Bounds on the Index of an Umbilic Point

John Guckenheimer
Mathematics Department
Cornell University

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Abstract

Umbilics are points of a surface embedded in three space where normal curvatures are independent of direction. The (in)famous Carathéodory Conjecture states that a compact simply connected embedded surface has at least two umbilic points. A counterexample to this conjecture would be a surface whose principal foliation has index two at a single umbilic. All (purported) proofs of the Carathéodory Conjecture are based on analyses of the index of an umbilic, concluding that it is at most one. This investigation gives a much simpler geometric argument that the index of an umbilic on an analytic surface cannot be an integer larger than one, providing new insight into the Carathéodory Conjecture. The results also establish lower bounds for the index of an umbilic based on its degeneracy.

Principal Curvatures and Foliations

Let $S \subset \mathbb{R}^3$ be a smooth oriented surface. If $p \in S$, $N(p)$ is the unit normal to S at p, $\gamma(s)$ is a planar curve on S parameterized by arc length with $\gamma(0) = p$ and $\frac{d\gamma}{ds}(0) = v$, then the normal curvature at p in the direction v is $\kappa(p, v) = \frac{d^2(\gamma)}{ds^2}(0) \cdot N(p)$. Points where the normal curvatures are independent of direction are *umbilics* (or umbilic points). Elsewhere, vectors v_1 and v_2 whose normal curvatures are maxima and minima of κ are *principal vectors*. *Rodrigues formula* states that the tangent vector $\frac{d\gamma}{dt}(p)$ to γ is a principal vector v if and only if $\frac{dN\circ\gamma}{dt}(p) = \kappa\frac{d\gamma}{dt}(p)$ for some κ [?], producing a system of equations whose solutions v,κ are principal unit vectors and curvatures:

$$\begin{aligned} N \cdot v &= 0 \\ v \cdot v &= 1 \\ \det(N, v, dN \cdot v) &= 0. \end{aligned} \tag{1}$$

The equations (??) express that v is a tangent vector to S, v has unit length and that $dN \cdot v$ lies in the plane spanned by N and v. The first of these equations is linear in v while the second two are quadratic and even in v. In the complement of the umbilics, the first and third of the equations define a *cross field* consisting of two orthogonal line fields. Integration of these line fields yields the *principal foliations* of the surface with singularities at the umbilics. The index theorem applies to the principal foliations: on a simply connected compact surface with isolated umbilics, the sum of their indices is two.[1] Umbilics of generic surfaces were classified by Darboux [?]. There are three types labelled *lemons*, *monstars* and *stars* with indices $\frac{1}{2}$, $\frac{1}{2}$ and $-\frac{1}{2}$ [?].

Sotomayor, Gutiérrez and Garcia developed a qualitative theory for the geometry of principal foliations analogous to dynamical systems theory on two dimensional manifolds [?, ?, ?, ?, ?, ?, ?, ?, ?]. They described properties of generic principal foliations and generic one parameter families of foliations. Their results were inspired by the work of Peixoto and Sotomayor on two dimensional vector fields [?, ?]. Here we pursue another aspect of principal foliations, namely the geometry of their principal foliations near umbilics

[1] The index of an isolated umbilic measures the change in the angle of a principal vector along a positively oriented closed curve surrounding that umbilic and no others.

that are degenerate. The geometry of two dimensional vector fields near non-hyperbolic equilibria of two dimensional dynamical systems has been studied extensively, e.g. [?].

Beyond bifurcations in generic one parameter families of principal foliations [?], existing literature on degenerate umbilics is dominated by the Carathéodory Conjecture. This conjecture states that a compact simply connected surface in $\mathbb{R}^3$ has at least two umbilics. The status of the Carathéodory Conjecture is controversial, with many papers that claim to prove the conjecture followed by work describing deficiencies of the proofs. In the list of papers below, the papers of Bol, Klotz, Scherbel and Ivanov give successive attempts to complete the arguments initiated by Hamburger. Smyth, Xavier and Guilfoyle use PDE methods and discuss smoothness extensively. Lasarovici and Smyth approach the problem topologically by formulating results that rule out "elliptic sectors" of an umbilic. Some papers show that "counterexamples" exist with relaxed smoothness or perturbations of the Euclidean metric.

- L. Bates, A weak counterexample to the Carathéodory conjecture, 2001 [?]
- G. Bol, Über Nabelpunkte auf einer Eifläche, 1943 [?]
- B. Guilfoyle, On isolated umbilic points, 2020 [?]
- B, Guilfoyle and W. Klingenberg, Isolated umbilical points on surfaces in $\mathbb{R}^3$, 2006 [?]
- C. Gutierrez, F. Mercuri and F. Sánchez-Bringas, On a conjecture of Carathéodory: analyticity versus smoothness, 1996 [?]
- C. Gutierrez and F. Sánchez-Bringas, On a Carathéodory's conjecture on umbilics: representing ovaloids, 1997 [?]
- C. Gutierrez and J. Sotomayor, Lines of curvature, umbilic points and Carathéodory conjecture, 1998 [?]
- H. Hamburger, Beweis einer Carathéodoryschen Vermutung I,II,III, 1940-41 [?, ?, ?]
- V. Ivanov, An analytic conjecture of Carathéodory, 2002 [?]
- T. Klotz, On G. Bol's proof of Carathéodory's conjecture, 1959 [?]
- L. Lazarovici, Elliptic sectors in surface theory and the Carathéodory-Loewner conjectures, 2000 [?]
- B. Smyth, The nature of elliptic sectors in the principal foliations of surface theory, 2005 [?]
- B. Smyth and F. Xavier, A sharp geometric estimate for the index of an umbilic on a smooth surface, 1992 [?]
- B. Smyth and F. Xavier, Real solvability of the equation $\partial^2_{\bar{z}}\omega = \rho g$ and the topology of isolated umbilics, 1998
- H. Scherbel, A new proof of Hamburger's Index Theorem on umbilical points, 1993 [?]
- J. Sotomayor and L. Mello, A note on some developments on Carathéodory conjecture on umbilic points, 1999 [?]
- C. Titus, A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points, 1973 [?]
- F. Xavier, An index formula for Loewner vector fields, 2007 [?]

In addition to these publications, Guilfoyle posted a manuscript on the ArXiv in 2008, that presents a different strategy for proving the Carathéodory conjecture.

Throughout this body of work, the goal has been to establish that the index of an isolated umbilic point is at most one. Since the sum of the indices of umbilics of a compact simply connected surface is two, this local result implies the Carathéodory conjecture. For analytic surfaces, the strategy initiated by Hamburger introduced special *Monge coordinates* that represent the surface locally as the graph of an analytic function. This paper also uses Monge coordinates as a starting point for studying analytic surfaces, but it adopts a geometric viewpoint that emphasizes bifurcations of principal foliations in families of surfaces.

Monge Coordinates at Umbilics

Denote the set of umbilics on the surface S by Λ. If $\lambda \in \Lambda$ is an umbilic, choose orthonormal coordinates (x, y, z) so that λ is the origin and the z axis is normal to S. The implicit function theorem implies that there is a neighborhood U of λ in which $S \cap U$ is the graph of a function $z = h(x, y)$ with $dh_{(0,0)} = (0, 0)$. Rodrigues formula gives an equation for the principal directions of S in terms of the derivatives of h. Denote the first and second partial derivatives of h by $h_x, h_y, h_{xx}, h_{xy}, h_{yy}$. The vector field $N = \alpha(-h_x, -h_y, 1)^{\mathsf{T}},\ \alpha^{-2} = 1 + h_x^2 + h_y^2$ is normal to S. Differentiating,

$$dN = d\alpha \begin{pmatrix} -h_x \\ -h_y \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} -h_{xx} & -h_{xy} & 0 \\ -h_{xy} & -h_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

A non-zero vector $(u, v, w)^{\mathsf{T}}$ is tangent to S if and only if $w = uh_x + vh_y$. Rodrigues Formula implies that it is a principal vector when

$$R(x, y, u, v, w) = h_{xy}h_y vw - h_x h_{yy} vw + h_{xy}v^2 + h_{xx}h_y uw - h_x h_{xy} uw - h_{yy}uv + h_{xx}uv - h_{xy}u^2 \tag{3}$$

vanishes with $w = uh_x + vh_y$.

