear to be an improvement of the traditional, while inconsistent, Black-Scholes model.

3. Hedge ratios determined by Black-Scholes model appear to be more reliable than those obtained from the deterministic volatility option valuation models.

The authors point out that the reason why “simpler is better” is because errors, from various sources, in the quoted option prices distort parameter estimates for deterministic volatility models and thus downgrade these models’ predictions.

4.2 The Autoregressive Conditional Heteroscedasticity (ARCH) Model⁴

The major flaw when using the m -windowed moving average method to estimate volatility is that it will eventually “forget” a particular observation once the observation is out of the window. This is best demonstrated by looking at events in the history. On October 23, 1997, stock markets around the world dipped in response to a 10 percent drop in the Hong Kong market. The crash resulted from a continuing monetary crisis in Southeast Asia that forced Hong Kong to raise its interest rates to stabilize its currency. Figure 4.1 shows the annualised volatility using a 90-days windowed method.

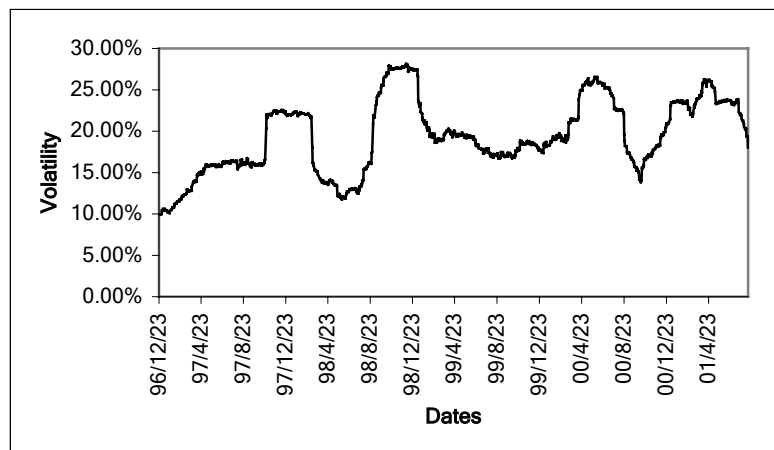


Figure 4.1: 90-day windowed volatility for S&P 500 index.

⁴ Hull [23] contains a thorough discussion of the ARCH family models (ARCH, EWMA, and GARCH), which are relevant in this research.

It can be seen, from Figure 4.1, that the sudden crash led to a rapid increase in volatility, as expected. The volatility level then remained high for a period of time. But around March 1998, about 90 trading days after the beginning of the crash, the volatility level suddenly dropped. This is because the 90-days window has already “forgotten” the crash completely. The same “skyscraper” pattern can also be seen from August 1998 to January 1999 and from April to August in 2000.

This feature of the m -windowed moving average method is contrary to our intuition. A key consideration of traders and risk managers is hedging performance, where they must know how much risk they are exposed to. As volatility measures risk, investors must know what the volatility level is at all times. Suppose an investor has suffered great loss during the 1997 crash, could he have forgotten about the crash after a period of time? No, because history is believed to repeat itself and the investor will certainly keep that in mind when predicting future volatility levels.

If we now define u_i as the proportional change in the asset price during day i i.e.

$$u_i = \frac{S_i - S_{i-1}}{S_i}$$

then \bar{u} is assumed to be zero and this allows us to use the maximum likelihood estimator rather than Equation (2.5), which is the unbiased estimator (with m replacing $m - 1$).

Thus,

$$\sigma_n^2 = \frac{1}{m} \sum_{i=1}^m u_{n-i}^2 \quad (4.1)$$

Equation (4.1) gives equal weights to all u_i^2 's. However, we may want to give more weight to more recent observations given that the objective is to monitor the current volatility level. Consider then the model

$$\sigma_n^2 = \sum_{i=1}^m \alpha_i u_{n-i}^2 \quad (4.2)$$

where $\alpha_i (> 0 \text{ for all } i)$ is the weight assigned to the observation i days ago. Since more recent observation should have more weight, we require $\alpha_i > \alpha_j$ if $i < j$. Also the sum of all weights must be one i.e.

$$\sum_{i=1}^m \alpha_i = 1$$

Next suppose that there exists a long-run average variance rate, V , which should be assigned with some weight γ . The model then become

$$\sigma_n^2 = \gamma V + \sum_{i=1}^m \alpha_i u_{n-i}^2 \quad (4.3)$$

The weights must still sum to one and thus

$$\gamma + \sum_{i=1}^m \alpha_i = 1$$

This is the ARCH(m) model introduced by Engle [16]. The term *autoregressive* in ARCH refers to the element of persistence in volatility, and the term *conditional heteroscedasticity* describes the presumed dependence of current volatility on the level of volatility realised in the past. The estimate of variance is based on a long-run average variance and m observations.

4.3 The Exponentially Weighted Moving Average (EWMA) Model

A special case of the model in Equation (4.2) is that α_i decreases exponentially as i increase. Specifically, $\alpha_{i+1} = \lambda \alpha_i$ where λ is a constant between zero and one (typically $0.9 < \lambda < 1$). This weighting scheme turns out to have a simple formula when updating volatility, which is

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2 \quad (4.4)$$

Therefore σ_n , the estimate of volatility for day n , is determined from σ_{n-1} , the estimate at day $n - 1$, and u_{n-1} , the proportional change in asset price at day $n - 1$.

With simple substitutions in Equation (4.4), it can be shown that

$$\sigma_n^2 = (1 - \lambda) \sum_{i=1}^m \lambda^{i-1} u_{n-i}^2 + \lambda^m \sigma_0^2$$

For large m , the term $\lambda^m \sigma_0^2$ is sufficiently small and become neglected so that Equation (4.4) is the same as Equation (4.2) with $\alpha_i = (1 - \lambda) \lambda^{i-1}$. The weights for u_i 's thus decline at a rate λ as i increases.

The parameter λ describes how responsive the estimates of daily variance with respect to the most recent observations on the u_i^2 's. A high value of λ (i.e. close to 1) implies that relatively

less weight is assigned to the u_i^2 's and thus the estimates of daily volatility respond relatively slowly to new information on the asset price. A low λ implies recent changes in the asset price have a great impact on volatility. In this case, the estimates of volatility on successive days will be volatile.

We now look at how the “memory” of EWMA model works. From Figure 4.2 below, it can be seen that the EWMA volatility increase rapidly on the day of the market crash, October 23, 1997, as the 90-days windowed method does. However, it does not suddenly “forget” about the crash after a period of time, like the 90-days windowed method does. Instead, it decreases slowly as there were no sudden big changes in the asset price while it “keeps the incident in its memory”.

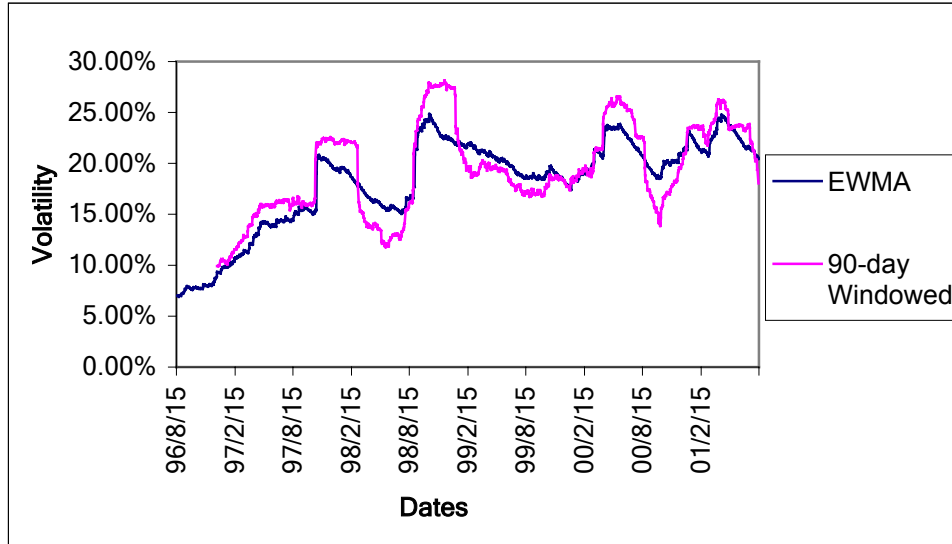


Figure 4.2: EWMA method vs. 90-day windowed method. The EWMA volatilities are calculated using $\lambda = 0.99$.

4.4 The Generalised Autoregressive Conditional Heteroscedasticity (GARCH) Model

There are many different types of modified ARCH models that have applications in economics and finance. In finance, one of the most popular ARCH models is the generalised ARCH (GARCH), which was introduced by Bollerslev [7]. In the general GARCH(p,q) model, σ_n^2 is

calculated from the most recent p observations on u^2 and the most recent q estimates of the variance rate.

4.4.1 GARCH(1,1)

In this research, we shall only look at GARCH(1,1) model, which is the most popular of GARCH models. In GARCH(1,1), σ_n^2 is based on the most recent observation on u^2 and the most recent estimate of the variance rate.

The difference between GARCH(1,1) model and EWMA model is analogous to the difference between Equation (4.2) and Equation (4.3). In a GARCH(1,1) model, there exists a long-run average variance rate, V . The model is

$$\sigma_n^2 = \gamma V + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \quad (4.5)$$

where γ , α , and β are weights and must sum to one. EWMA model is a special case of GARCH(1,1) model with $\gamma=0$, $\alpha=1-\lambda$, and $\beta=\lambda$.

GARCH(1,1) model can also be written as

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \quad (4.6)$$

where $\omega = \gamma V$. The parameters ω , α , and β can be estimated using a maximum likelihood method [23].

4.4.2 Using GARCH(1,1) to Forecast Future Volatility

Substituting $\gamma = 1 - \alpha - \beta$ in Equation (4.5), the variance rate estimated for day n is

$$\sigma_n^2 = (1 - \alpha - \beta)V + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

so that

$$\sigma_n^2 - V = \alpha(u_{n-1}^2 - V) + \beta(\sigma_{n-1}^2 - V)$$

On day $n+k$ in the future we have

$$\sigma_{n+k}^2 - V = \alpha(u_{n+k-1}^2 - V) + \beta(\sigma_{n+k-1}^2 - V)$$

The expected value of u_{n+k-1}^2 is σ_{n+k-1}^2 . Therefore

$$E[\sigma_{n+k}^2 - V] = (\alpha + \beta)E[\sigma_{n+k-1}^2 - V]$$

Using this equation repeatedly yields

$$E[\sigma_{n+k}^2 - V] = (\alpha + \beta)^k (\sigma_n^2 - V)$$

or

$$E[\sigma_{n+k}^2] = V + (\alpha + \beta)^k (\sigma_n^2 - V)$$

V can be calculated by ω / γ where $\gamma = 1 - \alpha - \beta$ and ω , α , and β are estimated using maximum likelihood methods. Suppose it is day n and an option matures at day $n + N$. To price this option we can use Black-Scholes formula with variance input

$$\frac{1}{N} \sum_{k=0}^{N-1} E[\sigma_{n+k}^2] \quad (4.7)$$

which is the expected variance rate during the life of the option. The longer the life of the option, the closer this estimate is to V , the long-run average variance.

ARCH and GARCH are discrete time stochastic difference equations. They have the attractive feature that virtually all financial time series data are recorded at discrete intervals.

4.5 The Constant Elasticity of Variance (CEV) model

One of the fundamental ideas in finance is that the asset price and its volatility should be negatively correlated⁵. One possible explanation: as a company's equity decreases in value, its leverage increases. This is because that its equity becomes more risky and thus volatility increases. Conversely, if a company's equity increases in value, its leverage decreases. As a result, the company's equity becomes less risky and its volatility decreases. This argument shows that we can expect volatility of equity to be a decreasing function of price and is consistent with Figure 3.2. This is known as the *leverage effect*.

Cox and Ross [13] attempted to model this inverse relationship between asset price and its volatility by considering asset price that follows the diffusion process:

$$dS = \mu S dt + \delta S^{\theta/2} dW_t \quad (4.8)$$

where W_t is a standard Brownian motion. This is known as the constant elasticity of variance (CEV) model.

⁵ This claim becomes highly disputable in the foreign exchange case, where the asset is a currency. See Wu [39].

In the CEV diffusion model, asset price and its volatility satisfy the following deterministic relationship:

$$\sigma(S, t) = \delta S^{(\theta-2)/2} \quad (4.9)$$

The elasticity of return variance with respect to asset price is $\theta-2$. θ is the elasticity coefficient and is restricted to be between 0 and 2. The instantaneous variance of the return is then given by $\sigma^2 = \delta^2 S^{(\theta-2)}$. When $\theta < 2$, this variance is a decreasing function of the asset price.

