Chapter 6: Set Theory

The first section of this chapter introduces additional terminology for sets and the concept of an element argument to prove that one set is a subset of another. The aim of this section is to provide a experience with a variety of types of sets and a basis for deriving the set properties discussed in the remainder of the chapter. The second and third sections show how to prove and disprove various proposed set properties of union, intersection, set difference and (general) complement using element arguments, algebraic arguments, and counterexamples. Section 5.4 introduces the concept of Boolean algebra, which generalizes both the algebra of sets with the operations of union and intersection and the properties of a set of statements with the operations of *or* and *and*. The section goes on to discuss Russell's paradox and shows that reasoning similar to Russell's can be used to prove an important property of computer algorithms.

Section 6.1

3. *c.* Yes. Every element in T is in S because every integer that is divisible by 6 is also divisible by 3. To see why this is so, suppose n is any integer that is divisible by 6. Then $n = 6m$ for some integer m. Since $6m = 3(2m)$ and since $2m$ is an integer (being a product of integers), it follows that $n = 3\cdot$ (some integer), and, hence, that n is divisible by 3.

6. *a.* $A \not\subseteq B$ because $2 \in A$ (because $2 = 5\cdot 0 + 2$) but $2 \notin B$ (because if $2 = 10b - 3$ for some integer b, then $10b = 5$, so $b = 1/2$, which is not an integer).

   b. $B \subseteq A$

   Proof:

   Suppose y is a particular but arbitrarily chosen element of B.

   [We must show that y is in A. By definition of A, this means that we must show that y = 5·(some integer) + 2.]

   By definition of B, $y = 10b - 3$ for some integer b.

   [Scratch work: Is there an integer, say a, such that $y = 5a+2$? If so, then $5a+2 = 10b-3$, which implies that $5a = 10b - 5$, or, equivalently, that $a = 2b - 1$. So give this value to a and see if it works.]

   Let $a = 2b - 1$. Then a is an integer and $5a + 2 = 5(2b - 1) + 2 = 10b - 5 + 2 = 10b - 3 = y$. Thus y is in A *[as was to be shown]*.

   c. $B = C$

   Proof:

   Part 1, Proof That $B \subseteq C$:

   Suppose y is a particular but arbitrarily chosen element of B.

   [We must show that y is in C. By definition of C, this means that we must show that y = 10·(some integer) + 7.]

   By definition of B, $y = 10b - 3$ for some integer b.

   [Scratch work: Is there an integer, say c, such that $y = 10c + 7$? If so, then $10c + 7 = 10b - 3$, which implies that $10c = 10b - 10$, or, equivalently, that $c = b - 1$. So give this value to c and see if it works.]

   Let $c = b - 1$. Then c is an integer and $10c + 7 = 10(b - 1) + 7 = 10b - 10 + 7 = 10b - 3 = y$. Thus y is in C *[as was to be shown]*.

Part 2, Proof That $C \subseteq B$:

Suppose z is a particular but arbitrarily chosen element of C.

[We must show that z is in B. By definition of B, this means that we must show that z = $10\cdot$(some integer) $- 3$.]

By definition of C, $z = 10c + 7$ for some integer c.

[Scratch work: Is there an integer, say b, such that $z = 10b - 3$? If so, then $10b - 3 = 10c + 7$, which implies that $10b = 10c + 10$, or, equivalently, that $b = c + 1$. So give this value to b and see if it works.]

9. *b.* $x \notin A$ or $x \notin B$ *c.* $x \notin A$ or $x \in B$

12.

a. $A \cup B = \{x \in \mathbf{R} \mid -3 \leq x < 2\}$ *b.* $A \cap B = \{x \in \mathbf{R} \mid -1 < x \leq 0\}$

c. $A^c = \{x \in \mathbf{R} \mid x < -3 \text{ or } x > 0\}$ *d.* $A \cup C = \{x \in \mathbf{R} \mid -3 \leq x \leq 0 \text{ or } 6 < x \leq 8\}$

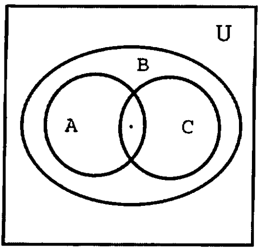
e. $A \cap C = \emptyset$ *f.* $B^c = \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x \geq 2\}$

g. $A^c \cap B^c = \{x \in \mathbf{R} \mid x < -3 \text{ or } x \geq 2\}$ *h.* $A^c \cup B^c = \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\}$

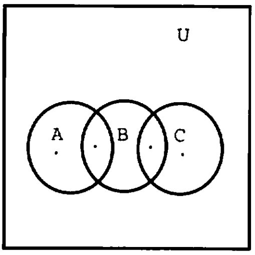
i. $(A \cap B)^c = \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\}$ *j.* $(A \cup B)^c = \{x \in \mathbf{R} \mid x < -3 \text{ or } x \geq 2\}$

Note that $(A \cap B)^c = A^c \cup B^c$ and that $(A \cup B)^c = A^c \cap B^c$.

15. *b.*



c.



18. *c.* Yes, because $\{\emptyset\}$ is the set that contains the one element $\emptyset$.

d. No, because $\emptyset$ has no elements and thus it cannot contain the element $\emptyset$.

27. *b.* Yes. Every element in $\{p, q, u, v, w, x, y, z\}$ is in one of the sets of the partition and no element is in more than one set of the partition.

c. No. The number 4 is in both sets $\{5, 4\}$ and $\{1, 3, 4\}$.

e. Yes. Every element in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is in one of the sets of the partition and no element is in more than one set of the partition.

30. Yes. By the quotient-remainder theorem, every integer can be represented in exactly one of the following forms: $4k$ or $4k+1$ or $4k+2$ or $4k+3$ for some integer k. Thus $\mathbf{Z} = A_0 \cup A_1 \cup A_2 \cup A_3$, $A_0 \cap A_1 = \emptyset$, $A_0 \cap A_2 = \emptyset$, $A_0 \cap A_3 = \emptyset$, $A_1 \cap A_2 = \emptyset$, $A_1 \cap A_3 = \emptyset$, and $A_2 \cap A_3 = \emptyset$.

33. *a.* $\mathscr{P}(\emptyset) = \{\emptyset\}$

c. $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Section 6.2

6. (2) **a.** $x \in A \cap (B \cup C)$ **b.** or **c.** and **d.** $A \cap (B \cup C)$ **e.** by definition of intersection, $x \in A$ and $x \in C$. Since $x \in C$, then, by definition of union, $x \in B \cup C$. Hence $x \in A$ and $x \in B \cup C$, and so, by definition of intersection, $x \in A \cap (B \cup C)$.

   (3) **a.** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

9. The proof that $(A - B) \cup (C - B) \subseteq (A \cup C) - B$ is in Appendix B.

   Proof that $\boldsymbol{(A \cup C) - B \subseteq (A - B) \cup (C - B)}$:

   Suppose that x is any element in $(A \cup C) - B$. *[We must show that $x \in (A - B) \cup (C - B)$.]*

   By definition of set difference, $x \in (A \cup C)$ and $x \notin B$.

   And, by definition of union, $x \in A$ or $x \in C$, and in both cases, $x \notin B$.

   Case 1 ($x \in A$ and $x \notin B$): Then, by definition of set difference, $x \in A - B$, and so by definition of union, $x \in (A - B) \cup (C - B)$.

   Case 2 ($x \in C$ and $x \notin B$): Then, by definition of set difference, $x \in C - B$, and so by definition of union, $x \in (A - B) \cup (C - B)$.

   In both cases, $x \in (A - B) \cup (C - B)$ *[as was to be shown].*

   So $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

   Because both subset containments have now been proved (one here and the other in Appendix B), we conclude that $(A - B) \cup (C - B) = (A \cup C) - B$.

15. Proof:

   Suppose A and B are sets and $A \subseteq B$. Let $x \in B^c$. *[We must show that $x \in A^c$.]*

   By definition of complement, $x \notin B$.

   It follows that $x \notin A$ *[because if $x \in A$ then $x \in B$ (since $A \subseteq B$), and this would contradict the fact that $x \notin B$].*

   Hence by definition of complement $x \in A^c$.

   [Thus $B^c \subseteq A^c$ by definition of subset.]

21. The "proof" claims that because $x \notin A$ or $x \notin B$, it follows that $x \notin A \cup B$. But it is possible for "$x \notin A$ or $x \notin B$" to be true and "$x \notin A \cup B$" to be false. For example, let $A = \{1, 2\}$, $B = \{2, 3\}$, and $x = 3$. Then since $3 \notin \{1, 2\}$, the statement "$x \notin A$ or $x \notin B$" is true. But since $A \cup B = \{1, 2, 3\}$ and $3 \in \{1, 2, 3\}$, the statement "$x \notin A \cup B$" is false.

36. *a.* Proof:

   Let A and B be any sets. *[We will show that $[(A - B) \cup (B - A) \cup (A \cap B] \subseteq A \cup B$ and that $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$.]*

   Proof that $\boldsymbol{(A - B) \cup (B - A) \cup (A \cap B \subseteq A \cup B)}$:

   Suppose $x \in (A - B) \cup (B - A) \cup (A \cap B)$. *[We must show that $x \in A \cup B$.]*

   By definition of union, $x \in A - B$ or $x \in B - A$ or $x \in A \cap B$.

   Case 1 ($x \in A - B$): In this case, by definition of set difference, $x \in A$ and $x \notin B$. In particular, $x \in A$. Then, by definition of union, $x \in A \cup B$.

   Case 2 ($x \in B - A$): In this case, by definition of set difference, $x \in B$ and $x \notin A$. In particular, $x \in B$. Then, by definition of union, $x \in A \cup B$.

   Case 3 ($x \in A \cap B$): In this case, by definition of intersection, $x \in A$ and $x \in B$. Then, by definition of union, $x \in A \cup B$.

   In all three cases, $x \in A \cup B$ *[as was to be shown].*

***Proof that* $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$:**

Suppose $x \in A \cup B$. *[We must show that $x \in (A - B) \cup (B - A) \cup (A \cap B)$.]*

By definition of union, $x \in A$ or $x \in B$.

***Case 1* ($x \in A$)**: In this case, either $x \in B$ or $x \notin B$. If $x \in B$, then, since x is also in A, $x \in A \cap B$ by definition of intersection. It follows by definition of union that $x \in (A - B) \cup (B - A) \cup (A \cap B)$. If $x \notin B$, then, since x is also in A, $x \in A - B$ by definition of set difference. It follows by definition of union that $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

***Case 2* ($x \in B$)**: In this case, either $x \in A$ or $x \notin A$. If $x \in A$, then, since x is also in B, $x \in A \cap B$ by definition of intersection. It follows by definition of union that $x \in (A - B) \cup (B - A) \cup (A \cap B)$. If $x \notin A$, then, since x is also in B, $x \in B - A$ by definition of set difference. It follows by definition of union that $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

In both cases, $x \in (A - B) \cup (B - A) \cup (A \cap B)$ *[as was to be shown]*.

[Since both subset containments have been proved, $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$ by definition of set equality.]

39. Proof:

Let $A_1, A_2, \ldots, A_n$ and B be any sets. *[We will show that $\bigcap_{i=1}^{n}(A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B)$.]*

***Proof that* $\bigcap_{i=1}^{n}(A_i - B) \subseteq \left(\bigcap_{i=1}^{n} A_i\right) - B$:**

Suppose $x \in \bigcap_{i=1}^{n}(A_i - B)$. *[Show that $x \in \left(\bigcap_{i=1}^{n} A_i\right) - B$.]*

By definition of general intersection, $x \in A_i - B$ for all integers $i = 1, 2, \ldots, n$.

And by definition of set difference, $x \in A_i$ and $x \notin B$ for all integers $i = 1, 2, \ldots, n$.

It follows by definition of general intersection that $x \in \left(\bigcap_{i=1}^{n} A_i\right)$, and it is also the case that $x \notin B$.

Hence $x \in \left(\bigcap_{i=1}^{n} A_i\right) - B$ by definition of set difference *[as was to be shown]*.

***Proof that* $\left(\bigcap_{i=1}^{n} A_i\right) - B \subseteq \bigcap_{i=1}^{n}(A_i - B)$:**

Suppose $x \in \left(\bigcap_{i=1}^{n} A_i\right) - B$. *[Show that $x \in \bigcap_{i=1}^{n}(A_i - B)$.]*

By definition of set difference, $x \in \left(\bigcap_{i=1}^{n} A_i\right)$ and $x \notin B$.

And by definition of general intersection, $x \in A_i$ for all integers $i = 1, 2, \ldots, n$, and it is also the case that $x \notin B$.

Hence $x \in (A_i - B)$ for all integers $i = 1, 2, \ldots, n$.

So $x \in \bigcap_{i=1}^{n}(A_i - B)$ by definition of general intersection *[as was to be shown]*.

[Since both subset containments have been proved, $\bigcap_{i=1}^{n}(A_i - B) = \left(\bigcap_{i=1}^{n} A_i\right) - B$ by definition of set equality.]

Section 6.3

12. True. Proof: Let A, B, and C be any sets. *[We must show that $A \cap (B-C) = (A \cap B)-(A \cap C)$.]*

    Proof that $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$:

    Suppose $x \in A \cap (B - C)$. *[We must show that $x \in (A \cap B) - (A \cap C)$.]*

    By definition of intersection, $x \in A$ and $x \in (B - C)$, and so

    $x \in A$ and, by definition of set difference, $x \in B$ and $x \notin C$.

    Now if x were in $A \cap C$, then x would be in C, which it is not.

    Thus $x \notin A \cap C$, and so $x \in A \cap B$ and $x \notin A \cap C$.

    Hence $x \in (A \cap B) - (A \cap C)$ by definition of set difference *[as was to be shown]*.

    [Therefore,$A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.]

    Proof that $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$:

    Suppose $x \in (A \cap B) - (A \cap C)$. *[We must show that $x \in A \cap (B - C)$.]*

    By definition of set difference, $x \in A \cap B$ and $x \notin A \cap C$, and so,

    by definition of intersection, $x \in A$ and $x \in B$, and also $x \notin A \cap C$.

    Now if x were in C then x would be in both A and C, and so x would be in $A \cap C$ which it is not.

    Thus $x \in A$ and $x \in B$ and $x \notin C$, and hence

    $x \in A$ and $x \in B - C$ by definition of set difference.

    Finally, by definition of intersection, $x \in A \cap (B - C)$ *[as was to be shown]*.

    [Therefore, $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$.]

    [Since both subset containments have been proved, $A \cap (B-C) = (A \cap B)-(A \cap C)$ by definition of set equality.]

15. True. Proof: Let A, B, and C be any sets such that $A \cap C = B \cap C$ and $A \cup C = B \cup C$. *[We must show that $A = B$.]*

    Proof that $A \subseteq B$

    Suppose $x \in A$. *[We must show that $x \in B$.]*

    Either $x \in C$ or $x \notin C$.

    Case 1 ($x \in C$): In this case, $x \in A$ and $x \in C$, and so, by definition of intersection, $x \in A \cap C$. But $A \cap C = B \cap C$ by hypothesis, and hence $x \in B \cap C$ by definition of subset. Thus, in particular, $x \in B$ *[as was to be shown]*.

    Case 2 ($x \notin C$): Since $x \in A$, by definition of union, $x \in A \cup C$. Now, by hypothesis, $A \cup C = B \cup C$. So $x \in B \cup C$, and, by definition of union, $x \in B$ or $x \in C$. But in this case $x \notin C$, and so $x \in B$ *[as was to be shown]*.

    [Therefore, $A \subseteq B$ by definition of subset.]

    Proof that $B \subseteq A$

    Suppose $x \in B$. *[We must show that $x \in A$.]*

    Either $x \in C$ or $x \notin C$.

    Case 1 ($x \in C$): In this case, $x \in B$ and $x \in C$, and so, by definition of intersection, $x \in B \cap C$. But $B \cap C = A \cap C$ by hypothesis, and hence $x \in A \cap C$ by definition of subset. Thus, in particular, $x \in A$ *[as was to be shown]*.

    Case 2 ($x \notin C$): Since $x \in B$, by definition of union, $x \in B \cup C$. Now, by hypothesis, $B \cup C = A \cup C$. So $x \in A \cup C$, and, by definition of union, $x \in A$ or $x \in C$. But in this case $x \notin C$, and so $x \in A$ *[as was to be shown]*.

[Therefore, $B \subseteq A$ by definition of subset.]

[Since both subset containments have been proved, $A = B$ by definition of set equality.]

21. False. The elements of $\mathscr{P}(A \times B)$ are subsets of $A \times B$, whereas the elements of $\mathscr{P}(A) \times \mathscr{P}(B)$ are ordered pairs whose first element is a subset of A and whose second element is a subset of B.

    Counterexample: Let $A = B = \{1\}$.

    Then $\mathscr{P}(A) = \{\emptyset, \{1\}\}$, $\mathscr{P}(B) = \{\emptyset, \{1\}\}$, and $\mathscr{P}(A) \times \mathscr{P}(B) = \{(\emptyset, \emptyset), (\emptyset, \{1\}), (\{1\}, \emptyset), (\{1\}, \{1\})\}$.

    On the other hand, $A \times B = \{(1,1)\}$, and so $\mathscr{P}(A \times B) = \{\emptyset, \{(1,1)\}\}$.

    By inspection $\mathscr{P}(A) \times \mathscr{P}(B) \neq \mathscr{P}(A \times B)$.

24. No. The sets S_a, S_b, S_c, and $S_\emptyset$ do not form a partition of $\mathscr{P}(S)$ because they are not mutually disjoint. For example, $\{a, b\} \in S_a$ and $\{a, b\} \in S_b$.

33. Proof: Let A and B be any sets. Then

$$\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B^c) \cap (A \cap B) && \text{by the set difference law} \\
&= A \cap [B^c \cap (A \cap B)] && \text{by the associative law for } \cap \\
&= A \cap [(A \cap B) \cap B^c] && \text{by the commutative law for } \cap \\
&= A \cap [A \cap (B \cap B^c)] && \text{by the associative law for } \cap \\
&= A \cap [A \cap \emptyset] && \text{by the complement law for } \cap \\
&= A \cap \emptyset && \text{by the identity law for } \cap \\
&= \emptyset && \text{by the identity law for } \cap.
\end{aligned}$$

39. Proof: Let A and B be any sets. Then

$$\begin{aligned}
&(A - B) \cup (B - A) \\
&= (A \cap B^c) \cup (B \cap A^c) && \text{by the set difference law (used twice)} \\
&= [(A \cap B^c) \cup B] \cap [(A \cap B^c) \cup A^c)] && \text{by the distributive law} \\
&= [B \cup (A \cap B^c)] \cap [A^c \cup (A \cap B^c)] && \text{by the commutative law for } \cup \text{ (used twice)} \\
&= [(B \cup A) \cap (B \cup B^c)] \cap [(A^c \cup A) \cap (A^c \cup B^c)] && \text{by the distributive law (used twice)} \\
&= [(A \cup B) \cap (B \cup B^c)] \cap [(A \cup A^c) \cap (A^c \cup B^c)] && \text{by the commutative law for } \cup \text{ (used twice)} \\
&= [(A \cup B) \cap U] \cap [U \cap (A^c \cup B^c)] && \text{by the complement law for } \cup \text{ (used twice)} \\
&= [(A \cup B) \cap U] \cap [(A^c \cup B^c) \cap U] && \text{by the commutative law for } \cap \\
&= (A \cup B) \cap (A^c \cup B^c) && \text{by the identity law for } \cap \text{ (used twice)} \\
&= (A \cup B) \cap (A \cap B)^c && \text{by De Morgan's law} \\
&= (A \cup B) - (A \cap B) && \text{by the set difference law.}
\end{aligned}$$

42. Let A and B be any sets. Then

$$\begin{aligned}
&(A - (A \cap B)) \cap (B - (A \cap B)) \\
&= (A \cap (A \cap B)^c) \cap (B \cap (A \cap B)^c) && \text{by the set difference law (used twice)} \\
&= A \cap ((A \cap B)^c \cap (B \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (((A \cap B)^c \cap B) \cap (A \cap B)^c) && \text{by the associative law for } \cap \\
&= A \cap ((B \cap (A \cap B)^c) \cap (A \cap B)^c) && \text{by the commutative law for } \cap \\
&= A \cap (B \cap ((A \cap B)^c \cap (A \cap B)^c)) && \text{by the associative law for } \cap \\
&= A \cap (B \cap (A \cap B)^c) && \text{by the idempotent law for } \cap \\
&= (A \cap B) \cap (A \cap B)^c && \text{by the associative law for } \cap \\
&= \emptyset && \text{by the complement law for } \cap.
\end{aligned}$$

45. *a.* Proof: Let A, B, and C be any sets.

    Proof that $(A - B) \cup (B - C) \subseteq (A \cup B) - (B \cap C)$: Suppose $x \in (A - B) \cup (B - C)$. By definition of union, $x \in A - B$ or $x \in B - C$.

    Case 1 ($x \in A - B$): In this case, by definition of set difference, $x \in A$ and $x \notin B$. Then since $x \in A$, by definition of union, $x \in A \cup B$. Also, since $x \notin B$, then $x \notin B \cap C$ (for otherwise, by definition of intersection, x would be in B, which it is not). Thus $x \in A \cup B$ and $x \notin B \cap C$, and so, by definition of set difference, $x \in (A \cup B) - (B \cap C)$.

Case 2 ($x \in B - C$): In this case, by definition of set difference, $x \in B$ and $x \notin C$. Then since $x \in B$, by definition of union, $x \in A \cup B$. Also, since $x \notin C$, then $x \notin B \cap C$ (for otherwise, by definition of intersection, x would be in C, which it is not). Thus $x \in A \cup B$ and $x \notin B \cap C$, and so, by definition of set difference, $x \in (A \cup B) - (B \cap C)$.

Hence, in both cases, $x \in (A \cup B) - (B \cap C)$, and so, by definition of subset,

$$(A - B) \cup (B - C) \subseteq (A \cup B) - (B \cap C).$$

Proof that $(A \cup B) - (B \cap C) \subseteq (A - B) \cup (B - C)$: Suppose $x \in (A \cup B) - (B \cap C)$. By definition of set difference, $x \in A \cup B$ and $x \notin B \cap C$. Note that either $x \in B$ or $x \notin B$.

Case 1 ($x \in B$): In this case $x \notin C$ because otherwise x would be in both B and C, which would contradict the fact that $x \notin B \cap C$. Thus, in this case, $x \in B$ and $x \notin C$, and so $x \in B - C$ by definition of set difference. Then $x \in (A - B) \cup (B - C)$ by definition of union.

Case 2 ($x \notin B$): In this case, since $x \in A \cup B$, then $x \in A$. Hence $x \in A$ and $x \notin B$, and so $x \in A - B$ by definition of set difference. Then $x \in (A - B) \cup (B - C)$ by definition of union.

Hence, in both cases, $x \in (A - B) \cup (B - C)$, and so, by definition of subset,

$$(A \cup B) - (B \cap C) \subseteq (A - B) \cup (B - C).$$

Therefore, since both set containments have been proved, we conclude that

$$(A - B) \cup (B - C) = (A \cup B) - (B \cap C)$$

by definition of set equality.

b. <u>Proof</u>: Let A, B, and C be any sets. Then

$$\begin{aligned}
&(A - B) \cup (B - C) \\
&\quad = (A \cap B^c) \cup (B \cap C^c) && \text{by the set difference law (used twice)} \\
&\quad = ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup C^c) && \text{by the distributive law} \\
&\quad = (B \cup (A \cap B^c)) \cap ((A \cap B^c) \cup C^c) && \text{by the commutative law for } \cup \\
&\quad = ((B \cup A) \cap (B \cup B^c)) \cap ((A \cap B^c) \cup C^c) && \text{by the distributive law} \\
&\quad = ((B \cup A) \cap U) \cap ((A \cap B^c) \cup C^c) && \text{by the complement law for } \cup \\
&\quad = (B \cup A) \cap ((A \cap B^c) \cup C^c) && \text{by the identity law for } \cap \\
&\quad = (A \cup B) \cap ((A \cap B^c) \cup C^c) && \text{by the commutative law for } \cup \\
&\quad = ((A \cup B) \cap (A \cap B^c)) \cup ((A \cup B) \cap C^c) && \text{by the distributive law} \\
&\quad = (((A \cup B) \cap A) \cap B^c) \cup ((A \cup B) \cap C^c) && \text{by the associative law for } \cap \\
&\quad = ((A \cap (A \cup B)) \cap B^c) \cup ((A \cup B) \cap C^c) && \text{by the commutative law for } \cap \\
&\quad = (A \cap B^c) \cup ((A \cup B) \cap C^c) && \text{by the absorption law} \\
&\quad = ((A \cap B^c) \cup \emptyset) \cup ((A \cup B) \cap C^c) && \text{by the identity law for } \cup \\
&\quad = ((A \cap B^c) \cup (B \cap B^c)) \cup ((A \cup B) \cap C^c) && \text{by the complement law for } \cap \\
&\quad = ((B^c \cap A) \cup (B^c \cap B)) \cup ((A \cup B) \cap C^c) && \text{by the commutative law for } \cap \\
&\quad = (B^c \cap (A \cup B)) \cup ((A \cup B) \cap C^c) && \text{by the distributive law} \\
&\quad = ((A \cup B) \cap B^c)) \cup ((A \cup B) \cap C^c) && \text{by the commutative law for } \cap \\
&\quad = (A \cup B) \cap (B^c \cup C^c) && \text{by the distributive law} \\
&\quad = (A \cup B) \cap (B \cap C)^c && \text{by De Morgan's law} \\
&\quad = (A \cup B) - (B \cap C) && \text{by the set difference law.}
\end{aligned}$$

c. Although writing down every detail of the element proof is somewhat tedious, its basic idea is not hard to see. In this case the element proof is probably easier than the algebraic proof.

51. Lemma: For any subsets A and B of a universal set U and for any element x,

(1) $x \in A \triangle B \Leftrightarrow (x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B)$

(2) $x \notin A \triangle B \Leftrightarrow (x \notin A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$.

Proof:

(1) Suppose A and B are any sets and x is any element. Then

$$\begin{array}{rcll} x \in A \triangle B & \Leftrightarrow & x \in (A - B) \cup (B - A) & \text{by definition of } \triangle \\ & \Leftrightarrow & x \in A - B \text{ or } x \in B - A & \text{by definition of } \cup \\ & \Leftrightarrow & (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A) & \text{by definition of set difference.} \end{array}$$

(2) Suppose A and B are any sets and x is any element.

Observe that there are only four mutually exclusive possibilities for the relationship of x to A and B: $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$ or $(x \in A$ and $x \in B)$ or $(x \notin A$ and $x \notin B)$.

By part (1), the condition that $x \in A \triangle B$ is equivalent to the first two possibilities. So the condition that $x \notin A \triangle B$ is equivalent to the second two possibilities.

In other words, $x \notin A \triangle B \Leftrightarrow (x \notin A$ and $x \notin B)$ or $(x \in A$ and $x \in B)$.

Theorem: For all subsets A, B, and C of a universal set U, if $A \triangle C = B \triangle C$ then $A = B$.

Proof: Let A, B, and C be any subsets of a universal set U, and suppose that $A \triangle C = B \triangle C$. *[We will show that $A = B$.]*

Proof that $A \subseteq B$: Suppose $x \in A$. Either $x \in C$ or $x \notin C$. If $x \in C$, then $x \in A$ and $x \in C$ and so by the lemma, $x \notin A \triangle C$. But $A \triangle C = B \triangle C$. Thus $x \notin B \triangle C$ either. Hence, again by the lemma, since $x \in C$ and $x \notin B \triangle C$, then $x \in B$. On the other hand, if $x \notin C$, then by the lemma, since $x \in A$, $x \in A \triangle C$. But $A \triangle C = B \triangle C$. So, again by the lemma, since $x \notin C$ and $x \in B \triangle C$, then $x \in B$. Hence in either case, $x \in B$ *[as was to be shown]*.

Proof that $B \subseteq A$: The proof is exactly the same as for $A \subseteq B$ with the letters A and B interchanged.

Since $A \subseteq B$ and $B \subseteq A$, by definition of set equality $A = B$.

54. Proof:

Suppose A and B are any subsets of a universal set U.

By the universal bound law for $\cap$, $B \cap \emptyset = \emptyset$, and so, by the commutative law for $\cap$, $\emptyset \cap B = \emptyset$.

Take the union with A of both sides to obtain $A \cup (\emptyset \cap B) = A \cup \emptyset$.

But the left-hand side of this equation is $A \cup (\emptyset \cap B) = (A \cup \emptyset) \cap (A \cup B) = A \cap (A \cup B)$ by the distributive law and the identity law for $\cup$.

And the right-hand side of the equation equals A by the by the identity law for $\cup$.

Hence $A \cap (A \cup B) = A$ *[as was to be shown]*.

Section 6.4

3. *a.* commutative law for $\cdot$ *b.* distributive law for $\cdot$ over $+$

c. idempotent law for $\cdot$ (exercise 1) *d.* identity law for $\cdot$

e. distributive law for $\cdot$ over $+$ *f.* commutative law for $+$ *g.* identity law for $\cdot$

Note that once Theorem 6.4.1(5b) has been proved (exercise 4), the proof of this property (Theorem 6.4.1(7a)) can be streamlined as shown below.