The goal here is to investigate principal foliations near degenerate umbilics. Equation (**??**) is satisfied identically when S is a sphere. (Spheres and planes are the only surfaces all of whose points are umbilics. [**?**]) Thus, we study the index of the origin for surfaces S that are graphs of functions

$$h(x, y) = 1 - \sqrt{1 - x^2 - y^2} + p(x, y) + q(x, y) \tag{4}$$

where p is a non-zero homogeneous polynomial of degree d and q is analytic function that is $o(d)$ at the origin.

For each (x, y), the expression $R(x, y, u, v, uh_x + vh_y)$ is a homogeneous quadratic polynomial of (u, v). This polynomial is either identically zero or has roots on the cross field of principal directions. Restricting to tangent vectors of unit length, R has four roots on $S - \Lambda$ representing the principal directions at (x, y). At umbilics, R itself is 0. Along an oriented closed curve $\sigma \subset S - \Lambda$ the cross field rotates by a total angle ψ that is a multiple of π. The winding number ω of the principal foliations around σ is $\frac{\psi}{2\pi}$, an integer multiple of 1/2. The index of an isolated umbilic λ is the winding number of a positively oriented simply closed curve σ that contains λ but no other umbilics in its interior. If S_a is an analytic family of surfaces, the umbilics Λ_a of S_a constitute an analytic variety defined by vanishing of the Rodrigues formula.

On circles C_r centered at the origin in the (x, y) plane, the equations (**??**) (appropriately scaled) have a limit that depends only on the lowest degree non-zero Taylor polynomial of R with respect to (x, y) as $x^2 + y^2 \to 0$. Using this limit simplifies calculation of the index of the umbilic. If this lowest degree d is odd, then the index is not an integer since following solutions of $R = 0$ around a simple closed curve enclosing the origin will reverse their orientation. Thus, umbilic points of index 2 could only occur for even d. The $d + 1$ dimensional space P_d of homogeneous polynomials $p_a(x, y)$ of degree d is parametrized by the coefficients a_k in

$$p_a(x, y) = \sum_{k=0}^{d} a_k x^{d-k} y^k,\ a = (a_0, \cdots, a_d).$$

Our primary focus is on the family of surfaces S_a^d that are graphs of $h(x, y) = 1 - \sqrt{1 - x^2 - y^2} + p_a(x, y)$. We study the index of the umbilic at the origin for $d > 2$ even. The next section presents a numerical investigation of S_a^8 that focuses upon the index of the degenerate umbilic at the origin. It computes functions that map the circles C_r to the angle of the maximal principal foliation along the circle. When the origin is the only umbilic inside C_r, the winding number of this map is the index of the origin. The final section presents theoretical results about S_a^d that give new insight into the Caratheodory conjecture.

A Numerical Study of S_a^8

This section investigates the family of surfaces S_a^8 which are graphs of the functions

$$h_a(x, y) = 1 - \sqrt{1 - x^2 - y^2} + \sum_{j=0}^{8} a_j x^{(8-j)} y^j. \tag{5}$$

with $a = (a_0, \cdots, a_8)$. When $a = 0$, the surface S_0 is a sphere on which all points are umbilics. When $a \neq 0$, the origin is a degenerate umbilic point due to the absence of terms of degree three in h_a. There is an 8 dimensional stratified set B in the 9 dimensional parameter space A so that the index of the umbilic at the origin of S_a is constant in each component of $A - B$. The results presented here suggest that these indices lie in the interval $[-3, 1]$.

The Rodrigues formula $R(x, y, u, v, uh_x + vh_y)$ for the surface S_a is

$$\begin{aligned} R(x, y, u, v) = &(-7a_7y^6 - 12a_6xy^5 - 15a_5x^2y^4 - 16a_4x^3y^3 - 15a_3x^4y^2 - 12a_2x^5y - 7a_1x^6)u^2 \\ &+((2a_6 - 56a_8)y^6 + (6a_5 - 42a_7)xy^5 + (12a_4 - 30a_6)x^2y^4 + (20a_3 - 20a_5)x^3y^3 \\ &+(30a_2 - 12a_4)x^4y^2 + (42a_1 - 6a_3)x^5y + (56a_0 - 2a_2)x^6)uv \\ &+(7a_7y^6 + 12a_6xy^5 + 15a_5x^2y^4 + 16a_4x^3y^3 + 15a_3x^4y^2 + 12a_2x^5y + 7a_1x^6)v^2 + R_7 \end{aligned} \tag{6}$$

where the minimal degree of terms in R_7 is 7. The truncation $\bar{R} = R - R_7$ is a homogeneous polynomial of degree 6 in (x, y) that does not depend on r. As $r \to 0$ along the ray $(r\cos(\theta), r\sin(\theta))$ in the (x, y) plane, $r^{-6}R \to r^{-6}\bar{R}$. As $(\cos(\theta), \sin(\theta))$ traverses the unit circle, solutions of $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$ approximate principal vectors of S_a in a neighborhood of the origin. When $\bar{R}$ is a regular function of (θ, ψ), the winding number of a smooth solution $\psi(\theta)$ to $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi) = 0$ gives the index of the umbilic of S_a at the origin. For fixed a_j and θ, the quadratic formula gives explicit solutions of $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$.

Writing $R(x, y, u, v, uh_x + vh_y) = c_{uu}(x, y, a)u^2 + c_{uv}(x, y, a)uv + c_{vv}(x, y, a)v^2$, (x, y, a) is an umbilic point when $c_{uu} = c_{uv} = c_{vv} = 0$. Note that the sum of the coefficients of u^2 and v^2 in $\bar{R}$ is 0. Setting

$$\begin{aligned} \beta = &7a_7y^6 + 12a_6xy^5 + 15a_5x^2y^4 + 16a_4x^3y^3 + 15a_3x^4y^2 + 12a_2x^5y + 7a_1x^6 \\ \gamma = &((2a_6 - 56a_8)y^6 + (6a_5 - 42a_7)xy^5 + (12a_4 - 30a_6)x^2y^4 + (20a_3 - 20a_5)x^3y^3 \\ &+ (30a_2 - 12a_4)x^4y^2 + (42a_1 - 6a_3)x^5y + (56a_0 - 2a_2)x^6)/2 \end{aligned} \tag{7}$$

the equation $\bar{R} = 0$ becomes $\beta(v^2 - u^2) + 2\gamma uv = 0$ with solutions

$$(u, v) = (\beta, -\gamma \pm \sqrt{\beta^2 + \gamma^2}).$$

Alternatively, in terms of (u, v) polar coordinates, if

$$(u, v) = \rho(\cos(\psi), \sin(\psi))$$

then

$$\frac{v}{u} = \tan(\psi),$$

and

$$\frac{2uv}{u^2 - v^2} = \tan(2\psi) = \frac{\beta}{\gamma}$$

so

$$\psi = \frac{1}{2}\tan^{-1}(\frac{\beta}{\gamma}). \tag{8}$$

For fixed θ, the equation $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$ defines ψ as a smooth function of θ for almost all values of $a = (a_0, \cdots, a_8)$. Equation (**??**) gives explicit solutions for 2ψ and determines ψ (mod π). Below, we calculate ψ as a solution of R rather than $\bar{R}$ in order to take account of the effects