The case where $\theta = 2$ reproduces the Black and Scholes [6] assumption of asset price following geometric Brownian motion, with variance rate σ^2 . There are other two special cases for the CEV model. The first one is the *absolute diffusion* process where $\theta = 0$ and the second one is the *square root diffusion* process where $\theta = 1$. See Figure 4.3 for sample paths of the three special cases of the CEV model.

Emanuel and MacBeth [15] compared deviations of market from model price for both Black-Scholes and CEV. Based on a daily sample of options on 6 stocks over two years, they have found that CEV model better explains market prices than the Black-Scholes model, when the prediction period is less than one month. This means that the parameter, θ , used to in the CEV model has to be estimated no more than one month earlier. As a result the optimal elasticity coefficient is not stable over time. Ang and Peterson [2] have found that optimal elasticity coefficient is highly volatile, and does not appear to be correlated with its past values.

Although the assumption of a constant elasticity coefficient in the CEV model is questionable, it is still more reasonable than Black-Scholes constant volatility assumption, which does not allow volatility to change at all. Cox [11] first developed the density function of asset price following a CEV diffusion process and the formula for valuing a European call option. Unfortunately, the general solution of a European call option price for an asset price process following a CEV diffusion process contains infinite summations. This may be problematic when the convergence of the series is slow. Approximations were developed to avoid these infinite summations for the *absolute diffusion* process and the *square root diffusion* process.

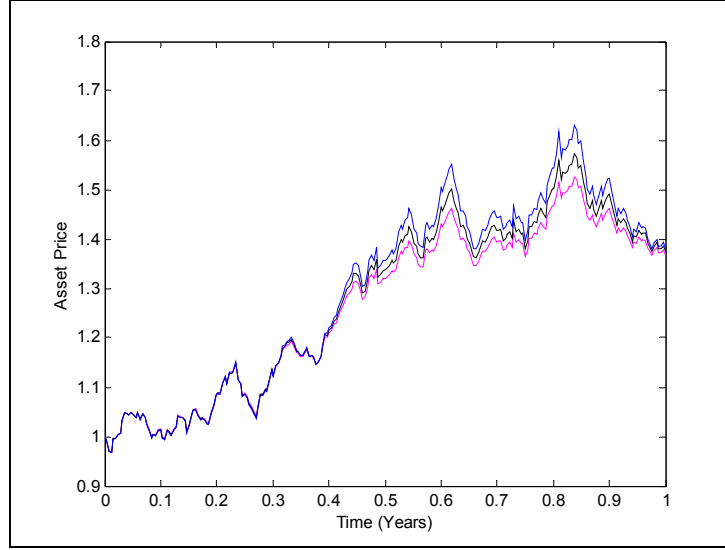


Figure 4.3: Three special cases of CEV diffusion processes. The purple curve is the absolute diffusion process where $\theta = 0$. The black curve is the square root diffusion process where $\theta = 1$. The blue curve is the geometric Brownian motion where $\theta = 2$. The simulation parameters are: $S_0 = 1$, $r = 0.1$, $q = 0$, and $\sigma = 0.2$.

4.5.1 The Absolute Diffusion Process

Cox and Ross [13] consider a special case in the CEV diffusion model where $\theta = 0$, the asset price process follows absolute diffusion process:

$$dS = \mu S dt + \delta dW_t \quad (4.10)$$

The price of a European call option with the underlying asset price follows a absolute diffusion process is:

$$C(S, \tau) = (Se^{-q\tau} - Ke^{-r\tau})N(y_1) + (Se^{-q\tau} + Ke^{-r\tau})N(y_2) + v(N'(y_1) + N'(y_2)) \quad (4.11)$$

where

$$v = \left(\frac{e^{-2q\tau} - e^{-2r\tau}}{2(r - q)} \right)^{\frac{1}{2}}$$

$$y_1 = \frac{Se^{-q\tau} - Ke^{-r\tau}}{v}$$

$$y_2 = \frac{-Se^{-q\tau} - Ke^{-r\tau}}{v}$$

and $N(\cdot)$ is the cumulative standard normal distribution function, $N'(\cdot)$ is the standard normal density function.

4.5.2 The Square Root Diffusion Process

Asset prices follow a square root diffusion process satisfy the stochastic differential equation:

$$dS = \mu S dt + \delta \sqrt{S} dW_t \quad (4.12)$$

This is a special case of the CEV model where $\theta = 1$. Cox [11] derived the price of European options under the square root diffusion process. For the purpose of this research, we are not going to discuss this in too much detail. See Cox [11].

4.5.3 The Non-Central Chi-Square Distribution

Schroder [34] showed that the general solution could be expressed as

$$C(S, \tau) = S e^{-q\tau} Q\left[2y; 2 + \frac{2}{2-\theta}, 2x\right] - K e^{-r\tau} \left(1 - Q\left[2x; \frac{2}{2-\theta}, 2y\right]\right) \quad (4.13)$$

where

$$\begin{aligned} x &= k S^{(2-\theta)} e^{(r-q)(2-\theta)\tau} \\ y &= k K^{2-\theta} \\ k &= \frac{2(r-q)}{\delta^2 (2-\theta) [e^{(r-q)(2-\theta)\tau} - 1]} \end{aligned}$$

$Q(z; \nu, \kappa)$ is the complementary non-central chi-square distribution function, evaluated at z , with ν degrees of freedom and non-central parameter κ . Schroder [34] also presented an efficient algorithm for computing the infinite summations in the general solution.

For odd degrees of freedom, $Q(z; \nu, \kappa)$ can be represented by the sum of normal distributions and elementary function. Denote $N(\cdot)$ the cumulative standard normal distribution function and $N'(\cdot)$ the standard normal density function, the formula for 1, 3, and 5 degrees of freedom are:

$$\begin{aligned}
Q(z;1,\kappa) &= N(\sqrt{\kappa} - \sqrt{z}) + N(-\sqrt{\kappa} - \sqrt{z}) \\
Q(z;3,\kappa) &= Q(z;1,\kappa) + \frac{N'(\sqrt{\kappa} - \sqrt{z}) + N'(\sqrt{\kappa} + \sqrt{z})}{\sqrt{\kappa}} \\
Q(z;5,\kappa) &= Q(z;1,\kappa) + \kappa^{-3/2} \left[(\kappa - 1 + \sqrt{\kappa z}) N'(\sqrt{\kappa} - \sqrt{z}) \right. \\
&\quad \left. - (\kappa - 1 - \sqrt{\kappa z}) N'(\sqrt{\kappa} + \sqrt{z}) \right]
\end{aligned} \tag{4.14}$$

The absolute diffusion process ($\theta = 0$) corresponds to 3 and 1 degrees of freedom in the two distributions in Equation (4.13). The simple formula in Equation (4.11) of Cox and Ross [13] can then be easily verified. Schroder [34] also suggested that the case where $\theta = 4/3$ (corresponding to 5 and 3 degrees of freedom) be combined with the formula for cases $\theta = 0$ and $\theta = 2$ (Black-Scholes formula) to interpolate CEV call option prices for any θ between zero and two. The details will be discussed in Chapter 6.

Chapter 5

Stochastic Volatility Models

Stochastic volatility models suggest that future volatility levels cannot be completely determined using the available information today. Empirical studies of Mandelbrot [25] and Fama [17] have shown that the distribution of asset returns exhibits fatter tails than that of a normal distribution. Since stochastic volatility models can be consistent with fat tails of the asset return distribution using stochastic volatility in option pricing is very popular.

If the volatility is either constant or deterministic, investors are only exposed to the risk from a randomly evolving asset price process. On the other hand, if the volatility is stochastic, investors are exposed to additional risk from the randomly evolving volatility process. Another distinctive feature, which stochastic volatility models do not share with deterministic volatility models, is that option values can change without any change in the price of the underlying asset. Change in volatility level alone will cause the value of the option to change.

5.1 The Two-Factor PDE

In the standard Black-Scholes set-up, the asset price process follows a geometric Brownian and is driven by a source of randomness, W_t :

$$dS = \mu S dt + \sigma S dW_t$$

Where μ is the drift rate and σ is the volatility of asset returns, which are both assumed to be constant.

In a stochastic volatility model, the volatility is driven by another source of randomness, which is different from W_t , but may be correlated with it. We thus have a coupled stochastic process:

$$\begin{aligned} dS &= \mu S dt + \sigma S dW_t^1 \\ dv &= p(S, v, t) dt + q(S, v, t) dW_t^2 \end{aligned} \quad (5.1)$$

where $v = \sigma^2$ is the variance rate of asset price, with p and q as its drift rate and the volatility respectively. W_t^1 and W_t^2 are standard Brownian motions with correlation coefficient ρ , i.e.

$$E(dW_t^1 dW_t^2) = \rho dt.$$

In this model, option prices depend on two random variables, namely asset price and volatility.

Next we attempt to derive a PDE for the two-factor model in a similar way as deriving the Black-Scholes PDE described in section 2.4.1. To create a risk-free portfolio, we need to eliminate the two sources of randomness, W^1 and W^2 . To eliminate W^1 , we can trade the underlying, S . For W^2 , since the volatility is not a traded asset, we need another derivative to do the trick. Therefore consider the portfolio, Π , containing:

$$\begin{aligned} &-1 : V \\ &+ \Delta : \text{shares} \\ &+ \Delta_1 : V_1 \end{aligned}$$

where V is the derivative we want to price, V_1 is the derivative we use to hedge the source of randomness of volatility. Δ and Δ_1 are the hedge ratios we require to make Π risk-free. Thus, the value of Π will be:

$$\Pi = -V + \Delta S + \Delta_1 V_1 \quad (5.2)$$

and the change of portfolio $d\Pi$ is:⁶

$$d\Pi = -dV + \Delta dS + \Delta_1 dV_1$$

Since V is a function of S , v , and t , then by Ito's lemma

⁶ According to Ito's lemma, we should have

$$d\Pi = -dV + \Delta(dS) + S(d\Delta) + (d\Delta)(dS) + \Delta_1(dV_1) + V_1(d\Delta_1) + (d\Delta_1)(dV_1).$$

The statement that

$$d\Pi = -dV + \Delta dS + \Delta_1 dV_1$$

is erroneous. Some authors attempt to justify this by claiming that Δ and Δ_1 are “instantaneously constant” or by claiming that the portfolio $(-1, +\Delta, +\Delta_1)$ is self-financing. Carr [10] pointed out that both of these claims are false. One possible fix would be to enforce the self-financing condition by varying also the option holding, not just the stock holding.

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv + \frac{1}{2} \left\{ \frac{\partial^2 V}{\partial S^2} (dS)^2 + \frac{\partial^2 V}{\partial v^2} (dv)^2 + 2 \frac{\partial^2 V}{\partial S \partial v} (dS)(dv) \right\}$$

From Equation (5.1),

$$\begin{aligned} (dS)^2 &= \sigma^2 S^2 dt \\ (dv)^2 &= q^2 dt \\ (dS)(dv) &= \sigma S q \rho dt \end{aligned}$$

Therefore the expression of dV becomes:

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} (\mu S dt + \sigma S dW_t^1) + \frac{\partial V}{\partial v} (p dt + q dW_t^2) \\ &\quad + \frac{1}{2} \left\{ \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt + q^2 \frac{\partial^2 V}{\partial v^2} dt + 2 \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} dt \right\} \\ &= \left\{ \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} \right\} dt \\ &\quad + \sigma S \frac{\partial V}{\partial S} dW_t^1 + q \frac{\partial V}{\partial v} dW_t^2 \end{aligned}$$

and similarly for V_1 .

Then

$$\begin{aligned} d\Pi &= -dV + \Delta dS + \Delta_1 dV_1 \\ &= - \left\{ \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} \right\} dt \\ &\quad - \sigma S \frac{\partial V}{\partial S} dW_t^1 - q \frac{\partial V}{\partial v} dW_t^2 + \Delta \mu S dt + \Delta \sigma S dW_t^1 \\ &\quad + \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \mu S \frac{\partial V_1}{\partial S} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} \right\} dt \\ &\quad + \Delta_1 \sigma S \frac{\partial V_1}{\partial S} dW_t^1 + \Delta_1 q \frac{\partial V_1}{\partial v} dW_t^2 \end{aligned}$$

$$\begin{aligned}
&= -\left\{ \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} \right\} dt \\
&\quad + \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \mu S \frac{\partial V_1}{\partial S} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} \right\} dt \\
&\quad + \Delta \mu S dt + \left(\Delta_1 \frac{\partial V_1}{\partial S} - \frac{\partial V}{\partial S} + \Delta \right) \sigma S dW_t^1 + \left(\Delta_1 \frac{\partial V_1}{\partial v} - \frac{\partial V}{\partial v} \right) q dW_t^2
\end{aligned}$$

To eliminate randomness, the dW^1 and dW^2 parts must be zero. Therefore

$$\begin{aligned}
\Delta_1 \frac{\partial V_1}{\partial S} - \frac{\partial V}{\partial S} + \Delta &= 0 \\
\Delta_1 \frac{\partial V_1}{\partial v} - \frac{\partial V}{\partial v} &= 0
\end{aligned}$$

defines the two hedge ratios Δ and Δ_1 .

