

chapter 6

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1 6.1

(a) We can get the likelihood function:

$$\log L(\mu, \sigma^2) = \sum_{i=1}^n \left(\log \frac{1}{\sqrt{2\pi}} + \frac{1}{2} \log \frac{1}{\sigma^2} + \frac{(x_i - \mu)^2}{2\sigma^2} \right) \quad (1)$$

So we can get the maximum likelihood variation.

$$\begin{aligned} \hat{\mu} &= \frac{1}{n} \sum_{i=1}^n x_i \\ \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2 \end{aligned} \quad (2)$$

(b)

$$\begin{aligned} E[\hat{\mu}] &= \frac{1}{n} \sum_{i=1}^n \left(\int \frac{x_i}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} dx_i \times_{j < i} \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_j - \mu)^2}{2\sigma^2}} dx_j \right) \\ &= \mu \end{aligned} \quad (3)$$

$$E[\hat{\sigma}^2] = \frac{n-1}{n} \sigma^2 \quad (4)$$

$$\begin{aligned} V[\hat{\mu}] &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 \\ &= \frac{\sigma^2}{n} \end{aligned} \quad (5)$$

$$\begin{aligned} V[\hat{\sigma}^2] &= E[\hat{\sigma}^4] - E[\hat{\sigma}^2]^2 \\ &= \frac{(n-1)^2}{n^3} (3\sigma^4 - \frac{n-3}{n-1} \sigma^3) \end{aligned} \quad (6)$$

(c)

$$V^{-1} = \begin{matrix} -\frac{n}{\sigma^2} & -\frac{1}{\sigma^4} \sum_{i=1}^n (x_i - \mu) \\ -\frac{1}{\sigma^4} \sum_{i=1}^n (x_i - \mu) & -\sum_{i=1}^n \left[\frac{(x_i - \mu)^2}{\sigma^6} - \frac{1}{2\sigma^4} \right] \end{matrix} \quad (7)$$

when the $n \rightarrow \infty$, the answer is same.

2 6.2

The likelihood function:

$$L = C_N^n p^n (1-p)^{N-n} \quad (8)$$

$$\hat{p} = \frac{n}{N} \quad (9)$$

$$E(\hat{p}) = \frac{E(n)}{N} = p \quad (10)$$

$$V(\hat{p}) = \frac{p(1-p)}{N} \quad (11)$$

According to the 6.16:

$$V(\hat{p}) > \frac{1}{E[-\frac{\partial^2 \log L}{\partial p^2}]} = \frac{N}{p(1-p)} \quad (12)$$

3 6.3

(a)

$$\hat{\alpha} = \frac{2n}{N} - 1 \quad (13)$$

$$\sigma_{\hat{\alpha}} = \sqrt{\frac{1-\alpha^2}{N}} \quad (14)$$

(b) $N > 9 * 10^6$

4 6.4

$$L = \frac{\nu^n}{n!} e^{-\nu} \quad (15)$$

So we can get $\hat{\nu} = n$

$$E(\hat{\nu}) = E(n) = \nu \quad (16)$$

$$V(\hat{\nu}) = V(n) = \nu \quad (17)$$

According to the 6.16:

$$V[\hat{\nu}] > \frac{1}{E[-\frac{\partial^2 \log L}{\partial \nu^2}]} = \frac{1}{E[\frac{n}{\nu^2}]} = \nu \quad (18)$$

5 6.5

$$\hat{\alpha} = \frac{n_R - n_L}{n_R + n_L} \quad (19)$$

Error transfer formula:

$$\begin{aligned} \sigma_{\hat{\alpha}}^2 &= \left(\frac{\partial \alpha}{\partial n_R}\right)^2 \sigma_{n_R}^2 + \left(\frac{\partial \alpha}{\partial n_L}\right)^2 \sigma_{n_L}^2 \\ &= \sqrt{\frac{1 - \alpha^2}{\nu_{tot}}} \end{aligned} \quad (20)$$

6 6.6

(a)

$$V[\alpha u + v] = \alpha^2 V[u] + V[v] + 2\alpha \text{Cov}(u, v) > 0 \quad (21)$$

We can let $\alpha^2 = \frac{V[v]}{V[u]}$:

$$V[v]V[u] > (\text{Cov}[u, v])^2 \quad (22)$$

(b) Using Cauchy-Schwarz :

$$V[\hat{\theta}]V\left[\frac{\partial}{\partial \theta} \log L\right] > (\text{Cov}[\hat{\theta}, \frac{\partial}{\partial \theta} \log L])^2 \quad (23)$$