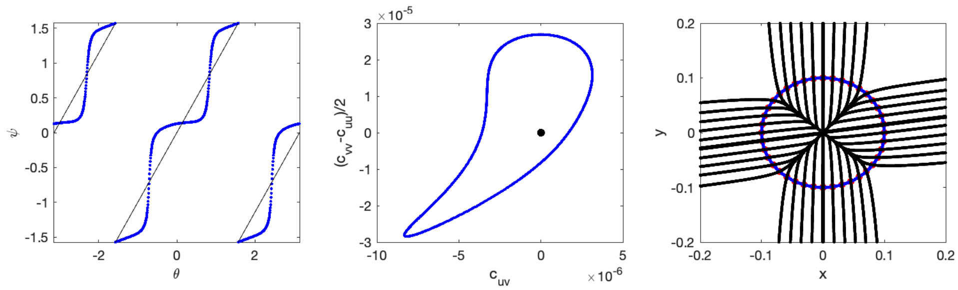


Figure 1: The index $I_0(a)$ for parameter $a = (1, 1, 1, 0, 0, 0, 1, 0, 1)$. The left panel shows 1000 solutions of $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$. The full set of solutions is a curve that winds once around both axes of the torus T^2 with positive orientation, so the index is 1. (Note that the angle of the principal directions is chosen in the interval $[-\pi, \pi]$ so that that there are jumps in value from π to $-\pi$ that are not discontinuities in the line field.) The middle panel plots c_{vv} vs. c_{uv} and $-c_{uu}$ vs c_{uv} along the solution curve. The origin is inside these curves, again implying that the index is 1. The right panel plots 40 lines of curvature that intersect the blue circle in equally spaced points. This also illustrates that the index is 1 because all of the lines of curvature are transverse to the circle.

of non-zero umbilic points on the winding numbers of curves. This enables us to study bifurcations of the umbilic index at the origin, which we will henceforth denote $I_0(a)$. Write

$$R(x, y, u, v) = c_{uu}(x, y)u^2 + c_{uv}(x, y)uv + c_{vv}(x, y)v^2. \tag{9}$$

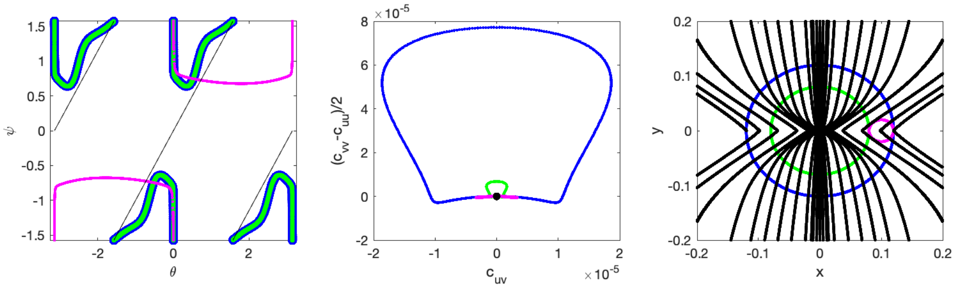
For fixed (x, y), the quadratic formula gives non-zero solutions of $R(x, y, \cos(\psi), \sin(\psi)) = 0$. Computation of normal curvatures in orthogonal coordinates aligned with the principal directions then determines which of the two principal directions has larger normal curvature. Results from these calculations are displayed in the figures of this section. The figures have rows of three plots which display the following information:

- The right plots show numerical integration of forty lines of curvature (black) projected onto the (x, y) plane. The initial points for these computations lie on blue circles of radius 0.1.
- The left plots show the function $\psi(\theta)$ with values in the interval $[-\pi, \pi]$, calculated along the circles displayed in the right plots. The colors in right and left plots match. The partitions of the circles were refined so that gaps between adjacent values of ψ are small.
- The middle plots show the curves c_{vv} vs. $(c_{uv} - c_{uu})/2$. Since $r^{-6}(c_{uu} + c_{vv}) \to 0$ as $r \to 0$, the value of $r^{-6}(c_{uu} + c_{vv})$ is relatively small when r is small. Where relevant detail happens on small scales, the plots are scaled to visualize this detail.

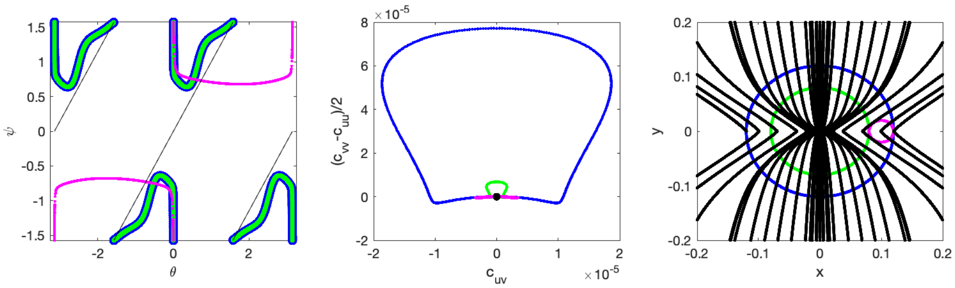
Figure **??** depicts the principal foliation and $I_0(a)$ for $a = (-1, -1, -1, 0, 0, 0, -1, 0, -1)$. The left panel plots solutions of $R(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$ on the circle of radius 0.1 showing that the index is 1. The middle panel plots the curves c_{uv} vs $(c_{uv} - c_{uu})/2$, again indicating that the index is ± 1. Since the coefficients are invariant under rotation of the (x, y) plane by π, the curve is traced twice corresponding to winding number 2 for 2ψ. The curve $(\beta(\theta), \gamma(\theta)), \theta \in [0, 2\pi]$ will henceforth be called the *BG curve* associated to the parameter a. The right hand panel plots 40 lines of curvature intersecting the blue circle at equally spaced points. The eight values of $\theta = \psi$ in the left panel correspond to *separatrices*, lines of curvature that approach the origin. Note that all the lines of curvature intersect the red circle transversally, again demonstrating the index is 1.

When $\beta(\theta) = \gamma(\theta) = 0$, $\bar{R}(\cos(\theta), \sin(\theta), u, v)$ vanishes identically as a function of (u, v). For fixed θ, this is a pair of linear equations in a, most of whose solutions constitute linear subspaces of dimension 7 and codimension 2. The closure of their union with varying θ is an 8 dimensional bifurcation subset B of the 9 dimensional parameter space A. In each component of $A - B$, $I_0(a)$ is continuous and therefore constant. Analyzing B as a stratified set and determining the changes of $I_0(a)$ when crossing B provides a strategy for bounding $I_0(a)$. The first step of this process is to study B at regular points where $\bar{R}(\cos(\theta), \sin(\theta), \cos(\psi), \sin(\psi)) = 0$ has nonzero gradient.

