NOTES ON 3D RIGID BODY PRESENTATION AND KINEMATICS

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Contents

1 Overview

	pros	cons	application
Rotation Matrix	Intuitive	suffer from numerical instability[1]	robotics
	straightforward matrix multiplication	redundant(9 num. for 3 dof)	
Axis-angle	Visually intuitive	hard to locate axis	IMU
Euler angles[2]	Intuitive	suffer from gimbal lock	aircraft control
	straightforward matrix multiplication		
Quaternions	Avoids problems with singularities	difficult to understand	graphics
	easy to interpolate between rotations		robotics, etc.

2 Rotation Matrix

2.1 Rotation Matrix in Plane

In order to express the body frame $\{b\}$ in fixed-frame coordinates $\{s\}$, two pieces of information are needed: the position vector p and the unit axis directions $\hat{x}_b$ and $\hat{y}_b$, as shown in Figure 2.1.

The unit axes $\hat{x}_b$ and $\hat{y}_b$ can be written as vectors, and packaged into a 2×2 **rotation matrix** P:

$$P = \begin{bmatrix} \hat{x}_b \\ \hat{y}_b \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

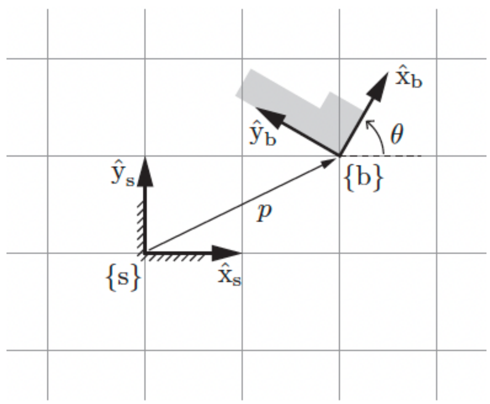

Figure 2.1: Rotation in plain.

In the standard right-hand[3] coordinate system, the positive direction for rotations is counterclockwise, its inverse/transpose represents a clockwise rotation. If the body frame $\{c\}$ rotates clockwise by an angle of θ, we can write

$$P_c = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P_b^T = P_b^{-1} \tag{2.2}$$

2.2 Rotation Matrix in 3D space

To extend the concepts of rigid-body motion to three dimensions, we can begin by focusing solely on orientation. Consider two frames, a fixed space frame $\{s\}$ and a body frame $\{b\}$, as illustrated in Figure 2.2.

[1] Even though rotation matrices themselves are orthogonal, the computation of rotation matrices can still suffer from numerical instability due to rounding errors. In particular, the computation of the inverse, which is used for undoing a rotation, can be sensitive to small changes in the matrix elements.

[2] There are twelve Euler angle conventions, and the most commonly used one is *yaw-pitch-row*, or ZYX, in a right-hand coordinate frame.

[3] the unit axes $\{\hat{x}, \hat{y}, \hat{z}\}$ always satisfy $\hat{x} \times \hat{y} = \hat{z}$

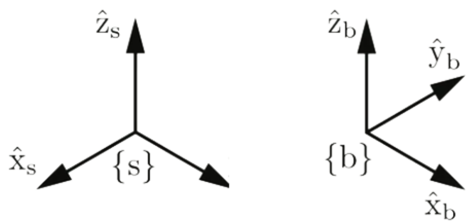

Figure 2.2: A fixed space frame $\{s\}$ and a body frame $\{b\}$.

