chapter 6

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1 6.1

(a) We can get the likelihood function:

$$logL(\mu,\sigma^2)=\sum_{i=1}^{n}(log\frac{1}{\sqrt{2\pi}}+\frac{1}{2}log\frac{1}{\sigma^2}+\frac{(x_i-\mu)^2}{2\sigma^2}) \tag{1}$$

So we can get the maximum likelihood variation.

$$\begin{aligned}\hat{\mu}&=\frac{1}{n}\sum_{i=1}^{n}x_i\\ \hat{\sigma^2}&=\frac{1}{n}\sum_{i=1}^{n}(x_i-\hat{\mu})^2\end{aligned} \tag{2}$$

(b)

$$\begin{aligned}E[\hat{\mu}]&=\frac{1}{n}\sum_{i=1}^{n}(\int\frac{x_i}{\sqrt{2\pi}\sigma}e^{-\frac{(x_i-\mu)}{2\sigma^2}}dx_i\times_{j<>i}\int\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x_j-\mu)}{2\sigma^2}}dx_j)\\ &=\mu\end{aligned} \tag{3}$$

$$E[\hat{\sigma^2}]=\frac{n-1}{n}\sigma^2 \tag{4}$$

$$\begin{aligned}V[\hat{\mu}]&=E[\hat{\mu}^2]-E[\hat{\mu}]^2\\ &=\frac{\sigma^2}{n}\end{aligned} \tag{5}$$

$$\begin{aligned}V[\hat{\sigma^2}]&=E[\hat{\sigma^2}^2]-E[\hat{\sigma^2}]^2\\ &=\frac{(n-1)^2}{n^3}(3\sigma^4-\frac{n-3}{n-1}\sigma^3)\end{aligned} \tag{6}$$

(c)

$$V^{-1}=\begin{matrix}-\frac{n}{\sigma^2} & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu)\\ -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu) & -\sum_{i=1}^{n}[\frac{(x_i-\mu)^2}{\sigma^6}-\frac{1}{2\sigma^4}]\end{matrix} \tag{7}$$

when the $n->\infty$, the answer is same.

2 6.2

The likelihood function:

$$L = C_N^n p^n (1-p)^{N-n} \tag{8}$$

$$\hat{p} = \frac{n}{N} \tag{9}$$

$$E(\hat{p}) = \frac{E(n)}{N} = p \tag{10}$$

$$V(\hat{p}) = \frac{p(1-p)}{N} \tag{11}$$

According to the 6.16:

$$V(\hat{p}) > \frac{1}{E[-\frac{\partial^2 logL}{\partial p^2}]} = \frac{N}{p(1-p)} \tag{12}$$

3 6.3

(a)

$$\hat{\alpha} = \frac{2n}{N} - 1 \tag{13}$$

$$\sigma_{\hat{\alpha}} = \sqrt{\frac{1-\alpha^2}{N}} \tag{14}$$

(b) $N > 9 * 10^6$

4 6.4

$$L = \frac{\nu^n}{n!} e^{-\nu} \tag{15}$$

So we can get $\hat{\nu} = n$

$$E(\hat{\nu}) = E(n) = \nu \tag{16}$$

$$V(\hat{\nu}) = V(n) = \nu \tag{17}$$

According to the 6.16:

$$V[\hat{\nu}] > \frac{1}{E[-\frac{\partial^2 logL}{\partial \nu^2}]} = \frac{1}{E[\frac{n}{\nu^2}]} = \nu \tag{18}$$

5 6.5

$$\hat{\alpha} = \frac{n_R - n_L}{n_R + n_L} \tag{19}$$

Error transfer formula:

$$\begin{aligned} \sigma_{\hat{\alpha}}^2 &= (\frac{\partial \alpha}{\partial n_R})^2 \sigma_{n_R}^2 + (\frac{\partial \alpha}{\partial n_L})^2 \sigma_{n_L}^2 \\ &= \sqrt{\frac{1-\alpha^2}{\nu_{tot}}} \end{aligned} \tag{20}$$

6 6.6

(a)

$$V[\alpha u + v] = \alpha^2 V[u] + V[v] + 2\alpha Cov(u, v) > 0 \tag{21}$$

We can let $\alpha^2 = \frac{V[v]}{V[u]}$:

$$V[v]V[u] > (Cov[u, v])^2 \tag{22}$$

(b) Using Cauchy-Schwarz :

$$V[\hat{\theta}]V[\frac{\partial}{\partial \theta} logL] > (Cov[\hat{\theta}, \frac{\partial}{\partial \theta} logL])^2 \tag{23}$$