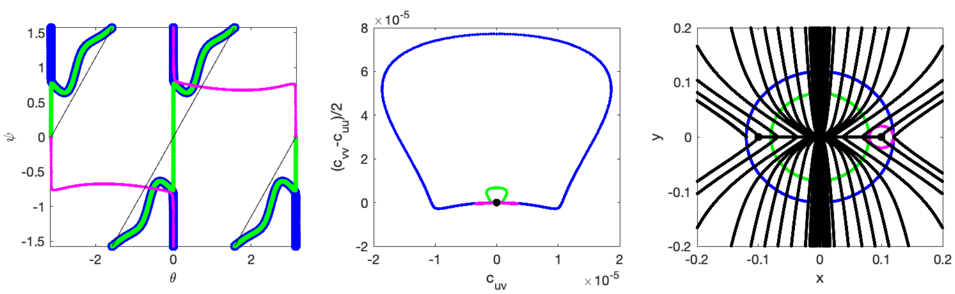
Rotation of a surface around the normal of an umbilic leaves the form of $\bar{R}$ unchanged, producing a linear transformation of the parameter space A. Thus, we can reduce the investigation of B to the case $\theta = 0$ or equivalently $(x, y) = (1, 0)$ in the truncation of (**??**). The parameters for which $(\beta, \gamma) = (0, 0)$ at $(x, y) = (1, 0)$ satisfy $a_1 = 0, a_2 = 28a_0$. Figure **??** shows the same information as Figure **??** for parameters $a = (1/28, 0, 1, 0, 0, 0, 2, 0, 1)$. The function $\bar{R}(1, 0, \cos(\psi), \sin(\psi))$ is identically zero, so does not determine the (approximate) principal directions at the point $(1, 0)$. We compute principal directions for values of a_3 slightly smaller and slightly larger than 1 in order to determine the change of index as the parameter vectors cross B.



(a) $a = (1/28, 0, 1, 0, 0, 0, 2, 0, 1)$: index undefined



(b) $a = (1/28, 0, 1.01, 0, 0, 0, 2, 0, 1)$: index 0



(c) $a = (1/28, 0, 0.987143, 0, 0, 0, 2, 0, 1)$: index 1

Figure 2: The index of the umbilic point at the origin jumps as the BG curve crosses the bifurcation curve B with varying parameter a. As a_2 decreases, the index jumps from 0 to 1. Note that there are a pair of star umbilics that emerge from the origin as a_2 decreases from 1. These are marked by large black dots.

Perturbing the parameter $a_2 = 1$, numerical computations show that the index is 1 when $a_2 < 1$ and that the index is 0 when $a_2 > 1$. They also show that two star umbilic points emerge from the origin as a_2 decreases from 1. Figure **??** displays results for $a = (1/28, 0, 1, 0, 0, 0, 2, 0, 1)$ lying on the bifurcation locus B. Figure **??** shows results for $a_2 = 1.01$ and Figure **??** shows results for $a_2 = 0.987143$, chosen so that two star umbilics are near the points with $(x, y) = (\pm 0.1, 0)$. For each value of a, the principal directions and c_{uv} vs $(c_{vv} - c_{uu})/2$ curves are computed along two circles centered at the origin of radius 0.12 (blue) and 0.08 (green). In the left hand panels, the angle ψ of the principal directions is chosen to lie in the interval $[-\pi, \pi]$ by principal vectors whose first coordinate is non-negative. Thus, the top and bottom of the left hand plots of principal directions should be considered as glued to one another with the unoriented principal directions having period π.

There is little difference between the results for $a_2 = 1$ and $a_2 = 1.01$. Undetectable at this scale is that the origin lies outside the c_{uv} vs $(c_{vv} - c_{uu})/2$ curve when $a_2 = 1.01$, but lies on the c_{uv} vs $(c_{vv} - c_{uu})/2$ curve for $a_2 = 1$. The index of the umbilic at the origin is 0 when $a_2 = 1.01$ because there are no horizontal principal directions. In the right panels, the circles used in calculating the principal directions and c_{uv} vs $(c_{vv} - c_{uu})/2$ curve are superimposed upon plots of 40 lines of curvature.

Figure **??** for $a_2 < 1$ is more complicated. To locate umbilic points near the origin in the one parameter family above, we substitute $(x, y) = (x, 0)$, and $a = (1/28, 0, a_2, 0, 0, 0, 2, 0, 1)$ into the Rodrigues formula $R(x, y, u, v, uh_x + vh_y)$. The result has the factor

$$4\,a_3\,x^{16} - 2\,x^8\,\sqrt{1-x^2} - 4\,a_3\,x^{14} - 49\,a_3 - 63\,x^2 - 28\,a_3\,x^8\,\sqrt{1-x^2} + 49$$

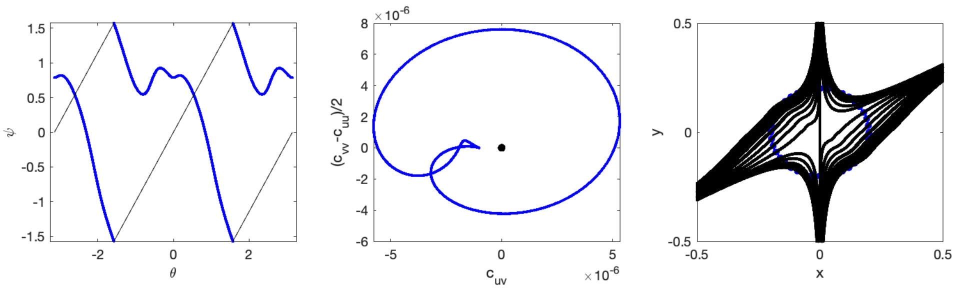
which can be solved for a_2 as a function of x. This function $a_2^u(x)$ is even and yields surfaces S_a with umbilic points at $(x, 0, h_a(x, 0))$. The value of $a_2^u(0.1)$, approximately 0.987143, is used in Figure **??**. The two star umbilic points near $(x, y) = (\pm 0.1, 0)$ are plotted as black dots and a magenta circle of radius 0.02 is drawn around $(0.1, 0)$. The indices of the star umbilics are $-\frac{1}{2}$ and there are three lines of curvature asymptotic to each. The index $-\frac{1}{2}$ of the star umbilic is highlighted by the magenta curve in the left panel which has a total variation of $-\pi$. $I_0(a)$ is now 1, as seen by the green curve in the left panel. The winding number of the principal foliation along the blue curve is 0, the sum of the indices at the three umbilic points. In the middle panel, the blue c_{uv} vs $(c_{vv} - c_{uu})/2$ curve passes below the origin (the value of y on the $y-$axis is approximately -3.4e-6) while the green c_{uv} vs $(c_{vv} - c_{uu})/2$ curve passes above the origin (the value of y on the $y-$axis is approximately 2.4e-7). These plots illustrate the changes that occur in the principal foliation as the parameter vector a undergoes a generic crossing of the bifurcation locus B consisting of parameters where $\bar{R} = 0$.

BG curves can be singular at points where their derivative vanishes. Again, without loss of generality, these can be analyzed by restricting attention to $\theta = 0$ by rotating the (x, y) plane. We find that the derivative vanishes to lowest order in x when $a_2 = 0$ and $a_3 = 7a_1$. This gives a pair of equations in the parameter a, so the locus C of parameters whose BG curves have a cusp is also a subset of codimension 1. Figure **??** shows the c_{uv} vs $(c_{vv} - c_{uu})/2$ curve for $a = -(0, 0.145, 0, 1, 1, 0, -2, 0, 0.2)$ which lies close to C. The BG curve at parameter $a = (0, 0.142, 0, 1, 1, 0, -2, 0, 0.2)$ shown in Figure **??** has a small loop near the cusp while the BG curve at parameter $a = (0, 0.146, 0, 1, 1, 0, -2, 0, 0.2)$ shown in Figure **??** does not. The middle plots of Figures **??** and **??** show only small segments of the c_{uv} vs $(c_{vv} - c_{uu})/2$ curves near the cusp.