Proof: For all elements a and b in B,

$$\begin{array}{rcll} (a + b) \cdot a & = & (a + b) \cdot (a + 0) & \text{by the identity law for } + \\ & = & a + (b \cdot 0) & \text{by the distributive law for } + \text{ over } \cdot \\ & = & a + 0 & \text{by exercise 4} \\ & = & a & \text{by the identity law for } +. \end{array}$$

6. *b.* Proof: By the uniqueness of the complement law, to show that $\overline{1} = 0$, it suffices to show that $1 + 0 = 1$ and $1 \cdot 0 = 0$. But the first equation is true by the identity law for +, and the second equation is true by exercise 4 (the universal bound law for ·). Thus $\overline{1} = 0$.

9. Proof 1: By exercise 8, we know that for all x and y in B, $\overline{x \cdot y} = \overline{x} + \overline{y}$. So suppose a and b are any elements in B. Substitute $\overline{a}$ and $\overline{b}$ in place of x and y in this equation to obtain $\overline{\overline{a} \cdot \overline{b}} = \overline{\overline{a}} + \overline{\overline{b}}$, and since$\overline{\overline{a}} + \overline{\overline{b}} = a + b$ by the double complement law, we have $\overline{\overline{a} \cdot \overline{b}} = a + b$. Hence by the uniqueness of the complement law, the complement of $\overline{a} \cdot \overline{b}$ is $a + b$. It follows by definition of complement that

$$(\overline{a} \cdot \overline{b}) + (a + b) = 1 \qquad \text{and} \qquad (\overline{a} \cdot \overline{b}) \cdot (a + b) = 0.$$

By the commutative laws for + and ·,

$$(a + b) + (\overline{a} \cdot \overline{b}) = 1 \qquad \text{and} \qquad (a + b) \cdot (\overline{a} \cdot \overline{b}) = 0,$$

and thus by the uniqueness of the complement law, the complement of $a + b$ is $\overline{a} \cdot \overline{b}$. In other words, $\overline{a + b} = \overline{a} \cdot \overline{b}$.

Proof 2: An alternative proof can be obtained by taking the proof for exercise 8 in Appendix B and changing every + sign to a · sign and every · sign to a + sign.

12. To avoid unneeded parentheses, assume that · takes precedence over +.

Lemma 1: The universal bound laws for a Boolean algebra can be derived from the Boolean algebra axioms without using the associative laws.

Proof: Suppose a is any element of a Boolean algebra B.

$$\begin{aligned} a + 1 &= (a + 1) \cdot 1 && \text{because 1 is an identity for } \cdot \\ &= (a + 1) \cdot (a + \overline{a}) && \text{by the complement law for } + \\ &= a + 1 \cdot \overline{a} && \text{by the distributive law for } + \text{ over } \cdot \\ &= a + \overline{a} \cdot 1 && \text{by the commutative law for } \cdot \\ &= a + \overline{a} && \text{because 1 is an identity for } \cdot \\ &= 1 && \text{by the complement law for } +. \end{aligned}$$

The proof that $a \cdot 0 = 0$ is obtained using the same sequence of steps but changing each · to + and each + to ·.

Lemma 2: The absorption laws for a Boolean algebra can be derived from the Boolean algebra axioms without using the associative laws.

Proof: Suppose a and b are any elements of a Boolean algebra B.

$$\begin{aligned} a \cdot b + a &= a \cdot b + a \cdot 1 && \text{because 1 is an identity for } \cdot \\ &= a \cdot (b + 1) && \text{by the distributive law for } \cdot \text{ over } + \\ &= a \cdot 1 && \text{by the universal bound law for } + \text{ (Lemma 1)} \\ &= a && \text{because 1 is an identity for } \cdot. \end{aligned}$$

The proof that $a \cdot (a + b) = a$ is obtained using the same sequence of steps but changing each · to + and each + to ·.

Note also that the proofs of the idempotent laws (Example 6.4.2 and the solution to exercise 1) use only the Boolean algebra axioms without the associative laws.

Theorem 1: The associative law for + can be derived from the other axioms in a Boolean algebra.

Proof:

Part 1: We first prove that for all x, y, and z in B, (1) $(x + (y + z)) \cdot x = x$ and (2) $((x + y) + z) \cdot x = x$.

So suppose x, y, and z are any elements in B. It follows immediately from an absorption law that

$$(1)\ (x + (y + z)) \cdot x = x$$

Also,

$$\begin{aligned}
((x + y) + z)) \cdot x &= x \cdot ((x + y) + z) && \text{by the commutative law for } \cdot \\
&= x \cdot (x + y) + x \cdot z && \text{by the distributive law for } \cdot \text{ over } + \\
&= (x + y) \cdot x + x \cdot z && \text{by the commutative law for } \cdot \\
&= x + x \cdot z && \text{by an absorption law} \\
&= x \cdot x + x \cdot z && \text{by the idempotent law for } \cdot \\
&= x \cdot (x + z) && \text{by the distributive law for } \cdot \text{ over } + \\
&= (x + z) \cdot x && \text{by the commutative law for } \cdot \\
&= x && \text{by an absorption law.}
\end{aligned}$$

Hence

$$(2)\ ((x + y) + z) \cdot x = x.$$

Part 2: By the commutative law for + and equation (2), for all x, y, and z in B,

$$((x + y) + z) \cdot y = ((y + x) + z) \cdot y = y.$$

And by the commutative law for + and equation (2), for all x, y, and z in B,

$$(x + (y + z)) \cdot y = ((y + x) + z) \cdot y = y.$$

Thus we have

$$(3)\ ((x + y) + z) \cdot y = y \quad \text{and} \quad (4)\ (x + (y + z)) \cdot y = y.$$

By similar reasoning we can also conclude that

$$(5)\ ((x + y) + z) \cdot z = z \quad \text{and} \quad (6)\ (x + (y + z)) \cdot z = z.$$

Part 3: We next prove that for all a, b, and c in B,

$$(7)\ a + (b + c) = ((a + b) + c) \cdot (a + (b + c)) \quad \text{and} \quad (8)\ (a + b) + c) = ((a + b) + c) \cdot (a + (b + c)).$$

To prove (7), suppose a, b, and c are any elements in B. Then

$$\begin{aligned}
&((a + b) + c) \cdot (a + (b + c)) \\
&\quad = ((a + b) + c) \cdot a + ((a + b) + c) \cdot (b + c)) && \text{by the distributive law for } \cdot \text{ over } + \\
&\quad = a + ((a + b) + c) \cdot (b + c)) && \text{by equation (2)} \\
&\quad = a + [((a + b) + c) \cdot b + ((a + b) + c) \cdot c] && \text{by the distributive law for } \cdot \text{ over } + \\
&\quad = a + (b + c) && \text{by equations (3) and (5).}
\end{aligned}$$

Similarly, if a, b, and c are any elements in B. Then we can prove equation (8) as follows:

$$\begin{aligned}
&((a + b) + c) \cdot (a + (b + c)) \\
&\quad = (a + (b + c)) \cdot ((a + b) + c) && \text{by the commutative law for } \cdot \\
&\quad = (a + (b + c)) \cdot (a + b) + (a + (b + c)) \cdot c && \text{by the distributive law for } \cdot \text{ over } + \\
&\quad = (a + (b + c)) \cdot (a + b) + c && \text{by equation (6)} \\
&\quad = [(a + (b + c)) \cdot a + (a + (b + c)) \cdot b] + c && \text{by the distributive law for } \cdot \text{ over } + \\
&\quad = (a + b) + c && \text{by equations (1) and (4).}
\end{aligned}$$

Therefore, since both $a + (b + c)$ and $(a + b) + c$ are equal to the same quantity, they are equal to each other: $a + (b + c) = (a + b) + c$.

Theorem 2: The associative law for $\cdot$ can be derived from the other axioms in a Boolean algebra.

Proof: Suppose a, b, and c are any elements in a Boolean algebra B. The proof that $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ is obtained using the same sequence of steps as in the proof of Theorem 1 but changing each $\cdot$ to $+$ and each $+$ to $\cdot$.

Alternative (Shorter) Proof for Theorem 1 (based on the outline in *Introduction to Boolean Algebra* by S. Givant and P. Halmos, Springer, 2010)

The Cancellation Law for $\cdot$: For all elements p, q, and r in a Boolean algebra,

$$\text{if } q\cdot p = r\cdot p \text{ and } q\cdot\overline{p} = r\cdot\overline{p}, \text{ then } q = r.$$

Lemma 3: The cancellation law for $\cdot$ can be derived from the other axioms in a Boolean algebra without using the associative law.

Proof: Suppose p, q, and r are any elements in a Boolean algebra B such that

$$q\cdot p = r\cdot p \quad\text{and}\quad q\cdot\overline{p} = r\cdot\overline{p}. \qquad (1)$$

We will show that $q = r$.

Now

$$\begin{aligned} q\cdot p + q\cdot\overline{p} &= q\cdot(p+\overline{p}) && \text{by the distributive law for } \cdot \text{ over } + \\ &= q\cdot 1 && \text{by the complement law for } + \\ &= q && \text{by the identity law for 1.} \end{aligned}$$

Similarly,

$$\begin{aligned} r\cdot p + r\cdot\overline{p} &= r\cdot(p+\overline{p}) && \text{by the distributive law for } \cdot \text{ over } + \\ &= r\cdot 1 && \text{by the complement law for } + \\ &= r && \text{by the identity law for 1.} \end{aligned}$$

But, by substitution from (1),

$$q\cdot p + q\cdot\overline{p} = r\cdot p + r\cdot\overline{p}.$$

Thus

$$q = r.$$

Theorem 1: The associative law for $+$ can be derived from the other axioms in a Boolean algebra.

Proof: Suppose a, b, and c are any elements in a Boolean algebra B.

Part 1: We first prove that $(a+(b+c))\cdot a = ((a+b)+c)\cdot a$.

$$(a+(b+c))\cdot a \quad = \quad a \quad \text{by an absorption law (Lemma 2).}$$

In addition:

$$\begin{aligned} ((a+b)+c)\cdot a &= a\cdot((a+b)+c) && \text{by the commutative law for } \cdot \\ &= a\cdot(a+b) + a\cdot c && \text{by the distributive law for } \cdot \text{ over } + \\ &= (a+b)\cdot a + a\cdot c && \text{by the commutative law for } + \\ &= a + a\cdot c && \text{by an absorption law} \\ &= a\cdot c + a && \text{by the commutative law for } + \\ &= a && \text{by an absorption law.} \end{aligned}$$

Since both quantities equal a, we conclude that

$$(a+(b+c))\cdot a = ((a+b)+c)\cdot a.$$

Part 2: We next prove that $(a + (b + c)) \cdot \overline{a} = ((a + b) + c) \cdot \overline{a}$

$$\begin{aligned}
(a + (b + c)) \cdot \overline{a} &= \overline{a} \cdot (a + (b + c)) && \text{by the commutative law for } \cdot \\
&= \overline{a} \cdot a + \overline{a} \cdot (b + c) && \text{by the distributive law for } \cdot \text{ over } + \\
&= 0 + \overline{a} \cdot (b + c) && \text{by the complement law for } \cdot \\
&= \overline{a} \cdot (b + c) + 0 && \text{by the commutative law for } + \\
&= \overline{a} \cdot (b + c) && \text{because 0 is an identity for } +.
\end{aligned}$$

In addition:

$$\begin{aligned}
((a + b) + c) \cdot \overline{a} &= \overline{a} \cdot ((a + b) + c) && \text{by the commutative law for } \cdot \\
&= \overline{a} \cdot (a + b) + \overline{a} \cdot c && \text{by the distributive law for } \cdot \text{ over } + \\
&= (\overline{a} \cdot a + \overline{a} \cdot b) + \overline{a} \cdot c && \text{by the distributive law for } \cdot \text{ over } + \\
&= (a \cdot \overline{a} + \overline{a} \cdot b) + \overline{a} \cdot c && \text{by the commutative law for } \cdot \\
&= (0 + \overline{a} \cdot b) + \overline{a} \cdot c && \text{by the complement law for } \cdot \\
&= (\overline{a} \cdot b + 0) + \overline{a} \cdot c && \text{by the commutative law for } + \\
&= \overline{a} \cdot b + \overline{a} \cdot c && \text{because 0 is an identity for } + \\
&= \overline{a} \cdot (b + c) && \text{by the distributive law for } \cdot \text{ over } +.
\end{aligned}$$

Since both quantities equal $\overline{a} \cdot (b + c)$, we conclude that

$$(a + (b + c)) \cdot \overline{a} = ((a + b) + c) \cdot \overline{a}.$$

Part 3: Parts (1) and (2) show that

$$(a + (b + c)) \cdot a = ((a + b) + c) \cdot a \quad \text{and} \quad (a + (b + c)) \cdot \overline{a} = ((a + b) + c) \cdot \overline{a}.$$

Thus, by the cancellation law

$$a + (b + c) = (a + b) + c.$$

15. This statement contradicts itself. If it were true, then because it declares itself to be a lie, it would be false. Consequently, it is not true. On the other hand, if it were false, then it would be false that "the sentence in this box is a lie," and so the sentence would be true. Consequently, the sentence is not false. Thus the sentence is neither true nor false, which contradicts the definition of a statement. Hence the sentence is not a statement.

18. In order for an *and* statement to be true, both components must be true. So if the given sentence is a true statement, the first component "this sentence is false" is true. But this implies that the sentence is false. In other words, the sentence is not true. On the other hand, if the sentence is false, then at least one component is false. But because the second component "$1 + 1 = 2$" is true, the first component must be false. Thus it is false that "this sentence is false," and so the sentence is true. In other words, the sentence is not false. Thus the sentence is neither true nor false, which contradicts the definition of a statement. Hence the sentence is not a statement.

24. Because the total number of strings consisting of 11 or fewer English words is finite, the number of such strings that describe integers must be also finite. Thus the number of integers described by such strings must be finite, and hence there is a largest such integer, say m. Let $n = m + 1$. Then n is "the smallest integer not describable in fewer than 12 English words." But this description of n contains only 11 words. So n is describable in fewer than 12 English words, which is a contradiction. (*Comment*: This contradiction results from the self-reference in the description of n.)

Review Guide: Chapter 6

Definitions and Notation:

- How can you express the definition of subset formally as a universal conditional statement? *(p. 337)*
- What is a proper subset of a set? *(p. 337)*
- How are the definitions of subset and equality of sets related? *(p. 339)*
- What are Venn diagrams? *(p. 340)*
- What are the union, intersection, and difference of sets? *(p. 341)*
- What is the complement of a set? *(p. 341)*
- What is the relation between sets and interval notation? *(p. 342)*
- How are unions and intersections defined for indexed collections of sets? *(p. 343)*
- What does it mean for two sets to be disjoint? *(p. 344)*
- What does it mean for a collection of sets to be mutually disjoint? *(p. 345)*
- What is a partition of a set? *(p. 345)*
- What is the power set of a set? *(p. 346)*
- What is an ordered n-tuple? *(p. 346)*
- What is the Cartesian product of n sets, where $n \geq 2$? *(p. 347)*

Set Theory

- How do you use an element argument to prove that one set is a subset of another set? *(p. 337-338)*
- What is the basic (two-step) method for showing that two sets are equal? *(pp. 339 and 356)*
- How are the procedural versions of set operations used to prove properties of sets? *(p. 352-353)*
- Are you familiar with the set properties in Theorems 6.2.1 and 6.2.2? *(pp. 352 and 354)*
- Why is the empty set a subset of every set? *(p. 362)*
- How is the element method used to show that a set equals the empty set? *(p. 362)*
- How do you find a counterexample for a proposed set identity? *(p. 367)*
- How do you find the number of subsets of a set with a finite number of elements? *(p. 369)*
- What is an "algebraic method" for proving that one set equals another set? *(p. 370-371)*
- What is a Boolean algebra? *(p. 375)*
- How do you deduce additional properties of a Boolean algebra from the properties that define it? *(p. 377)*
- What is Russell's paradox? *(p. 378)*
- What is the Halting Problem? *(p. 379)*

Chapter 7: Functions

The aim of Section 7.1 is to promote a broad view of the function concept and to give you experience with the wide variety of functions that arise in discrete mathematics. Representation of functions by arrow diagrams is emphasized to prepare the way for the discussion of one-to-one and onto functions in Section 7.2.

Section 7.2 focuses on function properties. As you are learning about one-to-one and onto functions in this section, you may need to review the logical principles such as the negation of $\forall$, $\exists$, and if-then statements and the equivalence of a conditional statement and its contrapositive. These logical principles are needed to understand the equivalence of the two forms of the definition of one-to-one and what it means for a function not to be one-to-one or onto.

Sections 7.3 and 7.4 go together in the sense that the relations between one-to-one and onto functions and composition of functions developed in Section 7.3 are used to prove the fundamental theorem about cardinality in Section 7.4. The proofs that a composition of one-to-one functions is one-to-one or that a composition of onto functions is onto (and the related exercises) will test the degree to which you have learned to instantiate mathematical definitions in abstract contexts, apply the method of generalizing from the generic particular in a sophisticated setting, develop mental models of mathematical concepts that are both vivid and generic enough to reason with, and create moderately complex chains of deductions.

When you read Section 7.4, try to see the connections that link Russell's paradox, the halting problem, and the Cantor diagonalization argument.

Section 7.1

3. *b.* False. The definition of function does not allow an element of the domain to be associated to two different elements of the co-domain, but it does allow an element of the co-domain to be the image of more than one element in the domain. For example, let $X = \{1, 2\}$ and $Y = \{a\}$ and define $f\colon X \to Y$ by specifying that $f(1) = f(2) = a$. Then f defines a function from X to Y for which a has two unequal preimages.

6. *b.* Define $F\colon \mathbf{Z}^{nonneg} \to \mathbf{R}$ as follows: for each nonnegative integer n, $F(n) = (-1)^n(2n)$.

9. *d.* $S(5) = 1 + 5 = 6$

e. $S(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$

f. $S(21) = 1 + 3 + 7 + 21 = 32$

12. *c.* $G(3, 2) = ((2\cdot 3 + 1)\ mod\ 5, (3\cdot 2 - 2)\ mod\ 5) = (7\ mod\ 5, 4\ mod\ 5) = (2, 4)$

d. $G(1, 5) = ((2\cdot 1 + 1)\ mod\ 5, 3\cdot 5 - 2)\ mod\ 5) = (3\ mod\ 5, 13\ mod\ 5) = (3, 3)$

18. *b.* $\log_2 1024 = 10$ because $2^{10} = 1024$

d. $\log_2 1 = 0$ because $2^0 = 1$

e. $\log_{10} \frac{1}{10} = -1$ because $10^{-1} = \frac{1}{10}$

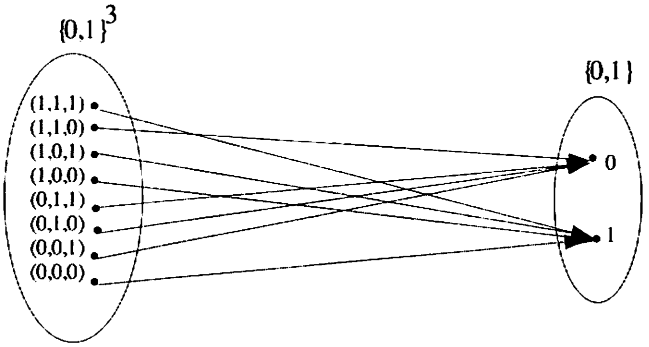
f. $\log_3 3 = 1$ because $3^1 = 3$

g. $\log_2 2^k = k$ because the exponent to which 2 must be raised to obtain 2^k is k
Alternative answer: $\log_2 2^k = k$ because $2^k = 2^k$

24. Since $\log_b y = 2$, then $b^2 = y$. Now, by properties of exponents, $(b^2)^1 = y$, and so $log_{b^2}(y) = 1$.

27. *b.* $g(aba) = aba$, $g(bbab) = babb$, $g(b) = b$ The range of g is the set of all strings of a's and b's, which equals S.

30. *b.*

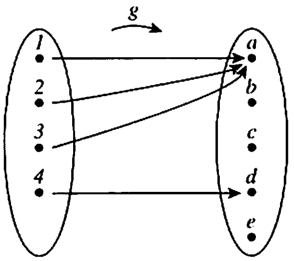


36. Student D is correct. Note that in order for S to be a function, $S(2)$ has to be that number y in the co-domain $J_4 - \{0\}$ such that $2y \textit{ mod } 4 = 1$. Now the only possible values for y are 1, 2, and 3, and

$$2 \cdot 1 \textit{ mod } 4 = 2 \neq 1, \quad 2 \cdot 2 \textit{ mod } 4 = 4 \textit{ mod } 4 = 0 \neq 1, \quad \text{and} \quad 2 \cdot 3 \textit{ mod } 4 = 6 \textit{ mod } 4 = 2 \neq 1.$$

Thus there is no number y in the co-domain $J_4 - \{0\}$ such that $S(2) = y$, and hence S does not satisfy property (1) of the definition of function.

39. *a.*



b. $g(A) = \{a\}$ $g(X) = \{a, d\}$ $g^{-1}(C) = \{1, 2, 3\}$ $g^{-1}(D) = \emptyset$ $g^{-1}(Y) = \{1, 2, 3, 4\} = X$

42. The property is true.

Proof: Suppose $y \in F(A \cap B)$. *[We must show that $y \in F(A) \cap F(B)$.]* By definition of image of a set, there exists an element x in $A \cap B$ such that $y = F(x)$. By definition of intersection, x is in A and x is in B., and so, by definition of image of an element, $F(x) \in F(A)$ and $F(x) \in F(B)$. Thus, by substitution, $y \in F(A)$ and $y \in F(B)$. It follows, by definition of intersection, that $y \in F(A) \cap F(B)$ *[as was to be shown]*.

48. The statement is true.

Proof 1: Let $F\colon X \to Y$ be any function, and suppose that $C \subseteq Y$ and $D \subseteq Y$. *[We must show that $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.]*

***Proof that $F^{-1}(C - D) \subseteq F^{-1}(C) - F^{-1}(D)$*:**

Suppose x is any element in $F^{-1}(C - D)$. *[We must show that $x \in F^{-1}(C) - F^{-1}(D)$.]*

By definition of inverse image, $F(x) \in C - D$, and so,

by definition of set difference, $F(x) \in C$ and $F(x) \notin D$.

Then, by definition of inverse image, $x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$, and so

by definition of set difference, $x \in F^{-1}(C) - F^{-1}(D)$ *[as was to be shown]*.

Proof that $\boldsymbol{F^{-1}(C) - F^{-1}(D) \subseteq F^{-1}(C - D)}$:

Suppose x is any element in $F^{-1}(C) - F^{-1}(D)$. *[We must show that $x \in F^{-1}(C - D)$.]*

By definition of set difference, $F^{-1}(C)$ and $x \notin F^{-1}(D)$, and so,

by definition of inverse image, $F(x) \in C$ and $F(x) \notin D$.

Then, by definition of set difference, $F(x) \in C - D$, and so,

by definition of inverse image $x \in F^{-1}(C - D)$ *[as was to be shown].*

Conclusion: Since both subset containments have been proved, we conclude that $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$ *[as was to be shown].*

Proof 2:*(This proof uses the logic of if-and-only-if statements.)*

Let $F\colon X \to Y$ be any function, and suppose that $C \subseteq Y$ and $D \subseteq Y$. *[We must show that $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.]*

Suppose x is any element in X. Then, by definition of inverse image and set difference,

$x \in F^{-1}(C - D) \Leftrightarrow F(x) \in C - D$

$\Leftrightarrow F(x) \in C$ and $F(x) \notin D$

$\Leftrightarrow x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$

$\Leftrightarrow x \in F^{-1}(C) - F^{-1}(D)$.

The preceding steps show that if $x \in F^{-1}(C - D)$ then $x \in F^{-1}(C) - F^{-1}(D)$ (which implies that $F^{-1}(C - D) \subseteq F^{-1}(C) - F^{-1}(D)$), and if $x \in F^{-1}(C) - F^{-1}(D)$ then $x \in F^{-1}(C - D)$ (which implies that $F^{-1}(C) - F^{-1}(D) \subseteq F^{-1}(C - D)$).

Hence $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.

51. *d.* $\phi(12) = 4$ *[because 1, 5, 7, and 11 have no common factors with 12 other than* ± 1*]*

e. $\phi(11) = 10$ *[because 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 have no common factors with 11 other than* ± 1*]*

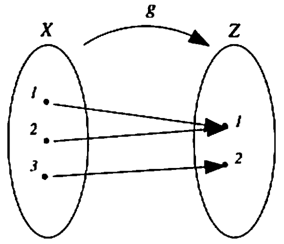
f. $\phi(1) = 1$ *[because 1 is the only positive integer which has no common factors with 1 other than* ± 1*]*

Section 7.2

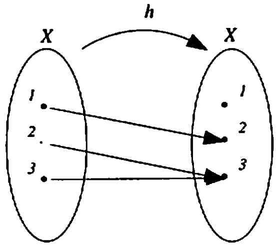
3. There are many counterexamples to the given statement. One is the function $f = \{(1,4),(2,4)\}$ with domain $\{1,2\}$ and co-domain $\{4\}$. Each element of the domain is related to exactly one element of the co-domain because 1 is related to 4 (and not to anything else), and 2 is related to 4 (and not to anything else). But f is not one-to-one because $f(1) = f(2)$ and $1 \neq 2$.

9. In each case below there are a number of correct answers.

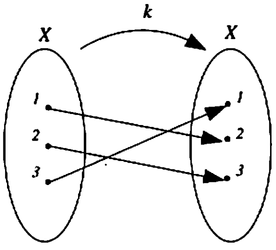
b.



c.



d.



12. *a.* (i) ***F is one-to-one***: Suppose n_1 and n_2 are in $\mathbf{Z}$ and $F(n_1) = F(n_2)$. *[We must show that $n_1 = n_2$.]* By definition of F, $2 - 3n_1 = 2 - 3n_2$. Subtracting 2 from both sides and dividing by -3 gives $n_1 = n_2$.

    (ii). ***F is not onto***: Let $m = 0$. Then m is in $\mathbf{Z}$ but $m \neq F(n)$ for any integer n. *[For if $m = F(n)$ then $0 = 2 - 3n$, and so $3n = 2$ and $n = 2/3$. But $2/3$ is not in $\mathbf{Z}$.]*

    b. ***G is onto***: Suppose y is any element of $\mathbf{R}$. *[We must show that there is an element x in $\mathbf{R}$ such that $G(x) = y$.]*

    [Scratch work: If such an x exists, then, by definition of G, $y = 2 - 3x$ and so $3x = 2 - y$, or, equivalently, $x = (2 - y)/3$. Let's check to see if this works.]

    Let $x = (2 - y)/3$. Then

    $$(1)\ x \in \mathbf{R} \quad \text{and} \quad (2)\ G(x) = G\left(\frac{2-y}{3}\right) = 2 - 3\left(\frac{2-y}{3}\right) = 2 - (2 - y) = 2 - 2 + y = y.$$

    [This is what was to be shown.]

18. ***f is one-to-one***:

    <u>Proof</u>: Let x_1 and x_2 be any real numbers other than -1, and suppose that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

    $$\frac{x_1+1}{x_1-1} = \frac{x_2+1}{x_2-1}.$$

    Cross-multiplying gives

    $$(x_1+1)(x_2-1) = (x_2+1)(x_1-1) \quad \text{or, equivalently,} \quad x_1x_2 - x_1 + x_2 - 1 = x_1x_2 - x_2 + x_1 - 1.$$

    Adding $1 - x_1x_2$ to both sides gives $-x_1 + x_2 = -x_2 + x_1$, or, equivalently, $2x_1 = 2x_2$. Dividing both sides by 2 gives $x_1 = x_2$ *[as was to be shown]*.

24. *a.* ***N is not one-to-one***: Let $s_1 = a$ and $s_2 = ab$. Then $N(s_1) = N(s_2) = 1$ but $s_1 \neq s_2$.

27. *a.* ***T is one-to-one***: $T(n)$ is the set of all the positive divisors of n. Observe that for all positive integers n, the largest element of $T(n)$ is n because n divides n and no integer larger than n divides n.

    So suppose n_1 and n_2 are positive integers and $T(n_1) = T(n_2)$. *[We must show that $n_1 = n_2$.]*

    Now $T(n_1)$ is the set of all the positive divisors of n_1 and $T(n_2)$ is the set of all the positive divisors of n_2.

    So since $T(n_1) = T(n_2)$, the largest element of $T(n_1)$, namely n_1, is the same as the largest element of $T(n_2)$, namely n_2.

    Hence $n_1 = n_2$ *[as was to be shown]*.

    b. ***T is not onto***: The set $\{2\}$ is a finite subset of positive integers, but there is no positive integer n such that $T(n) = \{2\}$. The reason is that the number 1 divides every positive integer, and so 1 must be an element of $T(n)$ for all positive integers n. But $1 \notin \{2\}$. (There are many other examples that show T is not onto.)

30. *a.* ***J is one-to-one***: Suppose (r_1, s_1) and (r_2, s_2) are in $\mathbf{Q} \times \mathbf{Q}$ and $J(r_1, s_1) = J(r_2, s_2)$. *[We must show that $(r_1, s_1) = (r_2, s_2)$.]* By definition of J,

    $$r_1 + \sqrt{2}s_1 = r_2 + \sqrt{2}s_2 \quad \text{and thus} \quad r_1 - r_2 = \sqrt{2}(s_2 - s_1).$$

    Note that both $r_1 - r_2$ and $s_2 - s_1$ are rational because they are differences of rational numbers (exercise 17 of Section 4.2).