(c) Prove:

$$\begin{aligned} E[\frac{\partial}{\partial \theta} logL] &= \int ... \int f_{joint}(x; \theta) \frac{\partial}{\partial \theta} log f_{joint}(x; \theta) dx_1 ... dx_n \\ &= \int ... \int \frac{\partial}{\partial \theta} f_{joint}(x; \theta) dx_1 ... dx_n \\ &= \frac{\partial}{\partial \theta} 1 \\ &= 0 \end{aligned} \tag{24}$$

And then

$$\begin{aligned} V[\hat{\theta}] &> \frac{(Cov[\hat{\theta}, \frac{\partial}{\partial \theta} logL])^2}{V[\frac{\partial}{\partial \theta} logL]} \\ &= \frac{(E[\hat{\theta} \frac{\partial}{\partial \theta} logL])^2}{E[(\frac{\partial}{\partial \theta} logL)^2]} \end{aligned} \tag{25}$$

(d) Prove a:

$$\begin{aligned} E[\hat{\theta} \frac{\partial}{\partial \theta} logL] &= \int ... \int f_{joint}(x; \theta) \hat{\theta} \frac{\partial}{\partial \theta} log f_{joint}(x; \theta) dx_1 ... dx_n \\ &= \int ... \int \hat{\theta} \frac{\partial}{\partial \theta} f_{joint}(x; \theta) dx_1 ... dx_n \\ &= \frac{\partial}{\partial \theta} E[\hat{\theta}] \\ &= 1 + \frac{\partial b}{\partial \theta} \end{aligned} \tag{26}$$

Prove b

$$\begin{aligned}E[\frac{\partial^2 logL}{\partial\theta^2}] &= \int...\int f_{joint}(x;\theta)\frac{\partial}{\partial\theta}(\frac{\partial f}{\partial\theta f})dx_1...dx_n \\ &= \int...\int \frac{\partial^2 f}{\partial\theta^2} - f_{joint}(x;\theta)(\frac{\partial f}{f\partial\theta})^2 dx_1...dx_n \\ &= -\int...\int f_{joint}(x;\theta)(\frac{\partial logL}{\partial\theta})^2 dx_1...dx_n \\ &= -E[(\frac{\partial logL}{\partial\theta})^2]\end{aligned} \tag{27}$$

So it is done.

7 6.7

(a)

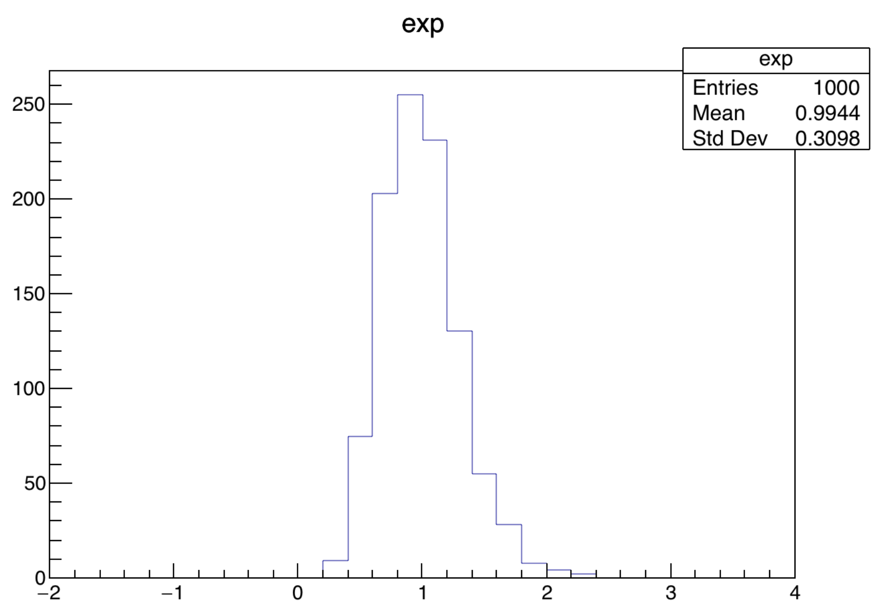


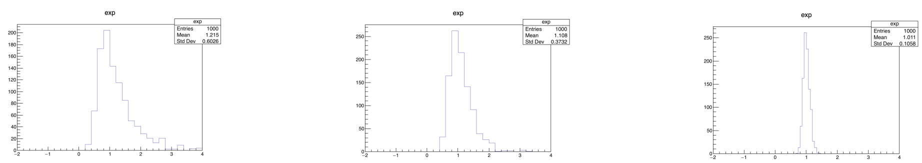
Figure 1: tau distribution

(b) We can get the maximum likelihood function:

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} \tag{28}$$

we use $\frac{\partial logL}{\partial\lambda} = 0$:

$$\lambda = \frac{n}{\sum t_i} \tag{29}$$



(a) n=5 (b) n=10 (c) n=100

Figure 2: lambda distribution

8 6.8

(a) The maximum likelihood function is :

$$L = \frac{1}{N^N} \tag{30}$$

When $N = 1$, the L is maximum.It is wrong. (b) We can take the n_{max} as the $\hat{N}_{taxi}$.