Now

$$\Delta \mu S dt = \left(-\Delta_1 \frac{\partial V_1}{\partial S} + \frac{\partial V}{\partial S} \right) \mu S dt = -\Delta_1 \mu S \frac{\partial V_1}{\partial S} dt + \mu S \frac{\partial V}{\partial S} dt$$

Thus

$$\begin{aligned}
d\Pi &= -\left\{ \frac{\partial V}{\partial t} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} \right\} dt \\
&\quad + \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} \right\} dt \\
&= r\Pi dt \\
&= r(-V + \Delta S + \Delta_1 V_1) dt \\
&= (-rV + r\Delta S + r\Delta_1 V_1) dt
\end{aligned}$$

since Π is now risk-free.

We therefore have

$$\begin{aligned}
& - \left\{ \frac{\partial V}{\partial t} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} \right\} \\
& + \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} \right\} \\
& = -rV + r\Delta S + r\Delta_1 V_1
\end{aligned}$$

which becomes

$$\begin{aligned}
& \frac{\partial V}{\partial t} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV + r\Delta S \\
& = \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} - rV_1 \right\}
\end{aligned}$$

But

$$r\Delta S = \left(-\Delta_1 \frac{\partial V_1}{\partial S} + \frac{\partial V}{\partial S} \right) rS = -\Delta_1 rS \frac{\partial V_1}{\partial S} + rS \frac{\partial V}{\partial S}$$

Therefore we have

$$\begin{aligned}
& \frac{\partial V}{\partial t} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV - \Delta_1 rS \frac{\partial V_1}{\partial S} + rS \frac{\partial V}{\partial S} \\
& = \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} - rV_1 \right\}
\end{aligned}$$

After re-arranging, the above equation becomes

$$\begin{aligned}
& \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV \\
& = \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + rS \frac{\partial V_1}{\partial S} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} - rV_1 \right\}
\end{aligned}$$

Now

$$\Delta_1 \frac{\partial V_1}{\partial v} - \frac{\partial V}{\partial v} = 0 \Rightarrow \Delta_1 = \frac{\partial V}{\partial v} / \frac{\partial V_1}{\partial v}$$

Thus

$$\begin{aligned}
& \frac{\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + p \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV}{\frac{\partial V}{\partial v}} \\
&= \frac{\frac{\partial V_1}{\partial t} + rS \frac{\partial V_1}{\partial S} + p \frac{\partial V_1}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V_1}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V_1}{\partial S \partial v} - rV}{\frac{\partial V_1}{\partial v}}
\end{aligned}$$

The left-hand side is a partial differential equation involving V only, and the right-hand side is a partial differential equation involving V_1 only. For the above equation to be possible, it suffices that both sides are equal to a function λ of only the independent variables S , v and t . In other words

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV = -(p - \lambda) \frac{\partial V}{\partial v}$$

for some function $\lambda(S, v, t)$. Reordering yields

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + (p - \lambda) \frac{\partial V}{\partial v} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial v^2} + \sigma S q \rho \frac{\partial^2 V}{\partial S \partial v} - rV = 0 \quad (5.3)$$

The function $\lambda(S, v, t)$ is called the market price of volatility risk or simply the risk premium of volatility. Since volatility is not a traded asset such as a stock, the volatility risk cannot be eliminated by arbitrage argument. Therefore its market price, λ , explicitly enters the PDE. Generally, λ is a measure of premium investors demand when taking on volatility risk and must be the same for all volatility dependent assets. Heston [18] has suggested that the parameter λ be determined by one volatility-dependent asset and then used to price all other volatility-dependent assets.

The form of the volatility risk premium λ is a weakness in all stochastic volatility models, because it cannot be deduced from the assumption that all investors prefer more wealth to less wealth. Investors have different tolerances toward risk (risk aversion) and are difficult to determine.

By the Feynman-Kac formula, V , satisfying Equation (5.3), has representation

$$V(S, v, t) = E_{t, S, v} \left[\Phi(S_T, v_T) \exp \left\{ - \int_t^T r(S_u, v_u, u) du \right\} \right] \quad (5.4)$$

where S and v follow the stochastic differential equations

$$\begin{aligned} dS &= rSdu + \sigma S d\hat{W}_u^1 \\ dv &= (p - \lambda)du + q d\hat{W}_u^2 \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} S_t &= S \\ v_t &= v \end{aligned}$$

\hat{W}_t^1 and \hat{W}_t^2 are standard Brownian motions in the risk-neutral world. The quantity $(p - \lambda)$ is the risk-neutral drift rate of the variance. When pricing derivatives, it is the risk-neutral drift rate that matters and not the real rate.

We assume the interest rate to be a constant. Let Φ be the payoff of the option. For example,

$$\Phi(S_T, v_T) = \max\{S_T - K, 0\} = (S_T - K)^+$$

where K is the strike price of a European call. Then Equation (5.4) can be written as

$$V(S, v, t) = e^{-r(T-t)} E_Q[(S_T - K)^+ | S_t = S, v_t = v] \quad (5.6)$$

E_Q represents expectation under the risk-neutral measure Q .

5.2 Stochastic Volatility Models

Most stochastic volatility models assume mean reversion, which is “regression to the mean” of volatility. This means that if the current variance level is below the long-run average level, θ , then variance will tend to “drift up” towards θ . If, on the other hand, the current variance level is above θ , variance will then drift down to θ . The long-run variance average level, θ , is also called the mean-reverting level. The rate at which variance is being pulled back towards θ is often referred to as the mean-reverting rate. The risk premium of volatility is typically assumed to be zero or a constant proportion of volatility.

5.2.1 Power Series Method of Hull and White

Hull and White [24] proposed the following stochastic volatility model

$$dv = \phi v dt + \xi v dW_t^2 \quad (5.7)$$

This is one of the first few models that tackled the stochastic volatility problem. However, they made simplifying assumptions that volatility risk is not priced and there is no correlation between W^1 and W^2 . In other words, $\lambda = 0$ and $\rho = 0$ in PDE (5.3). Hull and White [24] as well as Stein and Stein [36], which will be introduced in the next section, both their formulas for the option price rely on the distribution of the average variance \bar{V} of the asset price process over the life of the option defined by the stochastic integral

$$\bar{V} = \frac{1}{T-t} \int_t^T v(u) du$$

Hull and White [24] have found in their analysis that the conditional distribution of the terminal asset price, given the average variance, is lognormally distributed⁷. Moreover, the risk-neutral dynamics of the volatility do not depend on the asset price, i.e. $(p - \lambda)$ and q are independent of S . These results break down when the asset price and the volatility are correlated.

In the model of Hull and White, we have

$$\begin{aligned} p(S, v, t) &= \phi v \\ q(S, v, t) &= \xi v \\ \lambda &= 0 \\ \rho &= 0 \end{aligned}$$

Let $C(S, v, t)$ denote the value of a European call option at time t , with strike price K and maturing at time T . PDE (5.3) becomes:

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \phi v \frac{\partial C}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \xi^2 v^2 \frac{\partial^2 C}{\partial v^2} + \rho S \xi v^{\frac{3}{2}} \frac{\partial^2 C}{\partial S \partial v} - rC = 0$$

Since volatility risk is not priced, the value of C does not depend on investors' risk preferences, and thus $C(S, v, t)$ can be calculated using risk-neutral valuation:

⁷ See Lemma in Hull and White [24].

$$C(S, v, t) = e^{-r(T-t)} \int C(S_T, v_T, T) f(S_T | S, v) dS_T \quad (5.8)$$

where

T = option maturity;

S = asset price at time t ;

v = instantaneous variance rate at time t ;

$f(S_T | S, v)$ = the conditional distribution of S_T given the asset price and variance at time t ;

$C(S_T, v_T, T)$ = payoff function = $\max\{S_T - K, 0\}$.

For any three related random variable x , y , and z the conditional density functions must satisfy

$$f(x | y) = \int g(x | z) h(z | y) dz.$$

Using the average variance defined above, the distribution of S_T can be written as

$$f(S_T | v) = \int g(S_T | \bar{V}) h(\bar{V} | v) d\bar{V}.$$

Note that S is omitted for simplifying reason in the above expression.

Substituting this into Equation (5.8) yields

$$C(S, v, t) = e^{-r(T-t)} \iint C(S_T) g(S_T | \bar{V}) h(\bar{V} | v) d\bar{V} dS_T$$

which can be written as

$$C(S, v, t) = \int \left(e^{-r(T-t)} \int C(S_T) g(S_T | \bar{V}) dS_T \right) h(\bar{V} | v) d\bar{V}.$$

The inner term in the above equation is the Black-Scholes price for a European call option with average variance \bar{V} , which we shall denote by $BS(\bar{V})$. Thus we have

$$C(S, v, t) = \int BS(\bar{V}) h(\bar{V} | v) d\bar{V}.$$

Hull and White then expand $BS(\bar{V})$ in a Taylor series about its expected value $E(\bar{V})$ to get

$$\begin{aligned} C(S, v, t) &= BS(E(\bar{V})) + \frac{1}{2} \frac{\partial^2 BS}{\partial \bar{V}^2} \bigg|_{E(\bar{V})} \int (\bar{V} - E(\bar{V}))^2 h(\bar{V}) d\bar{V} + \dots \\ &= BS(E(\bar{V})) + \frac{1}{2} \frac{\partial^2 BS}{\partial \bar{V}^2} \bigg|_{E(\bar{V})} Var(\bar{V}) + \frac{1}{6} \frac{\partial^3 BS}{\partial \bar{V}^3} \bigg|_{E(\bar{V})} Skew(\bar{V}) + \dots \end{aligned}$$

where $Var(\bar{V})$ and $Skew(\bar{V})$ are the second and third central moments⁸ of \bar{V} . This series converges very quickly for sufficiently small values of $\xi^2(T-t)$. With $\phi = 0$:

$$\begin{aligned}
C(S, \sigma^2, t) = & BS(\sigma^2) \\
& + \frac{1}{2} \frac{S\sqrt{T-t}N'(d_1)(d_1d_2-1)}{4\sigma^3} \left\{ \frac{2\sigma^4(e^k - k - 1)}{k^2} - \sigma^4 \right\} \\
& + \frac{1}{6} \frac{S\sqrt{T-t}N'(d_1)[(d_1d_2-3)(d_1d_2-1) - (d_1^2 + d_2^2)]}{8\sigma^5} \\
& \times \sigma^6 \left\{ \frac{e^{3k} - (9+18k)e^k + (8+24k+18k^2+6k^3)}{3k^3} \right\} + \dots
\end{aligned} \tag{5.9}$$

where

$$k = \xi^2(T-t).$$

In Equation (5.9), $\sigma = \sqrt{v}$, which is the current volatility level of the asset return. d_1 and d_2 are calculated as usual by

$$\begin{aligned}
d_1 &= \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}} \\
d_2 &= d_1 - \sigma\sqrt{(T-t)}.
\end{aligned}$$

Hull and White [24] justified the use of $\phi = 0$ as for any nonzero ϕ , options of different maturities would exhibit different implied volatilities. Since this is not often observed empirically, Hull and White [24] concluded that ϕ is at least close to zero.

Hull and White [24] compared their results obtained from Equation (5.9) to the Black-Scholes model and have found that Black-Scholes model overvalues near-to-the-money options and undervalues deep-in-the-money and deep-out-of-the-money options⁹. Figure 5.1 shows the pricing biases for three different values of ξ .

⁸ These results are given in Hull and White [24] as well as in Boyle, P. P., and D. Emanuel, 1985, "Mean Dependent Options," Working Paper, Accounting Group, University of Waterloo.

⁹ What this means is really that Black-Scholes model produces higher prices than the Hull and White model for at-the-money option and lower prices for deep-in-the-money and deep-out-of-the-money options. Hull and White [24] did not compare their model price to the market price.