Suppose $s_2 - s_1 \neq 0$. Then $\sqrt{2}(s_2 - s_1)$ is a product of a nonzero rational number and an irrational number ($\sqrt{2}$), and so it is irrational (exercise 11 of Section 4.6). As a consequence, the rational number $(r_1 - r_2)$ equals the irrational number $(\sqrt{2}(s_2 - s_1))$. Because this is impossible, the supposition that $s_2 - s_1 \neq 0$ must be false, and therefore $s_2 - s_1 = 0$.

Thus, by substitution, $r_1 - r_2 = \sqrt{2}(s_2 - s_1) = \sqrt{2} \cdot 0 = 0$. So

$$r_1 - r_2 = 0 \quad \text{and} \quad s_2 - s_1 = 0 \quad \text{or, equivalently,} \quad r_1 = r_2 \quad \text{and} \quad s_2 = s_1.$$

Hence $(r_1, s_1) = (r_2, s_2)$ *[as was to be shown].*

b. ***J is not onto***: We show that J is not onto by giving an example of a real number that is not equal to $J(r,s)$ for any rational numbers r and s. For example, consider the number $\sqrt{3}$ and suppose there were rational numbers r and s such that

$$\sqrt{3} = r + \sqrt{2}s.$$

We will show that this supposition leads logically to a contradiction.]

Case 1 (s = 0): In this case, $\sqrt{3} = r$ where r is a rational number, which contradicts the fact that $\sqrt{3}$ is irrational (exercise 16, Section 4.7).

Case 2 (s ≠ 0): In this case,

$$\begin{array}{lll} & \sqrt{3} - \sqrt{2}s = r & \\ \Rightarrow & 3 + 2s^2 - 2s\sqrt{6} = r^2 & \text{by squaring both sides} \\ \Rightarrow & -2s\sqrt{6} = r^2 - 3 - 2s^2 & \text{by subtracting } 3 + 2s^2 \text{ from both sides} \\ \Rightarrow & \sqrt{6} = \dfrac{r^2 - 3 - 2s^2}{-2s} & \text{by dividing both sides by } -2s. \end{array}$$

But both r^2-3-2s^2 and $-2s$ are rational numbers because products and differences of rational numbers are rational (exercises 15 and 17, Section 4.2), and $-2s$ is nonzero because it is a product of -2 and s, which are both nonzero numbers (zero product property). Thus $\sqrt{6}$ is a quotient of a rational number and a nonzero rational number, which is rational (by the result of exercise 16 in Section 4.2). But this contradicts the fact that $\sqrt{6}$ is irrational (by the result of exercise 22, Section 4.7).

Conclusion: Since a contradiction was obtained in both cases, we conclude that the supposition is false. That is, there are no rational numbers r and s such that $\sqrt{3} = r + \sqrt{2}s$. Therefore J is not onto.

39. If $f: \mathbf{R} \to \mathbf{R}$ is onto and c is any nonzero real number, then $c \cdot f$ is also onto.

Proof: Suppose $f: \mathbf{R} \to \mathbf{R}$ is onto and c is any nonzero real number.

Let y be any element of $\mathbf{R}$. *[We must show that there exists an element x in $\mathbf{R}$ such that $c \cdot f(x) = y$.]*

Since $c \neq 0$, y/c is a real number, and since f is onto, there is an $x \in \mathbf{R}$ with $f(x) = y/c$. Then $y = c \cdot f(x) = (c \cdot f)(x)$. So $c \cdot f$ is onto *[as was to be shown].*

48. By the result of exercise 12a, F is not onto. Hence it is not a one-to-one correspondence.

51. Because D is not one-to-one, D is not a one-to-one correspondence.

54. By the result of exercise 17, f is one-to-one. f is also onto for the following reason. Given any real number y other than 3, let $x = \dfrac{1}{3-y}$. Then x is a real number (because $y \neq 3$) and

$$f(x) = f(\frac{1}{3-y}) = \frac{3\left(\frac{1}{3-y}\right) - 1}{\frac{1}{3-y}} = \frac{3\left(\frac{1}{3-y}\right) - 1}{\frac{1}{3-y}} \cdot \frac{(3-y)}{(3-y)} = \frac{3-(3-y)}{1} = 3 - 3 + y = y.$$

This calculation also shows that $f^{-1}(y) = \dfrac{1}{3-y}$ for all real numbers $y \neq 3$.

57. **Algorithm 7.2.1 Checking Whether a Function is One-to-One**

[For a given function F with domain $X = \{a[1], a[2], \ldots, a[n]\}$, this algorithm discovers whether or not F is one-to-one. Initially, answer is set equal to "one-to-one". Then the values of $F(a[i])$ and $F(a[j])$ are systematically compared for indices i and j with $1 \leq i < j \leq n$. If at any point it is found that $F(a[i]) = F(a[j])$ and $a[i] \neq a[j]$, then F is not one-to-one, and so answer is set equal to "not one-to-one" and execution ceases. If after all possible values of i and j have been examined, the value of answer is still "one-to-one", then F is one-to-one.]

Input: n *[a positive integer]*, $a[1], a[2], \ldots, a[n]$ *[a one-dimensional array representing the set X]*, F *[a function with domain X]*

Algorithm Body:

```
answer := "one-to-one"
i := 1
while (i ≤ n − 1 and answer = "one-to-one")
      j := i + 1
      while (j ≤ n and answer = "one-to-one")
            if (F(a[i]) = F(a[j]) and a[i] ≠ a[j]) then answer := "not one-to-one"
            j := j + 1
      end while
      i := i + 1
end while
```

Output: *answer [a string]*

Section 7.3

12. *b.* For all positive real numbers b and x, $\log_b x$ is the exponent to which b must be raised to obtain x. So if b is raised to this exponent, x is obtained. In other words, $b^{\log_b x} = x$.

15. *b.* $z/2 = t/2$ *c.* $f(x_1) = f(x_2)$

18. f must be one-to-one.

Proof:

Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and $g \circ f\colon X \to Z$ is one-to-one.

To show that f is one-to-one, suppose x_1 and x_2 are in X and $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]*

Then $g(f(x_1)) = g(f(x_2))$, and so $(g \circ f)(x_1) = (g \circ f)(x_2)$.

But $g \circ f$ is one-to-one. Hence $x_1 = x_2$ *[as was to be shown]*.

24. $g \circ f\colon \mathbf{R} \to \mathbf{R}$ is defined by $(g \circ f)(x) = g(f(x)) = g(x+3) = -(x+3)$ for all $x \in \mathbf{R}$.

Since $z = -(x+3)$ if, and only if, $x = -z-3$, $(g \circ f)^{-1}\colon \mathbf{R} \to \mathbf{R}$ is defined by $(g \circ f)^{-1}(z) = -z-3$ for all $z \in \mathbf{R}$.

Since $z = -y$ if, and only if, $y = -z$, $g^{-1}\colon \mathbf{R} \to \mathbf{R}$ is defined by $g^{-1}(z) = -z$ for all $z \in \mathbf{R}$.

Since $y = x+3$ if, and only if, $x = y-3$, $f^{-1}\colon \mathbf{R} \to \mathbf{R}$ is defined by $f^{-1}(y) = y-3$.

$f^{-1} \circ g^{-1}\colon \mathbf{R} \to \mathbf{R}$ is defined by $(f^{-1} \circ g^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(-z) = (-z)-3 = -z-3$ for all $z \in \mathbf{R}$.

By the above and the definition of equality of functions, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

27. The property is true.

Proof 1: Let X, Y, and Z be any sets, let $f : X \to Y$ and $g : Y \to Z$ be any functions, and let C be any subset of Z.

Proof that $((g \circ f)^{-1}(C) \subseteq f^{-1}(g^{-1}(C))$:

Suppose $x \in (g \circ f)^{-1}(C)$. *[We must show that* $x \in f^{-1}(g^{-1}(C))$.*]*

By definition of inverse image (for $g \circ f$), $(g \circ f)(x) \in C$, and so,

by definition of composition of functions, $g(f(x)) \in C$.

Then by definition of inverse image (for g), $f(x) \in g^{-1}(C)$, and

by definition of inverse image (for f), $x \in f^{-1}(g^{-1}(C))$.

So by definition of subset, $(g \circ f)^{-1}(C) \subseteq f^{-1}(g^{-1}(C))$.

Proof that $f^{-1}(g^{-1}(C)) \subseteq (g \circ f)^{-1}(C)$:

Suppose $x \in f^{-1}(g^{-1}(C))$. *[We must show that* $x \in (g \circ f)^{-1}(C)$.*]*

By definition of inverse image (for f), $f(x) \in g^{-1}(C)$, and so,

by definition of inverse image (for g), $g(f(x)) \in C$.

So by definition of composition of functions, $(g \circ f)(x) \in C$.

Then by definition of inverse image (for $g \circ f$), $x \in (g \circ f)^{-1}(C)$.

So by definition of subset, $f^{-1}(g^{-1}(C)) \subseteq (g \circ f)^{-1}(C)$.

Conclusion: Since each set is a subset of the other, the two sets are equal.

Proof 2 *(using the logic of if-and-only-if statements)*

Let X, Y, and Z be any sets, let $f : X \to Y$ and $g : Y \to Z$ be any functions, and let C be any subset of Z.

Then $x \in (g \circ f)^{-1}(C)$

$\Leftrightarrow (g \circ f)(x) \in C$ *[by definition of inverse image for* $g \circ f$*]*

$\Leftrightarrow g(f(x)) \in C$ *[by definition of composition of functions]*

$\Leftrightarrow f(x) \in g^{-1}(C)$ *[by definition of inverse image for* g*]*

$\Leftrightarrow x \in f^{-1}(g^{-1}(C))$ *[by definition of inverse image for* f*]*.

So both sets consist of the same elements, and thus, by definition of set equality, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Section 7.4

6. ***Part 1***: The function I: $2\mathbf{Z} \to \mathbf{Z}$ is defined as follows: $I(n) = n$ for all even integers n. I is clearly one-to-one because if $I(n_1) = I(n_2)$ then by definition of I, $n_1 = n_2$. But I is not onto because the range of I consists only of even integers. In other words, if m is any odd integer, then $I(n) \neq m$ for any even integer n.

Part 2: The function J: $\mathbf{Z} \to 2\mathbf{Z}$ is defined as follows $J(n) = 2\lfloor n/2 \rfloor$ for all integers n. Then J is onto because for any even integer m, $m = 2k$ for some integer k. Let $n = 2k$. Then $J(n) = J(2k) = 2\lfloor 2k/2 \rfloor = 2\lfloor k \rfloor = 2k = m$. But J is not one-to-one because, for example, $J(2) = 2\lfloor 2/2 \rfloor = 2 \cdot 1 = 2$ and $J(3) = 2\lfloor 3/2 \rfloor = 2 \cdot 1 = 2$, so $J(2) = J(3)$ but $2 \neq 3$.

(More generally, given any integer k, if $m = 2k$, then $J(m) = 2\lfloor m/2 \rfloor = 2\lfloor 2k/2 \rfloor = 2\lfloor k \rfloor = J(m)$ and $J(m+1) = 2\lfloor (m+1)/2 \rfloor = 2\lfloor (2k+1)/2 \rfloor = 2\lfloor k + 1/2 \rfloor = 2k$. So $J(m) = J(m+1)$ but $m \neq m+1$.)

9. Proof:

Define a function f: $\mathbf{Z}^+ \to \mathbf{Z}^{nonneg}$ as follows: $f(n) = n - 1$ for all positive integers n.

Observe that if $n \geq 1$ then $n - 1 \geq 0$, so f is well-defined.

In addition, f is one-to-one because for all positive integers n_1 and n_2, if $f(n_1) = f(n_2)$ then $n_1 - 1 = n_2 - 1$ and hence $n_1 = n_2$.

Moreover f is onto because if m is any nonnegative integer, then $m + 1$ is a positive integer and $f(m + 1) = (m + 1) - 1 = m$ by definition of f.

Thus, because there is a function f: $\mathbf{Z}^+ \to \mathbf{Z}^{nonneg}$ that is one-to-one and onto, $\mathbf{Z}^+$ has the same cardinality as $\mathbf{Z}^{nonneg}$.

It follows that $\mathbf{Z}^{nonneg}$ is countably infinite and hence countable.

12. Proof:

Define F: $S \to W$ by the rule $F(x) = (b - a)x + a$ for all real numbers x in S.

Then F is well-defined because if $0 < x < 1$, then $a < (b - a)x + a < b$.

In addition, F is one-to-one because if x_1 and x_2 are in S and $F(x_1) = F(x_2)$, then $(b-a)x_1 + a = (b - a)x_2 + a$ and so *[by subtracting a and dividing by b − a]* $x_1 = x_2$.

Furthermore, F is onto because if y is any element in W, then $a < y < b$ and so $0 < (y - a)/(b - a) < 1$.

Consequently, $(y - a)/(b - a) \in S$ and $h((y - a)/(b - a)) = (b - a)[(y - a)/(b - a)] + a = y$.

Hence F is a one-to-one correspondence, and so S and W have the same cardinality.

15. Proof:

Let B be the set of all bit strings (strings of 0's and 1's).

Define a function F: $\mathbf{Z}^+ \to B$ as follows: $F(1) = \epsilon$, $F(2) = 0$, $F(3) = 1$, $F(4) = 00$, $F(5) = 01$, $F(6) = 10$, $F(7) = 11$, $F(8) = 000$, $F(9) = 001$, $F(10) = 010$, and so forth.

At each stage, all the strings of length k are counted before the strings of length $k+1$, and the strings of length k are counted in order of increasing magnitude when interpreted as binary representations of integers.

Thus the set of all bit strings is countably infinite and hence countable.

Note: A more formal definition for F is the following:

$$F(n) = \begin{cases} \epsilon & \text{if } n = 1 \\ \text{the } k\text{-bit binary representation of } n - 2^k & \text{if } n > 1 \text{ and } \lfloor \log_2 n \rfloor = k. \end{cases}$$

For example, $F(7) = 11$ because $\lfloor \log_2 7 \rfloor = 2$ and the two-bit binary representation of $7 - 2^2$ $(= 3)$ is 11.

18. No. For instance, both $\sqrt{2}$ and $-\sqrt{2}$ are irrational (by Theorem 4.7.1 and exercise 23 in Section 4.6), and yet their average is $(\sqrt{2} + (-\sqrt{2}))/2$ which equals 0 and is rational.

More generally: If r is any rational number and x is any irrational number, then both $r+x$ and $r-x$ are irrational (by the result of exercise 12 in Section 4.6 or by the combination of Theorem 4.6.3 and exercise 9 in Section 4.6). Yet the average of these numbers is $((r+x)+(r-x))/2 = r$, which is rational.

21. *Two examples of many:* Define F: $\mathbf{Z} \to \mathbf{Z}$ by the rule $F(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. Then F is onto because given any integer m, $m = F(2m)$. But F is not one-to-one because, for instance, $F(1) = F(3) = 0$.

Define G: $\mathbf{Z} \to \mathbf{Z}$ by the rule $G(n) = \lfloor n/2 \rfloor$ for all integers n. Then G is onto because given any integer m, $m = \lfloor m \rfloor = \lfloor (2m)/2 \rfloor = G(2m)$. But G is not one-to-one because, for instance, $G(2) = \lfloor 2/2 \rfloor = 1$ and $G(3) = \lfloor 3/2 \rfloor = 1$ and $2 \neq 3$.

24. The proof given below is adapted from one in *Foundations of Modern Analysis* by Jean Dieudonné, New York: Academic Press, 1969, page 14.

Proof: : Suppose (a,b) and (c,d) are in $\mathbf{Z}^+ \times \mathbf{Z}^+$ and $(a,b) \neq (c,d)$.

Case 1, $a+b \neq c+d$: By interchanging (a,b) and (c,d) if necessary, we may assume that $a+b < c+d$. Then

$$\begin{array}{rlll} & H(a,b) & = & b+\dfrac{(a+b)(a+b+1)}{2} & \text{by definition of } H \\ \Rightarrow & H(a,b) & \leq & a+b+\dfrac{(a+b)(a+b+1)}{2} & \text{because } a \geq 0 \\ \Rightarrow & H(a,b) & < & (a+b+1)+\dfrac{(a+b)(a+b+1)}{2} & \text{because } a+b < a+b+1 \\ \Rightarrow & H(a,b) & < & \dfrac{2(a+b+1)}{2}+\dfrac{(a+b)(a+b+1)}{2} & \\ \Rightarrow & H(a,b) & < & \dfrac{(a+b+1)(a+b+2)}{2} & \text{by factoring out } (a+b+1) \\ \Rightarrow & H(a,b) & < & \dfrac{(c+d)(c+d+1)}{2} & \text{since } a+b<c+d \text{ and } a,\ b,\ c, \\ & & & & \text{and } d \text{ are integers, } a+b+1 \leq c+d \\ \Rightarrow & H(a,b) & < & d+\dfrac{(c+d)(c+d+1)}{2} & \text{because } d \geq 0 \\ \Rightarrow & H(a,b) & < & H(c,d) & \text{by definition of } H. \end{array}$$

Therefore, $H(a,b) \neq H(c,d)$.

Case 2, $a+b = c+d$: First observe that in this case $b \neq d$. For if $b = d$, then subtracting b from both sides of $a+b = c+d$ gives $a = c$, and so $(a,b) = (c,d)$, which contradicts our assumption that $(a,b) \neq (c,d)$. Hence,

$$H(a,b) = b+\frac{(a+b)(a+b+1)}{2} = b+\frac{(c+d)(c+d+1)}{2} \neq d+\frac{(c+d)(c+d+1)}{2} = H(c,d),$$

and so $H(a,b) \neq H(c,d)$.

Thus both in case 1 and in case 2, $H(a,b) \neq H(c,d)$, and hence H is one-to-one.

30. Proof by contradiction:

Suppose not. That is, suppose the set of all irrational numbers were countable.

Then the set of all real numbers could be written as a union of two countably infinite sets: the set of all rational numbers and the set of all irrational numbers.

By exercise 29 this union is countably infinite, and so the set of all real numbers would be countably infinite and hence countable.

But this contradicts the fact that the set of all real numbers is uncountable (which follows immediately from Theorems 7.4.2 and 7.4.3 or Corollary 7.4.4).

Hence the set of all irrational number is uncountable.

33. Proof:

First note that there are as many equations of the form $x^2+bx+c=0$ as there are pairs (b,c) where b and c are in $\mathbf{Z}$.

By exercise 32, the set of all such pairs is countably infinite, and so the set of equations of the form $x^2+bx+c=0$ is countably infinite.

Next observe that, by the quadratic formula, each equation $x^2+bx+c=0$ has at most two solutions (which may be complex numbers):

$$x = \frac{-b+\sqrt{b^2-4c}}{2} \quad \text{and} \quad x = \frac{-b-\sqrt{b^2-4c}}{2}.$$

Let

$$R_1 = \left\{ x \mid x = \frac{-b+\sqrt{b^2-4c}}{2} \quad \text{for some integers } b \text{ and } c \right\},$$

$$R_2 = \left\{ x \mid x = \frac{-b-\sqrt{b^2-4c}}{2} \quad \text{for some integers } b \text{ and } c \right\},$$

and $R = R_1 \cup R_2$. Then R is the set of all solutions of equations of the form $x^2 + bx + c = 0$ where b and c are integers.

Define functions F_1 and F_2 from the set of equations of the form $x^2 + bx + c = 0$ to the sets R_1 and R_2 as follows:

$$F_1(x^2 + bx + c = 0) = \frac{-b+\sqrt{b^2-4c}}{2} \quad \text{and} \quad F_2(x^2 + bx + c = 0) = \frac{-b-\sqrt{b^2-4c}}{2}.$$

Then F_1 and F_2 are onto functions defined on countably infinite sets, and so, by exercise 27, R_1 and R_2 are countable. Since any union of two countable sets is countable (exercise 31), $R = R_1 \cup R_2$ is countable.

36. Proof:

Let B be the set of all functions from $\mathbf{Z}^+$ to $\{0, 1\}$ and let D be the set of all functions from $\mathbf{Z}^+$ to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Elements of B can be represented as infinite sequences of 0's and 1's (for instance, 01101010110. . .) and elements of D can be represented as infinite sequences of digits from 0 to 9 inclusive (for instance, 20775931124. . .).

We define a function $H\colon B \to D$ as follows: For each function f in B, consider the representation of f as an infinite sequence of 0's and 1's.

Such a sequence is also an infinite sequence of digits chosen from 0 to 9 inclusive (one formed without using 2,3,. . . ,9), which represents a function in D. We define this function to be $H(f)$.

More formally, for each $f \in B$, let $H(f)$ be the function in D defined by the rule $H(f)(n) = f(n)$ for all $n \in \mathbf{Z}^+$.

It is clear from the definition that H is one-to-one.

We define a function $K\colon D \to B$ as follows: For each function g in D, consider the representation of g as a sequence of digits from 0 to 9 inclusive.

Replace each of these digits by its 4-bit binary representation adding leading 0's if necessary to make a full four bits. (For instance, 2 would be replaced by 0010.)

The result is an infinite sequence of 0's and 1's, which represents a function in B. This function is defined to be $K(g)$.

Note that K is one-to-one because if $g_1 \neq g_2$ then the sequences representing g_1 and g_2 must have different digits in some position m, and so the corresponding sequences of 0's and 1's will differ in at least one of the positions $4m - 3, 4m - 2, 4m - 1$, or $4m$, which are the locations of the 4-bit binary representations of the digits in position m.

It can be shown that whenever there are one-to-one functions from one set to a second and from the second set back to the first, then the two sets have the same cardinality. This fact is known as the Schröder-Bernstein theorem after its two discoverers. For a proof see, for example, *Set Theory and Metric Spaces* by Irving Kaplansky, *A Survey of Modern Algebra*, Third Edition, by Garrett Birkhoff and Saunders MacLane, *Naive Set Theory* by Paul Halmos, or *Topology* by James R. Munkres. The above discussion shows that there are one-to-one functions from B to D and from D to B, and hence by the Schröder-Bernstein theorem the two sets have the same cardinality.

Review Guide: Chapter 7

Definitions: How are the following terms defined?

- function f from a set X to a set Y *(p. 384)*
- Let f be a function from a set X to a set Y.
  - the domain, co-domain, and range of f *(p. 384)*
  - the value of f at x, where x is in X *(p. 384)*
  - the image of x under f, where x is in X *(p. 384)*
  - the output of f for the input x, where x is in X *(p. 384)*
  - the image of X under f *(p. 384)*
  - an inverse image of y, where y is in Y *(p. 384)*
  - the identity function on a set *(p. 387)*
  - the image of A, where $A \subseteq X$ *(p. 392)*
  - the inverse image of B, where $B \subseteq Y$ *(p. 392)*
- logarithm with base b of a positive number x and the logarithmic function with base b *(p. 388)*
- Hamming distance function *(p. 389)*
- Boolean function *(p. 390)*
- one-to-one function *(p. 397)*
- onto function *(p. 402)*
- exponential function with base b *(p. 405)*
- one-to-one correspondence *(p. 408)*
- inverse function *(p. 411)*
- composition of functions *(p. 417)*
- cardinality *(pp. 428-429)*
- countable and uncountable sets.*(p. 431)*

General Function Facts

- How do you draw an arrow diagram for a function defined on a finite set? *(p. 384)*
- Given a function defined by an arrow diagram or by a formula, how do you find values of the function, the range of the function, and the inverse image of an element in its co-domain? *(p. 385)*
- How do you show that two functions are equal? *(p. 386)*
- What is the relation between a sequence and a function? *(p. 387)*
- Can you give an example of a function defined on a power set? a function defined on a Cartesian product? *(p. 387-388)*
- What is an example of an encoding function? a decoding function? *(p. 389)*
- If the claim is made that a given formula defines a function from a set X to a set Y, how do you determine that the "function" is not well-defined? *(p. 391)*

One-to-one and Onto

- How do you show that a function is not one-to-one? *(pp. 397-400)*
- How do you show that a function defined on an infinite set is one-to-one? *(pp. 399-400)*
- How do you show that a function is not onto? *(pp. 402-405)*
- How do you show that a function defined on an infinite set is onto? *(pp. 403-405)*
- How do you determine if a given function has an inverse function? *(p. 411)*
- How do you find an inverse function if it exists? *(pp. 411-413)*

Exponents and Logarithms

- What are the four laws of exponents? *(p. 406)*
- What are the properties of logarithms that correspond to the laws of exponents? *(p. 406)*
- How can you use the laws of exponents to derive properties of logarithms? *(p. 407)*
- How are the logarithmic function with base b and the exponential function with base b related? *(p. 411)*

Composition of Functions

- How do you compute the composition of two functions? *(pp. 417-419)*
- What is the composition of a function with its inverse? *(p. 421)*
- Why is a composition of one-to-one functions one-to-one? *(pp. 421-422)*
- Why is a composition of onto functions onto? *(pp. 423-424)*

Applications of Functions

- What is a Hash function? *(p. 401)*
- How do you show that one set has the same cardinality as another? *(pp. 429-430)*
- How do you show that a given set is countably infinite? countable? *(p. 431)*
- How do you show that the set of all positive rational numbers is countable? *(p. 433)*
- How is the Cantor diagonalization process used to show that the set of real numbers between 0 and 1 is uncountable? *(pp. 433-435)*
- How do you show that the set of all computer programs in a given computer language is countable? *(pp. 437-438)*

Chapter 8: Relations

The first section of this chapter focuses on understanding equivalent ways to specify and represent relations, both finite and infinite. In Section 8.2 the reflexivity, symmetry, and transitivity properties of binary relations are introduced and explored, and in Section 8.3 equivalence relations are discussed. As you work on these sections, you will frequently use the fact that the same proof outlines are used to prove and disprove universal conditional statements no matter what their mathematical context.

Section 8.4 deepens and extends the discussion of congruence relations in Sections 8.2 and 8.3 through applications to modular arithmetic and cryptography. The section is designed to make it possible to give you meaningful practice with RSA cryptography without having to spend several weeks on the topic. After a brief introduction to the idea of cryptography, the first part of the section is devoted to helping you develop the facility with modular arithmetic that is needed to perform the computations for RSA cryptography, especially finding least positive residues of integers raised to large positive powers and using the Euclidean algorithm to compute positive inverses modulo a number. Proofs of the underlying mathematical theory are left to the end of the section.

Section 8.5 introduces another type of binary relation that is especially important in computer science: partial order relations

Section 8.1

3. *c. One possible answer*: 4, 7, 10, -2, -5

   d. One possible answer: 5, 8, 11, -1, -4

   e. Theorem:

   1. All integers of the form $3k$ are related by T to 0.
   2. All integers of the form $3k+1$ are related by T to 1.
   3. All integers of the form $3k+2$ are related by T to 2.

   Proof of (2): Let n be any integer of the form $n = 3k+1$ for some integer k. By substitution, $n-1 = (3k+1) - 1 = 3k$, and so by definition of divisibility, $3 \mid (n-1)$. Hence by definition of T, $n\,T\,1$.

   The proofs of (1) and (3) are identical to the proof of (2) with 0 and 2, respectively, substituted in place of 1.

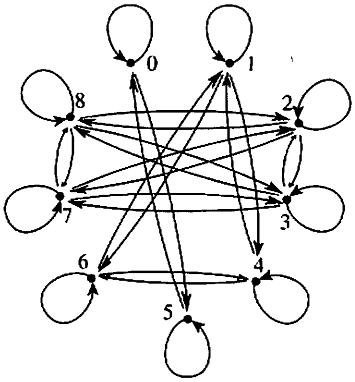
6. *b.* Yes, because $\{a,b\} \cap \{b,c\} = \{b\} \neq \emptyset$. *c.* Yes, because $\{a,b\} \cap \{a,b,c\} = \{a,b\} \neq \emptyset$.

9. *c.* No, because the sum of the characters in 2212 is 7 and the sum of the characters in 2121 is 6, and $7 \neq 6$.

   d. Yes, because the sum of the characters in 1220 is 5 and the sum of the characters in 2111 is 5, and $5 = 5$.

12. *b.* No. If $F\colon X \to Y$ is not one-to-one, then there exist x_1 and x_2 in X and y in Y such that $(x_1, y) \in F$ and $(x_2, y) \in F$ and $x_1 \neq x_2$. But this implies that there exist x_1 and x_2 in X and y in Y such that $(y, x_1) \in F^{-1}$ and $(y, x_2) \in F^{-1}$ and $x_1 \neq x_2$. Consequently, F^{-1} does not satisfy property (2) of the definition of function.

18.



24. *b.*	466581	Mary Lazars
	778400	Jamal Baskers

Section 8.2

18. ***Q is reflexive***: Suppose x is any real number. *[We must show that $x\ Q\ x$.]* By definition of Q, this means that $x - x$ is rational. But this is true because $x - x = 0$, and 0 is rational since $0 = 0/1$. *[So $x\ Q\ x$ as was to be shown.]*

Q is symmetric: Suppose x and y are any real numbers such that $x\ Q\ y$. *[We must show that $y\ Q\ x$.]* By definition of Q, $x - y$ is rational. Now $y - x = -(x - y)$ and the negative of any rational number is rational *[by exercise 13, Section 4.2]*. Hence $y - x$ is rational, and so $y\ Q\ x$ by definition of Q *[as was to be shown]*.

Q is transitive: Suppose x, y and z are any real numbers such that $x\ Q\ y$ and $y\ Q\ z$. *[We must show that $x\ Q\ z$.]* By definition of Q, $x - y$ is rational and $y - z$ is rational.