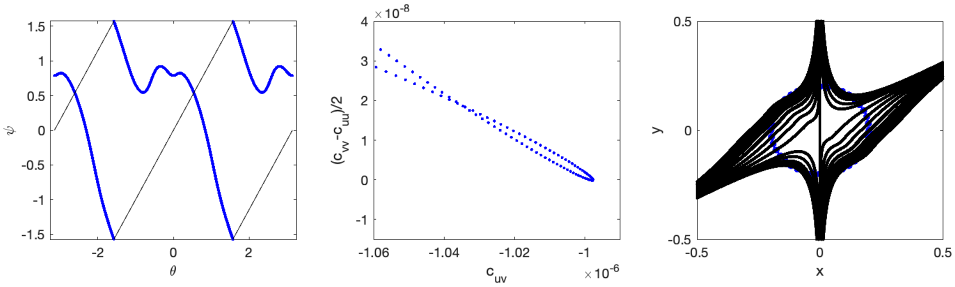
Singularities of B occur at parameter values for which the BG curve passes through the origin at more than one value of θ. Each specified value of θ gives rise to a pair of equations in the coefficients, so it seems likely that there are parameters for which the BG curve passes through the origin at four values of θ. Indeed, parameter $a = (1/28, 0, 1, 0, -3/2, 0, 1, 0, 1/28)$ gives a BG curve for $\bar{R}$ that passes through the origin at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. The Jacobian of the equations defining this four fold intersection is

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -56 \\ 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 56 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -7 & -12 & -15 & -16 & -15 & -12 & -7 & 0 \\ 56 & 42 & 28 & 14 & 0 & -14 & -28 & -42 & -56 \\ 0 & -7 & 12 & -15 & 16 & -15 & 12 & -7 & 0 \\ 56 & -42 & 28 & -14 & 0 & 14 & -28 & 42 & -56 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

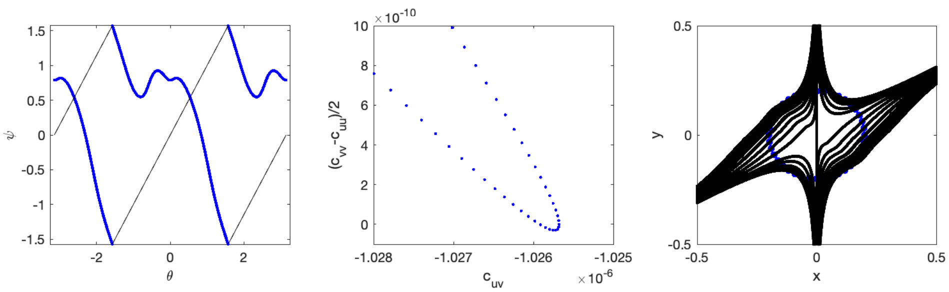
J has full rank 9, so this parameter is isolated among those whose BG curves pass through the origin at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. Since the equations that define these four fold normal crossings at the origin are polynomial equations in a and $y_j = \tan(\theta_j)$, $j = 0, 1, 2, 3$, the set of y_j for which their Jacobian J_y is singular is a proper subvariety of the four dimensional space of y (or the four dimensional θ torus). Clearly, J_y is singular when $y_i = y_j$, $i \neq j$ since J_y then has repeated rows. We used the computer program Macaulay2 [**?**] to analyze whether there are values of y with distinct coordinates for which J_y is singular. Using the equivariance of the equations with respect to rotations of the (x, y) plane, the size of this



(a) $a = (0, 0.145, 0, 1, 1, 0, -2, 0, 0.2)$: cusp in c_{uv} vs $(c_{vv} - c_{uu})/2$ curve



(b) $a = (0, 0.142, 0, 1, 1, 0, -2, 0, 0.2)$: c_{uv} vs $(c_{vv} - c_{uu})/2$ curve detail with loop

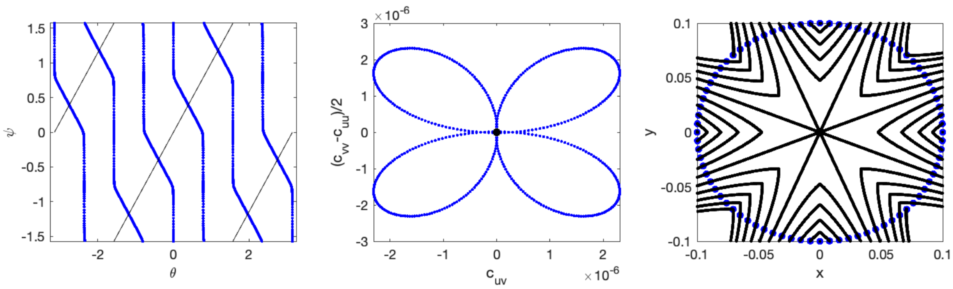


(c) $a = (0, 0.146, 0, 1, 1, 0, -2, 0, 0.2)$: c_{uv} vs $(c_{vv} - c_{uu})/2$ curve detail without loop

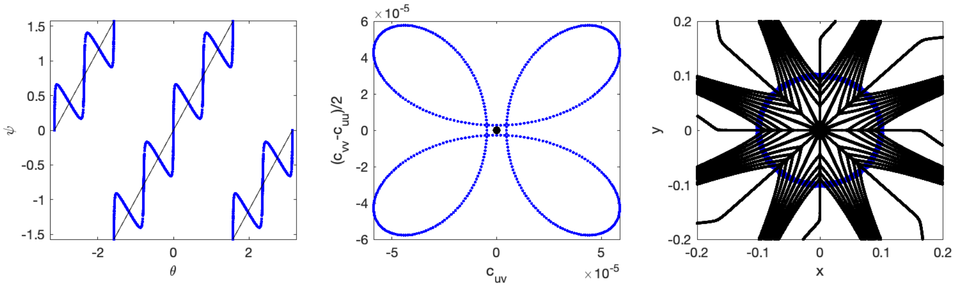
Figure 3: c_{uv} vs $(c_{vv} - c_{uu})/2$ curve with cusp and two perturbations

calculation was reduced by setting $y_0 = 0$. The computer algebra program Macaulay2 [**?**] then computed the primary decomposition of the ideal generated by the polynomial $p(y) = \det(J_y)$ in the ring $R[y_1, y_2, y_3]$. The degree of p is 28 and it has 762 terms with coefficients that are typically 10 or 11 digit integers. The ideal generated by p is the intersection of seven primary ideals, six of which are contained in the ideals generated by $y_1, y_2, y_3, (y_1 - y_2), (y_2 - y_3), (y_3 - y_1)$. The seventh primary ideal is generated by a polynomial $p_r(y_1, y_2, y_3)$ that has degree 16 and 268 terms. Numerical evaluations of p_r in Matlab strongly suggest that p_r is nonnegative, vanishing only along the line $y_1 = y_2 = y_3$. These calculations demonstrate (but do not rigorously prove) that J_y has full rank when y_1, y_2, y_3 are distinct and non-zero. This suggests that all four fold intersections of BG curves at the origin are continuations of one another. (From the symmetries of J_y with respect to y, every surface with a four fold crossing of the BG curve at the origin is represented by a y with $0 = y_0 < y_1 < y_2 < y_3$.)