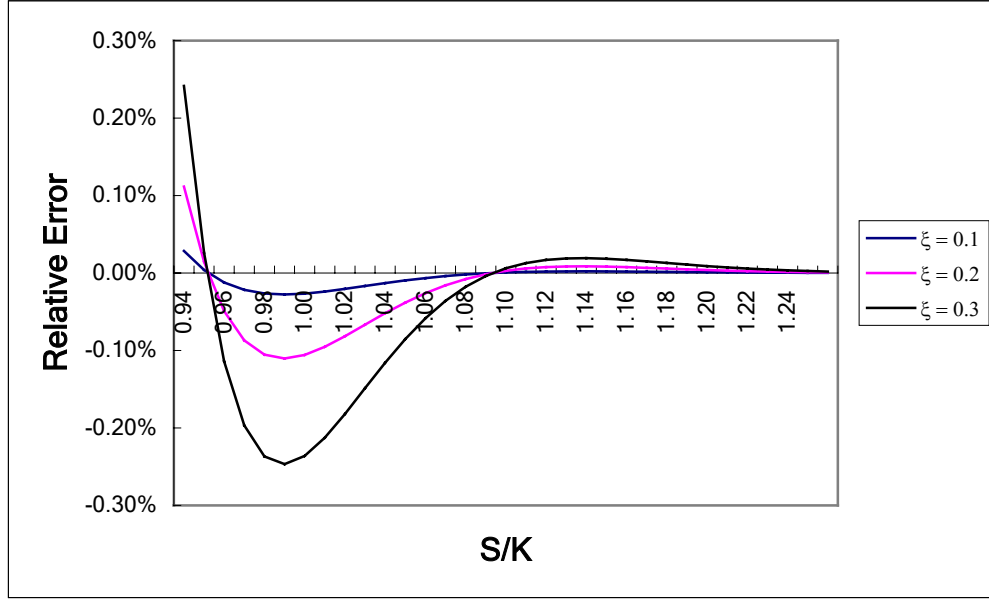


Figure 5.1: Relative errors of Hull and White prices with respect to the Black-Scholes price for varying ξ . The simulation parameters are: $r = 0$, $q = 0$, $\sigma_t = 15\%$, and $T - t = 180$ days.

Hull and White [24] have also developed a Monte Carlo procedure to simulate option prices for the cases of nonzero ϕ and nonzero correlation between asset price and its volatility. When the correlation is positive, the Black-Scholes model undervalues out-of-the-money options and overvalues in-the-money options. When the correlation is negative, the effect is reversed.

5.2.2 Fourier Inversion Method of Stein and Stein

Stein and Stein [36] model stochastic volatility as a mean-reverting process

$$d\sigma = -\delta(\sigma - \theta)dt + k dW_t^2 \quad (5.10)$$

which is also known as an arithmetic Ornstein-Uhlenbeck process. In Equation (5.10), σ (volatility) rather than v (variance rate) is modelled. Stein and Stein [36] also made the assumption that W^2 is not correlated with W^1 . As mentioned above, θ is the long-run average volatility or the mean-reverting level and δ is the mean-reverting rate. Stein and Stein [36], who considered the risk premium λ to be a constant, derived the density function of time- t asset price with stochastic volatility by inverting the Fourier transform for the convolution of

the lognormal density of S , conditional on \bar{V} and the density of \bar{V} . The details are given in the appendix of Stein and Stein [36].

The main advantage of the Stein and Stein model is that it provides a closed-form solution for the asset price distribution, rather than the option price itself. This feature gives us better understanding of the asset price dynamics when its volatility is stochastic. It is also computationally less expensive than other models or solving the PDE (5.3) by numerical procedures¹⁰.

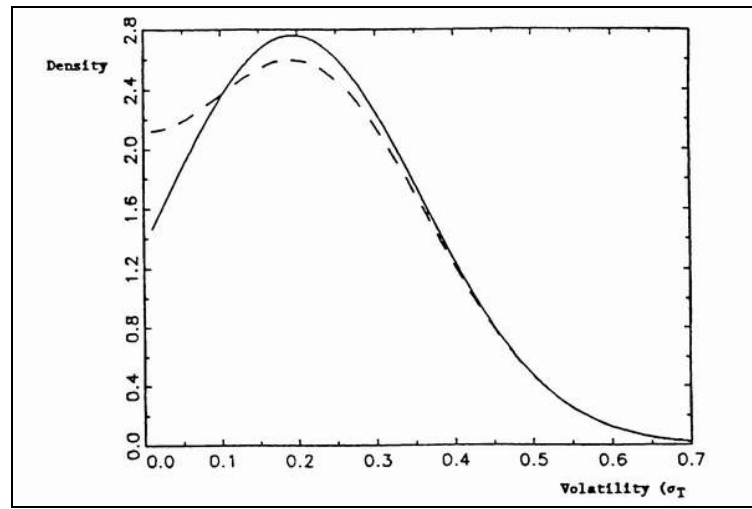


Figure 5.2: Probability density functions of absolute and reflected O-U processes. The solid curve is the density of the reflected O-U process, while the dashed curve is the density of the absolute of the O-U process. The parameter values used are $\sigma_0 = 0.15$, $\theta = 0.3$, $\delta = 2.0$, $k = 0.4$, and $T = 0.25$.

However, volatility process that follows Equation (5.10) may become negative. Stein and Stein [36] argued that since volatility enters their option pricing formula only as $\sigma^2(t)$, this is equivalent to imposing a reflecting barrier at 0 in Equation (5.10). Ball and Roma [4] have demonstrated that this is not the case. They outlined the subtle difference between the reflective

¹⁰ Well-known works includes Wiggins [37] using the hopscotch finite difference method, as well as Hull and White [24] and Scott [35] using Monte Carlo simulation.

Ornstein-Uhlenbeck process and the absolute value of the Ornstein-Uhlenbeck process, which is actually the model proposed by Stein and Stein [36]¹¹.

Figure 5.2 is taken from Ball and Roma [4] and the difference between the density functions of a reflected O-U process and an absolute value of the O-U process can be seen markedly.

5.2.3 Heston Model with Closed-Form Solutions

Heston [18] developed closed-form solution for European option prices as well as the hedge ratios. He modelled the variance process as a square-root mean-reverting process

$$dv = \kappa(\theta - v)dt + \zeta\sqrt{v}dW_t^2 \quad (5.11)$$

where W^2 has correlation ρ with W^1 . A similar model has been used to model interest rates by Cox, Ingersoll, and Ross [12]. Equation (5.11) ensures that variance is always positive for plausible parameter values.

Next we take a brief look at Heston's [18] approach. Let C be the value of a European call option with strike price K and maturing at time T . The volatility of the underlying asset price S follows the stochastic differential equation (5.11). In other words,

$$\begin{aligned} p(S, v, t) &= \kappa(\theta - v) \\ q(S, v, t) &= \zeta\sqrt{v} \end{aligned}$$

in Equation (5.1). PDE (5.3) then becomes

$$\begin{aligned} \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + (\kappa(\theta - v) - \lambda) \frac{\partial C}{\partial v} + \\ \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \zeta^2 v \frac{\partial^2 C}{\partial v^2} + \rho \zeta S v \frac{\partial^2 C}{\partial S \partial v} - rC = 0 \end{aligned} \quad (5.12)$$

C must satisfy PDE (5.12) subject to the following boundary condition

¹¹ However, Ball and Roma's [4] analysis of the Stein and Stein [36] model was not completely correct. Ball and Roma [4] correctly pointed out that the Stein and Stein model is not a reflected O-U process; but Ball and Roma [4] incorrectly claimed that it is the absolute value of an O-U process. See Schobel and Zhu [33].

$$\begin{aligned}
C(S, v, T) &= (S - K)^+, \\
C(0, v, t) &= 0, \\
\frac{\partial C}{\partial S}(\infty, v, t) &= 1, \\
\frac{\partial C}{\partial t}(S, 0, t) + rS \frac{\partial C}{\partial S}(S, 0, t) + \kappa \theta \frac{\partial C}{\partial v}(S, 0, t) - rC(S, 0, t) &= 0, \\
C(S, \infty, t) &= S.
\end{aligned} \tag{5.13}$$

By analogy with the Black-Scholes formula, Heston [18] guessed that the solution to PDE (5.12) is of the form

$$C(S, v, t) = SP_1 - e^{-r(T-t)} KP_2 \tag{5.14}$$

With

$$x = \ln(S)$$

and substituting the guessed the solution (5.14) into PDE (5.12) shows that P_1 and P_2 must satisfy PDEs

$$\frac{\partial P_j}{\partial t} + (r + u_j v) \frac{\partial P_j}{\partial x} + (a_j - b_j v) \frac{\partial P_j}{\partial v} + \frac{1}{2} v \frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2} \zeta^2 v \frac{\partial^2 P_j}{\partial v^2} + \rho \zeta v \frac{\partial^2 P_j}{\partial x \partial v} = 0 \tag{5.15}$$

for $j = 1, 2$, where

$$\mu_1 = \frac{1}{2}, \quad \mu_2 = -\frac{1}{2}, \quad a = \kappa \theta, \quad b_1 = \kappa + \lambda - \rho \zeta, \quad b_2 = \kappa + \lambda.$$

The probabilities are not immediately available in closed-form. However, Heston [18] showed that their characteristic functions, $f_1(x, v, T; \phi)$ and $f_2(x, v, T; \phi)$ respectively, satisfy the same PDE (5.15) subject to the final condition

$$f_j(x, v, T; \phi) = e^{ix\phi}$$

The characteristic functions are

$$f_j(x, v, t; \phi) = e^{A(T-t; \phi) + B(T-t; \phi)v + ix\phi} \tag{5.16}$$

where

$$A(\tau; \phi) = r\phi i \tau + \frac{a}{\zeta^2} \left\{ (b_j - \rho\zeta\phi i + d)\tau - 2 \ln \left(\frac{1 - ge^{dr}}{1 - g} \right) \right\}$$

$$B(\tau; \phi) = \frac{b_j - \rho\zeta\phi i + d}{\zeta^2} \left\{ \frac{1 - e^{dr}}{1 - ge^{dr}} \right\}$$

and

$$g = \frac{b_j - \rho\zeta\phi i + d}{b_j - \rho\zeta\phi i - d}$$

$$d = \sqrt{(\rho\zeta\phi i - b_j)^2 - \zeta^2(2\mu_j\phi i - \phi^2)}$$

The probabilities P_1 and P_2 can then be obtained by inverting the characteristic functions (5.16):

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[\frac{e^{-i\phi \ln(K)} f_j(x, v, T; \phi)}{i\phi} \right] d\phi \quad (5.17)$$

Equations (5.14), (5.16), and (5.17) gives the price of a European call option.

Volatility Dynamics in the Risk-Neutral World

Heston [18] suggested that the risk premium, λ , should be proportional to the variance v , i.e. $\lambda(S, v, t) = \lambda v$. Thus the risk-neutral drift of variance is

$$\kappa(\theta - v) - \lambda(S, v, t) = \kappa(\theta - v) - \lambda v = \kappa\theta - (\kappa + \lambda)v = (\kappa + \lambda) \left(\frac{\kappa\theta}{\kappa + \lambda} - v \right)$$

Define the risk-neutralised parameters

$$\kappa^* = \kappa + \lambda \quad \text{and} \quad \theta^* = \kappa\theta / (\kappa + \lambda) \quad (5.18)$$

Substitute κ^* and θ^* into Equation (5.12) to get

$$\begin{aligned} \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \kappa^* (\theta^* - v) \frac{\partial C}{\partial v} + \\ \frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \zeta^2 v \frac{\partial^2 C}{\partial v^2} + \rho \zeta S v \frac{\partial^2 C}{\partial S \partial v} - rC = 0 \end{aligned} \quad (5.19)$$

This suggests that the risk-neutralised variance process is

$$dv = \kappa^* (\theta^* - v)dt + \zeta \sqrt{v} dW_t^2 \quad (5.20)$$

instead of the real world variance process described in Equation (5.11). θ^* is the mean-reverting level (long-run average variance) and κ^* is the mean-reverting rate in the risk-neutral world. Equation (5.19) is equivalent to Equation (5.12), but without risk preference parameters. When modelling option prices using the Heston model, we should look at θ^* and κ^* rather than the original θ and κ .

Effects of parameters in Heston Model

The parameters ρ and ζ play important roles in determining the shape of the risk-neutral terminal asset price distribution and hence affect option prices directly. The sign and magnitude of ρ determine the sign and level of skewness in the asset price distribution. A negative correlation, *ceteris paribus*, implies that an increase in asset return is associated with a decrease in variance. Therefore, the left tail becomes thicker and right tail thinner than the lognormal distribution assumed by the Black-Scholes model. This results in a higher probability (than the lognormal distribution) for asset price to end up below a extreme low strike price and hence increases the prices of deep-out-of-the-money puts and deep-in-the-money calls and, on the other hand, decreases the prices of the deep-in-the-money puts and deep-out-of-the-money calls. However, since the kurtosis of the asset price distribution is not changed, prices of in-the-money calls and out-of-the-money puts are decreased, while out-of-the-money calls and in the money puts increase in value. A positive correlation has completely opposite effects. Figure 5.3 shows the effects of correlation parameter ρ have on option prices.

While the correlation parameter ρ affects the skewness of the implied asset price distribution, the parameter ζ controls the volatility of volatility. When ζ is zero, the volatility is deterministic; otherwise an increase in ζ increases the kurtosis of the implied asset price distribution, and makes the tails of the asset price distribution fatter. Thus the occurrences of extreme asset returns are more likely. As a result, options that are deep-in-the-money and deep-out-of-the-money increase in value, while prices of near-to-the-money options are lowered. Figure 5.4 shows effects of volatility of volatility ζ have on option prices.

