CPSC 121: Models of Computation

Unit 11: Sets

Based on slides by Patrice Belleville and Steve Wolfman

PART 1
REVIEW OF TEXT READING

These pages correspond to text reading and are not covered in the lectures.

Sets

A set is a collection of elements:

- the set of students in this class
- the set of lowercase letters in English
- the set of natural numbers (N)
- the set of all left-handed students in this class

An element is either in the set ($x \in S$) or not ($x \notin S$).

Is there a set of *everything*?

Quantifier Example

Someone in this class is left-handed (where C is the set of people in this class and L(p) means p is left-handed):

$\exists x \in C, L(x)$

What is a Set?

A set is an *unordered* collection of objects.

The objects in a set are called members.
($a \in S$ indicates a is a member of S;
$a \notin S$ indicates a is **not** a member of S)

A set contains its members.

Describing Sets (1/4)

Some sets…

$A = \{1, 3, 9\}$

$B = \{1, 3, 9, 27, snow\}$

$C = \{1, 1, 3, 3, 9, 9\}$

$D = \{A, B\}$

$D' = \{ \{1, 3, 9\}, \{1, 3, 9, 27, snow\} \}$

$E = \{ \}$

Describing Sets (2/4)

Some sets…

$A = \{1, 5, 25, 125, \ldots\}$

$B = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$C = \{1, 2, 3, \ldots, 98, 99, 100\}$

(The set of powers of 5, the set of integers, and the set of integers between 1 and 100.)

"…" is an ellipsis

Describing Sets (3/4)

Some sets, using set builder notation:

$A = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N},\ x = 5^y\}$

$B = \{2^i - 1 \mid i \text{ is a prime}\}$

$C = \{n \in \mathbb{Z} \mid 0 < n \le 100\}$

To read, start with "the set of all". Read "|" as "such that".

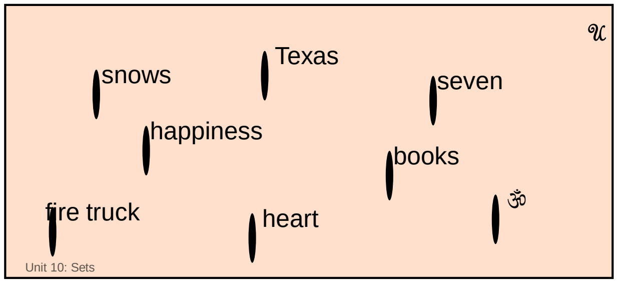
A: "*the set of all* natural numbers x *such that* x is a power of 5"
B: "*the set of all* numbers of the form 2^i-1 *such that* i is a prime"
C: "*the set of all* integers n *such that* $0 < n \le 100$"

Describing Sets (4/4)

Graphical depiction of sets: Venn diagrams.

Draw the set of all five-letter things.

All red things? All red, five-letter things?



$\mathcal{U}$ is the universal set of everything

Describing Sets (4/4)

Graphical depiction of sets: Venn diagrams.

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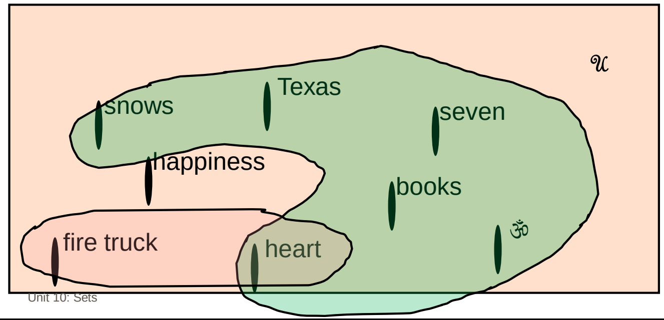


Containment

A set **A** is a subset of a set **B** iff
$\forall x \in \mathcal{U}, x \in A \rightarrow x \in B$.

We write A is a subset of B as $A \subseteq B$.

If $A \subseteq B$, can **B** have elements that are not elements of **A**?

Containment

A set **A** is a subset of a set **B** iff
$\forall x \in \mathcal{U}, x \in A \rightarrow x \in B$.

We write A is a subset of B as $A \subseteq B$.

If $A \subseteq B$, can **B** have elements that are not elements of **A**? Yes, but A can't have elements that are not elements of B.

Membership and Containment

$\mathbf{A} = \{1, \{2\}\}$

Is $1 \in \mathbf{A}$?
Is $\{1\} \subseteq \mathbf{A}$?