Then, since $x - p = (x - y) + (y - z)$, we have that $x - z$ is a sum of rational numbers, and hence $x - z$ is rational *[by Theorem 4.2.2]*. Thus, by definition of Q, $x\ Q\ z$ *[as was to be shown]*.

21. For each set X, let $N(X)$ be the number of elements in X.

L ***is not reflexive***: **L** is reflexive ⇔ for all sets A ∈ $\mathscr{P}(X)$, A **L** A. By definition of **L** this means that for all sets A in $\mathscr{P}(X)$, N(A) < N(A). But this is false for every set in $\mathscr{P}(X)$. For instance, let A = ∅. Then N(A) = 0, and 0 is not less than 0.

L ***is not symmetric***: For **L** to be symmetric would mean that for all sets A and B in $\mathscr{P}(X)$, if A **L** B then B **L** A. By definition of **L**, this would mean that for all sets A and B in $\mathscr{P}(X)$, if N(A) < N(B), then N(B) < N(A). But this is false for all sets A and B in $\mathscr{P}(X)$. For instance, take A = ∅ and B = $\{a\}$. Then N(A) = 0 and N(B) = 1. It follows that A is related to B by **L** (since $0 < 1$), but B is not related to A by **L** (since $1 \not< 0$).

L ***is transitive***: To prove transitivity of **L**, we must show that for all sets A, B, and C in $\mathscr{P}(X)$, if A **L** B and B **L** C then A **L** C. By definition of **L** this means that for all sets A, B, and C in $\mathscr{P}(X)$, if N(A) < N(B) and N(B) < N(C), then N(A) < N(C). But this is true by the transitivity property of order (Appendix A, T18).

24. **U** ***is not reflexive***: **U** is reflexive ⇔ for all sets A in $\mathscr{P}(X)$, A **U** A. By definition of **U** this means that for all sets A in $\mathscr{P}(X)$, A ≠ A. But this is false for every set in $\mathscr{P}(X)$. For instance, let A = ∅. It is not true that ∅ ≠ ∅.

U ***is symmetric***: **U** is symmetric ⇔ for all sets A and B in $\mathscr{P}(X)$, if A **U** B then B **U** A. By definition of **U**, this means that for all sets A and B in $\mathscr{P}(X)$, if A ≠ B, then B ≠ A. But this is true.

$\mathbf{U}$ ***is not transitive***: $\mathbf{U}$ is transitive $\Leftrightarrow$ for all sets A, B, and C in $\mathscr{P}(X)$, if A $\mathbf{U}$ B and B $\mathbf{U}$ C then A $\mathbf{U}$ C. By definition of $\mathbf{U}$ this means that for all sets A, B, and C in $\mathscr{P}(X)$, if A $\neq$ B and B $\neq Z$, then A $\neq$ C. But this is false as the following counterexample shows. Since $X \neq \emptyset$, there exists an element x in X. Let A $= \{x\}$, B $= \emptyset$, and C $= \{x\}$. Then A $\neq$ B and B $\neq Z$, but A $=$ C.

30. ***R is reflexive***: R is reflexive $\Leftrightarrow$ for all points p in A, $p\ R\ p$. By definition of R this means that for all elements p in A, p and p both lie on the same half line emanating from the origin. But this is true.

    R is symmetric:: *[We must show that for all points p_1 and p_2 in A, if $p_1 R\ p_2$ then $p_2 R\ p_1$.]* Suppose p_1 and p_2 are points in A such that $p_1 R\ p_2$. By definition of R this means that p_1 and p_2 lie on the same half line emanating from the origin. But this implies that p_2 and p_1 lie on the same half line emanating from the origin. So by definition of R, $p_2 R\ p_1$.

    R is transitive: *[We must show that for all points p_1, p_2 and p_3 in A, if $p_1 R\ p_2$ and $p_2 R\ p_3$ then $p_1 R\ p_3$.]* Suppose p_1, p_2, and p_3 are points in A such that $p_1 R\ p_2$ and $p_2 R\ p_3$. By definition of R, this means that p_1 and p_2 lie on the same half line emanating from the origin and p_2 and p_3 lie on the same half line emanating from the origin.

    Since two points determine a line, it follows that both p_1 and p_3 lie on the same half line determined by the origin and p_2. Thus p_1 and p_3 lie on the same half line emanating from the origin. So by definition of R, $p_1 R\ p_3$.

33. ***R is not reflexive***: R is reflexive $\Leftrightarrow$ for all lines l in A, $l\ R\ l$. By definition of R this means that for all lines l in the plane, l is perpendicular to itself. But this is false for every line in the plane.

    R is symmetric: *[We must show that for all lines l_1 and l_2 in A, if $l_1 R\ l_2$ then $l_2 R\ l_1$.]* Suppose l_1 and l_2 are lines in A such that $l_1 R\ l_2$. By definition of R this means that l_1 is perpendicular to l_2. But this implies that l_2 is perpendicular to l_1. So by definition of R, $l_2 R\ l_1$.

    R is not transitive: R is transitive $\Leftrightarrow$ for all lines l_1, l_2, and l_3 in A, if $l_1 R\ l_2$ and $l_2 R\ l_3$ then $l_1 R\ l_3$. But this is false. As a counterexample, take l_1 and l_3 to be the horizontal axis and l_2 to be the vertical axis. Then $l_1 R\ l_2$ and $l_2 R\ l_3$ because the horizontal axis is perpendicular to the vertical axis and the vertical axis is perpendicular to the horizontal axis. But $l_1 \not R\ l_3$ because the horizontal axis is not perpendicular to itself.

36. The statement is true.

    Proof: Suppose R is a transitive relation on a set A. To show that R^{-1} is transitive, we suppose that x, y, and z are any elements of A such that $x\ R^{-1}\ y$ and $y\ R^{-1}\ z$. *[We must show that $x\ R^{-1}\ z$.]* By definition of R^{-1}, $y\ R\ x$ and $z\ R\ y$, or, equivalently, $z\ R\ y$ and $y\ R\ x$. Since R is transitive, $z\ R\ x$. Thus, by definition of R^{-1}, $z\ R^{-1}\ x$ *[as was to be shown]*.

39. ***$R \cap S$ is transitive***: Suppose R and S are transitive. *[To show that $R \cap S$ is transitive, we must show that $\forall x, y, z \in A$, if $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$ then $(x, z) \in R \cap S$.]* So suppose x, y, and z are elements of A such that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. By definition of intersection, $(x, y) \in R$, $(x, y) \in S$, $(y, z) \in R$, and $(y, z) \in S$. It follows that $(x, z) \in R$ because R is transitive and $(x, y) \in R$ and $(y, z) \in R$. Also $(x, z) \in S$ because S is transitive and $(x, y) \in S$ and $(y, z) \in S$. Thus by definition of intersection $(x, z) \in R \cap S$.

42. ***$R \cup S$ is not necessarily transitive***: As a counterexample, let $R = \{(0, 1)\}$ and $S = \{(1, 2)\}$. Then both R and S are transitive (by default), but $R \cup S = \{(0, 1), (1, 2)\}$ is not transitive because $(0, 1) \in R \cup S$ and $(1, 2) \in R \cup S$ but $(0, 2) \notin R \cup S$. As another counterexample, let $R = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x < y\}$ and let $S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x > y\}$.

    Then both R and S are transitive because of the transitivity of order for the real numbers. But $R \cup S = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid x \neq y\}$ is not transitive because, for instance, $(1, 2) \in R \cup S$ and $(2, 1) \in R \cup S$ but $(1, 1) \notin R \cup S$.

Section 8.3

6. distinct equivalence classes: $\{0, 3, -3\}$, $\{1, 4, -2\}$, $\{2, 5, -1, -4\}$

9. distinct equivalence classes: $\{\emptyset, \{0\}, \{1, -1\}, \{-1, 0, 1\}\}$, $\{\{1\}, \{0, 1\}\}$, $\{\{-1\}, \{0, -1\}\}$

12. $[0] = \{x \in A \mid 5 \text{ divides } (x^2 - 0)\} = \{0\}$
    $[1] = \{x \in A \mid 5 \text{ divides } (x^2 - 1)\} = \{x \in A \mid 5 \text{ divides } x - 1)(x + 1)\} = \{1, -1, 4, -4\}$
    $[2] = \{x \in A \mid 5 \text{ divides } (x^2 - 2^2)\} = \{x \in A \mid 5 \text{ divides } (x - 2)(x + 2)\} = \{2, -2, 3, -3\}$

15. *b.* false *c.* true *d.* true

18. *b.* Let $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $x = 1$, $y = 2$, and $z = 3$. Then both x and y are in A_1 and both y and z are in A_2, but x and z are not both in either A_1 or A_2.

21. (1) Proof:

    F is reflexive: Suppose m is any integer. Since $m - m = 0$ and $4 \mid 0$, we have that $4 \mid (m - m)$. Consequently, $m\ F\ m$ by definition of F.

    F is symmetric: Suppose m and n are any integers such that $m\ F\ n$. By definition of F this means that $4 \mid (m - n)$, and so, by definition of divisibility, $m - n = 4k$ for some integer k. Now $n - m = -(m - n)$. Hence by substitution, $n - m = -(4k) = 4 \cdot (-k)$. It follows that $4 \mid n - m$ by definition of divisibility (since $-k$ is an integer), and thus $n\ F\ m$ by definition of F.

    F is transitive: Suppose m, n and p are any integers such that $m\ F\ n$ and $n\ F\ p$. By definition of F, this means that $4 \mid (m-n)$ and $4 \mid (n-p)$, and so, by definition of divisibility, $m-n = 4k$ for some integer k, and $n-p = 4l$ for some integer l. Now $m-p = (m-n)+(n-p)$. Hence by substitution, $m-p = 4k+4l = 4(k+l)$. It follows that $4 \mid (m-p)$ by definition of divisibility (since $k+l$ is an integer), and thus $m\ F\ p$ by definition of F.

    F is an equivalence relation because it is reflexive, symmetric, and transitive.

    (2) Four distinct classes: $\{x \in \mathbf{Z} \mid x = 4k \text{ for some integer } k\}$, $\{x \in \mathbf{Z} \mid x = 4k + 1 \text{ for some integer } k\}$, $\{x \in \mathbf{Z} \mid x = 4k + 2 \text{ for some integer } k\}$, $\{x \in \mathbf{Z} \mid x = 4k + 3 \text{ for some integer } k\}$

24. (1) Proof:

    R is reflexive because for each identifier x in A, x has the same memory location as x.

    R is symmetric because for all identifiers x and y in A, if x has the same memory location as y then y has the same memory location as x.

    R is transitive because for all identifiers x, y, and z in A, if x has the same memory location as y and y has the same memory location as z then x has the same memory location as z.

    R is an equivalence relation because it is reflexive, symmetric, and transitive.

    (2) There are as many distinct equivalence classes as there are distinct memory locations that are used to store variables during execution of the program. Each equivalence class consists of all variables that are stored in the same location.

27. (1) Proof:

    R is reflexive: Suppose m is any integer. Since $m^2 - m^2 = 0$ and $4 \mid 0$, we have that $4 \mid (m^2 - m^2)$. Consequently, $m\ R\ m$ by definition of R.

    R is symmetric: Suppose m and n are any integers such that $m\ R\ n$. By definition of R this means that $4 \mid (m^2 - n^2)$, and so, by definition of divisibility, $m^2 - n^2 = 4k$ for some integer k. Now $n^2 - m^2 = -(m^2 - n^2)$. Hence by substitution, $n^2 - m^2 = -(4k) = 4 \cdot (-k)$. It follows that $4 \mid (n^2 - m^2)$ by definition of divisibility (since $-k$ is an integer), and thus $n\ R\ m$ by definition of R.

R is transitive: Suppose m, n and p are any integers such that $m\ R\ n$ and $n\ R\ p$. By definition of R, this means that $4 \mid (m^2 - n^2)$ and $4 \mid (n^2 - p^2)$, and so, by definition of divisibility, $m^2 - n^2 = 4k$ for some integer k, and $n^2 - p^2 = 4l$ for some integer l. Now $m^2 - p^2 = (m^2 - n^2) + (n^2 - p^2)$. Hence by substitution, $m^2 - p^2 = 4k + 4l = 4(k + l)$. It follows that $4 \mid (m^2 - p^2)$ by definition of divisibility (since $k+l$ is an integer), and thus $m\ R\ p$ by definition of R.

R is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) If m is even, then $m = 2a$ for some integer a, and so $m^2 - 0^2 = (2a)^2 = 4a^2$, which is divisible by 4. Hence $m \in [0]$.

If m is odd, then $m = 2a+1$ for some integer a, and so $m^2 - 1^2 = (2a+1)^2 - 1 = 4a^2 + 4a + 1 - 1 = 4a^2 + 4a$, which is divisible by 4. Hence $m \in [1]$.

Thus there are two distinct equivalence classes:

$[0] = \{m \in \mathbf{Z} \mid m \text{ is even}\}$ and $[1] = \{m \in \mathbf{Z} \mid m \text{ is odd}\}$.

30. (1) Proof:

Q is reflexive because each ordered pair has the same second element as itself.

Q is symmetric for the following reason: Suppose (w, x) and (y, z) are ordered pairs of real numbers such that $(w, x)\ Q\ (y, z)$. Then, by definition of Q, $x = z$. By the symmetric property of equality, this implies that $z = x$, and so, by definition of Q, $(y, z)\ Q\ (w, x)$.

Q is transitive for the following reason: Suppose (u, v), (w, x), and (y, z) are ordered pairs of real numbers such that $(u, v)\ Q\ (w, x)$ and $(w, x)\ Q\ (y, z)$. Then, by definition of Q, $v = x$ and $x = z$. By the transitive property of equality, this implies that $v = z$, and so, by definition of Q, $(u, v)(y, z)\ Q\ (y, z)$.

Q is an equivalence relation because it is reflexive, symmetric, and transitive.

(2) There is one equivalence class for each real number. The distinct equivalence classes are all the sets of the form $\{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y = b\}$ where b is a real number. Equivalently, the distinct equivalence classes are all the vertical lines in the Cartesian plane.

33. The distinct equivalence classes can be identified with the points on a geometric figure, called a *torus*, that has the shape of the surface of a doughnut.

Each point in the interior of the rectangle $\{(x, y) \mid 0 < x < 1 \text{ and } 0 < y < 1\}$ is only equivalent to itself.

Each point on the top edge of the rectangle is in the same equivalence class as the point vertically below it on the bottom edge of the rectangle (so we can imagine identifying these points by gluing them together — this gives us a cylinder).

In addition, each point on the left edge of the rectangle is in the same equivalence class as the point horizontally across from it on the right edge of the rectangle (so we can also imagine identifying these points by gluing them together — this brings the two ends of the cylinder together to produce a torus).

39. Proof:

Suppose R is an equivalence relation on a set A, a and b are in A, and $[a] = [b]$.

Since R is reflexive, $a\ R\ a$, and so by definition of class, $a \in [a]$. *[Alternatively, one could reference exercise 36 here.]*

Since $[a] = [b]$, by definition of set equality, $a \in [b]$.

But then by definition of equivalence class, $a\ R\ b$.

42. *a.* Suppose $(a, b) \in A$. By commutativity of multiplication for the real numbers, $ab = ba$. But then by definition of R, $(a, b)R(a, b)$, and so R is reflexive.

b. Suppose $(a, b), (c, d) \in A$ and $(a, b)R(c, d)$. By definition of R, $ad = bc$, and so by commutativity of multiplication for the real numbers and symmetry of equality, $cb = da$. But then by definition of R, $(c, d)R(a, b)$, and so R is symmetric.

d. For example, (2,5), (4,10), (-2,-5), and (6,15) are all in [(2,5)].

45. The given argument assumes that from the fact that the statement "$\forall x$ in A, if $x\ R\ y$ then $y\ R\ x$" is true, it follows that given any element x in R, there must exist an element y in R such that $x\ R\ y$ and $y\ R\ x$. This is false. For instance, consider the following relation R defined on $A = \{1, 2\} : R = \{(1, 1)\}$. This relation is symmetric and transitive, but it is not reflexive. Given $2 \in A$, there is no element y in A such that $(2, y) \in R$. Thus we cannot go on to use symmetry to say that $(y, 2) \in R$ and transitivity to conclude that $(2, 2) \in R$.

Section 8.4

6. Proof:

Given any integer $n > 1$ and any integer a with $0 \leq a < n$, the notation $[a]$ denotes the equivalence class of a for the relation of congruence modulo n.

We first show that given any integer m, m is in one of the classes $[0], [1], [2], \ldots, [n - 1]$.

The reason is that, by the quotient-remainder theorem, $m = nk + a$, where k and a are integers and $0 \leq a < n$, and so, by Theorem 8.4.1, $m \equiv a \pmod{n}$. It follows by Lemma 8.3.2 that $[m] = [a]$.

Next we use an argument by contradiction to show that all the equivalence classes $[0], [1], [2], \ldots, [n-1]$ are distinct.

For suppose not. That is, suppose a and b are integers with $0 \leq a < n$ and $0 \leq b < n$, $a \neq b$, and $[a] = [b]$. Without loss of generality, we may assume that $a > b \geq 0$, which implies that $-a < -b \leq 0$. Adding a to all parts of the inequality gives $0 < a - b \leq a$. By Exercise 8.3.39, $[a] = [b]$ implies that $a \equiv b \pmod{n}$. Hence, by Theorem 8.4.1, $n \mid (a - b)$, and so, by Theorem 4.3.1, $n \leq a - b$.. But $a < n$. Thus $n \leq a - b \leq a < n$, which is contradictory. Therefore the supposition is false, and we conclude that all the equivalence classes $[0], [1], [2], \ldots, [n - 1]$ are distinct.

9. *b.* Proof:

Suppose a, b, c, d, and n are integers with $n > 1$, $a \equiv c \pmod{n}$, and $b \equiv d \pmod{n}$. *[We must show that $a - b \equiv (c - d) \pmod{n}$.]*

By definition, $a - c = nr$ and $b - d = ns$ for some integers r and s. Then

$$(a - b) - (c - d) = (a - c) - (b - d) = nr - ns = n(r - s).$$

But $r - s$ is an integer, and so, by definition, $a - b \equiv (c - d) \pmod{n}$.

12. *b.* Proof:

Suppose a is a positive integer. Then $a = \sum_{k=0}^{n} d_k 10^k$, for some nonnegative integer n and integers d_k where $0 \leq d_k < 10$ for all $k = 1, 2, \ldots, n$. By Theorem 8.4.3,

$$a = \sum_{k=0}^{n} d_k 10^k \equiv \sum_{k=0}^{n} d_k \cdot 1 \equiv \sum_{k=0}^{n} d_k \pmod{9}$$

because, by part (a), each $10^k \equiv 1 \pmod{9}$. Hence, by Theorem 8.4.1, both a and $\sum_{k=0}^{n} d_k$ have the same remainder upon division by 9, and thus if either one is divisible by 9, so is the other.

18. 48^1 *mod* $713 = 48$

48^2 *mod* $713 = 165$

48^4 *mod* $713 = 165^2$ *mod* $713 = 131$

48^8 *mod* $713 = 131^2$ *mod* $713 = 49$

48^{16} *mod* $713 = 49^2$ *mod* $713 = 262$

48^{32} *mod* $713 = 262^2$ *mod* $713 = 196$

48^{64} *mod* $713 = 196^2$ *mod* $713 = 627$

48^{128} *mod* $713 = 627^2$ *mod* $713 = 266$

48^{256} *mod* $713 = 266^2$ *mod* $713 = 169$

Hence, by Theorem 8.4.3,

$$48^{307} = 48^{256+32+16+2+1} = 48^{256}48^{32}48^{16}48^{2}48^{1} \equiv 169\cdot 196\cdot 262\cdot 165\cdot 48 \equiv 12 (\bmod 713),$$

and thus 48^{307} *mod* $713 = 12$.

21. The letters in EXCELLENT translate numerically into 05, 24, 03, 05,12, 12, 05, 14, 20. The solutions for exercises 19 (in Appendix B) and 20 (above) show that E, L, and C are encrypted as 15, 23, and 27, respectively. To encrypt X, we compute 24^3 *mod* $55 = 19$, to encrypt N, we compute 14^3 *mod* $55 = 49$, and to encrypt T, we compute 20^3 *mod* $55 = 25$. So the ciphertext is 15 19 27 15 23 23 15 49 25.

24. By Example 8.4.10, the decryption key is 27. Thus the residues modulo 55 for 51^{27}, 14^{27}, 49^{27}, and 15^{27} must be found and then translated into letters of the alphabet. Because $27 = 16 + 8 + 2 + 1$, we first perform the following computations:

$$\begin{array}{lll}
51^1 \equiv 51 \ (\bmod 55) & 14^1 \equiv 14 \ (\bmod 55) & 49^1 \equiv 49 \ (\bmod 55) \\
51^2 \equiv 16 \ (\bmod 55) & 14^2 \equiv 31 \ (\bmod 55) & 49^2 \equiv 36 \ (\bmod 55) \\
51^4 \equiv 16^2 \equiv 36 \ (\bmod 55) & 14^4 \equiv 31^2 \equiv 26 \ (\bmod 55) & 49^4 \equiv 36^2 \equiv 31 \ (\bmod 55) \\
51^8 \equiv 36^2 \equiv 31 \ (\bmod 55) & 14^8 \equiv 26^2 \equiv 16 \ (\bmod 55) & 49^8 \equiv 31^2 \equiv 26 \ (\bmod 55) \\
51^{16} \equiv 31^2 \equiv 26 \ (\bmod 55) & 14^{16} \equiv 16^2 \equiv 36 \ (\bmod 55) & 49^{16} \equiv 26^2 \equiv 16 \ (\bmod 55)
\end{array}$$

Then

51^{27} *mod* $55 = (26\cdot 31\cdot 16\cdot 51)$ *mod* $55 = 6$,

14^{27} *mod* $55 = (36\cdot 16\cdot 31\cdot 14)$ *mod* $55 = 9$,

49^{27} *mod* $55 = (16\cdot 26\cdot 36\cdot 49)$ *mod* $55 = 14$.

In addition, we know from the solution to exercise 23 above that 15^{27} *mod* $55 = 5$. But 6, 9, 14, and 5 translate into letters as F, I, N, and E. So the message is FINE.

27. *Step 1:* $4158 = 1568\cdot 2 + 1022$, and so $1022 = 4158 - 1568\cdot 2$

Step 2: $1568 = 1022\cdot 1 + 546$, and so $546 = 1568 - 1022$

Step 3: $1022 = 546\cdot 1 + 476$, and so $476 = 1022 - 546$

Step 4: $546 = 476\cdot 1 + 70$, and so $70 = 546 - 476$

Step 5: $476 = 70\cdot 6 + 56$, and so $56 = 476 - 70\cdot 6$

Step 6: $70 = 56\cdot 1 + 14$, and so $14 = 70 - 56$

Step 7: $56 = 14\cdot 4 + 0$, and so $\gcd(4158, 1568) = 14$,

which is the remainder obtained just before the final division.

Substitute back through steps 6–1:

$14 = 70 - 56 = 70 - (476 - 70\cdot 6) = 70\cdot 7 - 476$

$= (546 - 476) \cdot 7 - 476 = 7 \cdot 546 - 8 \cdot 476$

$= 7 \cdot 546 - 8 \cdot (1022 - 546) = 15 \cdot 546 - 8 \cdot 1022$

$= 15 \cdot (1568 - 1022) - 8 \cdot 1022 = 15 \cdot 1568 - 23 \cdot 1022$

$= 15 \cdot 1568 - 23 \cdot (4158 - 1568 \cdot 2) = 61 \cdot 1568 - 23 \cdot 4158$

(It is always a good idea to verify that no mistake has been made by verifying that the final expression really does equal the greatest common divisor. In this case, a computation shows that the answer is correct.)

30. Proof:

Suppose a and b are positive integers, $S = \{x \mid x \text{ is a positive integer and } x = as + bt \text{ for some integers } s \text{ and } t\}$, and c is the least element of S. We will show that $c \mid b$.

By the quotient-remainder theorem, $b = cq + r$ (*) for some integers q and r with $0 \leq r < c$.

Now because c is in S, $c = as + bt$ for some integers s and t. Thus, by substitution into equation (*),

$$r = b - cq = b - (as + bt)q = a(-sq) + b(1 - tq).$$

Hence, by definition of S, either $r = 0$ or $r \in S$.

But if $r \in S$, then $r \geq c$ because c is the least element of S, and thus both $r < c$ and $r \geq c$ would be true, which would be a contradiction.

Therefore, $r \notin S$, and thus by elimination, we conclude that $r = 0$.

It follows that $b - cq = 0$, or, equivalently, $b = cq$, and so $c \mid b$ *[as was to be shown]*.

33. Proof:

Suppose a, b, and c are integers such that $\gcd(a, b) = 1$, $a \mid c$, and $b \mid c$. We will show that $ab \mid c$.

By Corollary 8.4.6 (or by Theorem 8.4.5), there exist integers s and t such that $as + bt = 1$.

Also, by definition of divisibility, $c = au = bv$, for some integers u and v. Hence, by substitution,

$$c = asc + btc = as(bv) + bt(au) = ab(sv + tu).$$

But $sv + tu$ is an integer, and so, by definition of divisibility, $ab \mid c$ *[as was to be shown]*.

42. *b*. When $a = 8$ and $p = 11$,

$$a^{p-1} = 8^{10} = 1073741824 \equiv 1 (\bmod 11) \text{ because } 1073741824 - 1 = 11 \cdot 97612893.$$

Section 8.5

3. R is not antisymmetric.

Counterexample: Let $s = 0$ and $t = 1$. Then $s\ R\ t$ and $t\ R\ s$ because $l(s) \leq l(t)$ and $l(t) \leq l(s)$, since both $l(s)$ and $l(t)$ equal 1, but $s \neq t$.

6. R is a partial order relation.

Proof:

***R* is reflexive**: Suppose $r \in P$. Then $r = r$, and so by definition of R, $r\ R\ r$.

***R* is antisymmetric**: Suppose $r, s \in P$ and $r\ R\ s$ and $s\ R\ r$. *[We must show that $r = s$.]*

By definition of R, either r is an ancestor of s or $r = s$ and either s is an ancestor of r or $s = r$.

Now it is impossible for both r to be an ancestor of s and s to be an ancestor of r. Hence one of these conditions must be false, and so $r = s$ *[as was to be shown]*.

***R* is transitive**: Suppose $r, s, t \in P$ and $r\ R\ s$ and $s\ R\ t$. *[We must show that $r\ R\ t$.]*

By definition of R, either r is an ancestor of s or $r = s$ and either s is an ancestor of t or $s = t$.

In case r is an ancestor of s and s is an ancestor of t, then r is an ancestor of t, and so $r\ R\ t$.

In case r is an ancestor of s and $s = t$, then r is an ancestor of t, and so $r\ R\ t$.

In case $r = s$ and s is an ancestor of t, then r is an ancestor of t, and so $r\ R\ t$.

In case $r = s$ and $s = t$, then $r = t$, and so $r\ R\ t$. Thus in all four possible cases, $r\ R\ t$ *[as was to be shown]*.

Conclusion: Since R is reflexive, antisymmetric, and transitive, R is a partial order relation.

9. R is not a partial order relation because R is not antisymmetric.

Counterexample: Let $x = 2$ and $y = -2$. Then $x\ R\ y$ because $(-2)^2 \leq 2^2$, and $y\ R\ x$ because $2^2 \leq (-2)^2$. But $x \neq y$ because $2 \neq -2$.

12. Proof:

$\preceq$ *is reflexive*: Suppose s is in S. If $s = \epsilon$, then $s \preceq s$ by (3). If $s \neq \epsilon$, then $s \preceq s$ by (1). Hence in either case, $s \preceq s$.

$\preceq$ *is antisymmetric*: Suppose s and t are in S and $s \preceq t$ and $t \preceq s$. *[We must show that $s = t$.]*

By definition of S, either $s = \epsilon$ or $s = a_1a_2\ldots a_m$ and either $t = \epsilon$ or $t = b_1b_2\ldots b_n$ for some positive integers m and n and elements $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ in A.

It is impossible to have $s \preceq t$ by virtue of condition (2) because in that case there is no circumstance that would give $t \preceq s$.

[For suppose $s \preceq t$ by virtue of condition (2). Then for some integer k with $k \leq m$, $k \leq n$, and $k \geq 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, and $a_k\ R\ b_k$ and $a_k \neq b_k$. In this situation, it is clearly impossible for $t \preceq s$ by virtue either of condition (1) or (3), and so, if $t \preceq s$, then it must be by virtue of condition (2). But in that case, since $a_k \neq b_k$, it must follow that $b_k\ R\ a_k$, and so, since R is a partial order relation, $a_k = b_k$. However, this contradicts the fact that $a_k \neq b_k$. Hence it cannot be the case that $s \preceq t$ by virtue of condition (2).]

Similarly, it is impossible for $t \preceq s$ by virtue of condition (2).

Hence $s \preceq t$ and $t \preceq s$ by virtue either of condition (1) or of condition (3).

In case $s \preceq t$ by virtue of condition (1), then neither s nor t is the null string and so $t \preceq s$ by virtue of condition (1). Then by (1) $m \leq n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$ and $n \leq m$ and $b_i = a_i$ for all $i = 1, 2, \ldots, m$, and so, in this case, $s = t$.