In the coordinates of $\{s\}$, the axes of the body frame can be expressed as:

$$\hat{x}_b^T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \hat{y}_b^T = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \hat{z}_b^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

By concatenating the three unit vectors, we can construct the rotation matrix R_{sb}:

$$R_{sb} = \begin{bmatrix} \hat{x}_b \\ \hat{y}_b \\ \hat{z}_b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R \tag{2.3}$$

By denoting the unit axes of the fixed frame $\{\hat{x}_s, \hat{y}_s, \hat{z}_s\}$ and the unit axes of the body frame $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$, p can be expressed as

$$p = p_1\hat{x}_s + p_2\hat{y}_s + p_3\hat{z}_s \tag{2.4}$$

The axes of the body frame can be expressed as

$$\begin{aligned} \hat{x}_b &= r_{11}\hat{x}_s + r_{21}\hat{y}_s + r_{31}\hat{z}_s \\ \hat{y}_b &= r_{12}\hat{x}_s + r_{22}\hat{y}_s + r_{32}\hat{z}_s \\ \hat{z}_b &= r_{13}\hat{x}_s + r_{23}\hat{y}_s + r_{33}\hat{z}_s \end{aligned} \tag{2.5}$$

Therefore we can define $p \in \mathbb{R}^3$ and $R \in \mathbb{R}^{3\times3}$ as:

$$p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}, \quad R = \begin{bmatrix} \hat{x}_b^T & \hat{y}_b^T & \hat{z}_b^T \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.6}$$

As a rotation matrix $R \in \mathbb{R}^{3\times3}$ has nine parameters, despite having only three degree of freedoms, there are six explicit constraints that must be satisfied.

(a) **The unit norm condition:** $\hat{x}_b, \hat{y}_b, \hat{z}_b$ are all unit vectors.
(b) **The orthogonality condition:** $\hat{x}_b \cdot \hat{y}_b = \hat{x}_b \cdot \hat{z}_b = \hat{y}_b \cdot \hat{z}_b = 0$

These six constraints can be expressed more compactly as a single set of constraints on the matrix R,

$$R^T R = I \tag{2.7}$$

$R^T R = I$ implies $det(R^T R) = det(I) = 1$, and as R is a square matrix, it follows that $det(R^T)det(R) = 1$. Since $det(R^T) = det(R)$, it can be further simplified to:

$$det(R)^2 = 1 \iff det(R) = \pm 1 \tag{2.8}$$

In particular, for a proper rotation matrix with a right-handed coordinate system, it follows that $det(R) = 1$.

Definition 2.1. *The* ***special orthogonal group*** $SO(n)$[4], *is the set of all* $n \times n$ *real matrices* R *that satisfy (i)* $R^T R = I$ *and (ii)* $det(R) = 1$.

$$SO(n) = \{R \in \mathbb{R}^{n\times n} | RR^T = I, det(R) = 1\} \tag{2.9}$$

[4] $SO(n)$ groups are also known as matrix Lie groups, because the elements of the group form a differentiable manifold.

Properties of Rotation Matrices

Proposition 2.1. *The inverse of a rotation matrix $R \in SO(3)$ is also a rotation matrix, and it is equal to the transpose of R, i.e., $R^T = R^{-1}$.*

Proof. The condition $RR^T = I$ implies that $R^{-1}RR^T = R^{-1}I$ thus $R^T = R^{-1}$. Since R and R^T are square matrices, we have $\det(R^T) = \det(R) = 1$. Therefore, R^T is also a rotation matrix. □

Proposition 2.2. *The product of two rotation matrices is a rotation matrix.*

Proof. Given $R_1, R_2 \in SO(3)$, let $R = R_1R_2$. First, we have

$$\begin{aligned} RR^T &= (R_1R_2)(R_1R_2)^T \\ &= R_1R_2R_2^TR_1^T \\ &= R_1IR_1^T \\ &= R_1R_1^T \\ &= I \end{aligned}$$

Further,

$$\begin{aligned} \det(R) &= \det(R_1R_2) \\ &= \det(R_1)\det(R_2) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

Therefore, the product of two rotation matrices is a rotation matrix. □

Proposition 2.3. *Multiplication of rotation matrices is associative, $(R_1R_2)R_3 = R_1(R_2)(R_3)$, but generally not commutative, $R_1R_2 \neq R_2R_1$.*

Proof. Associativity and nocommutativity follows from the properties of matrix multiplication in linear algebra. Commutativity for planar rotations follows from a direct calculation. □

Proposition 2.4. *For any vector $x \in \mathbb{R}^3 and R \in SO(3)$, the vector $y = Rx$ has the same length as x.*

Proof. The length of y can be computed as follows:

$$\begin{aligned} ||y||^2 &= y^Ty \\ &= (Rx)^T(Rx) \\ &= x^TR^TRx \\ &= x^TIx \\ &= x^Tx \\ &= ||x||^2 \end{aligned}$$

□

Uses of Rotation Matrices

There are three major uses for a rotation matrix R:

(a) to represent an orientation;
(b) to change the reference frame;
(c) to rotate a vector or a frame.