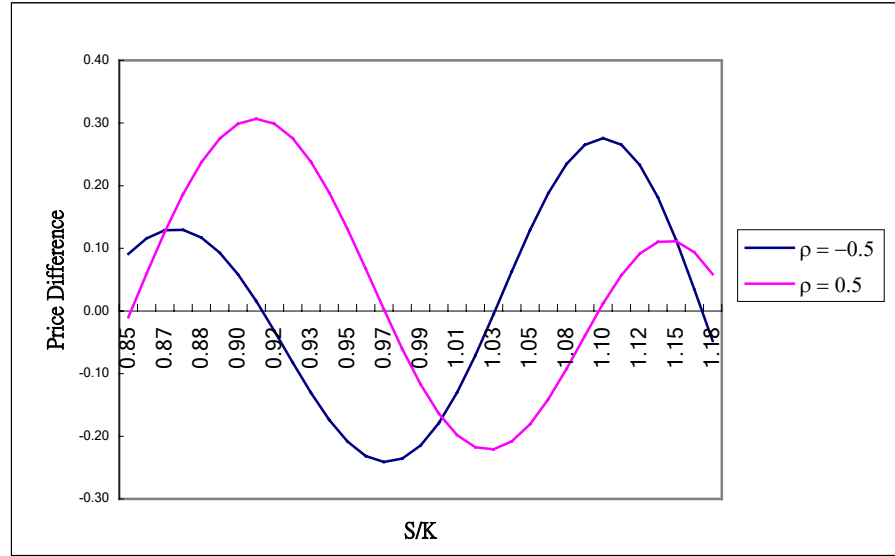


Figure 5.3: Price difference between Heston model and Black-Scholes model for positive and negative correlations. The parameters used for simulation are: $\kappa^* = 2$, $\theta^* = 0.01$, $\zeta = 0.1$, $T - t = 0.5$ years, $r = 0$, $q = 0$, and $K = 100$.

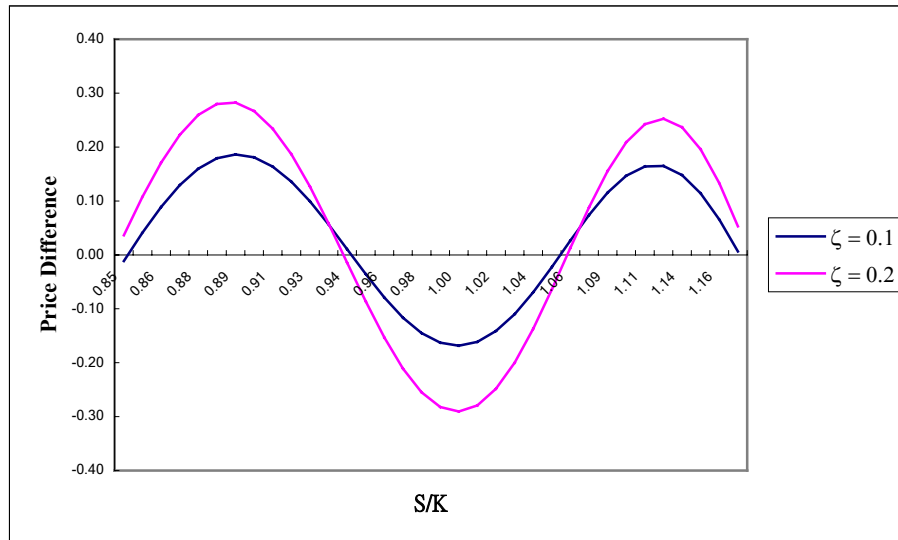


Figure 5.4: Price difference between Heston model and Black-Scholes model for different values of volatility of volatility ζ . The parameters used for simulation are: $\kappa^* = 2$, $\theta^* = 0.01$, $\rho = 0$, $T - t = 0.5$ years, $r = 0$, $q = 0$, and $K = 100$.

There were empirical works done on the Heston model. Bates [5] has found that prices produced by the Heston model are in closer agreement with the market option prices than those of the Black-Scholes model. Nandi [28] studied both pricing and hedging of Heston model using S&P 500 index options, and found that the returns of a hedge portfolio constructed using Heston model better matches the risk-free return than the Black-Scholes model.

Chapter 6

Data and Methodology

6.1 Data

The data (obtained from Bloomberg) is the S&P 500 index option prices recorded at approximately 11:00am on the 15th August 2001. Eight different maturity dates (ranging from 2001/8/18 to 2003/6/21), each with 18 different strike prices (except the one maturing on 2003/6/21, which has only 15 strike prices) are taken. The S&P 500 index closed at 1186.73 on 14th August 2001.

6.1.1 Background on the S&P 500 Stock Index

The *Standard and Poor's 500 (S&P 500) Index* is a market-value-weighted index based on a portfolio of 500 different stocks that are traded on the New York Stock Exchange (NYSE), American Stock Exchange (AMEX), and the Nasdaq National Market System. The 500 stocks contain 400 industrials, 40 utilities, 20 transportation companies, and 40 financial institutions. The weights of the stocks in the portfolio at any given time are proportional to their market capitalisations.

6.1.2 The S&P 500 Index Option

There are many different index options traded in the United States. One of the most popular ones is the S&P 500 index option traded on the Chicago Board Options Exchange (CBOE). The S&P 500 index option expires on the Saturday immediately following the third Friday of the expiry months, which are March, June, September, and December. The S&P 500 index option is European-style and can only be exercised on the last business day before expiry.

One contract is to buy or sell 100 times the index at the strike price. Settlement is in cash rather than by delivering the portfolio underlying of the index, which is inconvenient or impossible. The S&P 500 index option is selected for modelling because its underlying index consists of stocks representative of the entire stock market. It is also one of the most actively traded options in the world. Therefore liquidity is not a great concern when modelling.

6.1.3 Data Screening Procedure

All the call option prices taken from the market are checked whether they satisfy the lower boundary condition

$$S_t e^{-q(T-t)} - K e^{-r(T-t)} \leq C(S_t, t)$$

where S_t is the current asset price, K is the strike price, q is the continuous compounded dividend yield of the asset, r is the risk-free interest rate, and $C(S_t, t)$ is the call price at time t . If a call price from the market does not satisfy the lower boundary condition, it is considered as an invalid observation and discarded.

6.2 Methodology

Intuitively, the parameters of a good model should stay constant over time. Therefore, we assume that the parameters of the models that we implement are constants. This enables us to use the models to calculate option prices of one day based on the parameters estimated using the same day's data.

6.2.1 Parameter Estimation

Here parameters are not the option parameters (S , K , r , q , σ , and τ), but rather the primitive parameters of each of the models discussed earlier. They are summarised in Table 6.1.

Model	Primitive Parameters
GARCH(1,1)	ω, α, β
CEV	θ
Hull and White	ξ
Heston	$\kappa^*, \theta^*, \zeta, \rho$

Table 6.1: Primitive parameters in each model.

The parameters κ^* and θ^* in Heston model represent, respectively, the risk-neutral mean-reverting rate and the long-run average variance in the risk-neutral world. Since we are doing risk-neutral pricing, what we want to model is the risk-neutral world dynamics of variance, rather than the real-world dynamics.

Apart from the GARCH(1,1) model, in which the parameters are estimated from the history of the underlying, all parameters of the other models are estimated from data at hand. First, nine specific options are selected to best represent the data and are listed in Table 6.2:

	Short Maturity (3 days)			Medium Maturity (213 days)			Long Maturity (675 days)		
	im	nm	om	im	nm	om	im	nm	om
Strike Price	1100.00	1195.00	1270.00	1050.00	1175.00	1450.00	995.00	1200.00	1650.00
Market Price	91.00	6.80	0.90	176.60	87.50	4.20	286.80	156.30	16.40

Table 6.2: Options used for parameter estimations, where im, nm, and om are the abbreviations for in-the-money, near-to-the-money, and out-of-the-money options respectively.

The aim is to find the parameters by regression, i.e. find the parameters that produce the modelled prices, which best fit the market prices. This is done using the least-square method. The model should produce nine prices – call them C_{Model} and call the market prices C_{Market} . The difference between C_{Model} and C_{Market} is therefore a vector

$$\Delta = C_{Model} - C_{Market}$$

Thus all we need to do is to find the (set of) parameter(s) that minimise the inner product of Δ with itself.

6.2.2 Estimating Parameters in GARCH(1,1) Model¹²

For GARCH(1,1) model, the parameters ω , α , and β are estimated using historical data. The approach is known as *maximum likelihood method*, as mentioned earlier in this dissertation. The idea is to find parameter values that maximise the probability (or likelihood) of the data

¹² Source: Hull [23].

occurring. Suppose we have m observations u_1, u_2, \dots, u_m and the mean of the underlying distribution is zero (see section 4.2 in this dissertation). Define $v_i = \sigma_i^2$ as the variance estimated for day i . We assume that asset returns are normally distributed. Therefore the probability density for the i th observation, u_i , conditional on the i th variance, v_i , is normal. The likelihood function is then

$$L(v_1, v_2, \dots, v_m) = \prod_{i=1}^m \left\{ \frac{1}{\sqrt{2\pi v_i}} \exp\left(\frac{-u_i^2}{2v_i}\right) \right\} \quad (6.1)$$

This is equivalent to maximising the log-likelihood $l(v_1, v_2, \dots, v_m)$, where

$$\begin{aligned} l(v_1, v_2, \dots, v_m) &= \ln(L) \\ &= \prod_{i=1}^m \left\{ \frac{1}{\sqrt{2\pi v_i}} \exp\left(\frac{-u_i^2}{2v_i}\right) \right\} \\ &= -\frac{m}{2} \ln(2\pi) - \sum_{i=1}^m \ln(v_i) - \sum_{i=1}^m \frac{u_i^2}{2v_i} \end{aligned}$$

which is the same as maximising

$$\sum_{i=1}^m \left[-\ln(v_i) - \frac{u_i^2}{v_i} \right] \quad (6.2)$$

An iterative search can then be performed to find the parameters that maximise the expression (6.2).

6.2.3 Interpolating Option Prices in the CEV Model

The aim is to use Newton's interpolation formula to fit a second-degree polynomial to three CEV values where "closed-form" solutions exist. Let $f(\theta)$ denote the CEV European call price with θ . Then the interpolated call price for any θ between zero and two is

$$f(\theta) = f(0) + \frac{3}{4}\theta \left\{ f\left(\frac{4}{3}\right) - f(0) \right\} + \frac{1}{4}\theta \left(\theta - \frac{4}{3} \right) \left\{ 3f(0) - 9f\left(\frac{4}{3}\right) + 6f(2) \right\} \quad (6.3)$$

In equation (6.3), $f(0)$ is the CEV call price when $\theta = 0$, and hence can be calculated using formula given in Equation (4.11). $f(2)$ correspond to the case where $\theta = 2$, and thus the famous Black-Scholes formula can be used. From Equation (4.13), the CEV call price of the case where $\theta = 4/3$ is

$$f\left(\frac{4}{3}\right) = Se^{-q\tau}Q[2y;5,2x] - Ke^{-r\tau}(1 - Q[2x;3,2x])$$

where $Q(2y;5,2x)$ and $Q(2x;3,2y)$ can be evaluated using Equation (4.14).

6.2.4 Interest Rate and Dividend Yields

All of the option parameters (S , K , r , q , τ) are observable from the market except for the volatility of the underlying, which is, of course, what we are trying to model. Specific note should be taken of r , the risk-free interest rates, and q , the continuously compounded dividend yields, over the life of the options. They have been estimated and provided by Bloomberg and are summarised in Table 6.3.

Maturity Dates	Interest Rates	Dividend Yields
2001/08/18	3.400%	1.162%
2001/09/22	3.400%	1.071%
2001/10/20	3.400%	1.139%
2001/12/22	3.370%	1.253%
2002/03/16	3.310%	1.309%
2002/06/22	3.380%	1.281%
2002/12/21	3.510%	1.286%
2003/06/21	3.650%	1.290%

Table 6.3: Summary of interest rates and dividend yields used for option pricing, where interest rates and dividend yields are both quoted NACC (Nominal Annual Continuously Compounded) over the lives of the specific options.

6.2.5 Measuring Model Performances

To see how well a model performs, we look at the relative error generated by the model. Suppose C_{Model} is the call option prices generated by the model and C_{Market} is the actual price extracted from the market, the conventional relative error, $E_{Conventional}$, is calculated by

$$E_{Conventional} = \frac{|C_{Model} - C_{Market}|}{|C_{market}|}.$$

If the relative error (expressed in percentage) is small, it means the model gives a good approximation to the market. Conversely, if the relative error is big, then the model is considered to be a poor approximation to the market.

However, we do not only want to see how close the model price is to the market price, but also whether the model produces over- or underestimates to the market. In other words, we need to see when a model underprices and overprices an option.