(c) Prove:

$$\begin{aligned} E\left[\frac{\partial}{\partial \theta} \log L\right] &= \int \dots \int f_{joint}(x; \theta) \frac{\partial}{\partial \theta} \log f_{joint}(x; \theta) dx_1 \dots dx_n \\ &= \int \dots \int \frac{\partial}{\partial \theta} f_{joint}(x; \theta) dx_1 \dots dx_n \\ &= \frac{\partial}{\partial \theta} 1 \\ &= 0 \end{aligned} \quad (24)$$

And then

$$\begin{aligned} V[\hat{\theta}] &> \frac{(\text{Cov}[\hat{\theta}, \frac{\partial}{\partial \theta} \log L])^2}{V[\frac{\partial}{\partial \theta} \log L]} \\ &= \frac{(E[\hat{\theta} \frac{\partial}{\partial \theta} \log L])^2}{E[(\frac{\partial}{\partial \theta} \log L)^2]} \end{aligned} \quad (25)$$

(d) Prove a:

$$\begin{aligned} E\left[\hat{\theta} \frac{\partial}{\partial \theta} \log L\right] &= \int \dots \int f_{joint}(x; \theta) \hat{\theta} \frac{\partial}{\partial \theta} \log f_{joint}(x; \theta) dx_1 \dots dx_n \\ &= \int \dots \int \hat{\theta} \frac{\partial}{\partial \theta} f_{joint}(x; \theta) dx_1 \dots dx_n \\ &= \frac{\partial}{\partial \theta} E[\hat{\theta}] \\ &= 1 + \frac{\partial b}{\partial \theta} \end{aligned} \quad (26)$$

Prove b

$$\begin{aligned}
 E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] &= \int \dots \int f_{joint}(x; \theta) \frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial \theta} \right) dx_1 \dots dx_n \\
 &= \int \dots \int \frac{\partial^2 f}{\partial \theta^2} - f_{joint}(x; \theta) \left(\frac{\partial f}{f \partial \theta} \right)^2 dx_1 \dots dx_n \\
 &= - \int \dots \int f_{joint}(x; \theta) \left(\frac{\partial \log L}{\partial \theta} \right)^2 dx_1 \dots dx_n \\
 &= -E\left[\left(\frac{\partial \log L}{\partial \theta} \right)^2\right]
 \end{aligned} \tag{27}$$

So it is done.

7 6.7

(a)

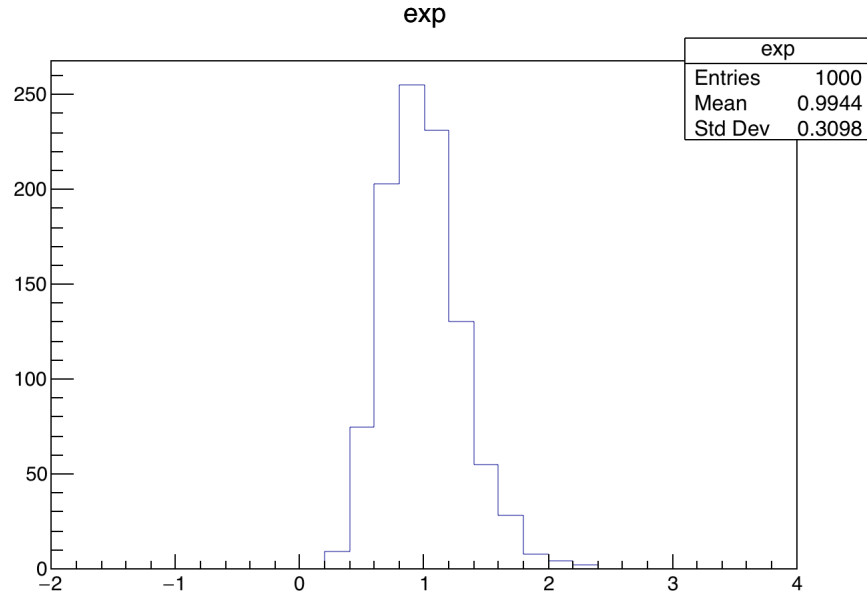


Figure 1: tau distribution

(b) We can get the maximum likelihood function:

$$L = \prod_{i=1}^n \lambda e^{-\lambda t_i} \tag{28}$$

we use $\frac{\partial \log L}{\partial \lambda} = 0$:

$$\lambda = \frac{n}{\sum t_i} \tag{29}$$

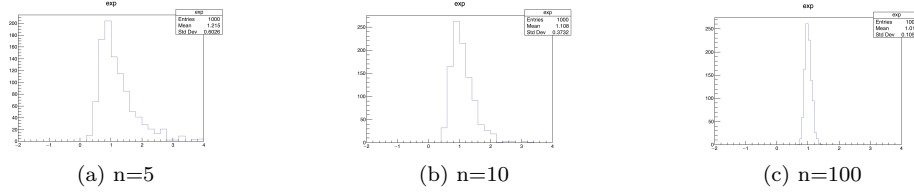


Figure 2: lambda distribution

8 6.8

(a) The maximum likelihood function is :

$$L = \frac{1}{N^N} \quad (30)$$

When $N = 1$, the L is maximum. It is wrong. (b) We can take the n_{max} as the \hat{N}_{taxi} .

$$\begin{aligned} E[\hat{N}_{taxi}] &= \sum_{i=N}^{N_{taxi}} \frac{i C_{i-1}^{N-1}}{C_{N_{taxi}}^N} \\ &= \sum_{i=N}^{N_{taxi}} N \frac{C_i^N}{C_{N_{taxi}}^N} \\ &= \frac{N(N_{taxi} + 1)}{N + 1} \end{aligned} \quad (31)$$

$$\begin{aligned} E[\hat{N}_{taxi}^2] &= \sum_{i=N}^{N_{taxi}} \frac{i^2 C_{i-1}^{N-1}}{C_{N_{taxi}}^N} \\ &= \sum_{i=N}^{N_{taxi}} N \frac{(i+1)C_i^N - C_i^N}{C_{N_{taxi}}^N} \\ &= \sum_{i=N}^{N_{taxi}} N \frac{(N+1)C_{i+1}^N - C_i^N}{C_{N_{taxi}}^N} \\ &= N(N+1) \frac{C_{N_{taxi}+2}^{N+2}}{C_{N_{taxi}}^N} - N \frac{C_{N_{taxi}+1}^{N+1}}{C_{N_{taxi}}^N} \end{aligned} \quad (32)$$

So we can get the variant:

$$V[\hat{N}_{taxi}] = \frac{(N_{taxi} + 1)(N_{taxi} - N)N}{(N + 2)(N + 1)^2} \quad (33)$$

9 6.9

The maximum likelihood function is :

$$L = \prod_{i=1}^N \frac{(\theta a(x_i))^{n_i}}{n_i!} e^{-\theta a(x_i)} \quad (34)$$

We use $\frac{\partial \log L}{\partial \theta} = 0$

$$\hat{\theta} = \frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N a(x_i)} \quad (35)$$

$$E[\hat{\theta}] = \frac{\sum_{i=1}^N \theta a(x_i)}{\sum_{i=1}^N a(x_i)} = \theta \quad (36)$$

$$V[\hat{\theta}] = \frac{\sum_{i=1}^N V[n_i]}{(\sum_{i=1}^N a(x_i))^2} = \frac{\theta}{\sum_{i=1}^N a(x_i)} \quad (37)$$

and the minimum border of variant is :

$$V[\hat{\theta}] > \frac{1}{E[-\frac{\partial^2 \log L}{\partial \theta^2}]} = \frac{1}{\sum_{i=1}^N E[\frac{n_i}{\theta^2}]} = \frac{\theta}{\sum_{i=1}^N a(x_i)} \quad (38)$$

10 6.10

According to the 6.9 :

$$\begin{aligned} \hat{\theta}_\nu &= \frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N E_i \epsilon(E_i) L_i} \\ \hat{\theta}_{\bar{\nu}} &= \frac{\sum_{i=1}^N \bar{n}_i}{\sum_{i=1}^N E_i \epsilon(E_i) L_i} \end{aligned} \quad (39)$$

So we can solve:

$$\begin{aligned} \langle q \rangle &= \frac{3\pi}{8G^2 M} (3\theta_\nu - \theta_{\bar{\nu}}) \\ \langle \bar{q} \rangle &= \frac{3\pi}{8G^2 M} (3\theta_{\bar{\nu}} - \theta_\nu) \end{aligned} \quad (40)$$

So

$$\langle \hat{g} \rangle = 1 - \frac{6\pi}{8G^2 M \sum_{i=1}^N E_i \epsilon(E_i) L_i} \left[\sum_{i=1}^N (n_i + \bar{n}_i) \right] \quad (41)$$