Representing an orientation Inspecting Figure 2.2, we can get

$$R_{sb} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{bs} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.10}$$

Proposition 2.5. *For any two frames* $\{a\}$ *and* $\{c\}$,

$$R_{ac} = R_{ca}^{-1} = R_{ca}^T$$

Proof. The length of y can be computed as follows:

$$\begin{aligned} ||y||^2 &= y^T y \\ &= (Rx)^T (Rx) \\ &= x^T R^T R x \\ &= x^T I x \\ &= x^T x \\ &= ||x||^2 \end{aligned}$$

□

Changing the reference frame Given rotation matrix $R_a b$ represents the orientation of $\{b\}$ in $\{a\}$, and $R_b c$ represents the orientation of $\{c\}$ in $\{b\}$. The orientation of $\{c\}$ in $\{a\}$ can be computed as

$$R_{ac} = R_{ab} R_{bc} \tag{2.11}$$

It is known as the *subscript cancellation rule*. As rotation matrix is just a collection of three unit vectors, the reference frame of a vector can also be changed by a rotation matrix following:

$$R_{ab} p_b = p_a \tag{2.12}$$

Rotating a vector or a frame Figure 2.3 shows a frame being rotated about a unit axis $\hat{\omega}$ by an amount θ, resulting in a new frame marked in light gray.

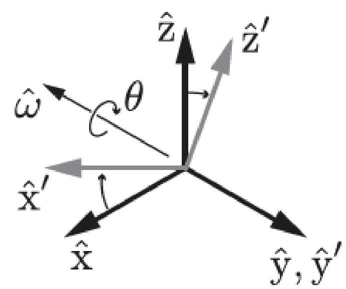

Figure 2.3: A coordinate frame with axes $\{\hat{x}, \hat{y}, \hat{z}\}$ is rotated by θ about a unit axis $\hat{\omega}$. The orientation of the final frame, with axes $\{\hat{x}', \hat{y}', \hat{z}'\}$, is written as R relative to the original frame.

This is known as the **axis-angle** representation, which can be written as:

$$R = Rot(\hat{\omega}, \theta),$$

Examples of rotation operations about coordinate frame axes are

$$Rot(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix},$$

$$Rot(\hat{y}, \theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$Rot(\hat{z}, \theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

More generally, as we will prove in the section ... , let $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$,

$$Rot(\hat{\omega}, \theta) = \begin{bmatrix} \cos\theta + \hat{\omega}_1^2(1-\cos\theta) & \hat{\omega}_1\hat{\omega}_2(1-\cos\theta) - \hat{\omega}_3\sin\theta & \hat{\omega}_1\hat{\omega}_3(1-\cos\theta) + \hat{\omega}_2\sin\theta \\ \hat{\omega}_2\hat{\omega}_1(1-\cos\theta) + \hat{\omega}_3\sin\theta & \cos\theta + \hat{\omega}_2^2(1-\cos\theta) & \hat{\omega}_2\hat{\omega}_3(1-\cos\theta) - \hat{\omega}_1\sin\theta \\ \hat{\omega}_3\hat{\omega}_1(1-\cos\theta) - \hat{\omega}_2\sin\theta & \hat{\omega}_3\hat{\omega}_2(1-\cos\theta) + \hat{\omega}_1\sin\theta & \cos\theta + \hat{\omega}_3^2(1-\cos\theta) \end{bmatrix}$$