Is $1 \subseteq \mathbf{A}$?
Is $\{1\} \in \mathbf{A}$?

Is $2 \in \mathbf{A}$?
Is $\{2\} \subseteq \mathbf{A}$?

Is $2 \subseteq \mathbf{A}$?
Is $\{2\} \in \mathbf{A}$?

Membership and Containment

$\mathbf{A} = \{1, \{2\}\}$

Is $1 \in \mathbf{A}$? Yes
Is $\{1\} \subseteq \mathbf{A}$? Yes

Is $1 \subseteq \mathbf{A}$?
Not meaningful since 1 is not a set.
Is $\{1\} \in \mathbf{A}$? No

Is $2 \in \mathbf{A}$? No
Is $\{2\} \subseteq \mathbf{A}$? No

Is $2 \subseteq \mathbf{A}$?
Not meaningful since 2 is not a set.
Is $\{2\} \in \mathbf{A}$? Yes

Thought Question

What if $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$?

Set Equality

Sets $\mathbf{A}$ and $\mathbf{B}$ are equal (denoted $\mathbf{A} = \mathbf{B}$) if and only if
$\forall \mathbf{x} \in \mathcal{U}, \mathbf{x} \in \mathbf{A} \leftrightarrow \mathbf{x} \in \mathbf{B}$.

Can we prove that that's equivalent to $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$?

Set Equality

Sets $\mathbf{A}$ and $\mathbf{B}$ are equal — denoted $\mathbf{A} = \mathbf{B}$ — if and only if $\forall \mathbf{x} \in \mathcal{U}, \mathbf{x} \in \mathbf{A} \leftrightarrow \mathbf{x} \in \mathbf{B}$.

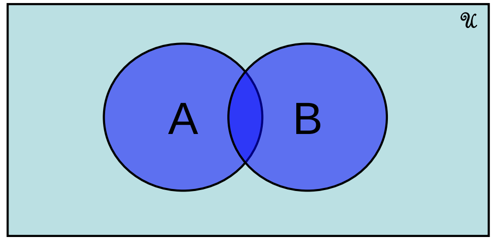
Can we prove that that's equivalent to $\mathbf{A} \subseteq \mathbf{B}$ and $\mathbf{B} \subseteq \mathbf{A}$?

Yes, using a standard predicate logic proof in which we note that $\mathbf{p} \leftrightarrow \mathbf{q}$ is logically equivalent to $\mathbf{p} \rightarrow \mathbf{q} \wedge \mathbf{p} \rightarrow \mathbf{q}$.

Set Union

The union of $\mathbf{A}$ and $\mathbf{B}$ — denoted $\mathbf{A} \cup \mathbf{B}$ — is $\{\mathbf{x} \in \mathcal{U} \mid \mathbf{x} \in \mathbf{A} \vee \mathbf{x} \in \mathbf{B}\}$.

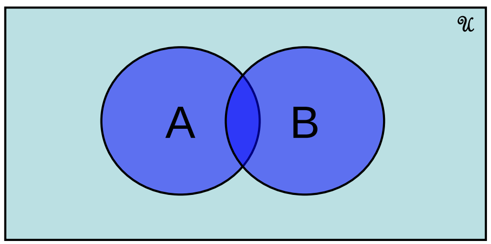
$\mathbf{A} \cup \mathbf{B}$ is the blue region...



Set Intersection

The intersection of $\mathbf{A}$ and $\mathbf{B}$ — denoted $\mathbf{A} \cap \mathbf{B}$ — is $\{\mathbf{x} \in \mathcal{U} \mid \mathbf{x} \in \mathbf{A} \wedge \mathbf{x} \in \mathbf{B}\}$.

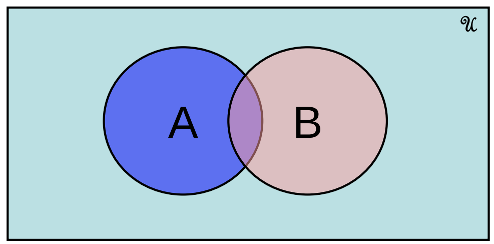
$\mathbf{A} \cap \mathbf{B}$ is the dark blue region...



Set Difference

The difference of $\mathbf{A}$ and $\mathbf{B}$ — denoted $\mathbf{A} - \mathbf{B}$ — is $\{\mathbf{x} \in \mathcal{U} \mid \mathbf{x} \in \mathbf{A} \wedge \mathbf{x} \notin \mathbf{B}\}$.

