CPSC 121: Models of Computation

Unit 9a

Mathematical Induction – Part 1

Based on slides by Patrice Belleville and Steve Wolfman

Pre-Class Learning Goals

- By the start of class, you should be able to
  - Convert sequences to and from explicit formulas that describe the sequence.
  - Convert sums to and from summation/Σ notation.
  - Convert products to and from product/Π notation.
  - Manipulate formulas in summation/product notation by adjusting their bounds, merging or splitting summations/products, and factoring out values.
  - Given a theorem to prove *and the insight into how to break the problem down in terms of smaller problems*, write out the skeleton of an inductive: the base case(s),the induction hypothesis, and the inductive step

Quiz 9 Feedback

- Generally:
- Issues:
- Essay Question:
  - As usual, we will revisit the open-ended question shortly.

In-Class Learning Goals

- By the end of this unit, you should be able to:
  - Formally prove properties of the non-negative integers (or a subset like integers that have appropriate self-referential structure) —including both equalities and inequalities—using either weak or strong induction as needed.
  - Critique formal inductive proofs to determine whether they are valid and where the error(s) lie if they are invalid.

? Addressing the Course Big Questions ?

- CPSC 121: the BIG questions:
  - How can we convince ourselves that an algorithm does what it's supposed to do?
  - How do we determine whether or not one algorithm is better than another one?
- Mathematical induction is a very useful tool when proving the correctness or efficiency of an algorithm.
- We will see several examples of this.

Outline

- **Introduction and Discussion**
  - **Example: single-elimination tournaments**.
  - Example: max swaps for sorting n items
- A Pattern for Induction
- Induction on Numbers

Example: Single-Elimination Tournament

- Problem: single-elimination tournament
  - Teams play one another in pairs
  - The winner of each pair advances to the next round
  - The tournament ends when only one team remains.

How do we start?

- Let's try some examples with small numbers

Example (cont`)

- What is the maximum number of teams in a ***0*-round** single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***1*-round** single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***2*-round** single-elimination tournament ?

  A. 0 teams

  B. 1 team

  C. 2 teams

  D. 3 teams

  E. None of the above.

Example (cont`)

- What is the maximum number of teams in a ***n*-round** single-elimination tournament ?

  A. n

  B. $2n$

  C. n^2

  D. 2^n

  E. None of the above.

How can we prove it?

- How can we prove it for every n ?
  - We will use a technique called mathematical induction.
- We show some basic cases first (for 0,1,2)
- Then we show that if the statement is true for case n-1 then it is true for case n

- Basic Cases how many we need?):
  - n= 0
  - n = 1
  - …

Case for n-1 → Case for n

- If at most 2^{n-1} teams can participate in a tournament with **n-1** rounds, then at most 2^n teams can participate in a tournament with **n** rounds?
- If we want to *prove* this statement, which of the following techniques might we use?

A. Antecedent assumption

B. Witness proof

C. WLOG

D. Proof by cases

E. None of the above.

Working out the proof:

- Proof (with holes):
  - Consider an unspecified tournament with **n-1** rounds. Assume that
  - How many teams we need to have a tournament with **n** rounds?
  - We can think of a tournament with **n** rounds as follows:
    - Two tournaments with n-1 rounds proceed in parallel.
    - The two winners then
  - Since each tournament with n-1 rounds has

Completing the proof:

- **Theorem**: if at most 2^{n-1} teams can play in an $(n-1)$-round tournament, then at most 2^n teams can play in an n-round tournament.
- **Proof:**
  - Assume at most 2^{n-1} teams can play in an $(n-1)$-round tournament.
  - An n-round tournament is two $(n-1)$-round tournaments where the winners play each other (since there must be a single champion).
  - By assumption, each of these may have at most 2^{n-1} teams. So, the overall tournament has at most $2 * 2^{n-1} = 2^n$ teams. QED!

Are We Done?

Here's the logical structure of our original theorem:
$\forall n \in \mathbf{N}$, $\mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Does that prove $\forall n \in \mathbf{N}$, $\mathbf{Max}(n, 2^n)$?

a. Yes.
b. No.
c. I don't know.

What More Do We Need?

We need to adjust it to
$\forall n \in \mathbf{N}$, $(n > 0) \wedge \mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Why doesn't this work for 0?

What do we do about the base case of our data definition?

Completing (?) the Proof (again)

Base Case Theorem: At most one team can play in a 0-round tournament.

Proof:

Every tournament must have one unique winner. A zero-round tournament has no games; so, it can only include one team: the winner. QED!

Now Are We Done?