$$\begin{aligned}
E[\hat{N}_{taxi}] &= \sum_{i=N}^{N_{taxi}} \frac{iC_{i-1}^{N-1}}{C_{N_{taxi}}^{N}} \\
&= \sum_{i=N}^{N_{taxi}} N \frac{C_i^N}{C_{N_{taxi}}^N} \\
&= \frac{N(N_{taxi}+1)}{N+1}
\end{aligned} \tag{31}$$

$$\begin{aligned}
E[\hat{N}_{taxi}^2] &= \sum_{i=N}^{N_{taxi}} \frac{i^2 C_{i-1}^{N-1}}{C_{N_{taxi}}^{N}} \\
&= \sum_{i=N}^{N_{taxi}} N \frac{(i+1)C_i^N - C_i^N}{C_{N_{taxi}}^N} \\
&= \sum_{i=N}^{N_{taxi}} N \frac{(N+1)C_{i+1}^{N+1} - C_i^N}{C_{N_{taxi}}^N} \\
&= N(N+1)\frac{C_{N_{taxi}+2}^{N+2}}{C_{N_{taxi}}^N} - N\frac{C_{N_{taxi}+1}^{N+1}}{C_{N_{taxi}}^N}
\end{aligned} \tag{32}$$

So we can get the variant:

$$V[\hat{N}_{taxi}] = \frac{(N_{taxi}+1)(N_{taxi}-N)N}{(N+2)(N+1)^2} \tag{33}$$

9 6.9

The maximum likelihood function is :

$$L = \prod_{i=1}^{N} \frac{(\theta a(x_i))^{n_i}}{n!} e^{-\theta a(x_i)} \tag{34}$$

We use $\frac{\partial logL}{\partial \theta} = 0$

$$\hat{\theta} = \frac{\sum_{i=1}^{N} n_i}{\sum_{i=1}^{N} a(x_i)} \tag{35}$$

$$E[\hat{\theta}] = \frac{\sum_{i=1}^{N} \theta a(x_i)}{\sum_{i=1}^{N} a(x_i)} = \theta \tag{36}$$

$$V[\hat{\theta}] = \frac{\sum_{i=1}^{N} V[n_i]}{(\sum_{i=1}^{N} a(x_i))^2} = \frac{\theta}{\sum_{i=1}^{N} a(x_i)} \tag{37}$$

and the minimum border of variant is :

$$V[\hat{\theta}] > \frac{1}{E[-\frac{\partial^2 logL}{\partial \theta^2}]} = \frac{1}{\sum_{i=1}^{N} E[\frac{n_i}{\theta^2}]} = \frac{\theta}{\sum_{i=1}^{N} a(x_i)} \tag{38}$$

10 6.10

According to the 6.9 :

$$\begin{aligned} \hat{\theta}_\nu &= \frac{\sum_{i=1}^{N} n_i}{\sum_{i=1}^{N} E_i \epsilon(E_i) L_i} \\ \hat{\theta}_{\bar{\nu}} &= \frac{\sum_{i=1}^{N} \bar{n}_i}{\sum_{i=1}^{N} E_i \epsilon(E_i) L_i} \end{aligned} \tag{39}$$

So we can solve:

$$\begin{aligned} \langle q \rangle &= \frac{3\pi}{8G^2 M}(3\theta_\nu - \theta_{\bar{\nu}}) \\ \langle \bar{q} \rangle &= \frac{3\pi}{8G^2 M}(3\theta_{\bar{\nu}} - \theta_\nu) \end{aligned} \tag{40}$$

So

$$\langle \hat{g} \rangle = 1 - \frac{6\pi}{8G^2 M \sum_{i=1}^{N} E_i \epsilon(E_i) L_i}[\sum_{i=1}^{N}(n_i + \bar{n}_i)] \tag{41}$$

11 6.11

(a)We use the maximum likelihood:

$$\begin{aligned} &\nu_0 = 1844.94 \pm 56.0739 \\ &k = (1.19863 \pm 0.0469766) \times 10^{-23} (J/K) \end{aligned} \tag{42}$$

(b) From the value you obtain for k, determine Avogadro's number using the relation:

$$N_A = R/k = 6.9412 \times 10^{23}/\mathrm{mol} \tag{43}$$

(c) Using chi-square way to estimate:

$$\begin{aligned}
&\nu_0 = 1844.94 \pm 39.6617 \\
&k = (1.19863 \pm 0.0332252) \times 10^{-23} (J/K) \\
&\chi^2 = 4.5873 \\
&M - N = 2
\end{aligned} \tag{44}$$

the goodness-of-fit $P = 0.1$