$\mathbf{A} - \mathbf{B}$ is the pure blue region.

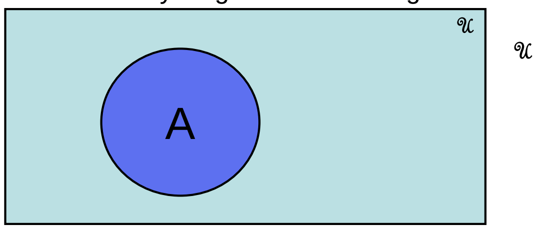


$\mathcal{U}$

Set Complement

The complement of $\mathbf{A}$ — denoted $\overline{\mathbf{A}}$ — is $\{x \in \mathcal{U} \mid x \notin A\}$.

$\overline{\mathbf{A}}$ is everything but the blue region.



Can we express this as a set difference?

Set Operation Equivalencies

Many logical equivalences have analogous set operation identities. Here are a few… read more in the text!

$A \cap B = B \cap A$	Commutative Law
$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$	Distributive Law
$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$	DeMorgan’s Law
$A \cap \mathcal{U} = A$	$\mathcal{U}$ as identity for $\cap$

. . .

PART 2
IN CLASS PAGES

Pre-Class Learning Goals

- By the start of class, you should be able to:
  - Define the set operations union, intersection, complement and difference, and the logical operations subset and set equality in terms of predicate logic and set membership.
  - Translate between sets represented explicitly (possibly using ellipses, e.g., { 4, 6, 8, … }) and using "set builder" notation (e.g., { x in Z^+ | $x^2 > 10$ and x is even }).
  - Execute set operations on sets expressed explicitly, using set builder notation, or a combination of these.
  - Interpret the empty set symbol $\varnothing$, including the fact that the empty set has no members and that it is a subset of any set.

Pre-Class Learning Goals

- By the start of class, you should be able to:
  - Define the terms domain, co-domain, range, image, and pre-image
  - Use appropriate function syntax to relate these terms (e.g., $f : A \to B$ indicates that f is a function mapping domain A to co-domain B).
  - Determine whether $f : A \to B$ is a function given a definition for f as an equation or arrow diagram.

Quiz 10 Feedback

- Generally:
- Issues:

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Define the power set and cartesian product operations in terms of predicate logic and set membership/subset relations.
  - Execute the power set, cartesian product, and cardinality operations on sets expressed through any of the notations discussed so far.
  - Apply your proof skills to proofs involving sets.
  - Relate DFAs to sets.

Outline

- What's the Use of Sets (history & DFAs)
- Cardinality (size)
- Power set (and an induction proof)
- Cartesian products (and application to DFAs)
- Set proofs.

Historical Notes on Sets

- Mathematicians formalized set theory to create a foundation for all of mathematics. Essentially all mathematical constructs can be defined in terms of sets.
- Hence sets are a powerful means of formalizing new ideas.
- But we have to be careful how we use them!

Russell's Paradox

- At the beginning of the 20th century Bertrand Russell discovered inconsistencies with the "naïve" set theory.
  - Russell focused on some special type of sets.
- Let S be the set of all sets that contain themselves:

  $S = \{ x \mid x \in x \}$.

  Does S contain itself?

  A. Yes, definitely.
  B. No, certainly not.
  C. Maybe (either way is fine).
  D. Cannot prove or disprove it.
  E. None of the above.
- So, no problem here.

Russell's Paradox (cont')

- Let R be the set of all sets that do not contain themselves. That is

  $R = \{ x \mid \sim x \in x \}$.
- Does R contain itself?
  A. Yes, definitely.
  B. No, certainly not.
  C. Maybe (either way is fine).
  D. Cannot prove or disprove it.
  E. None of the above.

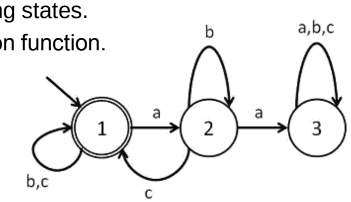
Same question, different form:

"Imagine a barber that shaves every man in town who does not shave himself. Does the barber shave himself?"

- Set theory has been restricted in a way that disallow this kind of sets.