In case $s \preceq t$ by virtue of condition (3), then $s = \epsilon$, and so since $t \preceq s$, $t \preceq \epsilon$. But the only situation that can give this result is condition (3) with $t = \epsilon$. Hence in this case, $s = t = \epsilon$.

Thus in all possible cases, if $s \preceq t$ and $t \preceq s$, then $s = t$ *[as was to be shown]*.

$\preceq$ *is transitive*: Suppose s and t are any elements of S such that $s \preceq t$ and $t \preceq u$. *[We must show that $s \preceq u$.]*

By definition of S, either $s = \epsilon$ or $s = a_1a_2\ldots a_m$, either $t = \epsilon$ or $t = b_1b_2\ldots b_n$, and either $u = \epsilon$ or $u = c_1c_2\ldots c_p$ for some positive integers m, n, and p and elements $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$, and $c_1, c_2, \ldots, c_p$ in A.

Case 1 ($s = \epsilon$): In this case, $s\ R\ u$ by (3).

Case 2 ($s \neq \epsilon$): In this case, since $s\ R\ t$, $t \neq \epsilon$, and since $t\ R\ u$, $u \neq \epsilon$.

Subcase a (s R t by condition (1) and t R u by condition (1)): Then $m \le n$ and $n \le p$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$ and $b_j = c_j$ for all $j = 1, 2, \ldots, n$. It follows that $a_i = c_i$ for all $i = i, 2, \ldots, m$, and so by (1), $s\ R\ u$.

Subcase b (s R t by condition (1) and t R u by condition (2)): Then $m \le n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$, and for some integer k with $k \le n$, $k \le p$, and $k \ge 1$, $b_j = c_j$ for all $j = 1, 2, \ldots, k-1$, $b_k\ R\ c_k$, and $b_k \ne c_k$.

If $k \le m$, then s and u satisfy condition (2) *[because $a_i = b_i$ for all $i = 1, 2, \ldots, m$ and so $k \le m$, $k \le p$, $k \ge 1$, $a_i = b_i = c_i$ for all $i = 1, 2, \ldots, k-1$, $a_k\ R\ c_k$, and $a_k \ne c_k$].*

If $k > m$, then s and u satisfy condition (1) *[because $a_i = b_i = c_i$ for all $i = 1, 2, \ldots, m$].* Thus in either case $s\ R\ u$.

Subcase c (s R t by condition (2) and t R u by condition (1)): Then for some integer k with $k \le m$, $k \le n$, $k \ge 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, $a_k\ R\ b_k$, and $a_k \ne b_k$, and $n \le p$ and $b_j = c_j$ for all $j = i, 2, \ldots, n$. Then s and u satisfy condition (2) *[because $k \le n$, $k \le p$ (since $k \le n$ and $n \le p$), $k \ge 1$, $a_i = b_i = c_i$ for all $i = 1, 2, \ldots, k-1$ (since $k - 1 < n$), $a_k Rc_k$ (since $b_k = c_k$ because $k \le n$), and $a_k \ne c_k$ (since $b_k = c_k$ and $a_k \ne b_k$)].* Thus $s\ R\ u$.

Subcase d (s R t by condition (2) and t R u by condition (2)): Then for some integer k with $k \le m$, $k \le n$, $k \ge 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, $a_k\ R\ b_k$, and $a_k \ne b_k$, and for some integer l with $l \le n$, $l \le p$, and $l \ge 1$, $b_j = c_j$ for all $j = 1, 2, \ldots, l-1$, $b_l\ R\ c_l$, and $b_l \ne c_l$.

If $k < l$, then $a_i = b_i = c_i$ for all $i = 1, 2, \ldots, k-1$, $a_k\ R\ b_k$, $b_k = c_k$ (in which case $a_k\ R\ c_k$), and $a_k \ne c_k$ (since $a_k \ne b_k$). Thus if, $k < l$, then $s \preceq u$ by condition (2).

If $k = l$, then $b_k\ R\ c_k$ (in which case $a_k\ R\ c_k$ by transitivity of R) and $b_k \ne c_k$. It follows that $a_k \ne c_k$ *[for if $a_k = c_k$, then $a_k\ R\ b_k$ and $b_k\ R\ a_k$, which implies that $a_k = b_k$ (since R is a partial order) and contradicts the fact that $a_k \ne b_k$].* Thus if $k = l$, then $s \preceq u$ by condition (2).

If $k > l$, then $a_i = b_i = c_i$ for all $i = 1, 2, \ldots, l-1$, $a_l\ R\ c_l$ (because $b_l\ R\ c_l$ and $a_l = b_l$), $a_l \ne c_l$ (because $b_l \ne c_l$ and $a_l = b_l$). Thus if $k > l$, then $s \preceq u$ by condition (2).

Hence, regardless of whether $k < l$, $k = l$, or $k > l$, we conclude that $s \preceq u$.

The above arguments show that in all possible situations, if $s \preceq t$ and $t \preceq u$ then $s \preceq u$ *[as was to be shown].* Hence $\preceq$ *is* transitive.

Conclusion: Since $\preceq$ *is* reflexive, antisymmetric, and transitive, $\preceq$ *is* a partial order relation.

15. Proof:

Suppose R is a relation on a set A and R is reflexive, symmetric, transitive, and anti-symmetric. We will show that R is the identity relation on A.

First note that for all x and y in A, if $x\ R\ y$ then, because R is symmetric, $y\ R\ x$. But then, because R is also anti-symmetric $x = y$. Thus for all x and y in A, if $x\ R\ y$ then $x = y$.

This argument, however, does not prove that R is the identity relation on A because the conclusion would also follow from the hypothesis (by default) in the case where $A \ne \emptyset$ and $R = \emptyset$.

But when $A \ne \emptyset$, it is impossible for R to equal $\emptyset$ because R is reflexive, which means that $x\ R\ x$ for every x in A.

Thus every element in A is related by R to itself, and no element in A is related to anything other than itself. It follows that R is the identity relation on A.

27. greatest element: (1,1)   least element: (0,0)

maximal elements: (1,1)   minimal elements: (0,0)

30. *c.* no greatest element and no least element

    d. greatest element: 9 least element: 1

33. A is not totally ordered by the given relation because $9 \nmid 12$ and $12 \nmid 9$.

36. {2, 4, 12, 24} or {3, 6, 12, 24}

45. One such total order is 3, 9, 2, 6, 18, 4, 12, 8.

48. One such total order is $\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}$.

51. *a.* 33 hours

Review Guide: Chapter 8

Definitions: How are the following terms defined?

- congruence modulo 2 relation *(p. 443)*
- inverse of a relation from a set A to a set B *(p. 444)*
- relation on a set *(p. 446)*
- directed graph of a relation on a set *(p. 446)*
- n-ary relation (and binary, ternary, quaternary relations) *(p. 447)*
- reflexive, symmetric, and transitive properties of a relation on a set *(p. 450)*
- congruence modulo 3 relation *(p. 455)*
- transitive closure of a relation on a set *(p. 457)*
- equivalence relation on a set *(p. 462)*
- equivalence class *(p. 465)*
- congruence modulo n relation *(p. 473)*
- representative of an equivalence class *(p. 472)*
- m is congruent to n modulo d *(p. 473)*
- plaintext and cyphertext *(p. 478)*
- residue of a modulo n *(p. 481)*
- complete set of residues modulo n *(p. 481)*
- d is a linear combination of a and b *(p. 486)*
- a and b are relatively prime; $a_1, a_2, \ldots, a_n$ are pairwise relatively prime *(p. 488)*
- an inverse of a modulo n *(p. 489)*
- antisymmetric relation *(p. 499)*
- partial order relation *(p. 500)*
- lexicographic order *(p. 502)*
- Hasse diagram *(p. 503)*
- a and b are comparable *(p. 505)*
- poset *(p. 506)*
- total order relation *(p. 506)*
- chain, length of a chain *(p. 506)*
- maximal element, greatest element, minimal element, least element *(p. 507)*
- topological sorting *(p. 507)*
- compatible partial order relations *(p. 508)*
- PERT and CPM *(p. 510)*
- critical path *(p. 512)*

Properties of Relations on Sets and Equivalence Relations

- How do you show that a relation on a finite set is reflexive? symmetric? transitive? *(pp. 450-452)*
- How do you show that a relation on an infinite set is reflexive? symmetric? transitive? *(pp. 453-456)*
- How do you show that a relation on a set is not reflexive? not symmetric? not transitive? *(pp. 451-454)*
- How do you find the transitive closure of a relation? *(p. 457)*
- What is the relation induced by a partition of a set? *(p. 460)*
- Given an equivalence relation on a set A, what is the relationship between the distinct equivalence classes of the relation and subsets of the set A? *(p. 469)*
- Given an equivalence relation on a set A and an element a in A, how do you find the equivalence class of a? *(pp. 465-467, 470-472)*
- In what way are rational numbers equivalence classes? *(pp. 473-474)*

Cryptography

- How does the Caesar cipher work? *(p. 478)*
- If a, b, and n are integers with $n > 1$, what are some different ways of expressing the fact that $n \mid (a - b)$? *(p. 480)*
- How do you reduce a number modulo n? *(p. 481)*
- If n is an integer with $n > 1$, is congruence modulo n an equivalence relation on the set of all integers? *(p. 481)*
- How do you add, subtract, and multiply integers modulo an integer $n > 1$? *(p. 482)*
- What is an efficient way to compute a^k where a is an integer with $a > 1$ and k is a large integer? *(pp. 484-485)*
- How do you express the greatest common divisor of two integers as a linear combination of the integers? *(p. 487)*
- When can you find an inverse modulo n for a positive integer a, and how do you find it? *(pp. 488-489)*
- How do you encrypt and decrypt messages using RSA cryptography? *(pp. 491-492)*
- What is Euclid's lemma? How is it proved? *(p. 492)*
- What is Fermat's little theorem? How is it proved? *(p. 494)*
- Why does the RSA cipher work? *(pp. 494-496)*

Partial Order Relations

- How do you show that a relation on a set is or is not antisymmetric? *(pp. 499-500)*
- If A is a set with a partial order relation R, S is a set of strings over A, and a and b are in S, how do you show that $a \preceq b$, where $\preceq$ denotes the lexicographic ordering of S? *(p. 502)*
- How do you construct the Hasse diagram for a partial order relation? *(p. 503)*
- How do you find a chain in a partially ordered set? *(p. 506)*
- Given a set with a partial order, how do you construct a topological sorting for the elements of the set? *(p. 508)*
- Given a job scheduling problem consisting of a number of tasks, some of which must be completed before others can be begun, how can you use a partial order relation to determine the minimum time needed to complete the job? *(pp. 511-512)*

Chapter 9: Counting and Probability

The primary aim of this chapter is to foster intuitive understanding for fundamental principles of counting and probability and an ability to apply them in a wide variety of situations. It is helpful to get into the habit of beginning a counting problem by listing (or at least imagining) some of the objects you are trying to count. If you see that all the objects to be counted can be matched up with the integers from m to n inclusive, then the total is $n - m + 1$ (Section 9.1). If you see that all the objects can be produced by a multi-step process, then the total can be found by counting the distinct paths from root to leaves in a possibility tree that shows the outcomes of each successive step (Section 9.2). And if each step of the process can be performed in a fixed number of ways (regardless of how the previous steps were performed), then the total can be calculated by applying the multiplication rule (Section 9.2).

If the objects to be counted can be separated into disjoint categories, then the total is just the sum of the subtotals for each category (Section 9.3). And if the categories are not disjoint, the total can be counted using the inclusion/exclusion rule (Section 9.3). If the objects to be counted can be represented as all the subsets of size r of a set with n elements, then the total is $\binom{n}{r}$ for which there is a computational formula (Section 9.5). Finally if the objects can be represented as all the multisets of size r of a set with n elements, then the total is $\binom{n+r-1}{r}$ (Section 9.6).

Section 9.4 introduces the pigeonhole principle, which provides a way to answer questions about how many of a certain object are needed to guarantee certain results and is used to show that certain results are guaranteed if a certain number of objects are present. The section includes the reasoning for why every rational number has a decimal expansion that either terminates or repeats.

Pascal's formula and the binomial theorem are discussed in Section 9.7. Each is proved both algebraically and combinatorially. Pascal's formula and the binomial theorem are discussed in Section 9.7. Each is proved both algebraically and combinatorially. Exercise 28 of Section 9.7 is intended to help you see how Pascal's formula is applied in the algebraic proof of the binomial theorem.

Sections 9.8 and 9.9 develop the axiomatic theory of probability through the concepts of expected value, conditional probability, independence, and Bayes' theorem. Exercise 20 of Section 9.1 can be solved directly by reasoning about the sample space, but it can also be solved using conditional probability, which is discussed in Section 9.9.

Section 9.1

6. $\{2\clubsuit, 3\clubsuit, 4\clubsuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 2\spadesuit, 3\spadesuit, 4\spadesuit\}$ Probability $= 12/52 = 3/13 \cong 23.1\%$

12. *b.* (ii) $\{GGB, GBG, BGG, GGG\}$ Probability $= 4/8 = 1/2 = 50\%$
    (iii) $\{BBB\}$ Probability $= 1/8 = 12.5\%$

15. The methods used to compute the probabilities in exercises 12, 13, and 14 are exactly the same as those in exercise 11. The only difference in the solutions are the symbols used to denote the outcomes; the probabilities are identical. These exercises illustrate the fact that computing various probabilities that arise in connection with tossing a coin is mathematically identical to computing probabilities in other, more realistic situations. So if the coin tossing model is completely understood, many other probabilities can be computed without difficulty.

27. Let k be the 62nd element in the array. By Theorem 9.1.1, $k - 29 + 1 = 62$, so $k = 62 + 29 - 1 = 90$. Thus the 62nd element in the array is $B[90]$.

30. $$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & 998 & 999 & 1000 & 1001 \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & \\ & 2\cdot 1 & & 2\cdot 2 & & 2\cdot 3 & & 2\cdot 499 & & 2\cdot 500 & \end{array}$$

    The diagram above shows that there are as many even integers between 1 and 1001 as there are integers from 1 to 500 inclusive. There are 500 such integers.

33. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$\text{The number of integers from } m \text{ to } n \text{ inclusive is } n - m + 1. \qquad \leftarrow P(n)$$

We will prove by mathematical induction that the property is true for all integers $n \geq m$.

Show that $P(m)$ is true: $P(m)$ is true because there is just one integer, namely m, from m to m inclusive. Substituting m in place of n in the formula $n - m + 1$ gives $m - m + 1 = 1$, which is correct.

Show that for all integers $k \geq m$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq m$ and suppose that

$$\text{The number of integers from } m \text{ to } k \text{ inclusive is } k - m + 1. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\text{The number of integers from } m \text{ to } k+1 \text{ inclusive is } (k+1) - m + 1. \qquad \leftarrow P(k+1)$$

Consider the sequence of integers from m to $k + 1$ inclusive:

$$\underbrace{m, \quad m+1, \quad m+2, \quad \ldots, \quad k}_{k-m+1}, \quad (k+1).$$

By inductive hypothesis there are $k - m + 1$ integers from m to k inclusive. So there are $(k - m + 1) + 1$ integers from m to $k + 1$ inclusive. But $(k - m + 1) + 1 = (k + 1) - m + 1$. So there are $(k + 1) - m + 1$ integers from m to $k + 1$ inclusive *[as was to be shown]*.

Section 9.2

12. *b.* Think of creating a string of hexadecimal digits that satisfies the given requirements as a 6-step process.

Step 1: Choose the first hexadecimal digit. It can be any hexadecimal digit from 4 through D (which equals 13). There are $13 - 4 + 1 = 10$ of these.

Steps 2-5: Choose the second through the fifth hexadecimal digits. Each can be any one of the 16 hexadecimal digits.

Step 6: Choose the last hexadecimal digit. It can be any hexadecimal digit from 2 through E (which equals 14). There are $14 - 2 + 1 = 13$ of these.

So the total number of the specified hexadecimal numbers is $10 \cdot 16 \cdot 16 \cdot 16 \cdot 16 \cdot 13 = 8{,}519{,}680$.

15. Think of creating combinations that satisfy the given requirements as multi-step processes in which each of steps 1-3 is to choose a number from 1 to 30, inclusive.

a. Because there are 30 choices of numbers in each of steps 1–3, there are $30^3 = 27{,}000$ possible combinations for the lock.

b. In this case we are given that no number may be repeated. So there are 30 choices for step 1, 29 for step 2, and 28 for step 3. Thus there are $30 \cdot 29 \cdot 28 = 24{,}360$ possible combinations for the lock.

18. *b.* Constructing a PIN that is obtainable by the same keystroke sequence as 5031 can be thought of as the following four-step process:

Step 1: Choose either the digit 5 or one of the three letters on the same key as the digit 5.

Step 2: Choose the digit 0.

Step 3: Choose the digit 3 or one of the three letters on the same key as the digit 3.

Step 4: Choose either the digit 1 or one of the two letters on the same key as the digit 1.

There are four ways to perform steps 1 and 3, one way to perform step 2, and three ways to perform step 4. So by the multiplication rule there are $4 \cdot 1 \cdot 4 \cdot 3 = 48$ different PINs that are keyed the same as 5031.

c. Constructing a numeric PIN with no repeated digit can be thought of as the following four-step process. Steps 1–4 are to choose the digits in position 1–4 (counting from the left). Because no digit may be repeated, there are 10 ways to perform step one, 9 ways to perform step two, 8 ways to perform step three, and 7 ways to perform step four. Thus the number of numeric PINs with no repeated digit is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

21. *a.* There are 2^{mn} relations from A to B because a relation from A to B is any subset of $A \times B$, $A \times B$ is a set with mn elements (since A has m elements and B has n elements), and the number of subsets of a set with mn elements is 2^{mn} (by Theorem 6.3.1).

b. In order to define a function from A to B we must specify exactly one image in B for each of the m elements in A. So we can think of constructing a function from A to B as an m-step process, where step i is to choose an image for the ith element of A (for $i = 1, 2, \ldots, m$). Because there are n choices of image for each of the m elements, by the multiplication rule, the total number of functions is $\underbrace{n \cdot n \cdot n \cdots n}_{m \text{ factors}} = n^m$.

c. The fraction of relations from A to B that are functions is $\dfrac{n^m}{2^{nm}} = \left(\dfrac{n}{2^n}\right)^m$

30. *a.* Call one of the integers r and the other s. Since r and s have no common factors, if p_i is a factor of r, then p_i is not a factor of s.

So for each $i = 1, 2, \ldots, m$, either $p_i{}^{k_i}$ is a factor of r or $p_i{}^{k_i}$ is a factor of s.

Thus, constructing r can be thought of as an m-step process in which step i is to decide whether $p_i{}^{k_i}$ is a factor of r or not.

There are two ways to perform each step, and so the number of different possible r's is 2^m.

Observe that once r is specified, s is completely determined because $s = n/r$.

Hence the number of ways n can be written as a product of two positive integers rs which have no common factors is 2^m. Note that this analysis assumes that order matters because, for instance, $r = 1$ and $s = n$ will be counted separately from $r = n$ and $s = 1$.

b. Each time that we can write n as rs, where r and s have no common factors, we can also write $n = sr$. So if order matters, there are twice as many ways to write n as a product of two integers with no common factors as there are if order does not matter. Thus if order does not matter, there are $2^m/2 = 2^{m-1}$ ways to write n as a product of two integers with no common factors.

33. *a.* The number of ways the 6 people can be seated equals the number or permutations of a set of 6 elements, namely, $6! = 720$.

b. Assuming that the row is bounded by two aisles, arranging the people in the row can be regarded as the following 2-step process:

Step 1: Choose the aisle seat for the doctor. *[There are 2 ways to do this.]*

Step 2: Choose an ordering for the remaining people. *[There are 5! ways to do this.]*

Thus, by the multiplication rule, the answer is $2\cdot 5! = 240$.

(If it is assumed that one end of the row is against a wall, then there is only one aisle seat and the answer is $5! = 120$.)

c. Each married couple can be regarded as a single item, so the number of ways to order the 3 couples is $3! = 6$.

36. *stu, stv, sut, suv, svt, svu, tsu, tsv, tus, tuv, tvs, tvu, ust, usv, uts, utv, uvs, uvt, vst, vsu, vts, vtu, vus, vut*

39. *b.* $P(9,6) = 9!/(9-6)! = 9!/3! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 60,480$

d. $P(7,4) = 7!/(7-4)! = 7!/3! = 7\cdot 6\cdot 5\cdot 4 = 840$

42. Proof 1: Let n be any integer such that $n \geq 3$. By the first version of the formula in Theorem 9.2.3,

$$\begin{aligned} P(n+1,3) - P(n,3) &= (n+1)n(n-1) - n(n-1)(n-2) \\ &= n(n-1)[(n+1)-(n-2)] \\ &= n(n-1)(n+1-n+2) \\ &= 3n(n-1) \\ &= 3P(n,2). \end{aligned}$$

Proof 2: Let n be any integer such that $n \geq 3$. By the second version of the formula in Theorem 9.2.3,

$$\begin{aligned} P(n+1,3) - P(n,3) &= \frac{(n+1)!}{((n+1)-3)!} - \frac{n!}{(n-3)!} \\ &= \frac{(n+1)!}{(n-2)!} - \frac{n!}{(n-3)!} \\ &= \frac{(n+1)\cdot n!}{(n-2)!} - \frac{(n-2)\cdot n!}{(n-2)\cdot(n-3)!} \\ &= \frac{n!((n+1)-(n-2))}{(n-2)!} \\ &= \frac{n!}{(n-2)!}\cdot 3 \\ &= 3P(n,2). \end{aligned}$$

45. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

The number of permutations of a set with n elements is $n!$. ← $P(n)$

We will prove by mathematical induction that the property is true for all integers $n \geq 1$.

Show that $P(1)$ is true: $P(1)$ is true because if a set consists of one element there is just one way to order it, and $1! = 1$.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$ and suppose that

The number of permutations of a set with k elements is $k!$. ← $P(k)$ inductive hypothesis

We must show that

number of permutations of a set with k elements is $(k+1)!$. ← $P(k+1)$

Let X be a set with $k+1$ elements. The process of forming a permutation of the elements of X can be considered a two-step operation as follows:

Step 1: Choose the element to write first.

Step 2: Write the remaining elements of X in some order.

Since X has $k+1$ elements, there are $k+1$ ways to perform step 1, and by inductive hypothesis there are $k!$ ways to perform step 2. Hence by the multiplication rule there are $(k+1)k! = (k+1)!$ ways to form a permutation of the elements of X. But this means that there are $(k+1)!$ permutations of X *[as was to be shown]*.

Section 9.3

6. *a.* For simplicity, start by assuming that a blank plate is allowed. Then the number of ways to construct a license plate can be thought of as a 2-step process, where step 1 is to choose the letters for the initial portion of the plate and Step 2 is to choose the digits for the second portion of the plate. Because anywhere from 0 to 3 letters may be chosen for the initial portion of the plate, by the addition rule, the number of ways to choose the letters for the initial portion of the plate is

$$\begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 0 letters}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 1 letter}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 2 letters}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 3 letters}\end{pmatrix}.$$

Because there are 26 letters in the alphabet, there is only one way to choose 0 letters, and 26 ways to choose one letter. The number of ways to choose two or three letters is computed using the multiplication rule. For example, choosing three letters can be thought of as a 3-step process: step 1 is to fill in the first letter, step 2 is to fill in the second letter, and step 3 is to fill in the third letter. Thus the number of ways to choose three letters is 26^3. Similarly, the number of ways to choose two letters is 26^2. It follows that the number of ways to choose from 0 to 3 letters is

$$1 + 26 + 26^2 + 26^3.$$

The same kind of reasoning can be applied to compute the number of ways to choose the digits for the second portion of the license plate, namely

$$\begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 0 digits}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 1 digit}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 2 digits}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 3 digits}\end{pmatrix} + \begin{pmatrix}\text{the number}\\ \text{of choices of}\\ \text{of 4 digits}\end{pmatrix}.$$

Because there are ten digits, there is only one way to choose 0 digits, and 10 ways to choose one digit. The number of ways to choose two, three, or four digits is computed using the multiplication rule. For example, choosing three digits can be thought of as a 3-step process: step 1 is to fill in the first digit, step 2 is to fill in the second digit, and step 3 is to fill in the third digit. Thus the number of ways to choose three digits is 26^3. Similarly, the number of ways to choose two digits is 26^2, and the number of ways to choose four digits is 26^4. It follows that the number of ways to choose from 0 to 4 digits is

$$1 + 10 + 10^2 + 10^3 + 10^4.$$

Since each choice of from 0 to 3 letters can be paired with each choice of from 0 to 4 digits, by the multiplication rule, the number of ways to choose from 0 to 3 letters to place in the initial portion of the license plate and from 0 to 4 digits to place in the final portion of the license plate is the product

$$(1 + 26 + 26^2 + 26^3)(1 + 10 + 10^2 + 10^3 + 10^4) = 203{,}097{,}969.$$

However, this number includes the blank plate, which is not allowed. So, by the difference rule, the total number of license plates is $203{,}097{,}969 - 1 = 203{,}097{,}968$.

9. *b.* On the ith iteration of the outer loop, there are i iterations of the inner loop, and this is true for each $i = 1, 2, \ldots, n$. Therefore, the total number of iterations of the inner loop is $1 + 2 + 3 + \cdots + n = n(n+1)/\ 2$.

12. *a.* The number of ways to arrange the 6 letters of the word *THEORY* in a row is $6! = 720$

    b. When the *TH* in the word *THEORY* are treated as an ordered unit, there are only 5 items to arrange, *TH, E, O, R,* and *Y.* and so there are 5! orderings. Similarly, there are 5! orderings for the symbols *HT, E, O, R,* and *Y.* Thus, by the addition rule, the total number of orderings is $5! + 5! = 120 + 120 = 240$.

15. The set of all possible identifiers may be divided into 30 non-overlapping subsets depending on the number of characters in the identifier. Constructing one of the identifiers in the kth subset can be regarded as a k-step process, where each step consists in choosing a symbol for one of the characters (say, going from left to right). Because the first character must be a letter, there are 26 choices for step 1, and because subsequent letters can be letters or digits or underscores there are 37 choices for each subsequent step. By the addition rule, we add up the number of identifiers in each subset to obtain a total. But because 82 of the resulting strings cannot be used as identifiers, by the difference rule, we subtract 82 from the total to obtain the final answer. Thus we have

    $(26 + 26 \cdot 37 + 26 \cdot 37^2 + \cdots + 26 \cdot 37^{29}) - 82 = 26(1 + 37 + 37^2 + \cdots + 37^{29}) - 82$

    $= 26 \cdot \sum_{k=0}^{29} 37^k - 82 = 26\left(\dfrac{37^{30} - 1}{37 - 1}\right) - 82 \cong 8.030 \times 10^{46} \cong.$

18. *b.* Proof: Let A and B be events in a sample space S. By the inclusion/exclusion rule (Theorem 9.3.3), $N(A \cup B) = N(A) + N(B) - N(A \cap B)$. So by the equally likely probability formula,

$$P(A \cup B) = \frac{N(A \cup B)}{N(S)} = \frac{N(A) + N(B) - N(A \cap B)}{N(S)} = \frac{N(A)}{N(S)} + \frac{N(B)}{N(S)} - \frac{N(A \cap B)}{N(S)}$$
$$= P(A) + P(B) - P(A \cap B).$$

21. Call the employees U, V, W, X, Y, and Z, and suppose that U and V are the married couple. Let A be the event that U and V have adjacent desks. Since the desks of U and V can be adjacent either in the order UV or in the order VU, the number of desk assignments with U and V adjacent is the same as the sum of the number of permutations of the symbols $\boxed{UV}$, W, X, Y, Z plus the number of permutations of the symbols $\boxed{VU}$, W, X, Y, Z. By the multiplication rule each of these is 5!, and so by the addition rule the sum is $2 \cdot 5!$. Since the total number of permutations of U, V, W, X, Y, Z is 6!,

$$P(A) = 2 \cdot \frac{5!}{6!} = \frac{2}{6} = \frac{1}{3}.$$

    Hence by the formula for the probability of the complement of an event,

$$P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}.$$

    So the probability that the married couple have nonadjacent desks is 2/3.