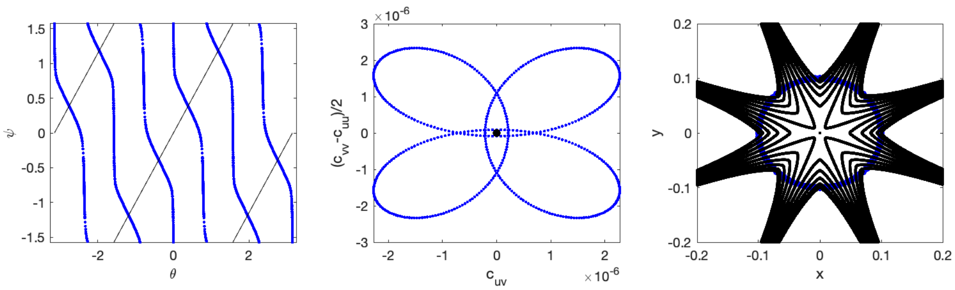
Figure **??** plots principal directions, c_{uv} vs $(c_{vv} - c_{uu})/2$ curves and lines of curvature for parameter $a = (1/28, 0, 1, 0, -3/2, 0, 1, 0, 1/28)$. The BG curve (on domain $\theta \in [0, \pi)$) in the middle panel has four branches that pass through the origin tangent to the coordinate axes. When a is perturbed from these values, each of the branches through the origin can be displaced independently. Each time a single branch passes through the origin, the index jumps by ± 1 as observed above. Thus the difference between the maximum index and the minimum index is at most 4. For perturbations whose BG curve does not pass through the origin, the index has a maximum of 1 and a minimum of -3. These are displayed in Figure **??** and Figure **??**. Since all four-fold crossings of BG curves at the origin are continuations of each other, this



(a) $a = (1/28, 0, 1, 0, -3/2, 0, 1, 0, 1/28)$: index undefined



(b) $a = (0.04, 0, 1, 0, -1.4, 0, 1, 0, 0.04)$: index 1



(c) $a = (0.03, 0, 1, 0, -1.6, 0, 1, 0, 0.03)$: index -3

Figure 4: BG curve with four branches passing through origin and two perturbations

provides evidence that umbilics at the origin in the family A have index in the interval $[-3, 1]$

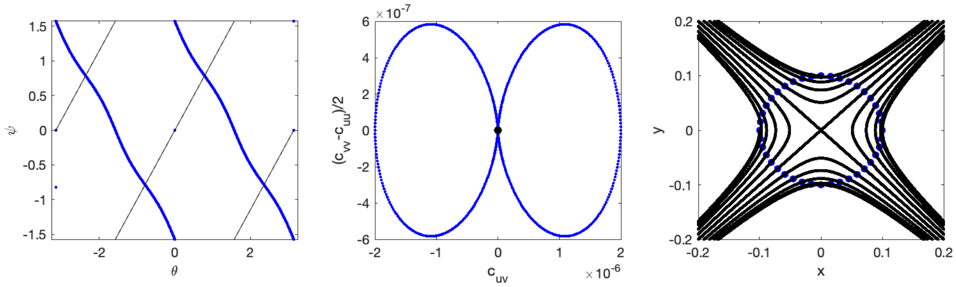
One final example analyzes the index for parameters a with a double cusp of their BG curves at the origin. Both $(\beta(0), \gamma(0)) = 0$ and $\frac{d}{d\theta}(\beta(\theta), \gamma(\theta))|_{(0,0)} = (0, 0)$ when $a_0 = a_1 = a_2 = a_3 = 0$. There is a cusp at the origin. If $a_5 = a_6 = a_7 = a_8 = 0$ also, there is a double cusp at the origin. Figure **??** plots data for $a = (0, 0, 0, 0, 1, 0, 0, 0, 0)$ and two perturbations with indices 1 and -3.

Theoretical Results

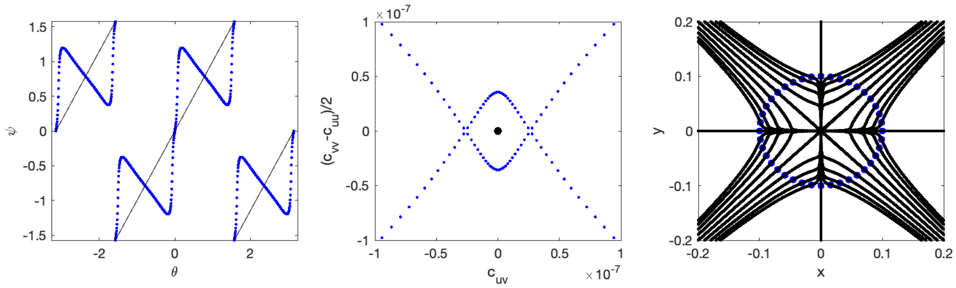
Throughout this section, $d > 3$ is a an even integer. The group $SO(2)$ of rotations of the (x, y) plane acts on the family S_a^d, preserving the index of the origin. If λ_a is a non-zero umbilic, there is a rotation that carries λ_a to the $x-$axis. The value $c_{uu}u^2 + c_{uv}uv + c_{vv}v^2 = R(x, 0, u, v, uh_x + vh_y)$ of Rodrigues formula (**??**) at $(x, 0)$ depends only on the coefficients (a_0, a_1, a_2) because the coefficients $a_k, k > 2$ appear only in terms divisible by y in the derivatives $h_x, h_y, h_{xx}, h_{xy}, h_{yy}$. Furthermore, the terms c_{uu} and c_{vv} are divisible by $a_1 x^{d-2}$ while

$$c_{uv} = (d(d-1)a_0 - 2a_2)x^{d-2} - \frac{2(da_0 + a_2)x^d}{1 - x^2} + O(a_1, x^{2d-2}).$$

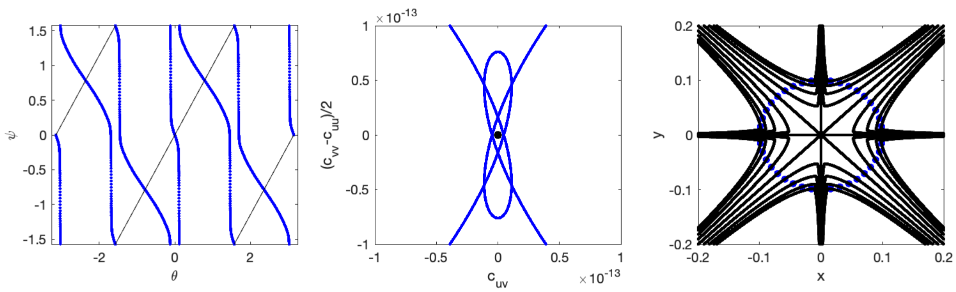
These facts imply that S_a has umbilics on the $x-$axis near the submanifold of the parameter space defined by $a_1 = 0,\ d(d-1)a_0 - 2a_2 - 2(da_0 + a_2)x^2 = 0$. When $d(d-1)a_0 = 2a_2$, these umbilics merge with the



(a) $a = (0, 0, 0, 0, 1, 0, 0, 0, 0)$: index undefined



(b) $a = (0.002, 0, 0.02, 0, 1, 0, 0.02, 0, 0.002)$: index 1



(c) $a = (0.002, 0, -0.02, 0, 1, 0, -0.02, 0, 0.002)$: index -3

Figure 5: BG curve with a double cusp at the origin and two perturbations. An expanded portion of the BG curve near the origin is displayed for each of the perturbations

umbilic at the origin. The index of the origin can change only at parameters in the $SO(2)$ orbits of these parameters where non-zero umbilics approach the origin along the x-axis.

There are some parameter values a for which the index of the origin is easily determined from symmetry considerations. First, the graph of $h(x, y) = 1 - \sqrt{1 - x^2 - y^2} + (x^2 + y^2)^{d/2}$ is a surface of revolution, so its principal foliations consist of circles centered at the origin and rays emanating from the origin. The index is $+1$. Second, the polynomial $((x + iy)^d + (x - iy)^d)/2$ is invariant under rotations of the (x, y) plane by angles that are integer multiples of $2\pi/d$. Thus the graphs of functions $h(x, y) = 1 - \sqrt{1 - x^2 - y^2} + \alpha_1(x^2 + y^2)^{d/2} + ((x + iy)^d + \alpha_2(x - iy)^d)/2$ are also invariant under rotation around the z axis by $2\pi/d$. The following result shows that $I_0 = 1 - d/2$.