To achieve this, we modify the conventional relative formula to

$$E = \frac{C_{Model} - C_{Market}}{C_{market}} .$$

A negative relative error then means that the model underprices the specific option, whereas a positive relative error means that the model overprices the specific option.

Chapter 7

Results

In this section, the prices generated by the models are checked against the market prices as well as those calculated using Black-Scholes formula.

7.1 Data Screening Result

As mentioned in the methodology section, all option prices extracted from the market must be checked whether they satisfy the lower boundary condition

$$S_t e^{-q(T-t)} - K e^{-r(T-t)} \leq C(S_t, t)$$

Those that do not must be discarded. Although some market option prices appear to be underpriced (with respect to Black-Scholes price), all market prices do satisfy the lower boundary condition.

7.2 Historical Volatility

Using the closing prices of S&P 500 index from 1996/8/15 to 2001/8/14, the historical volatility is estimated to be 0.194939 or 19.49% p.a. Unless otherwise stated, all the volatilities are annualised.

7.3 Option Categories

Each market option price that remains after the screening procedure is placed in one of 9 categories depending on their time to expiration and ratio of the asset price to the strike price. Three ranges of time to expiration are distinguished:

1. Short maturity (90– days or below 3 months)
2. Medium maturity (91 to 270 days or between 3 and 9 months)
3. Long maturity (271+ days or above 9 months)

In our case, the 3, 38, 66 days options belong into the short maturity category, the 129 and 213 days options belong to the medium maturity category, and 311, 493, 675 days options belong to the long maturity category.

The ratio of the current asset price to the strike price is also divided into three ranges:

1. Out-of-the-money (0.95–)
2. Near-to-the-money (0.95 to 1.05)
3. In-the-money (1.05+)

7.4 Model Performances

Matlab [26] scripts have been written to simulate the model prices in order to calculate the relative errors of each model. All programs are listed in Appendix A. A good model should produce small relative errors (of both signs) over different strike prices and different maturities.

7.4.1 GARCH(1,1) Model

The GARCH(1,1) parameters are estimated to be as follows:

$$\omega = 0.00000538659679$$

$$\alpha = 0.09960225692814$$

$$\beta = 0.86879882989990$$

Substituting the above parameter values into Equation (4.6) suggests that the time series of the S&P 500 variance rate is

$$\sigma_n^2 = 0.00000538659679 + 0.09960225692814u_{n-1}^2 + 0.86879882989990\sigma_{n-1}^2.$$

The long-run average variance $V = 0.00017046$ per day or 0.042957882 per annum, which yields an annualised volatility of 20.73%. This result is very close to the historical volatility of 19.49%, as can be expected. Table 7.1 shows the expected average variance over the lives of the options of the eight maturity dates calculated using Equation (4.7).

From Table 7.1, it can be seen that the longer it is to the maturity date, the closer the average variance is to the long-run average variance. We then use the predicted volatilities in Table 7.1 to price options according to their maturity dates.

Maturity Dates	Average Variance (per day)	Average Volatility (per day)	Predicted Volatility
2001/08/18	0.0000999412	0.009997059	15.87%
2001/09/22	0.0001277346	0.011301973	17.94%
2001/10/20	0.0001397527	0.011821706	18.77%
2001/12/22	0.0001528914	0.012364925	19.63%
2002/03/16	0.0001596626	0.012635768	20.06%
2002/06/22	0.0001630599	0.01276949	20.27%
2002/12/21	0.0001657944	0.012876118	20.44%
2003/06/21	0.0001670545	0.012924957	20.52%

Table 7.1: Predictive volatilities from GARCH(1,1). Note that the predicted volatilities in the right-most column are annualised.

It is found that GARCH(1,1) model generally underprices in-the-money and near-to-the-money options, and overprices out-of-the-money options. Since the GARCH(1,1) model is only a improvement of the Black-Scholes which uses different volatilities for options with different maturities, it uses the same volatility for different strikes¹³. This, however, does not comply with the implied volatility, which varies across different strike prices. Investors are often concerned about a stock market crash similar to that experienced in October 1987, and they price options accordingly [23]. This is the “crashophobia” argument of Mark Rubinstein.

For different maturities, the GARCH(1,1) model produces better results for medium maturity options than short and long maturity options.

¹³ A more consistent option-pricing approach would be to run Monte Carlo simulations of the asset price and volatility under (risk-neutral) GARCH dynamics. This would generate implied volatilities that vary across strike, not just maturity. This approach would, in principle, allow the GARCH parameters to be calibrated to option prices instead of the historical data. However, Monte Carlo at each time step of an optimisation procedure can be very time-consuming.

The option category where GARCH(1,1) performs best are the medium maturity, in-the-money and near-to-the-money options. In these categories, the relative errors of the model prices are constantly between -7% and -3% with a few exceptions of around -8% . This result gives one insights about how to predict the market price from the GARCH(1,1) price.

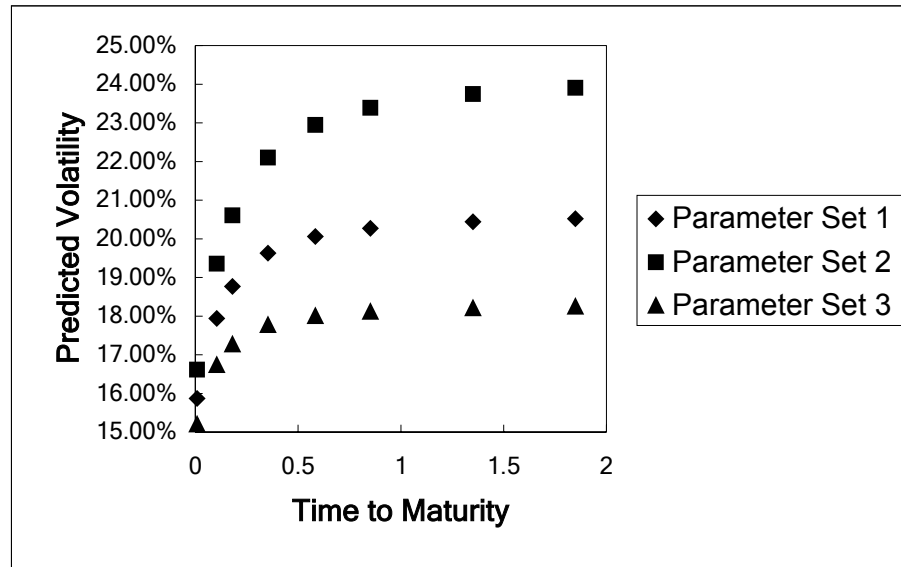


Figure 7.1: Effects of varying the GARCH(1,1) parameters have on the predicted volatilities. For all three parameter sets, ω and α are not changed. In Parameter Set 1, the original β is used. In Parameter Set 2, β is increased by 1%, whereas in Parameter Set 3, β is decreased by 1%.

It is observed that, the longer the life of the option, the more the predicted volatility is changed by varying parameter values. This result is shown in Figure 7.1.

The predicted volatilities are less sensitive to the changes in the parameters ω and α , but are more sensitive to the changes in β . For example, looking at predicted volatility for the furthest maturity, *ceteris paribus*, 1% increase in ω results in 0.1% increase in the predicted volatility; while 1% increase in α results in 0.32% increase in the predicted volatility. However, 1% increase in β results in 3.39% increase in the predicted volatility. Also increase in β cannot exceed 3.63%, or else the long-run average variance becomes negative.

The above result have direct impact on the sensitivity of option prices produced by the GARCH(1,1) to the changes in the parameters. The rate of option price change with respect to volatility of the underlying is determined by *vega*, which has two important properties:

1. For options with the same maturity, the closer the option is near-to-the-money, the higher the vega is associated with it.
2. For an at-the-money option, vega increases as time to maturity increases.

Thus we can deduce that long-life, near-to-the-money options have highest Vegas, and are therefore most sensitive to changes of underlying's volatility. As a result, they are also most sensitive to changes of the GARCH parameters, especially β .

7.4.2 The Constant Elasticity of Variance Model

The elasticity coefficient θ in the CEV model is estimated to be 1.729. Thus under the CEV assumption, asset price follows the stochastic differential equation

$$dS = \mu S dt + \delta S^{0.576} dW_t$$

with the instantaneous volatility given by

$$\sigma(S, t) = \delta S^{-0.1355}.$$

The prices generated by the CEV model appears to be a very good approximation for in-the-money and near-to-the-money options of short and medium maturities, with relative errors between -5% and $+5\%$.

To see how robust the CEV model is, we look at how change in the elasticity coefficient θ affects model option prices, using θ increased by 10% and decreased by 10%. A typical result is shown in Figure 7.2.

From Figure 7.2, it can be seen that the option prices produced using $\theta = 1.902$ and $\theta = 1.556$ do appear to be quite different from the original option prices with $\theta = 1.729$.

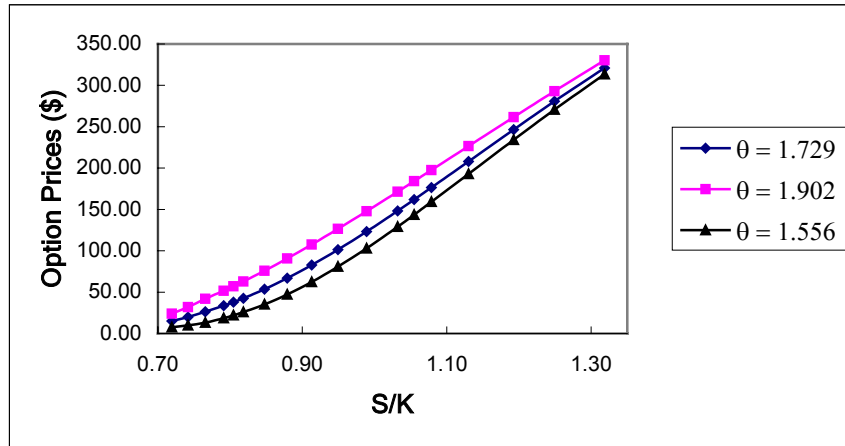


Figure 7.2: Option prices produced by CEV model with three different θ .

Figure 7.3 shows that the change in price decreases as the “moneyness” of option increases. Deep-in-the-money option experience least change in price when there is a change in θ . In fact, in our findings, we see that in-the-money options across all maturities have very small change in prices, while near-to-the-money and out-of-the-money options are sensitive to changes in θ .

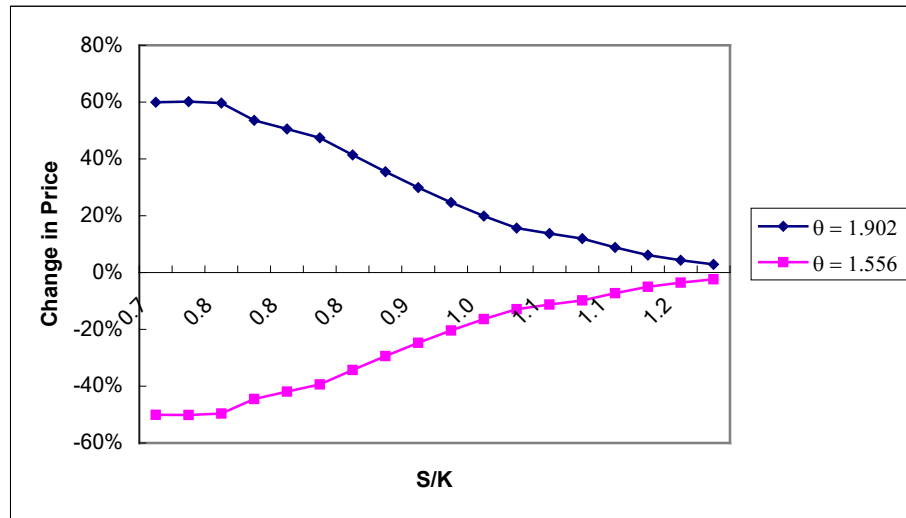


Figure 7.3: Change in CEV prices when varying the elasticity coefficient θ .

It is also found that, for in-the-money options, changes in θ do not cause changes in price to vary systematically with respect to different maturities.

7.4.3 Hull and White Model

The parameter of the Hull and White model is estimated to be $\xi = 0.002$. For near-to-the-money options of both short and long maturities, Hull and White model generally produces prices that better approximate the market than Black-Scholes. For medium maturity options, Hull and White model produces prices that are very close to Black-Scholes price.

For sensitivity analysis, we look at changes in the Hull and White price when we increase and decrease the value of the parameter ξ by 5%.