24. *a.* Let A and B be the sets of all integers from 1 through 1,000 that are multiples of 2 and 9 respectively. Then $N(A) = 500$ and $N(B) = 111$ (because $9 = 9 \cdot 1$ is the smallest integer in B and $999 = 9 \cdot 111$ is the largest). Also $A \cap B$ is the set of all integers from 1 through 1,000 that are multiples of 18, and $N(A \cap B) = 55$ (because $18 = 18 \cdot 1$ is the smallest integer in $A \cap B$ and $990 = 18 \cdot 55$ is the largest). It follows from the inclusion/exclusion rule that the number of integers from 1 through 1,000 that are multiples of 2 or 9 equals

$$N(A \cup B) = N(A) + N(B) - N(A \cup B) = 500 + 111 - 55 = 556.$$

b. The probability is $556/1000 = 55.6\%$.

c. $1000 - 556 = 444$

27. *a.* Let k be an integer with $k \geq 3$. The set of bit strings of length k that do not contain the pattern 101 can be partitioned into $k+1$ subsets: the subset of strings that start with 0 and continue with any bit string of length $k-1$ not containing 101 *[there are a_{k-1} of these]*, the subset of strings that start with 100 and continue with any bit string of length $k-3$ not containing 101 *[there are a_{k-3} of these]*, the subset of strings that start with 1100 and continue with any bit string of length $k-4$ not containing 101 *[there are a_{k-4} of these]*, the subset of strings that start with 11100 and continue with any bit string of length $k-5$ not containing 101 *[there are a_{k-5} of these]*, until the following subset of strings is obtained: $\{\underbrace{11\ldots1}_{k-3\ 1's}001, \underbrace{11\ldots1}_{k-3\ 1's}000\}$ *[there are 2 of these and a_1 equals 2]*. In addition, the three single-element sets $\{\underbrace{11\ldots1}_{k-2\ 1's}00\}$, $\{\underbrace{11\ldots1}_{k-1\ 1's}0\}$, and $\{\underbrace{11\ldots1}_{k-1\ 1's}1\}$ are in the partition, and since $a_0 = 1$ (because the only bit string of length zero that satisfies the condition is ϵ), $3 = a_0 + 2$. Thus by the addition rule,

$$a_k = a_{k-1} + a_{k-3} + a_{k-4} + \cdots + a_1 + a_0 + 2.$$

b. By part (a), if $k \geq 4$,

$$\begin{aligned} a_k &= a_{k-1} + a_{k-3} + a_{k-4} + \cdots + a_1 + a_0 + 2 \\ a_{k-1} &= a_{k-2} + a_{k-4} + a_{k-5} + \cdots + a_1 + a_0 + 2. \end{aligned}$$

Subtracting the second equation from the first gives

$$\begin{aligned} a_k - a_{k-1} &= a_{k-1} + a_{k-3} - a_{k-2} \\ \Rightarrow \qquad a_k &= 2a_{k-1} + a_{k-3} - a_{k-2}. \text{ (Call this equation (*).)} \end{aligned}$$

Note that $a_2 = 4$ (because all four bit strings of length 2 satisfy the condition) and $a_3 = 7$ (because all eight bit strings of length 3 satisfy the condition except 101). Thus equation (*) is also satisfied when $k = 3$ because in that case the right-hand side of the equation becomes $2a_2 + a_0 - a_1 = 2 \cdot 4 + 1 - 2 = 7$, which equals the left-hand side of the equation.

30. To get a sense of the problem, we compute s_4 directly. If there are four seats in the row, there can be a single student in any one of the four seats or there can be a pair of students in seats 1&3, 1&4, or 2&4. No other arrangements are possible because with more than two students, two would have to sit next to each other. Thus $s_4 = 4 + 3 = 7$. In general, if there are k chairs in a row, then

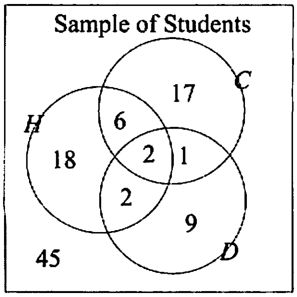
$s_k = s_{k-1}$ (the number of ways a nonempty set of students can sit in the row with no two students adjacent and chair k empty)

$+s_{k-2}$ (the number of ways students can sit in the row with chair k occupied, chair $k-1$ empty, and chairs 1 through $k-2$ occupied by a nonempty set of students in such a way that no two students are adjacent)

$+1$ (for the seating in which chair k is occupied and all the other chairs are empty

$= s_{k-1} + s_{k-2} + 1$ for all integers $k \geq 3$.

33. *c.*



e. 1 *f.* 17

36. *a.* by the double complement law and the difference rule *b.* by De Morgan's law
c. by the inclusion/exclusion rule

39. Imagine each integer from 1 through 999,999 as a string of six digits with leading 0's included. For each $i = 1, 2, 3$, let A_i be the set of all integers from 1 through 999,999 that do not contain the digit i. We want to compute $N(A_1{}^c \cap A_2{}^c \cap A_3{}^c)$. By De Morgan's law,

$$A_1{}^c \cap A_2{}^c \cap A_3{}^c = (A_1 \cup A_2)^c \cap A_3{}^c = (A_1 \cup A_2 \cup A_3)^c = U - (A_1 \cup A_2 \cup A_3),$$

and so, by the difference rule,

$$N(A_1{}^c \cap A_2{}^c \cap A_3{}^c) = N(U) - N(A_1 \cup A_2 \cup A_3).$$

By the inclusion/exclusion rule,
$N(A_1 \cup A_2 \cup A_3) = N(A_1)+N(A_2)+N(A_3)-N(A_1 \cap A_2)-N(A_1 \cap A_3)-N(A_2 \cap A_3)+N(A_1 \cap A_2 \cap A_3)$.
Now $N(A_1) = N(A_2) = N(A_3) = 9^6$ because in each case any of nine digits may be chosen for each character in the string (for A_i these are all the ten digits except i). Also each $N(A_i \cap A_j) = 8^6$ because in each case any of eight digits may be chosen for each character of the string (for $A_i \cap A_j$ these are all the ten digits except i and j). Similarly, $N(A_1 \cap A_2 \cap A_3) = 7^6$ because any digit except 1, 2, and 3 may be chosen for each character in the string. Thus

$$N(A_1 \cup A_2 \cup A_3) = 3 \cdot 9^6 - 3 \cdot 8^6 + 7^6,$$

and so, by the difference rule,

$$N(A_1{}^c \cap A_2{}^c \cap A_3{}^c) = N(U) - N(A_1 \cup A_2 \cup A_3) = 10^6 - (3 \cdot 9^6 - 3 \cdot 8^6 + 7^6) = 74,460.$$

42. *a.* $g_3 = 1$, $g_4 = 1$, $g_5 = 2$ (*LWLLL* and *WWLLL*)
b. $g_6 = 4$ (*WWWLLL, WLWLLL, LWWLLL, LLWLLL*)
c. If $k \geq 6$, then any sequence of k games must begin with exactly one of the possibilities: *W*, *LW*, or *LLW*. The number of sequences of k games that begin with *W* is g_{k-1} because the succeeding $k-1$ games can consist of any sequence of wins and losses except that the first sequence of three consecutive losses occurs at the end. Similarly, the number of sequences of k games that begin with *LW* is g_{k-2} and the number of sequences of k games that begin with *LLW* is g_{k-3}. Therefore, $g_k = g_{k-1} + g_{k-2} + g_{k-3}$ for all integers $k \geq 6$.

45. Proof (by mathematical induction): Let the property $P(k)$ be the sentence

> If a finite set A equals the union of k distinct mutually
> disjoint subsets subsets $A_1, A_2, \ldots, A_k$, then ← $P(k)$
> $N(A) = N(A_1) + N(A_2) + \cdots + N(A_k)$.

We will prove by mathematical induction that $P(k)$ is true for all integers $k \geq 1$.

Show that $P(1)$ is true: $P(1)$ is true because if a finite set A equals the "union" of one subset A_1, then $A = A_1$, and so $N(A) = N(A_1)$.

Show that for all integers $i \geq 1$, if $P(i)$ is true then $P(i+1)$ is true: Let i be any integer with $i \geq 1$ and suppose that

> If a finite set A equals the union of i distinct mutually disjoint subsets subsets $A_1, A_2, \ldots, A_i$, then $N(A) = N(A_1) + N(A_2) + \cdots + N(A_i)$. ← $P(i)$ inductive hypothesis

We must show that

> If a finite set A equals the union of $i+1$ distinct mutually disjoint subsets subsets $A_1, A_2, \ldots, A_{i+1}$, then $N(A) = N(A_1) + N(A_2) + \cdots + N(A_{i+1})$. ← $P(i+1)$

Let A be a finite set that equals the union of $i+1$ distinct mutually disjoint subsets $A_1, A_2, \ldots, A_{i+1}$. Then $A = A_1 \cup A_2 \cup \cdots \cup A_{i+1}$ and $A_i \cap A_j = \emptyset$ for all integers i and j with $i \neq j$.

Let B be the set $A_1 \cup A_2 \cup \cdots \cup A_i$. Then $A = B \cup A_{i+1}$ and $B \cap A_{i+1} = \emptyset$.

[For if $x \in B \cap A_{i+1}$, then $x \in A_1 \cup A_2 \cup \cdots \cup A_i$ and $x \in A_{i+1}$, which implies that $x \in A_j$, for some j with $1 \leq j \leq i$, and $x \in A_{i+1}$. But A_j and A_i are disjoint. Thus no such x exists.]

Hence A is the union of the two mutually disjoint sets B and A_{i+1}. Since B and A_{i+1} have no elements in common, the total number of elements in $B \cup A_{i+1}$ can be obtained by first counting the elements in B, next counting the elements in A_{i+1}, and then adding the two numbers together.

It follows that $N(B \cup A_{i+1}) = N(B) + N(A_{i+1})$ which equals $N(A_1) + N(A_2) + \cdots + N(A_i) + N(A_{i+1})$ by inductive hypothesis. Hence $P(i+1)$ is true *[as was to be shown]*.

48. Proof (by mathematical induction): Let the property $P(n)$ be the general inclusion/exclusion rule. We will prove by mathematical induction that $P(n)$ is true for all integers $n \geq 2$.

Show that $P(2)$ is true: $P(2)$ was proved in one way in the text preceding Theorem 9.3.3 and in another way in the solution to exercise 46.

Show that for all integers $r \geq 2$, if $P(r)$ is true then $P(r+1)$ is true: Let r be any integer with $r \geq 2$ and suppose that the general inclusion/exclusion rule holds for any collection of r finite sets. (This is the inductive hypothesis.) Let $A_1, A_2, \ldots, A_{r+1}$ be finite sets. Then

$$
\begin{aligned}
&N(A_1 \cup A_2 \cup \cdots \cup A_{r+1}) \\
&= N(A_1 \cup (A_2 \cup A_3 \cup \cdots \cup A_{r+1})) && \text{by the associative law for } \cup \\
&= N(A_1) + N(A_2 \cup A_3 \cup \cdots \cup A_{r+1}) - N(A_1 \cap (A_2 \cup A_3 \cup \cdots \cup A_{r+1})) \\
& && \text{by the inclusion/exclusion rule for two sets} \\
&= N(A_1) + N(A_2 \cup A_3 \cup \cdots \cup A_{r+1}) - N((A_1 \cap A_2) \cup (A_1 \cap A_3) \cup \cdots \cup (A_1 \cap A_{r+1})) \\
& && \text{by the generalized distributive law for sets} \\
& && \text{(exercise 37, Section 6.2)} \\
&= N(A_1) + \Big(\textstyle\sum_{2 \leq i \leq r+1} N(A_i) - \sum_{2 \leq i < j \leq r+1} N(A_i \cap A_j) \\
&\qquad + \textstyle\sum_{2 \leq i < j < k \leq r+1} N(A_i \cap A_j \cap A_k) - \cdots + (-1)^{r+1} N(A_2 \cap A_3 \cap \cdots \cap A_{r+1})\Big) \\
&\qquad - \Big(\textstyle\sum_{2 \leq i \leq r+1} N(A_1 \cap A_i) - \sum_{2 \leq i < j \leq r+1} N((A_1 \cap A_i) \cap (A_1 \cap A_j)) + \cdots \\
&\qquad + (-1)^{r+1} N((A_1 \cap A_2) \cap (A_1 \cap A_3) \cap \cdots \cap (A_1 \cap A_{r+1}))\Big) \\
& && \text{by inductive hypothesis}
\end{aligned}
$$

$$\begin{aligned}
&= N(A_1) + \Big(\textstyle\sum_{2\le i\le r+1} N(A_i) - \sum_{2\le i<j\le r+1} N(A_i \cap A_j) \\
&\qquad + \textstyle\sum_{2\le i<j<k\le r+1} N(A_i \cap A_j \cap A_k) - \cdots + (-1)^{r+1} N(A_2 \cap A_3 \cap \cdots \cap A_{r+1})\Big) \\
&\qquad - \Big(\textstyle\sum_{2\le i\le r+1} N(A_1 \cap A_i) - \sum_{2\le i<j\le r+1} N(A_1 \cap A_i \cap A_j) + \cdots \\
&\qquad\qquad + (-1)^{r+1} N(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{r+1})\Big) \\
&= \textstyle\sum_{1\le i\le r+1} N(A_i) - \sum_{1\le i<j\le r+1} N(A_i \cap A_j) + \sum_{1\le i<j<k\le r+1} N(A_i \cap A_j \cap A_k) \\
&\qquad - \cdots + (-1)^{r+2} N(A_1 \cap A_3 \cap \cdots \cap A_{r+1}).
\end{aligned}$$

[This is what was to be proved.]

Section 9.4

6. *a.* Yes.

   Solution 1: There are 6 possible remainders that can be obtained when an integer is divided by 7, namely 0, 1, 2, 3, 4, 5. Apply the pigeonhole principle, thinking of the 7 integers as the pigeons and the possible remainders as the pigeonholes. Each pigeon flies into the pigeonhole that is the remainder obtained when it is divided by 6. Since $7 > 6$, the pigeonhole principle says that at least two pigeons must fly into the same pigeonhole. So at least two of the numbers must have the same remainder when divided by 6.

   Solution 2: Let X be the set of seven integers and Y the set of all possible remainders obtained through division by 6, and consider the function R from X (the pigeons) to Y (the pigeonholes) defined by the rule: $R(n) = n \text{ } mod \text{ } 6$ (= the remainder obtained by the integer division of n by 6). Now X has 7 elements and Y has 6 elements (0, 1, 2, 3, 4, and 5). Hence by the pigeonhole principle, R is not one-to-one: $R(n_1) = R(n_2)$ for some integers n_1 and n_2 with $n_1 \neq n_2$. But this means that n_1 and n_2 have the same remainder when divided by 6.

   b. No. Consider the set $\{1, 2, 3, 4, 5, 6, 7\}$. This set has seven elements no two of which have the same remainder when divided by 8.

15. There are $n + 1$ even integers from 0 to $2n$ inclusive:

    $$0\,(= 2\cdot \underline{0}),\ \ 2\,(= 2\cdot \underline{1}),\ \ 4\,(= 2\cdot \underline{2}), \ldots,\ \ 2n\,(= 2\cdot \underline{n}).$$

    So a maximum of $n+1$ even integers can be chosen. Thus if at least $n+2$ integers are chosen, one is sure to be odd. Similarly, there are n odd integers from 0 to $2n$ inclusive, namely

    $$1\,(= 2\cdot \underline{1} - 1),\ \ 3\,(= 2\cdot \underline{2} - 1), \ldots,\ \ 2n - 1\,(= 2\cdot \underline{n} - 1).$$

    It follows that if at least $n + 1$ integers are chosen, one is sure to be even.

    (An alternative way to reach the second conclusion is to note that there are $2n + 1$ integers from 0 to $2n$ inclusive. Because $n + 1$ of them are even, the number of odd integers is $(2n + 1) - (n + 1) = n$.)

18. There are 15 distinct remainders that can be obtained through integer division by 15 (0, 1, 2, ..., 14). Hence at least 16 integers must be chosen in order to be sure that at least two have the same remainder when divided by 15.

21. The length of the repeating section of the decimal representation of 683/1493 is less than or equal to 1,492. The reason is that there are 1,492 nonzero remainders that can be obtained when a number is divided by 1,493. Thus, in the long-division process of dividing 683.0000... by 1,493, either some remainder is 0 and the decimal expansion terminates (in which case the

length of the repeating section is 0) or, only nonzero remainders are obtained and at some point within the first 1,492 successive divisions, a nonzero remainder is repeated. At that point the digits in the developing decimal expansion begin to repeat because the sequence of successive remainders repeats those previously obtained.

27. Yes. Let X be the set of 2,000 people (the pigeons) and Y the set of all 366 possible birthdays (the pigeonholes). Define a function $B\colon X \to Y$ by specifying that $B(x) = x$'s birthday. Now $2000 > 4 \cdot 366 = 1464$, and so by the generalized pigeonhole principle, there must be some birthday y such that $B^{-1}(y)$ has at least $4 + 1 = 5$ elements. Hence at least 5 people must share the same birthday.

30. Consider the maximum number of pennies that can be chosen without getting at least five from the same year. This maximum, which is 12, is obtained when four pennies are chosen from each of the three years. Hence at least thirteen pennies must be chosen to be sure of getting at least five from the same year.

33. <u>Proof</u>: Suppose A is a set of six positive integers each of which is less than 15. By Theorem 6.3.1, $\mathscr{P}(A)$, the power set of A, has $2^6 = 64$ elements, and so A has 63 nonempty subsets. Let k be the smallest number in the set A.

Given any nonempty subset of A, the sum of all the elements in the subset lies in the range from k through $k + 10 + 11 + 12 + 13 + 14 = k + 60$, and, by Theorem 9.1.1, there are $(k + 60) - k + 1 = 61$ integers in this range. Let S be the set of all possible sums of the elements that are in a nonempty subset of A. Then S has at most 61 elements.

Define a function F from the set of nonempty subsets of A to S as follows: For each nonempty subset X in A, let $F(X)$ be the sum of the elements of X. Because A has 63 nonempty subsets and S has 61 elements, the pigeonhole principle guarantees that F is not one-to-one. Thus there exist distinct nonempty subsets A_1 and A_2 of A such that $F(A_1) = F(A_2)$, which implies that the elements of A_1 add up to the same sum as the elements of A_2.

Note: In fact, it can be shown that it is always possible to find disjoint subsets of A with the same sum. To see why this is true, consider again the sets A_1 and A_2 found in the preceding proof. Then $A_1 \neq A_2$ and $F(A_1) = F(A_2)$. By definition of F, $F(A_1 - A_2) + F(A_1 \cap A_2) =$ the sum of the elements in $A_1 - A_2$ plus the sum of the elements in $A_1 \cap A_2$. But $A_1 - A_2$ and $A_1 \cap A_2$ are disjoint and their union is A_1. So $F(A_1 - A_2) + F(A_1 \cap A_2) = F(A_1)$. By the same reasoning, $F(A_2 - A_1) + F(A_1 \cap A_2) = F(A_2)$. Since $F(A_1) = F(A_2)$, we have that $F(A_1 - A_2) = F(A_1) - F(A_1 \cap A_2) = F(A_2) - F(A_1 \cap A_2) = F(A_2 - A_1)$. Hence the elements in $A_1 - A_2$ add up to the same sum as the elements in $A_2 - A_1$. But $A_1 - A_2$ and $A_2 - A_1$ are disjoint because $A_1 - A_2$ contains no elements of A_2 and $A_2 - A_1$ contains no elements of A_1.

36. <u>Proof</u>: Suppose that 101 integers are chosen from 1 to 200 inclusive. Call them $x_1, x_2, \ldots, x_{101}$. Represent each of these integers in the form $x_i = 2^{k_i} \cdot a_i$ where a_i is the uniquely determined odd integer obtained by dividing x_i by the highest possible power of 2. Because each x_i satisfies the condition $1 \leq x_i \leq 200$, each a_i satisfies the condition $1 \leq a_i \leq 199$. Define a function F from $X = \{x_1, x_2, \ldots, x_{101}\}$ to the set Y of all odd integers from 1 to 199 inclusive by the rule $F(x_i) =$ that odd integer a_i such that x_i equals $2^{k_i} \cdot a_i$. Now X has 101 elements and Y has 100 elements, namely

$$1 = 2 \cdot \underline{1} - 1,\ 3 = 2 \cdot \underline{2} - 1,\ 5 = 2 \cdot \underline{3} - 1, \ldots,\ 199 = 2 \cdot \underline{100} - 1.$$

Hence by the pigeonhole principle, F is not one-to-one: there exist integers x_i and x_j such that $F(x_i) = F(x_j)$ and $x_i \neq x_j$.

But $x_i = 2^{k_i} \cdot a_i$ and $x_j = 2^{k_j} \cdot a_j$ and $F(x_i) = a_i$ and $F(x_j) = a_j$. Thus $x_i = 2^{k_i} \cdot a_i$ and $x_j = 2^{k_j} \cdot a_i$. If $k_j > k_i$, then

$$x_j = 2^{k_j} \cdot a_i = 2^{k_j - k_i} \cdot 2^{k_i} \cdot a_i = 2^{k_j - k_i} \cdot x_i,$$

and so x_j is divisible by x_i. Similarly, if $k_j < k_i$, x_i is divisible by x_j. Hence, in either case, one of the numbers is divisible by another.

39. Let S be any set consisting entirely of integers from 1 through 100, and suppose that no integer in S divides any other integer in S. Factor out the highest power of 2 to write each integer in S as $2^k \cdot m$, where m is an odd integer.

    Now consider any two such integers in S, say $2^r \cdot a$ and $2^s \cdot b$. Observe that $a \neq b$. The reason is that if $a = b$, then whichever integer contains the fewer number of factors of 2 divides the other integer. (For example, $2^2 \cdot 3 \mid 2^4 \cdot 3$.)

    Thus there can be no more integers in S than there are distinct odd integers from 1 through 100, namely 50.

    Furthermore, it is possible to find a set T of 50 integers from 1 through 100 no one of which divides any other. For instance, $T = 51, 52, 53, \ldots, 99, 100$.

    Hence the largest number of elements that a set of integers from 1 through 100 can have so that no one element in the set is divisible by any other is 50.

Section 9.5

9. *a.* The number of committees of six that can be formed from the 40 members of the club is

$$\binom{40}{6} = 3,838,380.$$

12. The sum of two integers is even if, and only if, either both integers are even or both are odd *[see Example 4.2.3]*. Because $2 = 2 \cdot 1$ and $100 = 2 \cdot 50$, there are 50 even integers and thus 51 odd integers froml to 101 inclusive. Hence the number of distinct pairs is the number of ways to choose two even integers from the 50 plus the number of ways to choose two odd integers from the 51:

$$\binom{50}{2} + \binom{51}{2} = 1225 + 1275 = 2500.$$

18. An ordering for the letters in *MISSISSIPPI* can be created as follows:

    Step 1: Choose a subset of one position for the *M*

    Step 2: Choose a subset of four positions for the *I*'s

    Step 3: Choose a subset of four positions for the *S*'s

    Step 4: Choose a subset of two positions for the *P*'s

    Thus the total number of distinguishable orderings is

$$\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = \frac{11!}{1! \cdot 10!} \cdot \frac{10!}{4! \cdot 6!} \cdot \frac{6!}{4! \cdot 2!} \cdot \frac{2!}{2! \cdot 0!} = \frac{11!}{1! \cdot \cdot 4! \cdot 4! \cdot 2!} = 34,650,$$

    which agrees with the result in Example 9.5.10.

21. The number of symbols that can be represented in the Morse code using n dots and dashes is 2^n. Therefore, the number of symbols that can be represented in the Morse code using at most seven dots and dashes is

$$2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 = 2(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6) = 2\left(\frac{2^7-1}{2-1}\right) = 254.$$

24. *a.* Because $210 = 2 \cdot 3 \cdot 5 \cdot 7$, the distinct factorizations of 210 are $1 \cdot 210$, $2 \cdot 105$, $3 \cdot 70$, $5 \cdot 42$, $7 \cdot 30$, $6 \cdot 35$, $10 \cdot 21$, and $14 \cdot 15$. So there are 8 distinct factorizations of 210.

c. As in the answer to part (b), there are two different ways to look at the solution to this problem.

Solution 1: Separate the factorizations into categories: one category consists only of the factorization in which one factor is 1 and the other factor is the product of all five prime factors *[there is* $1 = \binom{5}{0}$ *such factorization]*, a second category consists of those factorizations in which one factor is a single prime and the other factor is the product of the four other primes *[there are* $\binom{5}{1}$ *such factorizations]*, and the third category contains those factorizations in which one factor is a product of two of the primes and the other factor is the product of the other three primes *[there are* $\binom{5}{2}$ *such factorizations]*. All possible factorizations are included among these categories, and so, by the addition rule, the answer is $\binom{5}{0} + \binom{5}{1} + \binom{5}{2}$ $= 1 + 5 + 10 = 16$.

Solution 2: Let $S = \{p_1, p_2, p_3, p_4, p_5\}$, let $p_1p_2p_3p_4p_5 = P$, and let f_1f_2 be any factorization of P. The product of the numbers in any subset $A \subseteq S$ can be used for f_1, with the product of the numbers in A^c being f_2.. Thus there are as many ways to write f_1f_2 as there are subsets of S, namely $2^5 = 32$ (by Theorem 6.3.1). But given any factors f_1 and f_2, we have that $f_1f_2 = f_2f_1$. Thus counting the number of ways to write f_1f_2 counts each factorization twice. So the answer is $\frac{32}{2} = 16$.

Note: In Section 9.7 we will show that $\binom{n}{r} = \binom{n}{n-r}$ whenever $n \geq r \geq 0$. Thus, for example, the answer can be written as

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} = \frac{1}{2}\left(\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right).$$

In Section 9.7 we will also show that for all integers $n \geq 0$,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = 2^n,$$

and so, in particular,

$$\frac{1}{2}\left[\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}\right] = \frac{1}{2} \cdot 2^5 = \frac{32}{2} = 16.$$

These facts illustrate the relationship between the two solutions to part (c) of this exercise.

d. Because the second solution given in parts (b) and (c) is the simplest, we give a general version of it as the answer to this part of the exercise. Let $S = \{p_1, p_2, p_3, \ldots, p_n\}$, let $p_1p_2p_3 \cdots p_n = P$, and let f_1f_2 be any factorization of P. The product of the numbers in any subset $A \subseteq S$ can be used for f_1, with the product of the numbers in A^c being f_2. Thus there are as many ways to write f_1f_2 as there are subsets of S, namely 2^n (by Theorem 6.3.1). But given any factors f_1 and f_2, we have that $f_1f_2 = f_2f_1$, and so counting the number of ways to write f_1f_2 counts each factorization twice. Hence the answer is $\frac{1}{2^n} = 2^{n-1}$.

27. *b.* A reflexive relation must contain (a,a) for all eight elements a in A. Any subset of the remaining 56 elements of $A \times A$ (which has a total of 64 elements) can be combined with these eight to produce a reflexive relation. Therefore, there are as many reflexive binary relations as there are subsets of a set of 56 elements, namely 2^{56}.

d. Form a relation that is both reflexive and symmetric by a two-step process: (1) pick all eight elements of the form (x,x) where $x \in A$, (2) pick a set of (distinct) pairs of elements of the form (a,b) and (b,a). There is just one way to perform step 1, and, as explained in the answer to part (c), there are 2^{28} ways to perform step 2. Therefore, there are 2^{28} binary relations on A that are reflexive and symmetric.

30. The error is that the "solution" overcounts the number of poker hands with two pairs. In fact, it counts every such hand twice. For instance, consider the poker hand $\{4\clubsuit, 4\diamondsuit, \text{J}\heartsuit, \text{J}\spadesuit, 9\clubsuit\}$. If the steps outlined in the false solution in the exercise statement are followed, this hand is first counted when the denomination 4 is chosen in step one, the cards $4\clubsuit$ and $4\diamondsuit$ are chosen in step two, the denomination J is chosen in step three, the cards J $\heartsuit$ and $\text{J}\spadesuit$ are chosen in step four, and $9\clubsuit$ is chosen in step five. The hand is counted a second time when the denomination J is chosen in step one, the cards $\text{J}\heartsuit$ and J $\spadesuit$ are chosen in step two, the denomination 4 is chosen in step three, the cards $4\clubsuit$ and $4\diamondsuit$ are chosen in step four, and $9\clubsuit$ is chosen in step five.

Section 9.6

6. $\binom{5+n-1}{5} = \binom{n+4}{5} = \dfrac{(n+4)(n+3)(n+2)(n+1)n}{120}$

9. The number of iterations of the inner loop is the same as the number of integer triples (i, j, k) where $1 \le k \le j \le i \le n$. As in Example 9.6.3,such triples can be represented as a string of $n-1$ vertical bars and three crosses indicating which three integers from 1 to n are included in the triple. Thus the number of such triples is the same as the number of strings of $(n-1)$ $|$'s and 3 $\times$'s, which is

$$\binom{n+2}{3} = \frac{n(n+1)(n+2)}{6}.$$

12. Think of the number 30 as divided into 30 individual units and the variables (y_1, y_2, y_3, y_4) as four categories into which these units are placed. The number of units in category y_i indicates the value of y_i in a solution of the equation. By Theorem 9.6.1, the number of ways to place 30 objects into four categories is

$$\binom{30+4-1}{30} = \binom{33}{30} = 5456.$$

So there are 5456 nonnegative integral solutions of the equation.

15. Any number from 1 through 99,999 whose digits add up to 9 can be thought of as a 5-digit number with leading zeroes included. Imagine that the 5 digits are categories into which we place 9 crosses. (For instance, $\times\times \mid \quad \mid \times \times \times \times \times \mid \times \mid \times\times$ corresponds to the number 20512.) By Theorem 9.6.1, there are $\binom{9+5-1}{9} = \binom{13}{9} = 715$ ways to place the crosses into the categories.

