chapter 6

Chenxi Gu
2017311017

May 7, 2018

1 6.1

(a) We can get the likelihood function:

$$logL(\mu, \sigma^2) = \sum_{i=1}^{n}(log\frac{1}{\sqrt{2\pi}} + \frac{1}{2}log\frac{1}{\sigma^2} + \frac{(x_i-\mu)^2}{2\sigma^2}) \tag{1}$$

So we can get the maximum likelihood variation.

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}\sum_{i=1}^{n} x_i \\ \hat{\sigma^2} &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu})^2 \end{aligned} \tag{2}$$

(b)

$$\begin{aligned} E[\hat{\mu}] = \frac{1}{n}\sum_{i=1}^{n}(\int \frac{x_i}{\sqrt{2\pi}\sigma} e^{-\frac{(x_i-\mu)}{2\sigma^2}} dx_i \times_{j<>i} \int \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x_j-\mu)}{2\sigma^2}} dx_j) \\ = \mu \end{aligned} \tag{3}$$

$$E[\hat{\sigma^2}] = \frac{n-1}{n}\sigma^2 \tag{4}$$

$$\begin{aligned} V[\hat{\mu}] &= E[\hat{\mu}^2] - E[\hat{\mu}]^2 \\ &= \frac{\sigma^2}{n} \end{aligned} \tag{5}$$

$$\begin{aligned} V[\hat{\sigma^2}] &= E[\hat{\sigma^2}^2] - E[\hat{\sigma^2}]^2 \\ &= \frac{(n-1)^2}{n^3}(3\sigma^4 - \frac{n-3}{n-1}\sigma^3) \end{aligned} \tag{6}$$

(c)

$$V^{-1} = \begin{matrix} -\frac{n}{\sigma^2} & -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu) \\ -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu) & -\sum_{i=1}^{n}[\frac{(x_i-\mu)^2}{\sigma^6} - \frac{1}{2\sigma^4}] \end{matrix} \tag{7}$$

when the $n->\infty$, the answer is same.

2 6.2

The likelihood function:

$$L = C_N^n p^n (1-p)^{N-n} \tag{8}$$

$$\hat{p} = \frac{n}{N} \tag{9}$$

$$E(\hat{p}) = \frac{E(n)}{N} = p \tag{10}$$

$$V(\hat{p}) = \frac{p(1-p)}{N} \tag{11}$$

According to the 6.16:

$$V(\hat{p}) > \frac{1}{E[-\frac{\partial^2 logL}{\partial p^2}]} = \frac{N}{p(1-p)} \tag{12}$$

3 6.3

(a)

$$\hat{\alpha} = \frac{2n}{N} - 1 \tag{13}$$

$$\sigma_{\hat{\alpha}} = \sqrt{\frac{1-\alpha^2}{N}} \tag{14}$$

(b) $N > 9 * 10^6$

4 6.4

$$L = \frac{\nu^n}{n!} e^{-\nu} \tag{15}$$

So we can get $\hat{\nu} = n$

$$E(\hat{\nu}) = E(n) = \nu \tag{16}$$

$$V(\hat{\nu}) = V(n) = \nu \tag{17}$$

According to the 6.16:

$$V[\hat{\nu}] > \frac{1}{E[-\frac{\partial^2 logL}{\partial \nu^2}]} = \frac{1}{E[\frac{n}{\nu^2}]} = \nu \tag{18}$$

5 6.5

$$\hat{\alpha} = \frac{n_R - n_L}{n_R + n_L} \tag{19}$$

Error transfer formula:

$$\begin{aligned}\sigma_{\hat{\alpha}}^2 &= (\frac{\partial \alpha}{\partial n_R})^2 \sigma_{n_R}^2 + (\frac{\partial \alpha}{\partial n_L})^2 \sigma_{n_L}^2 \\ &= \sqrt{\frac{1-\alpha^2}{\nu_{tot}}}\end{aligned} \tag{20}$$

