CPSC 121: Models of Computation

Unit 12: Functions

Based on slides by Patrice Belleville and Steve Wolfman

PART 1
REVIEW OF TEXT READING

These pages correspond to text reading and are not covered in the lectures.

What is a Function?

Mostly, a function is what you learned it was all through K-12 mathematics, with strange vocabulary to make it more interesting…

A function **f:A → B** maps values from its domain **A** to its co-domain **B**.

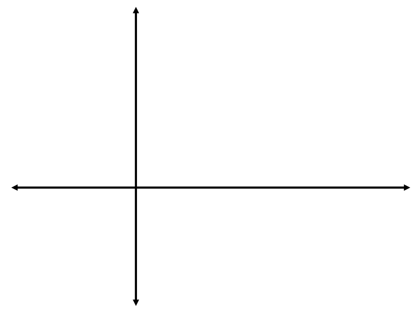
	<u>Domain</u>	<u>Co-domain</u>
f(x) = x^3		
f(x) = x mod 4		
f(x) = $\lfloor$**x**$\rfloor$		

Plotting Functions

f(x) = x^3

f(x) = x mod 4

f(x) = $\lfloor$**x**$\rfloor$



Not every function is easy to plot!

What is a Function?

Not every function has to do with numbers…

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.

	Domain	Co-domain
$\mathbf{f(x) = \sim x}$		
$\mathbf{f(x,y) = x \vee y}$		
$\mathbf{f(x)}$ = $\mathbf{x}$'s phone #		

What is a Function?

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.

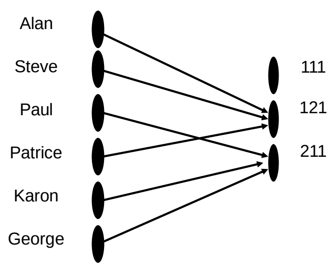
$$\mathbf{f(control, data1, data2) = (\sim control \wedge data1) \vee (control \wedge data2)}$$

Domain?

Co-domain?

What is a Function?

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.



Domain?

Co-domain?

Other examples?

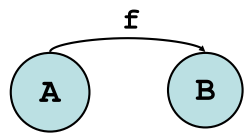
What is a Function?

A function $\mathbf{f:A \rightarrow B}$ maps values from its domain $\mathbf{A}$ to its co-domain $\mathbf{B}$.

$\mathbf{f}$ can't map one element of its domain to more than one element of its co-domain:

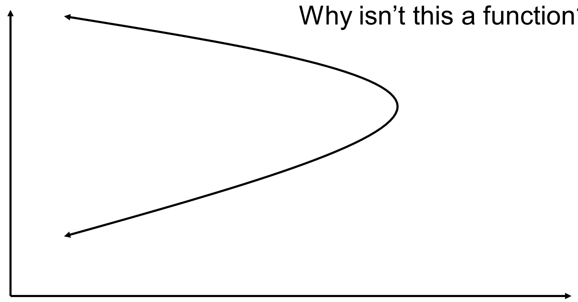
$$\forall \mathbf{x} \in \mathbf{A},\ \forall \mathbf{y_1, y_2} \in \mathbf{B},\ [(\mathbf{f(x)} = \mathbf{y_1}) \wedge (\mathbf{f(x)} = \mathbf{y_2})] \rightarrow (\mathbf{y_1} = \mathbf{y_2}).$$

Why insist on this?



Not a Function

Why isn't this a function?



(The Laffer Curve: a non-functional tax policy.)

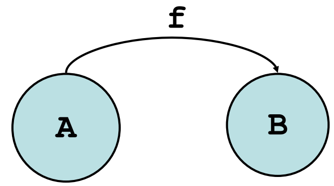
Function Terminology

A function **f:A → B** maps values from its domain **A** to its co-domain **B**.

For **f** to be a function, it must map every element in its domain:

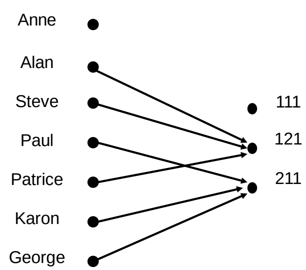
$\forall x \in A,\ \exists y \in B,\ f(x) = y.$

Why insist on this?