18. *a.* Think of the 4 kinds of coins as the n categories and the 30 coins to be chosen as the r objects. Each choice of 30 coins is represented by a string of $4-1=3$ vertical bars (to separate the categories) and 30 crosses (to represent the chosen coins). The total number of choices of 30 coins of the 4 different kinds is the number of strings of 33 symbols (3 vertical bars and 30 crosses), namely, $\binom{30+4-1}{30} = \binom{33}{30} = 5,456.$

b. Let T be the set of selections of 30 coins for which the coin's type is unrestricted, $Q_{\le 15}$ the set of selections containing at most 15 quarters, and $Q_{\ge 16}$ the set of selections containing at least 16 quarters. Then

$$T = Q_{\le 15} \cup Q_{\ge 16} \quad \text{and} \quad Q_{\le 15} \cap Q_{\ge 16} = \emptyset \quad \text{and so} \quad N(T) = N(Q_{\le 15}) + N(Q_{\ge 16}).$$

To compute $N(Q_{\ge 16})$, we reason as follows: If at least 16 quarters are included, we can choose the 30 coins by first selecting 16 quarters and then choosing the remaining 14 coins from the four different types. The number of ways to do this is

$$N(Q_{\ge 16}) = \binom{14+4-1}{14} = \binom{17}{14} = 680.$$

Then $N(T) = 5,456$ *[by part (a)]* and $N(Q_{\geq 16}) = 680$. Therefore, the number of selections containing at most 15 quarters is

$$N(Q_{\leq 15}) = N(T) - N(Q_{\geq 16}) = 5,456 - 680 = 4,776.$$

c. Let T be the set of selections of 30 coins for which the coin's type is unrestricted, $D_{\leq 20}$ the set of selections containing at most 20 dimes, and $D_{\geq 21}$ the set of selections containing at least 21 dimes. Then

$$T = D_{\leq 20} \cup D_{\geq 21} \quad \text{and} \quad D_{\leq 20} \cap D_{\geq 21} = \emptyset \quad \text{and so} \quad N(T) = N(D_{\leq 20}) + N(D_{\geq 21}).$$

To compute $N(D_{\geq 21})$, we reason as follows: If at least 21 dimes are included, we can choose the 30 coins by first selecting 21 dimes and then choosing the remaining nine coins from the four different types. The number of ways to do this is

$$N(D_{\geq 21}) = \binom{9+4-1}{9} = \binom{12}{9} = 220.$$

Then $N(T) = 5,456$ *[by part (a)]* and $N(D_{\geq 21}) = 220$. Therefore, the number of selections containing at most 20 dimes is

$$N(D_{\leq 20}) = N(T) - N(D_{\geq 21}) = 5,456 - 220 = 5,236.$$

d. As in parts (b) and (c), let T be the set of selections of 30 coins for which the coin's type is unrestricted, $Q_{\geq 16}$ the set of selections containing at least 16 quarters, $Q_{\leq 15}$ the set of selections containing at most 15 quarters, $D_{\geq 21}$ the set of selections containing at least 21 dimes, and $D_{\leq 20}$ the set of selections containing at most 20 dimes. If the pile has at most 15 quarters and at most 20 dimes, then the number of combinations of coins that can be chosen is $N(Q_{\leq 15} \cap D_{\leq 20})$, and, by the difference rule,

$$N(Q_{\leq 15} \cap D_{\leq 20}) = N(T) - N(Q_{\geq 16} \cup D_{\geq 21}).$$

In order to find $N(Q_{\geq 16} \cup D_{\geq 21})$, we first compute $N(Q_{\geq 16} \cap D_{\geq 21})$, which is the number of selections of coins containing at least 16 quarters and at least 21 dimes. However, 16 quarters plus 21 dimes would give a total of more than 30 coins. So there are no selections of this type. Thus

$$N(Q_{\leq 15} \cap D_{\leq 20}) = N(T) - N(Q_{\geq 16} \cup D_{\geq 21}) = 5,456 - 0 = 5,456.$$

Then, by the inclusion/exclusion rule,

$$N(Q_{\geq 16} \cup D_{\geq 21}) = N(Q_{\geq 16}) + N(D_{\geq 21}) - N(Q_{\geq 16} \cap D_{\geq 21}) = 680 + 220 - 0 = 900.$$

Therefore the answer to the question is

$$N(Q_{\leq 15} \cap D_{\leq 20}) = N(T) - N(Q_{\geq 16} \cup D_{\geq 21}) = 5,456 - 900 = 4,556.$$

21. Consider those columns of a trace table corresponding to an arbitrary value of k. The values of j go from 1 to k, and for each value of j, the values of i go from 1 to j.

k	k				
j	1	2	3	. . .	k
i	1	1 2	1 2 3	. . .	1 2 3 . . . k

So for each value of k, there are $1 + 2 + 3 + \cdots + k$ columns of the table. Since k goes from 1 to n, the total number of columns in the table is

$1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + n)$

$$\begin{aligned}
&= \sum_{k=1}^{1} k + \sum_{k=1}^{2} k + \cdots + \sum_{k=1}^{n-1} k + \sum_{k=1}^{n} k \\
&= \frac{1\cdot 2}{2} + \frac{2\cdot 3}{2} + \cdots + \frac{(n-1)\cdot n}{2} + \frac{n\cdot (n+1)}{2} \\
&= \frac{1}{2}[1\cdot 2 + 2\cdot 3 + \cdots + (n-1)\cdot n + n\cdot (n+1)] \\
&= \frac{1}{2}\left(\frac{n(n+1)(n+2)}{3}\right) \qquad \text{by exercise 13 of Section 5.2} \\
&= \frac{n(n+1)(n+2)}{6},
\end{aligned}$$

which agrees with the result of Example 9.6.4.

Section 9.7

9.
$$\begin{aligned}
\binom{2(n+1)}{2n} &= \binom{2n+2}{2n} \\
&= \frac{(2n+2)!}{(2n)!((2n+2)-2n)!} \\
&= \frac{(2n+2)(2n+1)(2n)!}{(2n)!2!} \\
&= \frac{(2n+2)(2n+1)}{2} \\
&= \frac{2(n+1)(2n+1)}{2} \\
&= (n+1)(2n+1)
\end{aligned}$$

12.
$$\begin{aligned}
\binom{n+3}{r} &= \binom{n+2}{r-1} + \binom{n+2}{r} \\
&= \left(\binom{n+1}{r-2} + \binom{n+1}{r-1}\right) + \left(\binom{n+1}{r-1} + \binom{n+1}{r}\right) \\
&= \binom{n+1}{r-2} + 2\cdot \binom{n+1}{r-1} + \binom{n+1}{r} \\
&= \left(\binom{n}{r-3} + \binom{n}{r-2}\right) + 2\cdot \left(\binom{n}{r-2} + \binom{n}{r-1}\right) + \left(\binom{n}{r-1} + \binom{n}{r}\right) \\
&= \binom{n}{r-3} + 3\cdot \binom{n}{r-2} + 3\cdot \binom{n}{r-1} + \binom{n}{r}
\end{aligned}$$

15. Proof (by mathematical induction): Let r be a fixed nonnegative integer, and let the property $P(n)$ be the formula

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}. \qquad \leftarrow Pn)$$

Show that $P(r)$ is true: To prove that $P(r)$ is true, we must show that

$$\sum_{i=r}^{r} \binom{i}{r} = \binom{r+1}{r+1}.$$

But the left-hand side of this equation is $\binom{r}{r} = 1$, and the right-hand side is $\binom{r+1}{r+1}$, which also equals 1. So $P(r)$ is true.

Show that for all integers $k \geq r$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq r$ and suppose that

$$\sum_{i=r}^{k} \binom{i}{r} = \binom{k+1}{r+1}. \quad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\sum_{i=r}^{k+1} \binom{i}{r} = \binom{(k+1)+1}{r+1} \quad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned} \sum_{i=r}^{k+1} \binom{i}{r} &= \sum_{i=r}^{k} \binom{i}{r} + \binom{k+1}{r} && \text{by writing the last term separately} \\ &= \binom{k+1}{r+1} + \binom{k+1}{r} && \text{by inductive hypothesis} \\ &= \binom{(k+1)+1}{r+1} && \text{by Pascal's formula,} \end{aligned}$$

and this is the right-hand side of $P(k+1)$ *[as was to be shown]*.

18. Proof (by mathematical induction): Let the property $P(n)$ be the equation

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+n}{n} = \binom{m+n+1}{n}. \quad \leftarrow P(n)$$

We will show by mathematical induction that the property is true for all integers $n \geq 0$.

Show that $P(0)$ is true: $P(0)$ is the equation $\binom{m}{0} = \binom{m+0+1}{0} = \binom{m+1}{0}$, is true because by exercise 1 both sides equal 1.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 0$ and suppose that

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+k}{k} = \binom{m+k+1}{k}. \quad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+(k+1)}{k+1} = \binom{m+(k+1)+1}{(k+1)}$$

or, equivalently,

$$\binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+k+1}{k+1} = \binom{m+k+2}{k+1}. \quad \leftarrow P(k+1)$$

But

$$\begin{aligned} \binom{m}{0} + \binom{m+1}{1} + \binom{m+2}{2} + \cdots + \binom{m+k+1}{k+1} &= \binom{m+k+1}{k} + \binom{m+k+1}{k+1} \\ &\quad \text{by inductive hypothesis} \\ &= \binom{m+k+2}{k+1} \\ &\quad \text{by Pascal's formula } (m+k+1 \text{ in place of } n \text{ and } k+1 \text{ in place of } r). \end{aligned}$$

[This is what was to be shown.]

24. *Solution 1*: $(u^2-3v)^4 = \binom{4}{0}(u^2)^4(-3v)^0 + \binom{4}{1}(u^2)^3(-3v)^1 + \binom{4}{2}(u^2)^2(-3v)^2$

$$+\binom{4}{3}(u^2)^1(-3v)^3+\binom{4}{4}(u^2)^0(-3v)^4$$

$$= u^8 - 12u^6v + 54u^4v^2 - 108u^2v^3 + 81v^4$$

Solution 2: An alternative solution is to first expand and simplify the expression $(a+b)^4$ and then substitute u^2 in place of a and $(-3v)$ in place of b and further simplify the result. Using this approach, we first apply the binomial theorem with $n=4$ to obtain

$$(a+b)^4 = \binom{4}{0}a^4b^0 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}a^1b^3 + \binom{4}{4}b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Substituting u^2 in place of a and $(-3v)$ in place of b gives

$$(u^2-3v)^4 = (u^2+(-3v))^4 = (u^2)^4 + 4(u^2)^3(-3v) + 6(u^2)^2(-3v)^2 + 4(u^2)(-3v)^3 + (-3v)^4$$
$$= u^8 - 12u^6v + 54u^4v^2 - 108u^2v^3 + 81v^4.$$

27. $\left(x^2-\frac{1}{x}\right)^5$

$$= (x^2)^5 + \binom{5}{1}(x^2)^4\left(-\frac{1}{x}\right)^1 + \binom{5}{2}(x^2)^3\left(-\frac{1}{x}\right)^2 + \binom{5}{3}(x^2)^2\left(-\frac{1}{x}\right)^3$$
$$+\binom{5}{4}(x^2)^1\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5$$
$$= x^{10} - 5x^7 + 10x^4 - 10x + \frac{5}{x^2} - \frac{1}{x^5}$$

30. Term is $\binom{10}{3}(2x)^7 3^3$. Coefficient is $\frac{10!}{3!\cdot 7!}\cdot 2^7\cdot 3^3 = 120\cdot 128\cdot 27 = 414{,}720$.

39. Proof: Let n be an integer with $n \geq 0$. Apply the binomial theorem with $a=3$ and $b=-1$ to obtain

$$\begin{aligned} 2^n &= (3+(-1))^n \\ &= \binom{n}{0}3^n(-1)^0 + \binom{n}{1}3^{n-1}(-1)^1 + \cdots + \binom{n}{i}3^{n-i}(-1)^i + \cdots + \binom{n}{n}3^{n-n}(-1)^n \\ &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}3^{n-i}. \end{aligned}$$

42. Proof (by mathematical induction): Let the property $P(n)$ be the sentence

> For any set S with n elements, S has 2^{n-1} subsets with an even number of elements and 2^{n-1} subsets with an odd number of elements. ← $P(n)$

We will prove by mathematical induction that the property is true for all integers $n \geq 1$.

Show that $P(1)$ is true: $P(1)$ is true because any set S with just 1 element, say x, has two subsets: $\emptyset$, which has 0 elements, and $\{x\}$, which has 1 element. Since 0 is even and 1 is odd, the number of subsets of S with an even number of elements equals the number of subsets of S with an odd number of elements, namely, 1, and $1 = 2^0 = 2^{1-1}$.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$ and suppose that

> For any set S with k elements, S has 2^{k-1} subsets with an even number of elements and 2^{k-1} subsets with an odd number of elements. ← $P(k)$ inductive hypothesis

We must show that

> For any set S with $k+1$ elements, S has $2^{(k+1)-1}$ subsets with an even number of elements and $2^{(k+1)-1}$ subsets with an odd number of elements.

or, equivalently,

> For any set S with $k+1$ elements, S has 2^k subsets with an even number of elements and 2^k subsets with an odd number of elements. ← $P(k+1)$

Call the elements of $S = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$. By inductive hypothesis, $\{x_1, x_2, \ldots, x_k\}$ has 2^{k-1} subsets with an even number of elements and 2^{k-1} subsets with an odd number of elements. Now every subset of $\{x_1, x_2, \ldots, x_k\}$ is also a subset of S, and the only other subsets of S are obtained by taking the union of a subset of $\{x_1, x_2, \ldots, x_k\}$ with $\{x_{k+1}\}$. Moreover, if a subset of $\{x_1, x_2, \ldots, x_k\}$ has an even number of elements, then the union of that subset with $\{x_{k+1}\}$ has an odd number of elements. So 2^{k-1} of the subsets of S that are obtained by taking the union of a subset of $\{x_1, x_2, \ldots, x_k\}$ with $\{x_{k+1}\}$ have an even number of elements and 2^{k-1} have an odd number of elements. Thus the total number of subsets of S with an even number of elements is

$$2^{k-1} + 2^{k-1} = 2 \cdot 2^{k-1} = 2^{1+(k-1)} = 2^k.$$

Similarly, the total number of subsets of S with an odd number of elements is also

$$2^{k-1} + 2^{k-1} = 2^k$$

[as was to be shown].

Alternative justification for the identity in exercise 36: Let n be any positive integer, let E be the largest even integer less than or equal to n, and let O be the largest odd integer less than or equal to n. Let S be any set with n elements. Then the number of subsets of S with an even number of elements is $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{E}$, and the number of subsets of S with an odd number of elements is $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{O}$. But there are as many subsets with an even number of elements as there are subsets with an odd number of elements, so if we subtract the second of these quantities from the first we obtain 0:

$$\begin{aligned} 0 &= \left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{E}\right] - \left[\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{O}\right] \\ &= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}. \end{aligned}$$

48. Let n be an integer with $n \geq 0$. Then

$$\begin{aligned} \sum_{r=0}^{n} \binom{n}{r} x^{2r} &= \sum_{r=0}^{n} \binom{n}{r} 1^{n-r}(x^2)^r \\ &\qquad \text{by the laws of exponents and because 1 raised to any power is 1} \\ &= (1+x^2)^n \\ &\qquad \text{by the binomial theorem with } a = 1 \text{ and } b = x^2. \end{aligned}$$

52. Let n be an integer with $n \geq 0$. Then

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} 3^{2n-2k} 2^{2k} &= \sum_{k=0}^{n} \binom{n}{k} (3^2)^{n-k} (2^2)^k \\
&\qquad \text{by the laws of exponents} \\
&= \sum_{k=0}^{n} \binom{n}{k} 9^{n-k} 4^k \\
&\qquad \text{because } 3^2 = 9 \text{ and } 2^2 = 4 \\
&= (9+4)^n \\
&\qquad \text{by the binomial theorem with } a = 9 \text{ and } b = 4 \\
&= 13^n.
\end{aligned}$$

Section 9.8

3. *a.* $P(A \cup B) = 0.4 + 0.2 = 0.6$

   b. By the formula for the probability of a general union and because $S = A \cup B \cup C$,

   $$P(S) = ((A \cup B) \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C).$$

   Suppose $P(C) = 0.2$. Then, since $P(S) = 1$,

   $$1 = 0.6 + 0.2 - P((A \cup B) \cap C) = 0.8 - P((A \cup B) \cap C).$$

   Solving for $P((A \cup B) \cap C)$ gives $P((A \cup B) \cap C) = -0.2$, which is impossible. Hence $P(C) \neq 0.2$.

6. First note that we can apply the formula for the probability of the complement of an event to obtain $0.3 = P(U^c) = 1 - P(U)$. Solving for $P(U)$ gives $P(U) = 0.7$. Second, observe that by De Morgan's law $U^c \cup V^c = (U \cap V)^c$. Thus

   $$0.4 = P(U^c \cup V^c) = P((U \cap V)^c) = 1 - P(U \cap V).$$

   Solving for $P(U \cap V)$ gives $P(U \cap V) = 0.6$. So, by the formula for the union of two events,

   $$P(U \cup V) = P(U) + P(V) - P(U \cap V) = 0.7 + 0.6 - 0.6 = 0.7.$$

9. *b.* By part (a), $P(A \cup B) = 0.7$. So, since $C = (A \cup B)^c$, by the formula for the probability of the complement of an event,

   $$P(C) = 1 - P(A \cup B) = 1 - 0.7 = 0.3.$$

   c. By the formula for the probability of the complement of an event,

   $$P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6.$$

   e. By De Morgan's law $A^c \cup B^c = (A \cap B)^c$. Thus, the formula for the probability of the complement of an event,

   $$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - 0.2 = 0.8.$$

   f. Solution 1: Because $C = S - (A \cup B)$, we have that $C = (A \cup B)^c$. Then

   $$\begin{aligned}
   B^c \cap C &= B^c \cap (A \cup B)^c && \text{by substitution} \\
   &= B^c \cap (A^c \cap B^c) && \text{by De Morgan's law} \\
   &= (B^c \cap A^c) \cap B^c && \text{by the associative law for } \cap \\
   &= (A^c \cap B^c) \cap B^c && \text{by the commutative law for } \cap \\
   &= A^c \cap (B^c \cap B^c) && \text{by the associative law for } \cap \\
   &= A^c \cap B^c && \text{by the idempotent law for } \cap \\
   &= (A \cup B)^c && \text{by De Morgan's law} \\
   &= C && \text{by substitution.}
   \end{aligned}$$

Hence, by part (b), $P(B^c \cap C) = P(C) = 0.3$.

Solution 2: Because $C = S - (A \cup B)$, we have that $C = (A \cup B)^c$. Thus by De Morgan's law, $C = A^c \cap B^c$. Now $A^c \cap B^c \subseteq B^c$ *[by Theorem 9.2.1(1)b]* and hence $B^c \cap C = C$ *[by Theorem 9.2.3a]*. Therefore $P(B^c \cap C) = P(C) = 0.3$.

12. Proof 1: Suppose S is any sample space and U and V are any events in S. First note that by the set difference, distributive, universal bound, and identity laws,

$$(V \cap U) \cup (V - U) = (V \cap U) \cup (V \cap U^c) = V \cap (U \cup U^c) = V \cap S = V.$$

Next, observe that if $x \in (V \cap U) \cap (V - U)$, then, by definition of intersection, $x \in (V \cap U)$ and $x \in (V - U)$, and so, by definition of intersection and set difference, $x \in V$, $x \in U$, $x \in V$, and $x \notin U$, and hence, in particular, $x \in U$ and $x \notin U$, which is impossible. It follows that $(V \cap U) \cap (V - U) = \emptyset$. Thus, by substitution and by probability axiom 3 (the formula for the probability of mutually disjoint events),

$$P(V) = P((V \cap U) \cup (V - U)) = P(V \cap U) + P(V - U).$$

Solving for $P(V - U)$ gives

$$P(V - U) = P(V) - P(U \cap V).$$

Proof 2: Suppose S is any sample space and U and V are any events in S. First note that by the set difference, distributive, universal bound, and identity laws,

$$U \cup (V - U) = U \cup (V \cap U^c) = (U \cup V) \cap (U \cup U^c) = (U \cup V) \cap S = U \cup V.$$

Also by the set difference law, and the associative, commutative, and universal bound laws for $\cap$,

$$U \cap (V - U) = U \cap (V \cap U^c) = U \cap (U^c \cap V) = (U \cap U^c) \cap V = \emptyset \cap V = \emptyset.$$

Thus, by probability axiom 3 (the formula for the probability of mutually disjoint events),

$$P(U \cup V) = P(U \cup (V - U)) = P(U) + P(V - U).$$

But also by the formula for the probability of a general union,

$$P(U \cup V) = P(U) + P(V) - P(U \cap V).$$

Equating the two expressions for $P(U \cup V)$ gives

$$P(U) + P(V - U) = P(U) + P(V) - P(U \cap V).$$

Subtracting $P(U)$ from both sides gives

$$P(V - U) = P(V) - P(U \cap V).$$

15. *Solution 1*: The net gain for the first prize winner is \$10,000,000 − \$0.60 = \$9,999,999.40, that for the second prize winner is \$1,000,000 − \$0.60 = \$999,999.40, and that for the third prize winner is \$50,000 − \$0.60 = \$49,999.40. Each of the other 29,999,997 million people who mail back an entry form has a net loss of \$0.60. Because all of the 30 million entry forms have an equal chance of winning the prizes, the expected gain or loss is

$$\$9999999.40\cdot \frac{1}{30000000} + \$999999.40\cdot \frac{1}{30000000} + \$49999.40\cdot \frac{1}{30000000} - \$0.60\cdot \frac{29999997}{30000000}) \cong -\$0.23,$$

or an expected loss of about 23 cents per person.

Solution 2: The total amount spent by the 30 million people who return entry forms is $30,000,000\cdot \$0.60 = \$18,000,000$. The total amount of prize money awarded is \$10,000,000+ \$1,000,000+ \$50,000 = \$11,050,000. Thus the net loss is \$18,000,000 − \$11,050,000 = \$6,950,000, and so the expected loss per person is 6950000/30000000 ≅ −\$0.23, or about 23 cents per person.

18. Let 2_1 and 2_2 denote the two balls with the number 2, let 8_1 and 8_2 denote the two balls with the number 8, and let 1 denote the other ball. There are $\binom{5}{3} = 10$ subsets of 3 balls that can be chosen from the urn. The following table shows the sums of the numbers on the balls in each set and the corresponding probabilities:

Subset	Sum s	Probability of s
$\{1, 2_1, 2_2\}$	5	1/10
$\{1, 2_1, 8_1\}, \{1, 2_2, 8_1\}, \{1, 2_1, 8_2\}, \{1, 2_2, 8_2\}$	11	4/10
$\{2_1, 2_2, 8_1\}, \{2_1, 2_2, 8_2\}$	12	2/10
$\{1, 8_1, 8_2\}$	17	1/10
$\{2_1, 8_1, 8_2\}, \{2_2, 8_1, 8_2\}$	18	2/10

Thus the expected value is $5 \cdot \frac{1}{10} + 11 \cdot \frac{4}{10} + 12 \cdot \frac{2}{10} + 17 \cdot \frac{1}{10} + 18 \cdot \frac{2}{10} = \frac{126}{10} = 12.6.$

21. When a coin is tossed 4 times, there are $2^4 = 16$ possible outcomes and there are $\binom{4}{h}$ ways to obtain exactly h heads (as shown by the technique illustrated in Example 9.5.9). The following table shows the possible outcomes of the tosses, the amount gained or lost for each outcome, the number of ways the outcomes can occur, and the probabilities of the outcomes.

Number of Heads	Net Gain (or Loss)	Number of Ways	Probability
0	−\$3	$\binom{4}{0} = 1$	1/16
1	−\$2	$\binom{4}{1} = 4$	4/16
2	−\$1	$\binom{4}{2} = 6$	6/16
3	\$2	$\binom{4}{3} = 4$	4/16
4	\$3	$\binom{4}{4} = 1$	1/16

Thus the expected value is $(-\$3)\cdot\frac{1}{16}+(-\$2)\cdot\frac{4}{16}+(-\$1)\cdot\frac{6}{16}+\$2\cdot\frac{4}{16}+\$3\cdot\frac{1}{16} = -\$\frac{6}{16} = -\$0.375.$ So this game has an expected loss of 37.5 cents.

Section 9.9

3. Of the students who received A's on the first test, the percent who also received A's on the second test is

$$\frac{\text{the percent of students who received } A\text{'s on both tests}}{\text{the percent of students who received } A\text{'s on the first test}} = \frac{15\%}{25\%} = 0.6 = 60\%.$$

Thus the probability that a person who has the condition tests positive for it is 99%.

9. Proof: Suppose that a sample space S is a union of two disjoint events B_1 and B_2, that A is an event in S with $P(A) \neq 0$, and that $P(B_k) \neq 0$ for $k = 1$ and $k = 2$. Because B_1 and B_2 are disjoint, the same reasoning as in Example 9.9.5 establishes that

$$A = (A \cap B_1) \cup (A \cap B_2) \quad \text{and} \quad (A \cap B_1) \cap (A \cap B_2) = \emptyset.$$

Thus

$$P(A) = P(A \cap B_1) + P(A \cap B_2).$$

Moreover, for each $k = 1$ or 2, by definition of conditional probability, we have both

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)} = \frac{P(A \cap B_k)}{P(A)} \quad \text{and} \quad P(A \cap B_k) = P(A \mid B_k)P(B_k).$$

Putting these results together gives that for each $k = 1$ or 2,

$$P(B_k \mid A) = \frac{P(A \cap B_k)}{P(A)} = \frac{P(A \mid B_k)P(B_k)}{P(A \cap B_1) + P(A \cap B_2)} = \frac{P(A \mid B_k)P(B_k)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)},$$

which is Bayes' theorem for $n = 2$.

12. *a.* Let B_1 be the event that the first urn is chosen, B_2 the event that the second urn is chosen, and A the event that the chosen ball is blue. Then

$$P(A \mid B_1) = \frac{4}{20} \quad \text{and} \quad P(A \mid B_2) = \frac{10}{19}.$$

$$P(A \cap B_1) = P(A \mid B_1)P(B_1) = \frac{4}{20} \cdot \frac{1}{2} = \frac{1}{10}.$$

Also

$$P(A \cap B_2) = P(A \mid B_2)P(B_2) = \frac{10}{19} \cdot \frac{1}{2} = \frac{5}{19}.$$

Now A is the disjoint union of $A \cap B_1$ and $A \cap B_2$. So

$$P(A) = P(A \cap B_1) + P(A \cap B_2) = \frac{1}{10} + \frac{5}{19} = \frac{69}{190} \cong 36.3\%.$$

Thus the probability that the chosen ball is blue is approximately 36.3%.

b. Solution 1 (using Bayes' theorem): Given that the chosen ball is blue, the probability that it came from the first urn is $P(B_1 \mid A)$. By Bayes' theorem and the computations in part (a),

$$P(B_1 \mid A) = \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{5}{19}} = \frac{19}{69} \cong 27.5$$

Solution 2 (without explicit use of Bayes' theorem): Given that the chosen ball is blue, the probability that it came from the first urn is $P(B_1 \mid A)$. By the results of part (a),

$$P(B_1 \mid A) = \frac{P(A \cap B_1)}{P(A)} = \frac{\frac{1}{10}}{\frac{69}{190}} = \frac{19}{69} \cong 27.5.$$

15. Let B_1 be the event that the part came from the first factory, B_2 the event that the part came from the second factory, and A the event that a part chosen at random from the 180 is defective.

a. The probability that a part chosen at random from the 180 is from the first factory is $P(B_1) = \frac{100}{180}$.

b. The probability that a part chosen at random from the 180 is from the second factory is $P(B_2) = \frac{80}{180}$.

c. The probability that a part chosen at random from the 180 is defective is $P(A)$. Because 2% of the parts from the first factory and 5% of the parts from the second factory are defective, $P(A \mid B_1) = \frac{2}{100}$ and $P(A \mid B_2) = \frac{5}{100}$. By definition of conditional probability,

$$P(A \cap B_1) = P(A \mid B_1)P(B_1) = \frac{2}{100} \cdot \frac{100}{180} = \frac{1}{90}$$

$$P(A \cap B_2) = P(A \mid B_2)P(B_2) = \frac{5}{100} \cdot \frac{80}{180} = \frac{2}{90}.$$

Now because B_1 and B_2 are disjoint and because their union is the entire sample space, A is the disjoint union of $A \cap B_1$ and $A \cap B_2$. Thus the probability that

$$P(A) = P(A \cap B_1) + P(A \cap B_2) = \frac{1}{90} + \frac{2}{90} = \frac{3}{90} \cong 3.3\%.$$

d. Solution 1 (using Bayes' theorem): Given that the chosen part is defective, the probability that it came from the first factory is $P(B_1 \mid A)$. By Bayes' theorem and the computations in part (a),

$$P(B_1 \mid A) = \frac{P(A \mid B_1)P(B_1)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)} = \frac{\frac{1}{90}}{\frac{1}{90} + \frac{2}{90}} = \frac{1}{3} \cong 33.3\%.$$

Solution 2 (without explicit use of Bayes' theorem): Given that the chosen ball is green, the probability that it came from the first urn is $P(B_1 \mid A)$. By the results of part (a),

$$P(B_1 \mid A) = \frac{P(A \cap B_1)}{P(A)} = \frac{\frac{1}{90}}{\frac{3}{90}} = \frac{1}{3} \cong 33.3\%.$$

18. Proof: Suppose A and B are events in a sample space S, and $P(A \cap B) = P(A)\,P(B)$, $P(A) \neq 0$, and $P(B) \neq 0$. Applying the hypothesis to the definition of conditional probability gives

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$$

and

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)\,P(B)}{P(A)} = P(B).$$

21. If A and B are events in a sample space and $A \cap B = \emptyset$ and A and B are independent, then (by definition of independence) $P(A \cap B) = P(A)P(B)$, and (because $A \cap B = \emptyset$) $P(A \cap B) = 0$. Hence $P(A)\,P(B) = 0$, and so (by the zero product property) either $P(A) = 0$ or $P(B) = 0$.