Here's the logical structure of our theorems:
(1) $\mathbf{Max}(0, 1)$.
(2) $\forall n \in \mathbf{Z}^0$, $(n > 0) \wedge \mathbf{Max}(n-1, 2^{n-1}) \rightarrow \mathbf{Max}(n, 2^n)$.

Do these prove $\forall n \in \mathbf{Z}^0$, $\mathbf{Max}(n, 2^n)$?

a. Yes.
b. No.
c. I don't know.

One Extra Step We'll Do

Really, we are done.

But just to be thorough, we'll add:

Termination: *n* is a non-negative integer, and each application of the inductive step reduces it by 1. Therefore, it must reach our base case (0) in a finite number of steps.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbf{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Do these prove **Max**$(1,2^1)$?

a. Yes.

b. No.

c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbf{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Plus, we know **Max**$(1,2^1)$.

Do all of these prove **Max**$(2,2^2)$?

a. Yes.

b. No.

c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

Max$(0,2^0)$.

$\forall n \in \mathbf{Z}^0$, $(n > 0) \wedge$ **Max**$(n-1, 2^{n-1}) \rightarrow$ **Max**$(n, 2^n)$.

Plus, we know **Max**$(1,2^1)$ and **Max**$(2,2^2)$.

Do all of these prove **Max**$(3,2^3)$?

a. Yes.

b. No.

c. I don't know.

Step-by-Step?

Here's the logical structure of our theorems:

`Max(0,2⁰).` — $\mathtt{Max}(0, 2^0)$.

$\forall n \in \mathbb{Z}^0,\ (n > 0) \land \mathtt{Max}(n-1, 2^{n-1}) \rightarrow \mathtt{Max}(n, 2^n)$.

From this, can we prove $\mathtt{Max}(n, 2^n)$ for any particular integer n?

a. Yes.

b. No.

c. I don't know.

Tournament Proof Summary

Theorem: At most 2^n teams play in an n-round tournament.

Proof: We proceed by induction.

Base Case: A zero-round tournament has no games and so can only include one (that is, 2^0) team: the winner. So, at most 2^0 teams play in a 0-round tournament. ✓

Induction Hypothesis: WLOG, for an arbitrary integer n > 0, assume at most 2^{n-1} teams play in an (n-1)-round tournament.

Inductive Step: We'll show it is true for n. An n-round tournament is two (n-1)-round tournaments where the winners play each other. By the IH, each of these has at most 2^{n-1} teams. So, the overall tournament has at most $2 * 2^{n-1} = 2^n$ teams. ✓

[Termination: n is a non-negative integer, and each application of the inductive step reduces it by 1. Therefore, it must reach our base case (0) in a finite number of steps.]✓

QED

Outline

- **Introduction and Discussion**
  - **Example: single-elimination tournaments**.
  - **Example: max swaps for sorting n items**
- A Pattern for Induction
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Example 2: Sorting n items

- How many swaps do we need to sort n items?
  - Suppose we place items from left to right.
    - The items already placed are ordered.
    - We swap each new item with its neighbour until it is at the right place.
  - The i-th item may be swapped with all previous i-1 items.
  - So the total number of swaps is

$$\sum_1^n (i-1) = \sum_0^{n-1} j = \frac{n(n-1)}{2}$$

  - Hence we need to prove that $\sum_0^n i = \frac{n(n+1)}{2}$

Example 2: Sorting n items

- Which facts do we need to prove?

  A. $\sum_0^0 i = 0$

  B. For every $n \geq 0$ if $\sum_0^{n-1} i = \frac{(n-1)n}{2}$, then $\sum_0^n i = \frac{n(n+1)}{2}$

  C. For every $n > 0$ if $\sum_0^{n-1} i = \frac{(n-1)n}{2}$, then $\sum_0^n i = \frac{n(n+1)}{2}$

  D. Both (a) and (c)

  E. None of the above.

Example 1: Sorting n items

- Proof:
  - ***Base case***: n = 0
    - Clearly : $\sum_0^0 i = 0 = \frac{n(n+1)}{2}$
  - ***Induction step***:
    - Pick an unspecified n > 0. Assume that (***inductive hypothesis)***: $\sum_0^{n-1} i = \frac{(n-1)n}{2}$
    - Then
      - $\sum_0^n i = (\sum_0^{n-1} i) + n$
      - $= \frac{(n-1)n}{2} + n$ (by the inductive hypothesis)
      - $= \frac{2n + (n-1)n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$
  - Hence by the principle of M.I., the theorem holds. QED

Outline

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  - Example: max swaps for sorting n items
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An Induction Proof Pattern

Type of Problem: Prove some property of a structure that is naturally defined in terms of itself.