Theorem: In the family S^d_a, $1 - d/2 \leq I_a(0)$ whenever this index is well defined.

Proof: Fix a and examine S_a, the graph of the function

$$h_a(x, y) = 1 - \sqrt{1 - x^2 - y^2} + \sum_{j=0}^{d} a_j x^{(d-j)} y^j.$$

The Rodrigues formula $R(x, y, u, v, uh_x + vh_y)$ is a homogeneous polynomial in (u, v) and an analytic function in (x, y) whose lowest degree terms have degree $(d - 2)$. Thus the truncation $\bar{R}$ to terms of degree $d - 2$ can be viewed as a polynomial equation in projective coordinates on $\mathbb{P} \times \mathbb{P}$ of degree 2 in the (u, v) coordinate and degree $d-2$ in the (x, y) coordinate. When the solutions to this equation form a smooth curve, $I_a(0)$ is defined and determined by the curve. The curve has winding number 2 along the (u, v) coordinate and winding number in the interval $[2-d, d-2]$ along the (x, y) coordinate. Geometrically, the solutions represent the winding of the principal foliations of S along a infinitesimal curve that encircles the origin twice. This implies that $1 - d/2 \leq I_a(0) \leq d/2 - 1$.

The main theoretical result of this paper is the following:

Theorem: Let $S_{a_\epsilon} \subset S_a^d, \epsilon \in [0, 1]$ be a one parameter family of surfaces with $I_0(S_{a_\epsilon}) = 1$ for $\epsilon < \epsilon_*$ but $I_0(S_{a_\epsilon}) \neq 1$ for $\epsilon > \epsilon_*$. Then $I_0(S_{a_\epsilon}) < 1$ for $\epsilon > \epsilon_*$. Generic umbilics λ_ϵ emerging from the origin at $\epsilon = \epsilon_*$ are stars.

Proof: Berry and Hannay [**?**] write the Monge form of a surface at an umbilic as

$$f(x, y) = \frac{1}{2}k(x^2 + y^2) + \frac{1}{6}(\alpha x^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3) + O(4).$$

They further show that the index of generic umbilics is determined by the quantity

$$J(\alpha, \beta, \gamma, \delta) = \alpha\gamma - \gamma^2 + \beta\delta - \beta^2.$$

If J is positive, the umbilic index is $\frac{1}{2}$, while if J is negative, the umbilic index is $-\frac{1}{2}$. Thus to prove the theorem, it suffices to compute J at a non-zero umbilic λ_ϵ with ϵ slightly larger than ϵ_*.

Without loss of generality, consider the surface S_a which is the graph of $h_a(x, y) = 1 - \sqrt{1 - x^2 - y^2} + \sum_{j=0}^d a_j x^{(d-j)} y^j$ with $a = a_\epsilon$ and umbilic $\lambda = (x_u, 0, H_a(x_u, 0, 0)$. Set

$$H_a(x, y, z) = z - (1 - \sqrt{1 - x^2 - y^2} + \sum_{j=0}^{d} a_j x^{(d-j)} y^j)$$

to be a function of three variables whose zero level set is S_a. Compute the Monge form $\hat{h}$ of the surface S_a in three steps:

- Set $H^1(x, y, z) = H^1(x - x_u, y, z - h_a(x_u, 0, 0)$ to translate λ to the origin.
- Set $H = H^1 \circ T$ where T is the rotation of the (x, z) plane so that the normal $dH(0, 0, 0) = d(H^1 \circ T)(0, 0, 0) = (0, 0, 1)$ is in the vertical direction. Denote the new coordinates by $(\hat{x}, \hat{y}, \hat{z}) = T(x - x_u, y, z - h_a(x_u, 0, 0))$.
- Replace H by a function $\hat{H}$ that has the same zero level set, but is in Monge form: $\hat{H}(\hat{x}, \hat{y}, \hat{z}) = \hat{z} - \hat{h}(\hat{x}, \hat{y})$. The function $\hat{H}$ will have the form $g(\hat{x}, \hat{y}, \hat{z})H((\hat{x}, \hat{y}, \hat{z})$ with g determined by solving for Taylor series coefficients that put gH into the desired form $\hat{z} - \hat{h}(\hat{x}, \hat{y})$. The following terms of degrees 1, 2 and 3 are non-zero in the Taylor series of H: $z, x^2, y^2, xz, z^2, x^3, x^2 z, xy^2, xz^2, y^2 z, z^3$. Setting $g(\hat{x}, \hat{y}, \hat{z}) = b_x \hat{x} + b_z \hat{z} + b_{xx} \hat{x}^2 + b_{xz} \hat{x}\hat{z} + b_{zz} \hat{z}^2 + b_{yy} \hat{y}^2$ with
$$b_x = -\hat{H}_{xz}/\hat{H}_z, b_z = -\hat{H}_z z/\hat{H}_z, b_{xx} = -\hat{H}_{xxz}/\hat{H}_z + \hat{H}_{xz}^2/\hat{H}_z^2 + \hat{H}_{xx}\hat{H}_{zz}/\hat{H}_z^2,$$
$$b_{xz} = -\hat{H}_{xzz}/\hat{H}_z + 2\hat{H}_{zz}\hat{H}_{xz}/\hat{H}_z^2, b_{zz} = -\hat{H}_{zzz}/\hat{H}_z + \hat{H}_{zz}^2/\hat{H}_z^2, b_{yy} = -\hat{H}_{yyz}/\hat{H}_z + \hat{H}_{yy}\hat{H}_{zz}/\hat{H}_z^2$$
eliminates the terms $xz, z^2, x^2 z, xz^2, z^3$ from $\hat{H}$ and replaces the coefficients of the terms $\hat{H}_{xxx}, \hat{H}_{xyy}$ by $\hat{H}_{xxx} - \hat{H}_{xx}\hat{H}_{zz}/\hat{H}_z, \hat{H}_{xyy} - \hat{H}_{xz}\hat{H}_{yy}/\hat{H}_z$. Finally, divide $\hat{H}$ by $\hat{H}_z$ to obtain the desired Monge form.

The result of applying this computation to $H_a((x, y, z) = z - (1 - \sqrt{1 - x^2 - y^2} + a_0 x^d + a_2 x^{d-2} y^2)$ yields to leading order in x_u the Monge form

$$\hat{H}((\hat{x}, \hat{y}, \hat{z}) = \frac{1}{2}(\hat{x}^2 + \hat{y}^2) - a_0 x_u^{d-3} d(d-1)(d-2)(\frac{1}{6}\hat{x}^3 + \frac{1}{2}\hat{x}\hat{y}^2)$$

The value of J to leading order in x_u is $-d^2(d^2-1)(d-2)/2$ which is clearly negative for $d > 2$. Consequently, λ is a star umbilic with index $-\frac{1}{2}$. When λ_ϵ emerges from the origin, the winding number of a curve σ surrounding the origin remains 1. Since the winding number of σ is the sum of the indices of umbilics inside σ, $I_0(S_{a_\epsilon}) < 1$ for $\epsilon > \epsilon_*$ small.

Centers and Symmetry

The index of a generic umbilic is $\pm\frac{1}{2}$. Umbilics with index 1 must be degenerate. Moreover, they are not found in generic one parameter families of surfaces. Here we discuss briefly how umbilics with index 1 appear in generic two parameter families of surfaces and the qualitative changes in principal foliations that occur when these surfaces are perturbed. There seems to be little (if any) systematic discussion of this issue in the literature about principal foliations. Our results are fragmentary, but they do outline a conjectural picture of how principal foliations bifurcate in generic two parameter families of surfaces near an umbilic of index 1.