Warning: some mathematicians would say that makes **f** "total".

Not a Function

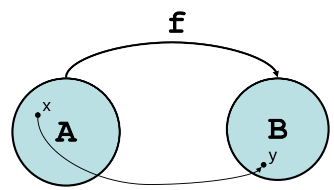


Function Terminology

A function **f:A → B** maps values from its domain **A** to its co-domain **B**.

f(x) is called the image of **x** (under **f**).

x is called the pre-image of **f(x)** (under **f**).

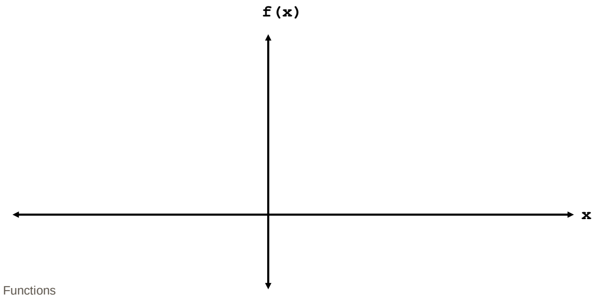


Trying out Terminology

$f(x) = x^2$

What is the image of 16?

What is the range of f?



PART 2

IN CLASS PAGES

Pre-Class Learning Goals

- By the start of class, you should be able to:
  - Define the terms domain, co-domain, range, image, and pre-image
  - Use appropriate function syntax to relate these terms (e.g., f : A → B indicates that f is a function mapping domain A to co-domain B).
  - Determine whether f : A → B is a function given a definition for f as an equation or arrow diagram.

Quiz 10

- In General:
- Specific issues:

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Define the terms injective (one-to-one), surjective (onto), bijective (one-to-one correspondence), and inverse.
  - Determine whether a given function is injective, surjective, and/or bijective.
  - Determine whether the inverse of a given function is a function.

Outline

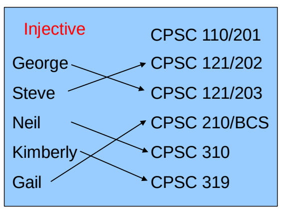
- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

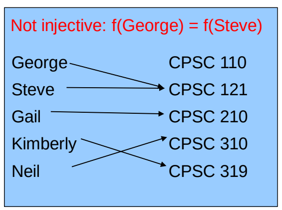
Injective Functions

Some special types of functions:

- A function $f : A \to B$ is injective (one-to-one) if

  $\forall x \in A, \forall y \in A, x \neq y \to f(x) \neq f(y)$.
- In the arrow diagram: at most one arrow points to each element of B.





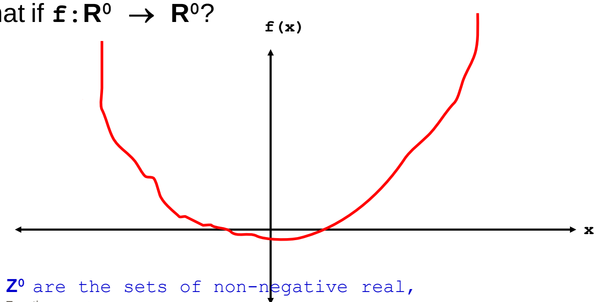
Trying out Terminology

$\mathbf{f:R \to R^0}$

$\mathbf{f(x) = x^2}$

Injective? No

What if $\mathbf{f:R^0 \to R^0}$?