Part 1: Insight: how does the problem “break down” in terms of smaller pieces? Induction doesn’t help you with this part. It is **not** a technique to figure out patterns, only to prove them.

Part 2: Proof. Establish that the property is true for your base case(s). Establish that it is true at each step of construction of a more complex structure. Establish that you could create a finite proof out of these steps for any value of interest (*termination*).

A Pattern For Induction

Theorem: At most 2^n teams play in an n-round tournament.

Proof: We proceed by induction.

Base Case(s): A zero-round tournament has no games and so can only include one (that is, 2^0) team: the winner. So, at most 2^0 teams play in a 0-round tournament.

Induction Hypothesis: WLOG, for an arbitrary integer n > 0, assume at most 2^{n-1} teams play in an (n-1)-round tournament.

Inductive Step: An n-round tournament is two (n-1)-round tournaments where the winners play each other. By the IH, each of these has at most 2^{n-1} teams. So, the overall tournament has at most $2 \cdot 2^{n-1} = 2^n$ teams.

[Check Termination: n is a non-negative integer, and each application of the inductive step reduces it by 1. Therefore, it must reach our base case (0) in a finite number of steps.]

QED

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A Pattern For Induction

Theorem: _P(n)_ is true for all n ≥ _some smallest number_.

Proof: We proceed by induction on n.

Base Case(s) (_P(?)_ is true for a few simple cases)**:** Prove with our other techniques. Which cases? Almost certainly the _some smallest number_. Maybe more. Make notes on which you need, as with a witness proof!

Induction Hypothesis: For an arbitrary n > _the largest of your base cases_: assume P(?) is true.

Inductive Step (Prove that _P(n)_ is true):

WLOG, let n be greater than _the largest of your base cases_.

Assume _P(?)_ is true for _whichever cases you need_.

Break _P(n)_ down in terms of the smaller case(s). The smaller cases are true, by assumption. Build back up to show that: _P(n)_ is true.

[Check Termination: n is a ___________, and each application of the inductive step ___________. Therefore, it must reach (one of) our base case(s) in a finite number of steps (without "jumping past")].

QED

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A Pattern For Induction

P(n) is ______________.

Theorem: P(n) is true for all n ≥ _______.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for _______)**:**
Prove each base case via your other techniques.

Induction Hypothesis:
For an arbitrary n > _______, assume P(·) is true for ___________.

Inductive Step:

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by the IH.

Build back up to show that P(n) is true.

[Termination: n is a ___________, and each application of the inductive step ___________. Therefore, it must reach (one of) our base case(s) in a finite number of steps (without "jumping past")].

QED

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A Pattern For Induction

P(n) is true for all n ≥ _______

Which base cases? Almost certainly the smallest n. Otherwise, you don't know yet. Do the *rest* of the proof now. Come back to the base case(s) when you know which one(s) you need!

assume P(·) is true for ___________.

Which values are we going to assume P(·) is true for? Whichever we need. How do we know the ones we need? We don't, yet. So… do the *rest* of the proof now. Come back to the assumption when you know which one(s) you need!

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A Pattern For Induction

Build back up to show that P(n) is true

What must n be larger than? The largest of your base cases. (Why? So you don't assume the theorem true for something that's too small, like a *negative one* round tournament.) But, you don't know all your base cases yet. So…do the *rest* of the proof now. Come back to this bound once you know your base case(s).

Break P(n) down in terms of the smaller case(s)

How do we break the problem down in terms of smaller cases? *THIS* is the real core of every induction problem. Figure this out, and you're ready to fill in the rest of the blanks!

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

This is the most common style of insight, and the same as we had in our "# teams in a tournament" question.

(It's called "weak induction".)

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ _______.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for _______**):**
Prove each base case via your other techniques.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **1**)**:**
Prove each base case via your other techniques. We only need n=1 because n=2 works based on n=1, and all subsequent cases also eventually break down into the n=1 case.

Inductive Step (for n > _______, if P(·) is true for ____________, then P(n) is true):

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **1**)**:**
Prove each base case via your other techniques.

Inductive Step (for n > **1**: if P(·) is true for **n-1**, then P(n) is true):

WLOG, let n be greater than ____________.

Assume P(·) is true for __________________.

Examples: Breaking down into a problem one smaller

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **1**)**:**
Prove each base case via your other techniques.

Inductive Step (for n > **1**: if P(·) is true for **n-1**, then P(n) is true):

WLOG, let n be greater than **1**.

Assume P(·) is true for **n-1**.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Rephrased a bit (to get rid of P(·))

You want to prove P(n) for all n ≥ 1. You know that P(n) is true if P(n-1) is true. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **1**.