We start with an analysis of paraboloids $S_{(a,b,c)}$ in Monge form; i.e. the graphs of functions

$$h_{(a,b,c)}(x,y) = ax^2 + 2bxy + cy^2$$

with $b^2 - ac < 0$. When $a = c$ and $b = 0$, $S_{(a,0,a)}$ is a surface of revolution, so its principal foliations consist of (1) its intersections with planes contain the z axis of symmetry and (2) circles that are its intersections with planes orthogonal to the z-axis. Observe that rotation of the (x,y) plane by angle $\frac{1}{2}\arctan(\frac{2b}{c-a})$ eliminates the middle term xy in the function h. This reduces the search for umbilics to the case $b = 0$. Calculation of the Rodrigues determinant for the zero level set of $h(x,y,z) = z - ax^2 - cy^2$, $a, c > 0$ finds umbilics at $(\pm((\frac{c-a}{4a^2c})^{1/2}, 0)$ when $a < c$ and at $(0, \pm((\frac{a-c}{4ac^2})^{1/2})$ when $c < a$. As $a - c$ changes sign from negative to positive, a pair of umbilics on the x axis coalesce at the origin and then reemerge along the y axis. The geometry resembles that of complex square roots of a real variable. In two dimensional families of surfaces $S_{(a,b,c)}$ with fixed $\frac{a+c}{2}$ and varying b and $\frac{a-c}{2}$, the surface swept out by umbilics is a double cover of this parameter plane which is ramified at the origin.

The surfaces $S_{(a,0,c)}$ are symmetric with respect to reflections along the x and y axes since $h_{(a,0,c)}(x,y) = h_{(a,0,c)}(-x,y)$ and $h_{(a,0,c)}(x,y) = h_{(a,0,c)}(x,-y)$. In the case $0 < a < c$, the umbilic points lie in the plane $y = 0$ and the intersection of $S_{(a,0,c)}$ with this plane contains a line of curvature connecting the two umbilic points. Furthermore, this connecting line of curvature is surrounded by a family of closed lines of curvature that are symmetric with respect to the reflection $h_{(a,0,c)}(x,y) = h_{(a,0,c)}(x,-y)$. Figure **??** displays principal foliations of paraboloids. Applying rotations as described above, all of the surfaces $S_{(a,b,c)}$, $(a,b,c) \neq (0,0,0)$ have a line of curvature connecting its two umbilics, surrounded by a family of closed lines of curvature. These geometric properties are not generic within the space of smooth, analytic or even polynomial surfaces.

Generic perturbations of a surface in $S_{(a,b,c)}$ lack lines of curvature connecting umbilics and have isolated closed lines of curvature. Figure **??** depicts principal foliations of three surfaces belonging to a one parameter family of surfaces in Monge form obtained by adding the cubic term $-0.002x^3 + 0.002x^2y + 0.001xy^2 - 0.001y^3$ to $h_{(1/2+\lambda,0,1/2-\lambda)}(x,y)$. The parameter λ takes the values -0.01 in the left panel of the figure and the umbilics lie close to the y-axis. (Note the difference in scales of the x and y axes.) The separatrices of the two umbilics are disjoint and plotted as green and blue curves. Each makes a tight turn around the opposite umbilic and spirals outward. The right panel of Figure **??** is similar to the left with the roles of the x and y axes interchanged. The middle panel displays twenty consecutive intersections of a single line of curvature with the ray $x = 0$; $y < 0$ on the surface with parameter $\lambda = 0$. In contrast to the principal foliation in the center panel of Figure **??**, this line of curvature is not closed: it spirals slowly in the radial direction. A more complete bifurcation analysis of surfaces with an umbilic of index 1 would characterize which surfaces have lines of curvature that connect umbilics and surfaces where the number of closed lines of curvature changes. That analysis is complementary to the main results of this paper related to the Caratheodory conjecture.

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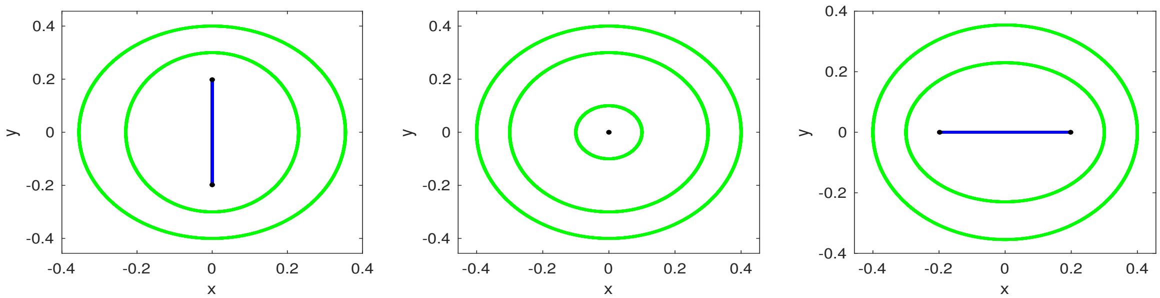
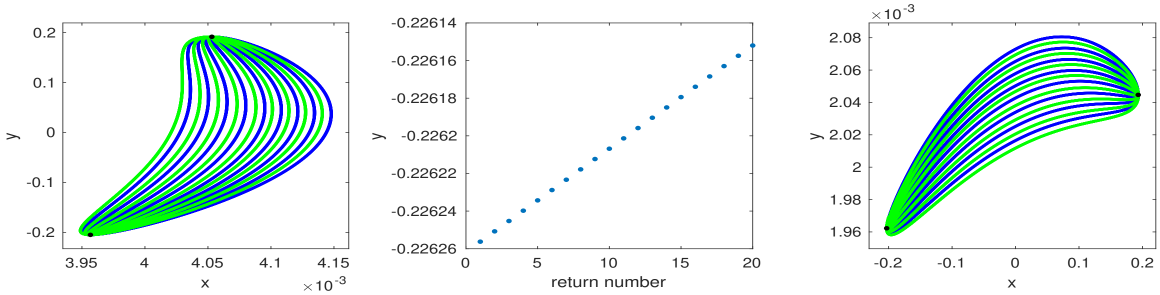


(a) Principal foliations of the graphs of $h(x,y) = x^2/2 + y^2/2 - \lambda(x^2 - y^2)$ for (left) $\lambda = -0.01$, (center) $\lambda = 0$ and (right) $\lambda = 0.01$. Apart from umbilics plotted as black dots and connecting lines of curvature plotted blue, all other lines of curvature are closed curves plotted green. As λ increases through 0, the two umbilic on the y-axis merge at the origin and remerge along the x-axis.



(b) Principal foliation of the graphs of $h(x,y) = x^2/2 + y^2/2 + \lambda * (x^2 - y^2) - 0.002x^3 + 0.002 * x^2 * y + 0.001 * x * y^2 - 0.001y^3$ for (left) $\lambda = -0.01$, (center) $\lambda = 0$ and (right) $\lambda = 0.01$. The left and right panels plot the umbilics together with their separatrix lines of curvature. These separatrices no longer connect umbilics as in Figure **??**, instead making tight turns around the opposite umbilic and then spiraling outward. The center panel plots successive values of y for intersections of a single line of curvature with the x-axis to the left of the origin. The line of curvature is not closed: it spirals around the origin slowly.

Figure 6: Bifurcations of a principal foliation at a degenerate umbilic of index 1 that splits into two lemon umbilics of index $\frac{1}{2}$.

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Matlab codes for producing the figures in this paper can be found in the Github repository umbilic_index.