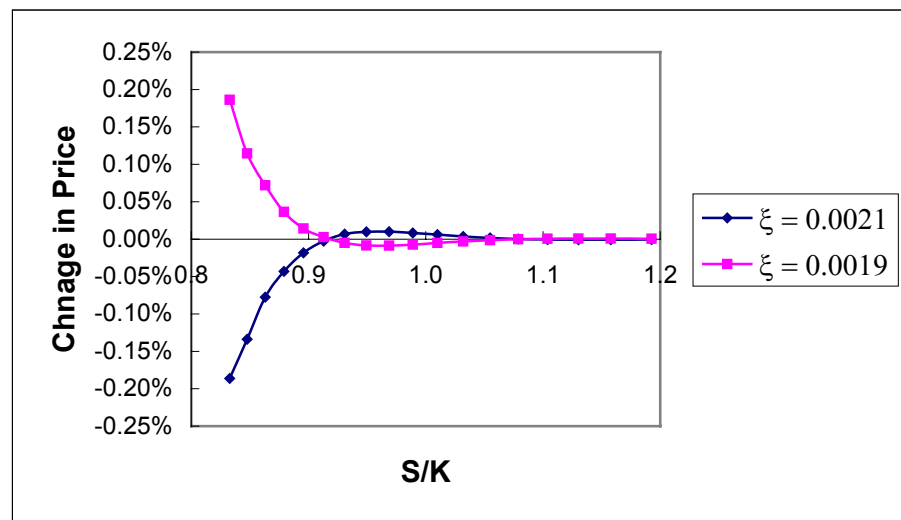


Figure 7.4: Changes in the Hull and White price when the value of the parameter ξ varies.

Out-of-the-money option prices do change quite significantly with respect to changes in ξ . On the other hand, near-to-the-money and in-the-money options are not sensitive to changes in ξ . It is also found that, the longer the life of the option is, the more sensitive the option price is.

Since ξ is the volatility of volatility in the Hull and White model, it has similar effects on option prices as the parameter ζ in the Heston model. Thus increase in ξ results in increase of

deep-in-the-money and deep-out-of-the-money options, and decrease of near to maturity options. Decreasing ξ has completely opposite effects.

7.4.4 Heston Model

The parameters in the Heston model are estimated to be

$$\kappa^* = 0.719$$

$$\theta^* = 0.0164$$

$$\zeta = 0.0183$$

$$\rho = -1$$

This means the risk-neutral rate of mean-reversion is 0.719; the risk-neutral long-run average variance is 0.0164. $\rho = -1$ means that the asset price and the variance/volatility is perfectly negatively correlated. This complies with the *leverage effect*. Suppose that we know the real world long-run average variance to be the square of the historical volatility estimated earlier i.e. $\theta = 0.194939^2 = 0.038$, we can then calculate the real world parameters κ and λ using Equation (6.3). The results are $\kappa = 0.3103$ and $\lambda = 0.4807$. Thus the risk-neutral world dynamic of variance is given by

$$dv = 0.719(0.0164 - v)dt + 0.0183\sqrt{v}dW_t^2$$

while the real world dynamic is

$$dv = 0.3103(0.038 - v)dt + 0.0183\sqrt{v}dW_t^2.$$

And $\lambda(S, v, t) = 0.4807v$ is the measure of volatility risk premium.

Using the set of parameters described above to price the S&P 500 index options produces some very impressive results. Except for options of the shortest and the longest maturities, the Heston model gives good approximations for in-the-money and near-to-the-money options of other maturities, with some of the relative errors below 1%.

To see how the Heston prices change with respect to changes in parameter values, we select arbitrarily to vary one or some parameters values as follows:

1. 10% increase in κ^* , *ceteris paribus*.
2. 10% increase in κ^* as well as 10% increase in θ^* , *ceteris paribus*.
3. 10% increase in κ^* , 10% increase in θ^* , and 10% increase in ζ , *ceteris paribus*.

4. 10% increase in θ^* and 10% increase in ζ , *ceteris paribus*.
5. 10% increase in ζ , *ceteris paribus*.
6. 10% increase in ρ , *ceteris paribus*.

The results are shown in Figure 7.5.

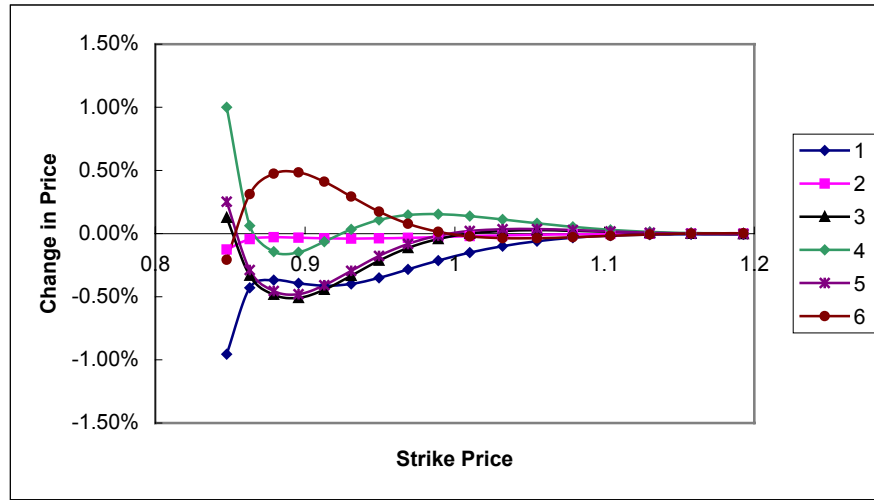


Figure 7.5: Change in the Heston prices with respect to changes in parameter values.

It can be seen that changes in Heston price for out-of-the-money options are bigger relative to near-to-the-money and in-the-money options. Nevertheless, all price changes remains between -1% and $+1\%$, which are only small changes. Thus the Heston model is the most robust in all the models we have looked at in terms of sensitivity to parameter changes.

7.5 Choosing between Models

The simulation results for all four models can be found in the file RESULTS.XLS in the diskette provided with this dissertation. Because all four models produce relative errors with respect to the market in all option categories, we suggest that there is no one model which can fully approximate the market. The possible reasons are:

1. Parameter values are incorrectly estimated.
2. Models cannot fully capture the variations in the market.

The first possibility is highly likely in this research. Since the data set that we used in this research is the option prices taken in one day, the estimated parameter values may be incorrect. To obtain more precise parameter estimates, the daily option prices over a period of time are required.

We consider the second possibility because in all models, it is assumed that option prices are completely determined by the underlying assets. However, this is not the case in practice. In the market, option prices are determined by much more factors, such as demand and supply, than just the underlying assets. Therefore, the models cannot fully describe option prices in the market.

When a cross-comparison is performed on all models, it is found that none of the four models dominates the others. In other words, no one model can produce prices that give relative errors lower than the other models for all strikes and all maturities. Thus our original goal, which is to find the best model, cannot be achieved. The next best alternative is then to find which models are more appropriate when pricing a particular category of options, recalling that options are divided into nine categories (See section 7.3).

Both the CEV and the Heston model appear to be pricing better than the others in the short maturity and in-the-money category. For short maturity and near-to-the-money category, the Hull and White as well as the GARCH(1,1) model proves to be least in error. However, the simple Black-Scholes model, which assumes constant volatility, also comes in handy. For short maturity and out-of-the-money category, it is the GARCH(1,1) model that produces best prices with respect to the market.

For medium and long maturity options of all “moneyness” categories, the results are not as diverse as for short maturity options. Although all other models are good approximations, the Heston model appears to be better by producing much smaller relative errors. Therefore for medium and long maturities options, the best model to be used for option pricing is the Heston model. Table 7.2 summarises the results.

Of the three models with their parameters calibrated to the option prices (CEV, Hull and White, and Heston), the Heston model’s advantage compared to the other two is not

surprising, given that it has four parameters, whereas the other two models are free to choose only one parameter.

	Short Maturity	Medium Maturity	Long Maturity
In-the-money	CEV, Heston	Heston	Heston
Near-to-the-money	BS, HW, GARCH(1,1)	Heston	Heston
Out-of-the-money	GARCH(1,1)	Heston	Heston

Table 7.2: Best models to be used for pricing the S&P 500 Index option.

Chapter 8

Conclusion

In this dissertation, we attempted to find the best model for the non-constant volatility problem. We have considered two deterministic volatility and two stochastic volatility models. The former two are the GARCH(1,1) model, and the Constant Elasticity of Variance model; and the latter two are the Hull and White model, and the Heston model. To achieve our goal, we have looked at the relative errors of the model prices with respect to the market prices. The sensitivities of the model prices to model parameters are also considered.

Findings

From the results of the simulation, we have found that, for S&P 500 index option:

1. None of the models can fully reproduce the market prices since all of them produce relative errors with respect to the market.
2. There is no model that dominates the others by producing prices that are in closer agreement with the market prices in all option categories.
3. Of all the models, the Constant Elasticity of Variance model is one of the best for pricing the short maturity, in-the-money options.
4. The GARCH(1,1) model, comparing to other models, is good in for both short maturity, near-to-the-money and short maturity, out-of-the-money option categories.
5. The Hull and White model is another good model in the short maturity, near-to-the-money option category.
6. Lastly, the Heston model is good for pricing the short maturity, near-to-the-money options, as well as all medium and long maturities options.

Future Work

As discussed in Chapter 7, we argue that the data consists of only one day's option prices, which may not be sufficient to estimate the parameter values for the models. Future work should repeat the simulation with daily option prices over a longer period of time (e.g. a year) for estimating and comparing models.

Work can also be done comparing additional models of volatility, e.g. the Stein and Stein model [36]. This model has a distinctive feature that it provides the closed-form solution for terminal asset price distribution given stochastic volatility.

For the purpose of this research, we made the assumption that model parameters stay constant over time. This enables us to estimate model parameters from one day's option prices, and then calculate option prices of the same day based on the parameter estimates. Thus the results are of limited significance. In future work, this assumption should be relaxed and existing data should be used to estimate option prices.

Appendix A

Program Listing

All programs written in Matlab [26] to simulate model prices are listed here.

A.1 Black-Scholes Program: BS.m

```
S = 1186.73;
rate = 0.034;
div = 0.01162;
K = 1100;
T = 3/365;
vol = 0.194938616;
GARCH = (S.*exp(-div.*(T)).*erfc(-d1./sqrt(2))./2 -
K.*exp(-rate.*(T)).*erfc(-d2./sqrt(2))./2)
```

A.2 CEV Program: f.m

```
function c = f(theta)
% Calculationg option prices using a Constant Elasticity of Variance model
% with any elasticity coefficient theta between 0 and 2
global r q K T
r = [0.034 0.034 0.034 0.0331 0.0331 0.0331 0.0365 0.0365 0.0365];
q = [0.01162 0.01162 0.01162 0.01309 0.01309 0.01309 0.0129 0.0129 0.0129];
K = [1100 1195 1270 1050 1175 1450 995 1200 1650];
T = [0.00822 0.00822 0.00822 0.5836 0.5836 0.5836 1.8493 1.8493 1.8493];
% Calculate CEV prices for theta = 4/3
```



```

f4b3 = CEV(4/3);
% Get CEV prices for theta = 0 and 2
f0 = [86.9241 4.9501 0.0002 167.0460 82.8748 6.3574 267.3965 139.3724 14.4004];
f2 = [86.92 4.98 0.0003 164.22 82.47 9.39 259.94 139.67 25.22];
% Interpolating for any theta between 0 and 2
CEVP = f0 + (3/4).*theta.*(f4b3-f0) +
(1/4).*theta.*(theta-4/3).*(3.*f0-9.*f4b3+6*f2);
c = CEVP;

```

A.3 CEV Program: CEV.m

```

function c = CEV(theta)
% Calculationg option prices using a Constant Elastcity of Variance model
% with elasticity coefficient theta = 4/3

% Get option parameters
global r q K T
S0 = 1186.73;
r = [0.034 0.034 0.034 0.0331 0.0331 0.0331 0.0365 0.0365 0.0365];
q = [0.01162 0.01162 0.01162 0.01309 0.01309 0.01309 0.0129 0.0129 0.0129];
K = [1100 1195 1270 1050 1175 1450 995 1200 1650];
T = [0.00822 0.00822 0.00822 0.5836 0.5836 0.5836 1.8493 1.8493 1.8493];
sig = 0.194938616;

% determine delta based on current asset price and instantaneous volatility
delta = sig/(S0^((theta-2)/2));
z = exp((r-q).*(2-theta).*T);
k = 2.*(r-q)./delta./delta./(2-theta)./(z-1);
x = k.*(S0.^(2-theta)).*z;
y = k.*(K.^(2-theta));
CEVP = S0.*exp(-q.*T).*Q3(2.*y,2.*x) - K.*exp(-r.*T).*(1-Q1(2.*x,2.*y));
c = CEVP;

```

A.4 CEV Program: Q1.m

```

function c = Q1(z,kappa)

```

```
% Compute non-central chi-square distribution
% with 1 degrees of freedom
c = normcdf(sqrt(kappa)-sqrt(z)) + normcdf(-sqrt(kappa)-sqrt(z));
```