Proof: We proceed by induction on n.

Base Case(s) (P(1) is true)**:**
Prove each base case via your other techniques.

Inductive Step (for n > **1**: if P(·) is true for **n-1**, then P(n) is true):

WLOG, let n be greater than **1**.

Assume P(n-1) is true.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

This is the second most common style of insight, and the same as we had in our "prove the height algorithm correct" question.

(It's called "strong induction"; technically, we did "structural induction".)

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ _______.

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for _______**):**
Prove each base case via your other techniques.

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **22**)**:**
Prove each base case via your other techniques. For n=23, we just need n=22. For n=24, we just need n=22 and n=23… and n=23 breaks down in terms of n=22, and so on.

Inductive Step (for n > _______, if P(·) is true for ____________, then P(n) is true):

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **22**):
Prove each base case via your other techniques.

Inductive Step (for n > **22**: if P(·) is true for **every integer from 22 up to n-1**, then P(n) is true):

WLOG, let n be greater than ____________.

Assume P(·) is true for __________________.

Examples: Breaking down into all smaller problems

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **22**):
Prove each base case via your other techniques.

Inductive Step (for n > **22**: if P(·) is true for **every integer from 22 up to n-1**, then P(n) is true):

WLOG, let n be greater than **22**.

Assume P(·) is true for **every integer from 22 up to n-1**.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Rephrased a bit (to be more predicate logic-y)

You want to prove P(n) for all n ≥ 22. You know that P(n) is true if P(·) is true for every integer from 22 up to n-1. How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **22**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for **22**):
Prove each base case via your other techniques.

Inductive Step (for n > **22**: if P(·) is true for **every integer from 22 up to n-1**, then P(n) is true):

WLOG, let n be greater than **22**.

Assume **for all integers i where 22 ≤ i < n, P(i) is true**.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P($\lfloor n/2 \rfloor$) and P($\lceil n/2 \rceil$) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

But, your insight may come in *any* form.
Maybe you need problems half as large or one-third.
Maybe you need problems that are 7 smaller.
Maybe you need the problems that are 1, 2, and 3 smaller.

Regardless, the pattern is the same!

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ _______.

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for _______**):**
Prove each base case via your other techniques.

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for n = **7, 8, 9, 10, 11, 12, 13):**
Prove each base case via your other techniques. (We need all the way up to 13 because only at 14/2 do we reach a base case. From 15 on, we always eventually hit a base case.)

Inductive Step (for n > _______, if P(·) is true for ____________, then P(n) is true):

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P(⌊n/2⌋) and P(⌈n/2⌉) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for n = **7, 8, 9, 10, 11, 12, 13**)**:**
Prove each base case via your other techniques.

Inductive Step (for n > **13**: if P(·) is true for **⌊n/2⌋ and ⌈n/2⌉**, then P(n) is true):

WLOG, let n be greater than ____________.

Assume P(·) is true for __________________.

Examples: breaking down into a problem half as big

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P($\lfloor n/2 \rfloor$) and P($\lceil n/2 \rceil$) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (P(·) is true for n = **7, 8, 9, 10, 11, 12, 13**): Prove each base case via your other techniques.

Inductive Step (for n > **13**: if P(·) is true for $\lfloor$**n/2**$\rfloor$ **and** $\lceil$**n/2**$\rceil$, then P(n) is true):

WLOG, let n be greater than **13**.

Assume P(·) is true for $\lfloor$**n/2**$\rfloor$ **and** $\lceil$**n/2**$\rceil$.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED

Rephrased slightly (to get rid of P(·))

You want to prove P(n) for all n ≥ 7. You know that P(n) is true if P($\lfloor n/2 \rfloor$) and P($\lceil n/2 \rceil$) are both true (i.e., P(·) is true for n/2 rounded down and n/2 rounded up). How do we fill in the blanks?

Theorem: P(n) is true for all n ≥ **7**.

Proof: We proceed by induction on n.

Base Case(s) (**P(7), P(8), P(9), P(10), P(11), P(12), P(13)**): Prove each base case via your other techniques.

Inductive Step (for n > **13**: if P(·) is true for $\lfloor$**n/2**$\rfloor$ **and** $\lceil$**n/2**$\rceil$, then P(n) is true):

WLOG, let n be greater than **13**.

Assume **P($\lfloor$ n/2$\rfloor$) and P($\lceil$n/2$\rceil$)** are both true.

Break P(n) down in terms of the smaller case(s).

The smaller cases are true, by assumption.

Build back up to show that P(n) is true.

This completes our induction proof. QED