Sets and Functions are Very Useful

- Despite this, sets (and functions) are incredibly useful.
- E.g. We can definite valid DFAs formally: a DFA is a 5-tuple (I, S, s0, F, N) where
  - I is a finite set of characters (input alphabet).
  - S is a finite set of states.
  - s0 $\in$ S is the initial state.
  - F $\subseteq$ S is the set of accepting states.
  - N: S x I $\rightarrow$ S is the transition function.



Example: Testing the DFA Formalism

I the (finite) set of letters in the input language.

S the (finite) set of states.

s_0 the initial state; $s_0 \in S$.

F the set of accepting ("final") states; $F \subseteq S$.

N the next-state function, **to be described**.

Must a DFA have an initial state?

Can a DFA have more than one initial state?

Must a DFA have an accepting state?

Can all states in a DFA be accepting?

Can the initial state be accepting?

a. Yes b. No c. Not enough information.

Testing the DFA Formalism

Must a DFA have an initial state?

Can a DFA have more than one initial state?

Must a DFA have an accepting state?

Can all states in a DFA be accepting?

Can the initial state be accepting?

s_0 is the initial state; so, yes.

s_0 is the only initial state; so, no.

$F \subseteq S$. Is $\varnothing \subseteq S$?

$F \subseteq S$. Is $S \subseteq S$?

$s_0 \in S$ and $F \subseteq S$. Can $s_0 \in F$?

A Note on Set Notation

- There is usually confusion between $\in$ and $\subseteq$.
  - The $\in$ predicate takes in an element x and a set S, and returns true if the element x belongs to S.
  - The $\subseteq$ predicate takes in two sets S and T, and returns true if **every** element of S is also an element of T. That is, if

    $\forall x \in U, x \in S \rightarrow x \in T$

- Note: We use U to denote the Universal set.

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Set Cardinality

- Cardinality: the number of elements of a set S, denoted by |S|.
- What is the cardinality of the following set:

  { 1, 2, 3, { a, b, c }, snow, rain } ?

  A. 3

  B. 6

  C. 8

  D. Some other integer

  E. The cardinality of the set is undefined.

Cardinality Exercises

Given the definitions:

A = {1, 2, 3}

B = {2, 4, 6, 8}

What are:

$\lvert A \rvert$	= 3		$\lvert A - B \rvert$	= 2	
$\lvert B \rvert$	= 4		$\lvert B - A \rvert$	= 3	
$\lvert A \cup B \rvert$	= 6		$\lvert \{\{\}\} \rvert$	= 1	
$\lvert A \cap B \rvert$	= 1		$\lvert \{\varnothing\} \rvert$	= 1	
			$\lvert \{\{\varnothing\}\} \rvert$	= 1	

a. 0 b. 1 c. 2 d. 3 e. None of these

Set Cardinality

- This gets trickier with infinite sets.
- Consider Z^+, Z, Q (the rational numers). Which of these sets have the smaller and larger cardinality?

  A. Smaller: Z^+, larger: Q

  B. Smaller: Z^+ and Z, larger: Q

  C. All three sets have the same cardinality

  D. None of the above.

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Power Sets

- Suppose we have a set S with n students, and we want to form a group to go watch a movie.
  - What are the different groups that can we form?

  OR:
- We have a circuit with n inputs (so S is the set of inputs).
  - What combinations of inputs might be true?
- In both cases, we are looking for all the subsets of S.

  That is, we want to find $\{ T \mid T \subseteq S \}$

Power Sets

- The power set of a set S, denoted P (S), is the set whose elements are all subsets of S.
- Given the definitions

  A = { a, b, f }, B = { b, c },

  which of the following are correct:

  A. P (B) = { {b}, {c}, {b, c} }

  B. P (A - B) = { ∅, {a}, {f}, {a, f} }

  C. |P (A ∩ B)| = 1

  D. |P (A ∪ B)| = 4

  E. None of the above

Cardinality of a *Finite* Power Set

- Theorem :

  If S is a finite set then $|P(S)| = 2^{|S|}$
- We prove this theorem by induction on the cardinality of the set S
- Base case:
  - Base case: |S| = 0. What is S in this case?

Cardinality of a *Finite* Power Set

- Theorem :

  If S is a finite set then $|P(S)| = 2^{|S|}$
- We prove this theorem by induction on the cardinality of the set S
- Base case:
  - Base case: |S| = 0. Then S = ∅ , P(S) = {∅ } and |S| = 1
- Inductive step:
  - Let S be any set with cardinality k > 0.
  - Assume for any set T with |T| < k, $|P(T)| = 2^{|T|}$.
    We'll prove it for S.