6 6.6

(a)

$$V[\alpha u + v] = \alpha^2 V[u] + V[v] + 2\alpha Cov(u, v) > 0 \tag{21}$$

We can let $\alpha^2 = \frac{V[v]}{V[u]}$:

$$V[v]V[u] > (Cov[u, v])^2 \tag{22}$$

(b) Using Cauchy-Schwarz :

$$V[\hat{\theta}]V[\frac{\partial}{\partial \theta} logL] > (Cov[\hat{\theta}, \frac{\partial}{\partial \theta} logL])^2 \tag{23}$$

(c)

$$\begin{aligned}V[\hat{\theta}] &> \frac{(Cov[\hat{\theta}, \frac{\partial}{\partial \theta} logL])^2}{V[\frac{\partial}{\partial \theta} logL]} \\ &= \frac{(E[\hat{\theta}\frac{\partial}{\partial \theta} logL])^2}{E[(\frac{\partial}{\partial \theta} logL)^2]}\end{aligned} \tag{24}$$

7 6.8

(a) The maximum likelihood function is :

$$L = \frac{1}{N^N} \tag{25}$$

When $N = 1$, the L is maximum.It is wrong. (b) We can take the n_{max} as the $\hat{N}_{taxi}$.

$$\begin{aligned}E[\hat{N}_{taxi}] &= \sum_{i=N}^{N_{taxi}} \frac{iC_{i-1}^{N-1}}{C_{N_{taxi}}^N} \\ &= \sum_{i=N}^{N_{taxi}} N\frac{C_i^N}{C_{N_{taxi}}^N} \\ &= \frac{N(N_{taxi}+1)}{N+1}\end{aligned} \tag{26}$$

$$
\begin{aligned}
E[\hat{N}_{taxi}^2] &= \sum_{i=N}^{N_{taxi}} \frac{i^2 C_{i-1}^{N-1}}{C_{N_{taxi}}^N} \\
&= \sum_{i=N}^{N_{taxi}} N \frac{(i+1)C_i^N - C_i^N}{C_{N_{taxi}}^N} \\
&= \sum_{i=N}^{N_{taxi}} N \frac{(N+1)C_{i+1}^{N+1} - C_i^N}{C_{N_{taxi}}^N} \\
&= N(N+1)\frac{C_{N_{taxi}+2}^{N+2}}{C_{N_{taxi}}^N} - N\frac{C_{N_{taxi}+1}^{N+1}}{C_{N_{taxi}}^N}
\end{aligned} \tag{27}
$$

So we can get the variant:

$$
V[\hat{N}_{taxi}] = \frac{(N_{taxi}+1)(N_{taxi}-N)N}{(N+2)(N+1)^2} \tag{28}
$$

8 6.9

The maximum likelihood function is :

$$
L = \prod_{i=1}^{N} \frac{(\theta a(x_i))^{n_i}}{n!} e^{-\theta a(x_i)} \tag{29}
$$

We use $\frac{\partial logL}{\partial \theta} = 0$

$$
\hat{\theta} = \frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N a(x_i)} \tag{30}
$$

$$
E[\hat{\theta}] = \frac{\sum_{i=1}^N \theta a(x_i)}{\sum_{i=1}^N a(x_i)} = \theta \tag{31}
$$

$$
V[\hat{\theta}] = \frac{\sum_{i=1}^N V[n_i]}{(\sum_{i=1}^N a(x_i))^2} = \frac{\theta}{\sum_{i=1}^N a(x_i)} \tag{32}
$$

and the minimum border of variant is :

$$
V[\hat{\theta}] > \frac{1}{E[-\frac{\partial^2 logL}{\partial \theta^2}]} = \frac{1}{\sum_{i=1}^N E[\frac{n_i}{\theta^2}]} = \frac{\theta}{\sum_{i=1}^N a(x_i)} \tag{33}
$$