A.5 CEV Program: Q3.m

```
function c = Q3(z,kappa)
% Compute non-central chi-square distribution
% with 3 degrees of freedom
c = Q1(z,kappa) + (n(sqrt(kappa)-sqrt(z))-n(sqrt(kappa)+sqrt(z)))./sqrt(kappa);
```

A.6 Hull and White Program: HW.m

```
function c = HW(xi)
% Compute the Hull and White model price with parameter xi

% Get option parameters
S0 = 1186.73;
r = [0.034 0.034 0.034 0.0331 0.0331 0.0331 0.0365 0.0365 0.0365];
q = [0.01162 0.01162 0.01162 0.01309 0.01309 0.01309 0.0129 0.0129 0.0129];
OP = [91 6.8 0.9 176.6 87.5 4.2 286.8 156.3 16.4];
K = [1100 1195 1270 1050 1175 1450 995 1200 1650];
T = [0.00822 0.00822 0.00822 0.5836 0.5836 0.5836 1.8493 1.8493 1.8493];
sig = 0.194938616;

% Calculate k
k = xi*xi.*T;
d1 = (log(S0./K)+(r-q+0.5*sig*sig).*T)./(sig.*sqrt(T));
d2 = d1-sig*sqrt(T);

% Calculate Black-Scholes price
BSP = S0.*exp(-q.*T).*erfc(-d1./sqrt(2))./2 -
K.*exp(-r.*T).*erfc(-d2./sqrt(2))./2;
% Power series approximation
```

```

Term2 =
(S0.*sqrt(T).*(1/sqrt(2*pi)).*exp(-0.5.*(d1.^2)).*(d1.*d2-1))./(8*sig^3).*(2
*(sig^4)*(exp(k)-k-1)./(k.^2)-(sig^4));
Term3a =
(S0.*sqrt(T).*(1/sqrt(2*pi)).*exp(-0.5.*(d1.^2)).*((d1.*d2-1).*(d1.*d2-3)-(d
1.^2+d2.^2)))/(48*(sig^5));
Term3b =
(sig^6).*(exp(3.*k)-(9+18.*k).*exp(k)+(8+24.*k+18.*(k.^2)+6.*(k.^3)))/(3.*(k
.^3));
HWP = BSP + Term2 + Term3a.*Term3b;
c = HWP;

```

A.7 Heston Program: Heston.m

```

function c = Heston(x)
%Compute Heston price
global kappa theta zeta rho
kappa = x(1); % risk-neutral rate of mean-reversion
theta = x(2); % risk-neutral long-run average variance
zeta = x(2); % volatility of volatility
rho = x(3); % correlation between S and v
global S0 r q K T
global mu1 mu2 a b1 b2
% get option parameters
S0 = 1186.73;
r = 0.034;
q = [0.01162 0.01162 0.01162 0.01309 0.01309 0.01309 0.0129 0.0129 0.0129];
OP = [91 6.8 .9 176.6 87.5 4.2 286.8 156.3 16.4];
K = [1100 1195 1270 1050 1175 1450 995 1200 1650];
T = [0.00822 0.00822 0.00822 0.5836 0.5836 0.5836 1.8493 1.8493 1.8493];
x0 = log(S0);
v0 = 0.194938616^2;
mu1 = 0.5;
mu2 = -0.5;
a = kappa*theta;
b1 = kappa - rho*zeta;

```

```

b2 = kappa;
k = 6;
N = 2^k;
Iodd = 1:2:(N-1);
Ieven = 2:2:(N-2);
% intergration using Simpson's method
a = 0.0000001;
b = 100;
dphi = (b-a)/N;
phi_odd = a + Iodd*dphi;
phi_even = a + Ieven*dphi;
odd_sum1 = 0;
odd_sum2 = 0;
even_sum1 = 0;
even_sum2 = 0;
for i = 1:length(phi_odd)
    odd_sum1 = odd_sum1 + P1_integrand(phi_odd(i),x0,v0,T);
    odd_sum2 = odd_sum2 + P2_integrand(phi_odd(i),x0,v0,T);
end
for i = 1:length(phi_even)
    even_sum1 = even_sum1 + P1_integrand(phi_even(i),x0,v0,T);
    even_sum2 = even_sum2 + P2_integrand(phi_even(i),x0,v0,T);
end
sum1 =
(P1_integrand(a,x0,v0,T)+4*odd_sum1+2*even_sum1+P1_integrand(b,x0,v0,T))*dph
i/3;
sum2 =
(P2_integrand(a,x0,v0,T)+4*odd_sum2+2*even_sum2+P2_integrand(b,x0,v0,T))*dph
i/3;
P1 = 0.5 + sum1./pi;
P2 = 0.5 + sum2./pi;
HP = S0*exp(-q.*T).*P1 - exp(-r.*T).*K.*P2;
c = HP;

```

A.8 Heston Program: P1_integrand.m

```

function c = P1_integrand(phi,x,v,t)
% Integrand of P1
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d1 g1
global x0 v0
c = real(exp(-i*phi.*log(K)).*f1(phi,x,v,t)/(i*phi));

```

A.9 Heston Program: P2_integrand.m

```

function c = P2_integrand(phi,x,v,t)
% Integrand of P2
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d2 g2
global x0 v0
c = real(exp(-i*phi.*log(K)).*f2(phi,x,v,t)/(i*phi));

```

A.10 Heston Program: f1.m

```

function c = f1(phi,x,v,t)
% Compute characteristic function f1
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d1 g1
d1 = sqrt((rho*zeta*phi*i-b1)^2-zeta*zeta*(2*mu1*phi*i-phi*phi));
g1 = (b1-rho*zeta*phi*i+d1)/(b1-rho*zeta*phi*i-d1);
c = exp(A1(T,phi)+B1(T,phi).*v+i.*x.*phi);

```

A.11 Heston Program: f2.m

```

function c = f2(phi,x,v,t)

```

```

% Compute characteristic function f2
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d2 g2
d2 = sqrt((rho*zeta*phi*i-b2)^2-zeta*zeta*(2*mu2*phi*i-phi*phi));
g2 = (b2-rho*zeta*phi*i+d2)/(b2-rho*zeta*phi*i-d2);
c = exp(A2(T,phi)+B2(T,phi).*v+i.*x.*phi);

```

A.12 Heston Program: A1.m

```

function c = A1(t,phi)
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d1 g1
c = r.*phi.*i.*t +
a/(zeta^2).*((b1-rho*zeta*phi*i+d1).*t-2.*log((1-g1.*exp(d1.*r))./(1-g1)));

```

A.13 Heston Program: A2.m

```

function c = A2(t,phi)
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d2 g2
c = r.*phi.*i.*t + a/(zeta^2)*((b2-rho*zeta*phi*i+d2).*t -
2.*log((1-g2.*exp(d2.*r))./(1-g2)));

```

A.14 Heston Program: B1.m

```

function c = B1(t,phi)
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2

```

```

global d1 g1
c = (b1-rho*zeta*phi*i+d1)./(zeta*zeta).*((1-exp(d1.*r))./(1-g1.*exp(d1.*r)));

```

A.15 Heston Program: B2.m

```

function c = B2(t,phi)
global kappa theta zeta lambda rho
global S0 r q K T
global mu1 mu2 a b1 b2
global d2 g2
c = (b2-rho*zeta*phi*i+d2)./(zeta*zeta).*((1-exp(d2.*r))./(1-g2.*exp(d2.*r)));

```

Bibliography

1. Abken, P. A., and S. Nandi, 1996, "Options and Volatility," *Economic Review*, 81, 21-35.
2. Ang, J. S., and D. R. Peterson, 1984, "Empirical Properties of the Elasticity Coefficient in the Constant Elasticity of Variance Model," *Finance Review*, 1, 372-380.
3. Baaquie, B.E., 1997, "A Path Integral Approach to Option Pricing with Stochastic Volatility: Some Exact Results," *J. Phys. I France*, 7, 1733-1753.
4. Ball, C. A. and A. Roma, 1994, "Stochastic Volatility Option Pricing," *Journal of Financial and Quantitative Analysis*, 29, 589-607.
5. Bates, D., 1995, "Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutschemark Option," *Review of Financial Studies*, 9, 69-107.
6. Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.
7. Bollerslev, T., 1986, "Generalised Autoregressive Conditional Heteroscedasticity," *Journal of Econometrics*, 31, 307-327.
8. Briys, E., M. Bellalah, H. M. Mai, and F. De Varenne, 1998, *Options, Futures, and Exotic Derivatives. Theory, Application and Practice*, John Wiley.
9. Brockman, P., and M. Chowdhury, 1997, "Deterministic versus Stochastic Volatilities: Implications for Option Pricing Models," *Applied Financial Economics*, 7, 499-505.
10. Carr, P., 1999, "FAQ's in Option Pricing Theory," Working Paper, Banc of Americas Securities.
11. Cox, J. C., 1975, "Notes on Option pricing I: Constant Elasticity of Variance Diffusions," Unpublished Notes, Graduate School of Business, Stanford University.
12. Cox, J. C., J. E. Ingersoll, and S. A. Ross, 1985, "A Theory of the Term Structure of Interest Rates," *Econometrica*, 53, 385-408.
13. Cox, J. C., and S. A. Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics*, 3, 145-166.

14. Dumas, B., J. Fleming, and R. E. Whaley, 1998, "Implied Volatility functions: Empirical Tests," *Journal of Finance*, 53, 2059-2106.
15. Emanuel, D. C., and J. D. MacBeth, 1982, "Further results on the Constant Elasticity of Variance Call Option Pricing Model." *Journal of Financial and Quantitative Analysis*, 17, 533-554.
16. Engle, R. F., 1982, "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of U.K. inflation," *Econometrica*, 50, 987-1008.
17. Fama, E. F., 1965, "The Behaviour of Stock Market prices," *Journal of Business*, 8, 34-105.
18. Heston, S. L., 1993, "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6, 327-343.
19. Heston, S. L., and S. Nandi, 2000, "A Closed-Form GARCH Option Valuation Model," *Review of Financial Studies*, 13, 585-625.
20. Heston, S. L., and S. Nandi, 2000, "Derivatives on Volatility: Some Simple Solutions Based on Observables," Federal Reserve Bank of Atlanta Working Paper 2000-20.
21. Hobson, D., 1996, "Stochastic Volatility," Preprint, University of Bath.
22. Hobson, D. G., and L. C. G. Rogers, 1996, "Complete Models with Stochastic Volatility." University of Bath, School of Mathematical Sciences. Unpublished paper.
23. Hull, J. C., 2000, *Options, Futures, and Other Derivatives*. 4th ed. Prentice Hall.
24. Hull, J., and A. White, 1987, "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281-300.
25. Mandelbrot, B., 1963, "The Variation of certain Speculative prices," *Journal of Business*, 36, 394-419.
26. The Mathworks Inc. <http://www.mathworks.com>.
27. Melino, A., and S. M. Turnbull, 1990, "Pricing Foreign Currency Options with Stochastic Volatility," *Journal of Econometrics*, 45, 239-265.
28. Nandi, S., 1996, "Pricing and Hedging Index Options under Stochastic Volatility: an Empirical Examination," Federal Reserve Bank of Atlanta Working Paper 96-9.
29. Nelson, D. B., 1990, "ARCH Models as Diffusion Approximations," *Journal of Econometrics*, 45, 7-38.
30. Oztukel, A., and P. Wilmott, 1996, "Uncertain parameters, an Empirical Stochastic Volatility Model and Confidence Limits," Working paper, Mathematical Institute, Oxford and Department of Mathematics, Imperial College, London.

31. Rubinstein, M., 1978, "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classed from August 23, 1976 through August 31, 1978," *Journal of Finance*, 40, 455-480.
32. Rubinstein, M., 1994, "Implied Binomial Trees," *Journal of Finance*, 49, 771-818.
33. Schobel, R., and J. Zhu, 1999, "Stochastic Volatility with an Ornstein-Uhlenbeck Process: An Extension," *European Finance Review*, 3, 23-46.
34. Schroder, M., 1989, "Computing the Constant Elasticity of Variance Option pricing Formula," *Journal of Finance*, 44, 211-219.
35. Scott, L. O., 1987, "Option Pricing When the Variance Changes Randomly: Theory, Estimation, and an Application," *Journal of Financial and Quantitative Analysis*, 22, 419-438.
36. Stein, E. M., and J. C. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: An Analytic Approach," *Review of Financial Studies*, 4, 727-752.
37. Wiggins, J. B., 1987, "Option Values under Stochastic Volatilities. Theory and Empirical Estimates," *Journal of Financial Economics*, 19, 351-372.
38. Wilmott, P., 2000, *Paul Wilmott on Quantitative Finance*, 2 Volumes, John Wiley.
39. Wu, G., 2001, "The Determinants of Asymmetric Volatility", *Review of Financial Studies*, 14, 837-859.