Cardinality of a *Finite* Power Set

- Theorem :
  If S is a finite set then $|P(S)| = 2^{|S|}$
- Inductive step (continue):
  - Let x be an arbitrary element of S.
  - Consider $S - \{x\}$. $|S - \{x\}| = k-1$.
    So, $|P(S - \{x\})| = 2^{k-1}$ by the inductive hypothesis.
  - Furthermore $P(S - \{x\})$ is the set of all subsets of S that do not include x.

Cardinality of a *Finite* Power Set

- Theorem :
  If S is a finite set then $|P(S)| = 2^{|S|}$
- Inductive step (continue):
  - However, there are exactly as many subsets of S that include x as do not include x.
  - (Because each subset of S that **does** include x can be matched up with exactly one of the subsets that does not include x that is the same but for x.)
  - So,
    $|P(S)| = 2|P(S - \{x\})|$
    $= 2*2^{k-1} = 2^k = 2^{|S|}$

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- What's the Use of Sets (history & DFAs)
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Tuples

- An ordered tuple (or just tuple) is an ordered collection of elements.
  (An n-tuple is a tuple with n elements.)
- Two tuples are equal when their corresponding elements are equal.
- Example:
  $(a, 1, \varnothing) = (a, 5 - 4, A \cap \overline{A})$
  $(a, b, c) \neq (a, c, b)$

Cartesian Product

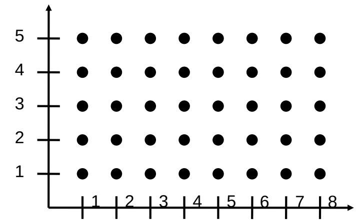
- The cartesian product of two sets S and T, denoted S x T, is the set of all tuples whose first element is drawn from S and whose second element is drawn from **T**
- In other words,

$$S \times T = \{ (s, t) \mid s \in S \wedge t \in T \}.$$

  - Each element of S x T is called a 2-tuple or a pair.

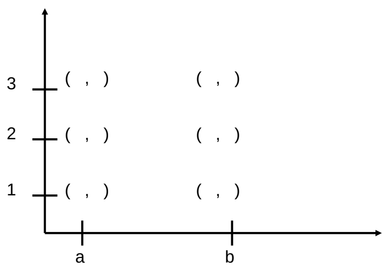
Cartesian Product

- We can visualize $Z^+ \times Z^+$ as follows:



Cartesian Product

- What is `{a,b} × {1,2,3}`:



Example DFA, Arrows : $S \times I \rightarrow S$

For *this* DFA:

I = { a, b }

S = { t, m, b, g}

s_0 = t

F = { t, b }

N: (t, a) → m

(t, b) → b

(m, a) → g

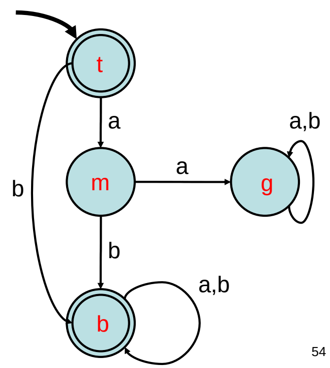
(m, b) → b

(b, a) → b

(b, b) → b

(g, a) → g

(g, b) → g



Outline

- What's the Use of Sets (history & DFAs)
- Cardinality (size)
- Power set (and an induction proof)
- Cartesian products
- Examples of Set proofs.

Example of a proof with Sets

a) **Prove that:** $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

Pick an arbitrary $x \in \overline{A \cap B}$,

Then $x \notin A \cap B$. Def'n of $\overline{}$

$\sim(x \in A \wedge x \in B)$ Def'n of $\cap$

$x \notin A \vee x \notin B$ De Morgan's

$x \in \overline{A} \vee x \in \overline{B}$ Def'n of $\overline{}$

$x \in (\overline{A} \cup \overline{B})$ Def'n of $\cup$

Example of a proof with Sets

b) **Prove that:** $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Pick an arbitrary $x \in \overline{A} \cup \overline{B}$

Then,

$x \in \overline{A} \vee x \in \overline{B}$

$x \notin A \vee x \notin B$

$\sim(x \in A \wedge x \in B)$

$x \notin A \cap B$

$x \in \overline{A \cap B}$