

homework 2

Chenxi Gu
2017311017

April 26, 2018

1 2.2

$A(g_a)$ is a representation.

$$A(g_1)A(g_2) = A(g_1g_2) \quad (1)$$

So we can get :

$$A^T(g_1)^{-1}A^T(g_2)^{-1} = [[A(g_1)A(g_2)]^T]^{-1} = A^T(g_1g_2)^{-1} \quad (2)$$

$$A^\dagger(g_1)^{-1}A^\dagger(g_2)^{-1} = [[A(g_1)A(g_2)]^\dagger]^{-1} = A^\dagger(g_1g_2)^{-1} \quad (3)$$

$A^T(g_a)^{-1}, A^\dagger(g_a)^{-1}$ are also representation.

$A^T(g_a), A^\dagger(g_a)$ are not representation, but Abel group. if $A(g_a)$ is a Unitary representation:

$$\begin{aligned} A^\dagger(g) &= A(g^{-1}) \\ A^{T\dagger}(g)^{-1} &= A^T(g) \\ A^{\dagger\dagger}(g)^{-1} &= A(g)^{-1} \end{aligned} \quad (4)$$

So $A^T(g)^{-1}, A^\dagger(g)^{-1}$ are unitary representation. It is easy to prove they are irreducible representation.

2 2.3

It is easy to find the matrix commute with all elements in group $A(g)$:

$$[A(g_a), \sum_C A(g_b)] = 0 \quad (5)$$

Using Schur lemma :

$$\sum_C A(g_b) = \lambda E \quad (6)$$

3 2.7

We choose natural basic :

$$e = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad a = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad a^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad a^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (7)$$

So we can get the left representation:

$$L(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad L(a) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (8)$$

The right presentation:

$$R(e) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad R(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

4 2.8

In the representation $A^p(g_a)$, the character is $\chi^p(g_a)$. In the representation $A^r(g_a)$, the character is $\chi^r(g_a)$.

Since $A^p(g_a)$ and $A^r(g_a)$ are Irreducible unequivalent representation.

$$\sum_a \chi^p(g_a) \chi^{r*}(g_a) = 0 \quad (10)$$

this equation also mean the identical representation and $A^p(g_a) \otimes A^{r*}(g_a)$ are orthogonal.

$$\sum_a \chi^p(g_a) \chi^{p*}(g_a) = 1 \quad (11)$$

For the same reason, $A^p(g_a) \otimes A^{p*}(g_a)$ only have 1 identical representation.

5 2

In the active view:

$$SO(2) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (12)$$

In the passive view:

$$SO(2) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (13)$$

6 3

We can get the $D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$

Secondly, we can find the class in the group:

(a) $n = 2k$

$$\begin{aligned} & \{e\} \\ & \{a, a^{n-1}\} \\ & \{a^2, a^{n-2}\} \\ & \dots \\ & \{a_k\} \\ & \{b, b^2, \dots, b^{2k}\} \\ & \{ba, ba^3, \dots, b^{2k-1}\} \end{aligned} \quad (14)$$

So we can deduce there are 4 1-dimension representation.

$$\begin{pmatrix} a=1 \\ b=1 \end{pmatrix} \quad \begin{pmatrix} a=1 \\ b=-1 \end{pmatrix} \quad \begin{pmatrix} a=-1 \\ b=1 \end{pmatrix} \quad \begin{pmatrix} a=-1 \\ b=-1 \end{pmatrix} \quad (15)$$

And $\frac{n}{2} - 1$ 2-dimensions representation, the i_{th} 2-dimensions representation is:

$$A_i^{(2)} = \left\{ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} \cos(\frac{2\pi i}{n}) & -\sin(\frac{2\pi i}{n}) \\ \sin(\frac{2\pi i}{n}) & \cos(\frac{2\pi i}{n}) \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (16)$$

(b) $n = 2k + 1$

$$\begin{aligned} & \{e\} \\ & \{a, a^{n-1}\} \\ & \{a^2, a^{n-2}\} \\ & \dots \\ & \{a_k, a_{k+1}\} \\ & \{b, b^2, \dots, b^n\} \end{aligned} \quad (17)$$

So we can deduce there are 2 1-dimension representation.

$$\begin{pmatrix} a=1 \\ b=1 \end{pmatrix} \quad \begin{pmatrix} a=1 \\ b=-1 \end{pmatrix} \quad (18)$$

And $\frac{n-1}{2}$ 2-dimensions representation, the i_{th} 2-dimensions representation is:

$$A_i^{(2)} = \left\{ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad a = \begin{pmatrix} \cos(\frac{2\pi i}{n}) & -\sin(\frac{2\pi i}{n}) \\ \sin(\frac{2\pi i}{n}) & \cos(\frac{2\pi i}{n}) \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (19)$$

7 4

We know the pauli group:

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i\epsilon_{ijk}\sigma_k \\ \sigma_i^2 &= 1 \end{aligned} \quad (20)$$

So we can get the group element:

$$G = \{\pm 1, \pm i, \pm \sigma_x, \pm i\sigma_x, \pm \sigma_y, \pm i\sigma_y, \pm \sigma_z, \pm i\sigma_z\} \quad (21)$$

It is easy to find there are 10 classes. 8 1-dimension representation and 2 2-dimensions representation.

2-dimensions representation:

$$G_1^{(2)} = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (22)$$

$$G_2^{(2)} = \left\{ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \quad (23)$$

8 5

According to the definition of project operator:

$$\begin{aligned} P_1^2 &= \frac{1}{3}(T_e - \frac{1}{2}T_d - \frac{1}{2}T_f - T_a + \frac{1}{2}T_b + \frac{1}{2}T_c) \\ P_{21}^2 &= \frac{1}{3}(\frac{\sqrt{3}}{2}T_d - \frac{\sqrt{3}}{2}T_f + \frac{\sqrt{3}}{2}T_b - \frac{\sqrt{3}}{2}T_c) \\ P_2^2 &= \frac{1}{3}(T_e - \frac{1}{2}T_d - \frac{1}{2}T_f + T_a - \frac{1}{2}T_b - \frac{1}{2}T_c) \\ P_{12}^2 &= \frac{\sqrt{3}}{6}(-T_d + T_f + T_b - T_c) \end{aligned} \quad (24)$$