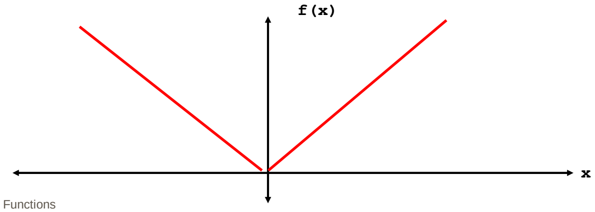
$\mathbf{R^0}$, $\mathbf{Z^0}$ are the sets of non-negative real, integer numbers

Trying out Terminology

`f(x) = |x|` (the absolute value of `x`)

Injective?

a. Yes, if `f:R` → $\mathbf{R}^0$
b. Yes, if `f:`$\mathbf{R}^0$ → **R**
c. Yes, for some other domain/co-domain
d. No, not for any domain/co-domain
e. None of these is correct



Trying out Terminology

`f:{s|s is a 121 student} → {A+, A, …, D, F}`
`f(s) = s's mark in 121`

Is `f` injective?

a. Yes
b. No
c. Not enough information

Trying out Terminology

- **`f:{s|s is a 121 student} → {A+, A, …, D, F}`**
- What if we didn't know what `f` represented, only its "type" and the fact that there are 300 CPSC 121 students:

Is `f` injective?

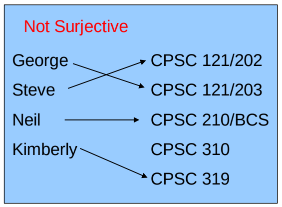
a. Yes
b. No
c. Not enough information

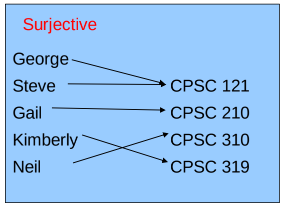
Outline

- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

Surjective Functions

- A function f : A → B is surjective (onto) if
  ∀y ∈ B, ∃x ∈ A, f(x) = y.
  Can we define it in terms of range and co-domain?
- In the arrow diagram: at least one arrow points to each element of B.



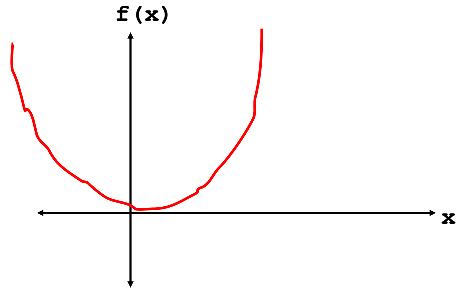


Trying out Terminology

f:R → R⁰
f(x) = x²

Surjective? ✓

What if **f:R → R**?
What if **f:Z → Z⁰**?



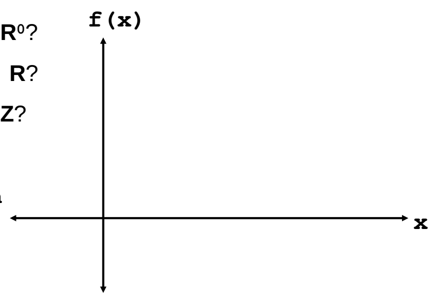
R⁰, Z⁰ are the sets of non-negative real, integer numbers

Trying out Terminology

f(x) = ⌊x⌋

Is **f** surjective?

a. Yes, for f:R → R⁰?
b. Yes, for f:R⁰ → R?
c. Yes, for f:R → Z?
d. No, not for any domain/co-domain
e. None of these is correct



Trying out Terminology

f:{s|s is a 121 student} → {A+, A, …, D, F}

f(s) = s's mark in 121

- Is **f** surjective?

a. Yes
b. No
c. Not enough information

- Could we *ever* know that f was surjective just by knowing **f**'s domain and co-domain?

Hash Functions

- A hash function maps its input onto the indexes of an array so we can store arbitrary data in an array

  e.g. h(x) = x mod k
  where k is the array size

- If it's not surjective, then we "waste" entries in the array that are never mapped to!

Surjective Functions So Far

- Which combinational circuits with one output are surjective?

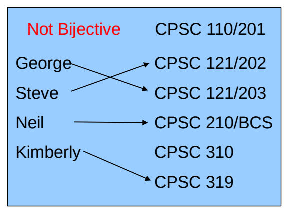
a. Every such circuit.
b. Any such circuit that represents a contingency (neither a tautology nor contradiction).
c. Only the ones equivalent to an inverter.
d. No such circuit is surjective.
e. None of these is correct.

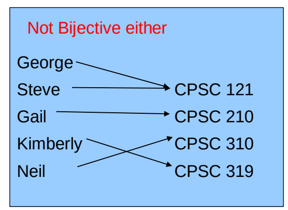
Outline

- Injective Functions
- Surjective Functions
- Bijective Functions
- Inverse Operations.

Bijective Functions

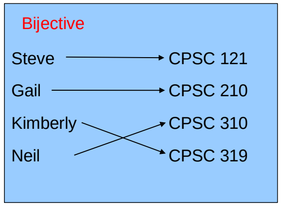
- A function $f : A \rightarrow B$ is bijective (also one-to-one correspondence) if it is both one-to-one and onto (both injective and surjective).
- In the arrow diagram: exactly one arrow points to each element of B.





Bijective Functions

- This is bijective

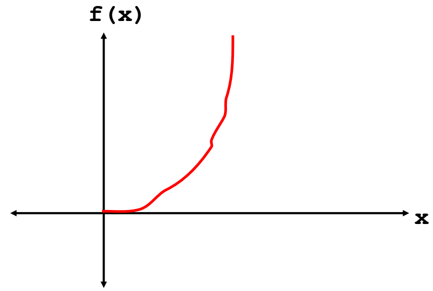


Trying out Terminology

f(x) = x^2
f:? → ?

Bijective for what domain/co-domain?

$R^0 \to R^0$



Outline

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Inverse of a Function

- The inverse of a function f: A → B, denoted f^{-1}, is
  $f^{-1}: B \to A$.
  $f^{-1}(y) = x \leftrightarrow f(x) = y$.

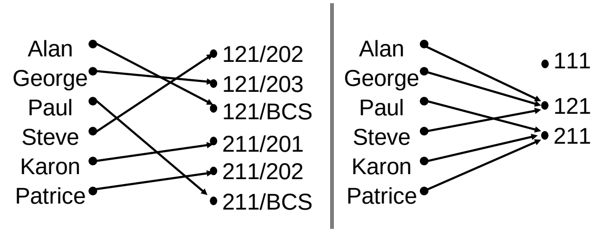
- In other words:
  - If we think of a function as a list of pairs.
    E.g. $f(x) = x^2$: { (1, 1), (2, 4), (3, 9), (4, 16), ... }
  - Then f^{-1} is obtained by swapping the elements of each pair:
    f^{-1} = { (1, 1), (4, 2), (9, 3), (16, 4), ... }

Inverse of a Function

- Is f^{-1} a function?
  - A. Yes, always.
  - B. No, never.
  - C. Yes, but only if f is injective.
  - D. Yes, but only if f is surjective.
  - E. Yes, but only if f is bijective.
- Can we prove it?

Trying out Terminology

What's the inverse of each of these **f**s?

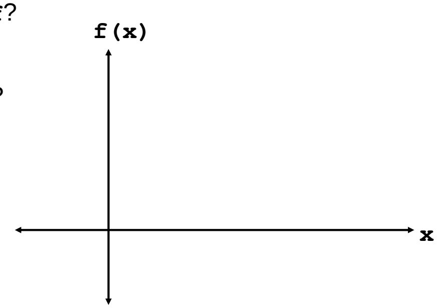


Trying out Terminology

f(x) = x²

What's the inverse of **f**?

What should the domain/co-domain be?



Appendix 3: An Inverse Proof

- **Theorem**: If **f : A → B** is bijective, then **f⁻¹ : B → A** is a function.
- **Proof:** We proceed by antecedent assumption.
  - Assume **f : A → B** is bijective.
  - Consider an arbitrary element **y** of **B**. Because **f** is surjective, there is some **x** in **A** such that **f(x) = y**. Because **f** is injective, that is the only such **x**.
  - **f⁻¹(y) = x** by definition; so, **f⁻¹** maps every element of **B** to exactly one element of **A**.

QED