24. Let A be the event that a randomly chosen error is missed by proofreader X, and let B be the event that the error is missed by proofreader Y. Then $P(A) = 0.12$ and $P(B) = 0.15$.

a. Because the proofreaders work independently, $P(A \cap B) = P(A)\,P(B)$. Hence the probability that the error is missed by both proofreaders is

$$P(A \cap B) = P(A)\,P(B) = (0.12)(0.15) = 0.018 = 1.8\%.$$

b. Assuming that the manuscript contains 1000 typographical errors, the expected number of missed errors is $1000 \cdot 0.018\% = 18$.

27. *Solution*: The family could have two boys, two girls, or one boy and one girl.

Let the subscript 1 denote the firstborn child (understanding that in the case of twins this might be by only a few moments), and let the subscript 2 denote the secondborn child.

Then we can let (B_1G_2, B_1) denote the outcome that the firstborn child is a boy, the secondborn is a girl, and the child you meet is the boy.

Similarly, we can let (B_1B_2, B_2) denote the outcome that both the firstborn and the secondborn are boys and the child you meet is the secondborn boy.

When this notational scheme is used for the entire set of possible outcomes for the genders of the children and the gender of the child you meet, all outcomes are equally likely and the sample space is denoted by

$\{(B_1B_2, B_1), (B_1B_2, B_2), (B_1G_2, B_1), (B_1G_2, G_2), (G_1B_2, G_1), (G_1B_2, B_2), (G_1G_2, G_1), (G_1G_2, G_2)\}$.

The event that you meet one of the children and it is a boy is

$$\{(B_1B_2, B_1), (B_1B_2, B_2), (B_1G_2, B_1), (G_1B_2, B_2)\}.$$

The probability of this event is $4/8 = 1/2$.

Discussion: An intuitive way to see this conclusion is to realize that the fact that you happen to meet one of the children and see that it is a boy gives you no information about the gender of the other child. Because each of the children is equally likely to be a boy, the probability that the other child is a boy is 1/2.

Consider the following situation in which the probabilities are identical to the situation described in the exercise. A person tosses two fair coins and immediately covers them so that you cannot see which faces are up. The person then reveals one of the coins, and you see that it is heads. This action on the person's part has given you no information about the other coin; the probability that the other coin has also landed heads up is 1/2.

30. *a.* $P(\text{0 false positives}) = \begin{bmatrix}\text{the number of ways 0 false}\\ \text{positives can be obtained}\\ \text{over a ten-year period}\end{bmatrix}\left(P\begin{pmatrix}\text{false}\\ \text{positive}\end{pmatrix}\right)^0\left(P\begin{pmatrix}\text{not a false}\\ \text{positive}\end{pmatrix}\right)^{10}$

$= \binom{10}{0}0.96^{10} = 1 \cdot 0.96^{10} \cong 0.665 = 66.5\%$

c. $P(\text{2 false positives}) = \begin{bmatrix}\text{the number of ways 2 false}\\ \text{positives can be obtained}\\ \text{over a ten-year period}\end{bmatrix}\left(P\begin{pmatrix}\text{false}\\ \text{positive}\end{pmatrix}\right)^2\left(P\begin{pmatrix}\text{not a false}\\ \text{positive}\end{pmatrix}\right)^{8}$

$= \binom{10}{2}0.04^2 \cdot 0.96^8 = 45 \cdot 0.04^2 \cdot 0.96^8 = 0.05194 \cong 5.2\%$

d. Let T be the event that a woman's test result is positive one year, and let C be the event that the woman has breast cancer.

(i) By Bayes' formula, the probability of C given T is

$$\begin{aligned}P(C\,|\,T) &= \frac{P(T\,|\,C)P(C)}{P(T\,|\,C)P(C) + P(T\,|\,C^c)P(C^c)}\\ &= \frac{(0.98)(0.0002)}{(0.98)(0.0002) + (0.04)(0.9998)}\\ &\cong 0.00488 = 4.88\%.\end{aligned}$$

(ii) The event that a woman's test result is negative one year is T^c. By Bayes formula, the probability of C given T^c is

$$\begin{aligned}P(C\,|\,T^c) &= \frac{P(T^c\,|\,C)P(C)}{P(T^c\,|\,C)P(C) + P(T^c\,|\,C^c)P(C^c)}\\ &= \frac{(0.02)(0.0002)}{(0.02)(0.0002) + (0.96)(0.9998)}\\ &\cong 0.000004 = 0.0004\%.\end{aligned}$$

33. Suppose a gambler starts with $\$k$. Rolling a fair die leads to one of two disjoint outcomes: winning \$1 or losing \$1. Let A_k be the event that the gambler is ruined when he has $\$k$. Then A_k is the disjoint union of the following two events: C_k and D_k, where

C_k is the event that the gambler has $\$k$, wins the next roll, and eventually gets ruined
and D_k is the event that the gambler has $\$k$, loses the next roll, and eventually gets ruined.

Now P_k is the probability that the gambler eventually gets ruined when he has \$$k$. By probability axiom 3,

$$P_k = P(C_k) + P(D_k).$$

Let W be the event that the gambler wins on any given roll. Then

$$P(W) = \frac{1}{6} \quad \text{and} \quad P(W^c) = \frac{5}{6}.$$

For each integer k with $1 \leq k \leq 300$, the definition of conditional probability can be used to find $P(C_k)$ and $P(D_k)$:

$$\begin{aligned}
P(C_k) &= P(A_k \cap W) \\
&\qquad \text{by definition of } C_k,\ A_k, \text{ and } W \\
&= P(A_k \,|\, W)P(W) \\
&\qquad \text{by definition of conditional probability} \\
&= P(A_{k+1}) \cdot \frac{1}{6} \\
&\qquad \text{because if the gambler wins on a roll when he has \$}k \\
&\qquad \text{then on the next roll he has \$}(k+1) \\
&= P_{k+1} \cdot \frac{1}{6}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(D_k) &= P(A_k \cap W^c) \\
&\qquad \text{by definition of } C_k,\ A_k, \text{ and } W \\
&= P(A_k \,|\, W^c)P(W^c) \\
&\qquad \text{by definition of conditional probability} \\
&= P(A_{k-1}) \cdot \frac{5}{6} \\
&\qquad \text{because if the gambler loses on a roll when he has \$}k \\
&\qquad \text{then on the next roll he has \$}(k-1) \\
&= P_{k-1} \cdot \frac{5}{6}.
\end{aligned}$$

Thus,

$$P_k = P(C_k) + P(D_k) = P_{k+1} \cdot \frac{1}{6} + P_{k-1} \cdot \frac{5}{6}.$$

Review Guide: Chapter 9

Probability

- What is the sample space of an experiment? *(p. 518)*
- What is an event in the sample space? *(p. 518)*
- What is the probability of an event when all the outcomes are equally likely? *(p. 518)*

Counting

- If m and n are integers with $m \leq n$, how many integers are there from m to n inclusive? *(p. 521)*
- How do you construct a possibility tree? *(p. 525)*
- What are the multiplication rule, the addition rule, and the difference rule? *(pp. 527, 540, 541)*
- What is the inclusion/exclusion rule? *(p. 545 and exercise 48 on p. 553)*
- What is a permutation? an r-permutation? *(pp. 531, 533)*
- What is $P(n,r)$? *(p. 533)*
- How does the multiplication rule give rise to $P(n,r)$? *(pp. 533-534)*
- When should you use the multiplication rule and when should you use the addition rule? *(p. 577)*
- What are some situations where both the multiplication and the addition or difference rule must be used? *(pp. 540-545)*
- What is the formula for the probability of the complement of an event? *(p. 543)*
- How are IP addresses created? *(p. 544)*
- How is the inclusion/exclusion rule used? *(pp. 546-549)*
- What is an r-combination? *(p. 566)*
- What is an unordered selection of elements from a set? *(p. 566)*
- What is complete enumeration? *(p. 567)*
- What formulas are used to compute $\binom{n}{r}$ by hand? *(p. 568)*
- What are some situations where both r-combinations and the addition or difference rule must be used? *(pp. 569-571)*
- What are some situations where r-combinations, the multiplication rule, and the addition rule are all needed? ? *(pp. 573-574)*
- How can r-combinations be used to count the number of permutations of a set with repeated elements? *(pp. 575-576)*
- What are some formulas for the number of permutations of a set of objects when some of the objects are indistinguishable from each other? *(p. 577)*
- What are Stirling numbers of the second kind? How do you find a recurrence relation for the number of ways a set of size n can be partitioned into r subsets? *(pp. 578-580)*
- What is an r-combination with repetition allowed (or a multiset of size r)? *(p. 584)*
- How many r-combinations with repetition allowed can be selected from a set of n elements? *(p. 586)*

The Pigeonhole Principle

- What is the pigeonhole principle? *(p. 554)*
- How is the pigeonhole principle used to show that rational numbers have terminating or repeating decimal expansions? *(pp. 557-559)*

- What is the generalized pigeonhole principle? *(p. 559)*
- What is the relation between one-to-one and onto for a function defined from one finite set to another of the same size? *(p. 562)*

Pascal's Formula and the Binomial Theorem

- What is Pascal's formula? Can you apply it in various situations? *(p. 593)*
- What is the algebraic proof of Pascal's formula? *(p. 595)*
- What is the combinatorial proof of Pascal's formula? *(pp. 595-596)*
- What is the binomial theorem? Can you apply it in various situations? *(p. 598)*
- What is the algebraic proof of the binomial theorem? *(p. 598-600)*
- What is the combinatorial proof of the binomial theorem? *(pp. 600-601)*

Probability Axioms and Expected Value

- What is the range of values for the probability of an event? *(p. 605)*
- What is the probability of an entire sample space? *(p. 605)*
- What is the probability of the empty set? *(p. 605)*
- If A and B are disjoint events in a sample space S, what is $P(A \cup B)$? *(p. 605)*
- If A is an event in a sample space S, what is $P(A^c)$? *(p. 605)*
- If A and B are any events in a sample space S, what is $P(A \cup B)$? *(p. 606)*
- How do you compute the expected value of a random experiment or process, if the possible outcomes are all real numbers and you know the probability of each outcome? *(p. 608)*
- What is the conditional probability of one event given another event? *(p. 612)*
- What is Bayes' theorem? *(p. 616)*
- What does it mean for two events to be independent? *(p. 618)*
- What is the probability of an intersection of two independent events? *(p. 618)*
- What does it mean for events to be mutually independent? *(p. 620)*
- What is the probability of an intersection of mutually independent events? *(p. 621)*

Chapter 10: Graphs and Trees

The first section of this chapter introduces the terminology of graph theory, illustrating it in a variety of different instances. Several exercises are designed to clarify the distinction between a graph and a drawing of a graph. You might point out to students the advantage of the formal definition over the informal drawing for computer representation of graphs. Other exercises explore the use of graphs to solve problems of various sorts. In some cases, students may be able to solve the given problems, such as the wolf, the goat, the cabbage and the ferryman, more easily without using graphs than using them. The point to make is that such problems *can* be solved using graphs and that for more complex problems involving, say, hundreds of possible states, a graphical representation coupled with a computer path-finding algorithm makes it possible find a solution that could not be discovered by trial-and-error alone. The variety of solutions for exercise 33, on the number of edges of a complete graph illustrates the relations among different branches of discrete mathematics. The rest of the exercises in this section are intended to give you practice in applying the theorem that relates the total degree of a graph to the number of its edges, especially for exploring properties of simple graphs, complete graphs, and bipartite graphs.

In Section 10.2 the general topic of trails, paths and circuits is discussed, including the notion of connectedness and Euler and Hamiltonian circuits. As in the rest of the chapter, an attempt is made to balance the presentation of theory and application.

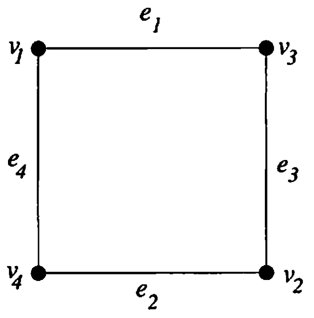
Section 10.3 introduces the concept of the adjacency matrix of a graph. The main theorem of the section states that the ijth entry of the kth power of the adjacency matrix equals the number of walks of length k from the ith to the jth vertices in the graph. Matrix multiplication is defined and explored in this section in a way that is intended to be adequate even if you have never seen the definition before.

The concept of graph isomorphism is discussed in Section 10.4. In this section the main theorem gives a list of isomorphic invariants that can be used to determine the non-isomorphism of two graphs.

The last three sections of the chapter deal with the subject of trees. Section 10.5 focuses on basic definitions, examples, and theorems giving necessary and sufficient conditions for graphs to be trees, and Section 10.6 contains the definition of rooted tree, binary tree, and the theorems that relate the number of internal to the number of terminal vertices of a full binary tree and the maximum height of a binary tree to the number of its terminal vertices. Section 10.7 on spanning trees and shortest paths contains Kruskal's, Prim's, and Dijkstra's algorithms and proofs of their correctness, as well as applications of minimum spanning trees and shortest paths.

Section 10.1

6.



9. (i) e_1, e_2, e_7 are incident on e_1.
(ii) v_1 and v_2 are adjacent to v_3.
(iii) e_2 and e_7 are adjacent to e_1.
(iv) e_1 and e_3 are loops.
(v) e_4 and e_5 are parallel.
(vi) v_4 is an isolated vertex.
(vii) degree of $v_3 = 2$
(viii) total degree of the graph $= 14$

27. *b.* Yes. Each could be friends with all three others.

30. Let t be the total degree of the graph. Since the degree of each vertex is at least $d_{\min}$ and at most $d_{\max}$, $d_{\min}\cdot v \le t \le d_{\max}\cdot v$. But by Theorem 10.1.1, t equals twice the number of edges. So by substitution, $d_{\min}\cdot v \le 2e \le d_{\max}\cdot v$. Dividing each part of the inequality by 2 produces the required result:

$$\frac{1}{2}d_{\min}\cdot v \ \le e \ \le \ \frac{1}{2}d_{\max}\cdot v.$$

33. *b.* Proof 1: Suppose n is an integer with $n \ge 1$ and K_n is a complete graph on n vertices. If $n = 1$, then K_n has one vertex and 0 edges and $\frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$, and so K_n has $\frac{n(n-1)}{2}$ edges. If $n \ge 2$, then since each pair of distinct vertices of K_n is connected by exactly one edge, there are as many edges in K_n as there are subsets of size two of the set of n vertices. By Theorem 9.5.1, there are $\binom{n}{2}$ such sets. But

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}.$$

Hence there are $\frac{n(n-1)}{2}$ edges in K_n.

Proof 2 (by mathematical induction: Let the property $P(n)$ be the sentence

$$\text{A complete graph on } n \text{ vertices has } \frac{n(n-1)}{2} \text{ edges.} \qquad \leftarrow P(n)$$

We will prove that $P(n)$ is true for all integers $n \ge 1$.

Show that $P(1)$ is true: $P(1)$ is true because a complete graph on one vertex has 0 edges and the quantity $\frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$ also.

Show that for all integers $m \ge 1$, if $P(m)$ is true then $P(m+1)$ is true: Let m be any integer with $m \ge 1$, and suppose

$$\text{A complete graph on } m \text{ vertices has } \frac{m(m-1)}{2} \text{ edges.} \qquad \leftarrow \begin{array}{l} P(m) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\text{A complete graph on } m+1 \text{ vertices has } \frac{(m+1)((m+1)-1)}{2} \text{ edges,}$$

or, equivalently,

$$\text{A complete graph on } m+1 \text{ vertices has } \frac{m(m+1)}{2} \text{ edges.} \qquad \leftarrow P(m+1)$$

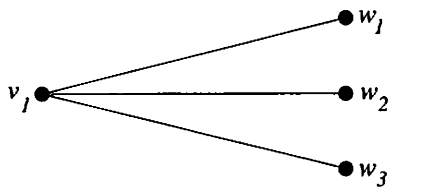
Let K_{m+1} be a complete graph on $m+1$ vertices. Temporarily remove one vertex, v, together with all the edges joining this vertex to the other vertices of the graph. In the graph thus obtained, each vertex is connected to each other vertex by exactly one edge, and so the graph is a complete graph on m vertices. By inductive hypothesis this graph has $\frac{m(m-1)}{2}$ edges. Connecting v to each of the m other vertices adds another m edges. Hence the total number of edges of K_{m+1} is

$$\frac{m(m-1)}{2}+m=\frac{m(m-1)}{2}+\frac{2m}{2}=\frac{m^2-m+2m}{2}=\frac{m(m+1)}{2}$$

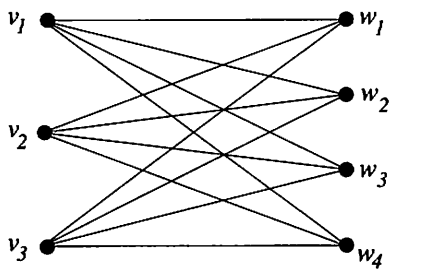
[as was to be shown].

Proof 3: Suppose n is an integer with $n \geq 1$ and K_n is a complete graph on n vertices. Because each vertex of K_n is connected by exactly one edge to each of the other $n-1$ vertices of K_n, the degree of each vertex of K_n is $n-1$. Thus the total degree of K_n equals the number of vertices times the degree of each vertex, or $n(n-1)$. Let e be the number of edges of K_n. By Theorem 10.1.1, the total degree of K_n equals $2e$, and so $n(n-1) = 2e$. Equivalently, $e = n(n-1)/2$ *[as was to be shown].*

36. *b.* $K_{1,3}$



c. $K_{3,4}$

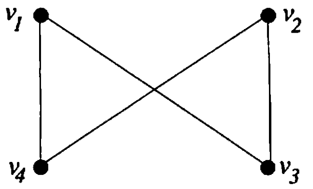


d. If $n \neq m$, the vertices of $K_{m,n}$ are divided into two groups: one of size m and the other of size n. Every vertex in the group of size m has degree n because each is connected to every vertex in the group of size n. So $K_{m,n}$ has n vertices of degree m. Similarly, every vertex in the group of size n has degree m because each is connected to every vertex in the group of size m. So $K_{m,n}$ has n vertices of degree m. Note that if $n = m$, then all $n+m = 2n$ vertices have the same degree, namely n.

e. The total degree of $K_{m,n}$ is $2mn$ because $K_{m,n}$ has m vertices of degree n (which contribute mn to its total degree) and n vertices of degree m (which contribute another mn to its total degree)

f. The number of edges of $K_{m,n} = mn$. The reason is that the total degree of $K_{m,n}$ is $2mn$, and so, by Theorem 10.1.1, $K_{m,n}$ has $2mn/2 = mn$ edges. Another way to reach this conclusion is to say that $K_{m,n}$ has n edges coming out of each of the group of m vertices (each leading to a vertex in the group of n vertices) for a total of mn edges. Equivalently, $K_{m,n}$ has m edges coming out of each of the group of n vertices (each leading to a vertex in the group of m vertices) for a total of mn edges.

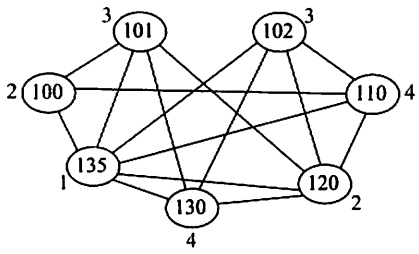
39. *b.*



42. The graph obtained by taking all the vertices and edges of G together with all the edges of G' is K_n. Therefore, by exercise 33b, the number of edges of G plus the number of edges of G' equals $n(n-1)/2$.

45. Yes. Suppose that in a group of two or more people, each person is acquainted with a different number of people. Then the acquaintance graph representing the situation is a simple graph in which all the vertices have different degrees. But by exercise 44(c) such a graph does not exist. Hence the supposition is false, and so in a group of two or more people there must be at least two people who are acquainted with the same number of people within the group.

48. In the following graph each course number is represented as a vertex. Vertices are joined if, and only if, the corresponding courses have a student in common.



Vertex 135 has maximal degree, so use color #1 for it. All vertices share edges with vertex 135, and so color #1 cannot be used on any other vertex.

From the remaining uncolored vertices, only vertex 120 has maximal degree. So use color #2 for it. Because vertex 100 does not share an edge with vertex 120, color #2 may also be used for it.

From the remaining uncolored vertices, all of 101, 102, 110, and 130 have maximal degree. Choose any one of them, say vertex 101, and use color #3 for it. Neither vertex 102 nor vertex 110 shares an edge with vertex 101, but they do share an edge with each other. So color #3 may be used for only one of them. If color #3 is used for vertex 110, then, since the remaining vertices 130 and 102 are connected, two additional colors would be needed for them to have different colors. On the other hand, if color #3 is used for vertex 102, then, since the remaining vertices, 110 and 130, are not connected to each other, color 4 may be used for both. Therefore, to minimize the number of colors, color #3 should be used for vertex 102 and color #4 for vertices 110 and 130. The result is indicated in the annotations on the graph.

To use the results for scheduling exams, let color n correspond to exam time n. Then

Time 1: MCS135

Time 2: MCS 100 and MCS120

Time 3: MCS101 and MCS102

Time 4: MCS110 and MCS130

Note that because, for example, MSC135, MSC102, MSC110, and MSC 120 are all connected to each other, they must all be given different colors, and so the schedule for the seven exams must use at least four time periods.

Section 10.2

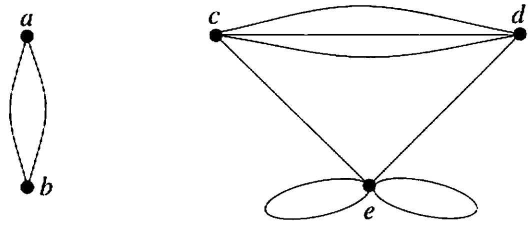
3. *b.* No, because e_1e_2 could refer either to $v_1e_1v_2e_2v_1$ or to $v_2e_1v_1e_2v_2$.

6. *b.* $\{v_7, v_8\}$, $\{v_1, v_2\}$, $\{v_3, v_4\}$

   c. $\{v_2, v_3\}$, $\{v_6, v_7\}$, $\{v_7, v_8\}$, $\{v_9, v_{10}\}$

9. *b.* Yes, by Theorem 10.2.3 since G is connected and every vertex has even degree.

   c. Not necessarily. It is not specified that G is connected. For instance, the following graph satisfies the given conditions but does not have an Euler circuit:



15. One Euler circuit is the following: *stuvwxyzrsuwyuzs*.

18. Yes. One Euler circuit is $ABDEACDA$.

21. One Euler path from u to w is $uv_1v_2v_3uv_0v_7v_6v_3v_4v_6wv_5v_4w$.

24. One Hamiltonian circuit is *balkjedcfihgb*. The only other one traverses this circuit in the opposite direction.

27. Call the given graph G and suppose G has a Hamiltonian circuit. Then G has a subgraph H that satisfies conditions (1) – (4) of Proposition 10.2.6. Since the degree of B in G is five and every vertex in H has degree two, three edges incident on B must be removed from G to create H. Edge $\{B, C\}$ cannot be removed because doing so would result in vertex C having degree less than two in H. Similar reasoning shows that edges $\{B, E\}$, $\{B, F\}$, and $\{B, A\}$ cannot be removed either. It follows that the degree of B in H must be at least four, which contradicts the condition that every vertex in H has degree two in H. Hence no such subgraph H can exist, and so G does not have a Hamiltonian circuit.

30. One Hamiltonian circuit is $v_0v_1v_5v_4v_7v_6v_2v_3v_0$.

33. Other such graphs are those shown in exercises 17, 21, 23, 24, 29 and 30.

36. It is clear from the map that only a few routes have a chance of minimizing the distance. For instance, one must go to either Düsseldorf or Luxembourg just after leaving Brussels or just before returning to Brussels, and one must either travel from Berlin directly to Munich or the reverse. The possible minimizing routes are those shown below plus the same routes traveled in the reverse direction.

Route	Total Distance (in km)
Bru-Lux-Düss-Ber-Mun-Par-Bru	$219 + 224 + 564 + 585 + 832 + 308 = 2732$
Bru-Düss-Ber-Mun-Par-Lux-Bru	$223 + 564 + 585 + 832 + 375 + 219 = 2798$
Bru-Düss-Lux-Ber-Mun-Par-Bru	$223 + 224 + 764 + 585 + 832 + 308 = 2936$
Bru-Düss-Ber-Mun-Lux-Par-Bru	$223 + 564 + 585 + 517 + 375 + 308 = 2572$

The routes that minimize distance, therefore, are the bottom route shown in the table and that same route traveled in the reverse direction.

39. Proof:

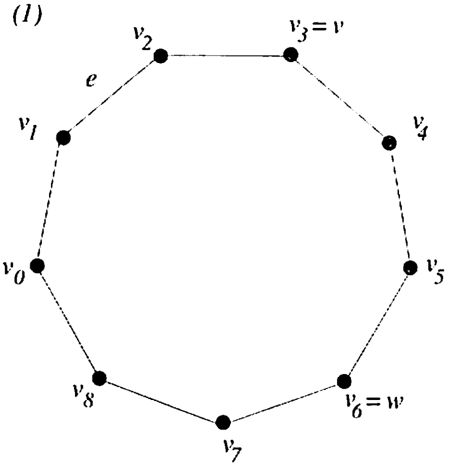
Suppose vertices v and w are part of a circuit in a graph G and one edge e is removed from the circuit. Without loss of generality, we may assume the v occurs before the w in the circuit, and we may denote the circuit by $v_0e_1v_1e_2\ldots e_{n-1}v_{n-1}e_nv_0$ with $v_i = v$, $v_j = w$, $i < j$, and $e_k = e$.

In case either $k \leq i$ or $k > j$, then $v = v_ie_{i+1}v_{i+1}\ldots v_{j-1}e_jv_j = w$ is a trail in G from v to w that does not include e.

In case $i < k \leq j$, then $v = v_ie_iv_{i-1}e_{i-1}\ldots v_1e_1v_0e_nv_{n-1}\ldots e_{j+1}v_j = w$ is a trail in G from v to w that does not include e.

These possibilities are illustrated by examples (1) and (2) in the diagram below. In both cases there is a trail in G from v to w that does not include e.

(1)

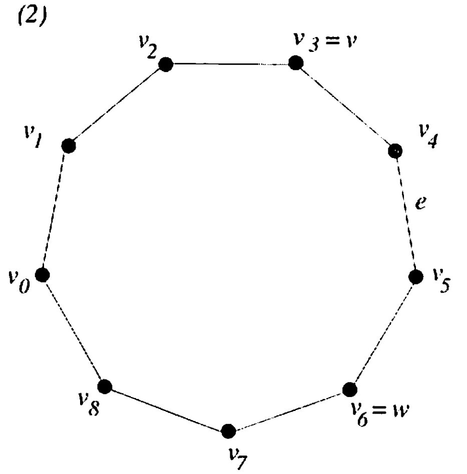


$i = 3, j = 6$, e is deleted

trail from v to w :

$v = v_3v_4v_5v_6 = w$

(2)



$i = 3, j = 6$, e is deleted

trail from v to w :

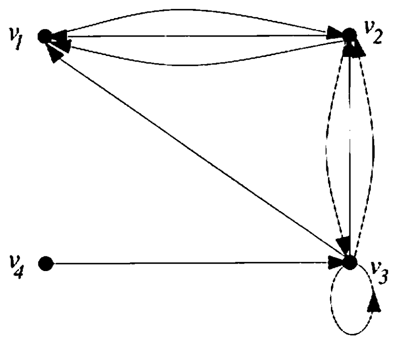
$v = v_3v_2v_1v_0v_8v_7v_6 = w$

48. *a.* Let m and n be positive integers and let $K_{m,n}$ be a complete bipartite graph on (m, n) vertices. Since $K_{m,n}$ is connected, by Theorem 10.2.4 it has an Euler circuit if, and only if, every vertex has even degree. But $K_{m,n}$ has m vertices of degree n and n vertices of degree m. So $K_{m,n}$ has an Euler circuit if, and only if, both m and n are even.

b. Let m and n be positive integers, let $K_{m,n}$ be a complete bipartite graph on (m, n) vertices, and suppose $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$ are the disjoint sets of vertices such that each vertex in V_1 is joined by an edge to each vertex in V_2 and no vertex within V_1 or V_2 is joined by an edge to any other vertex within the same set. If $m = n \geq 2$, then $K_{m,n}$ has the following Hamiltonian circuit: $v_1w_1v_2w_2\ldots v_mw_mv_1$. If $K_{m,n}$ has a Hamiltonian circuit, then $m = n$ because the vertices in any Hamiltonian circuit must alternate between V_1 and V_2 (since no edges connect vertices within either set) and because no vertex, except the first and last, appears twice in a Hamiltonian circuit. If $m = n = 1$, then $K_{m,n}$ does not have a Hamiltonian circuit because $K_{1,1}$ contains just one edge joining two vertices. Therefore, $K_{m,n}$ has a Hamiltonian circuit if, and only if, $m = n \geq 2$.

Section 10.3

3. *b.*



Any labels may be applied to the edges because the adjacency matrix does not determine edge labels.

6. *b.* The graph is not connected; the matrix shows that there are no edges joining the vertices from the set $\{v_1, v_2\}$ to those in the set $\{v_3, v_4\}$.

9. *b.*

$$\begin{bmatrix} 0 & 8 \\ -5 & 4 \end{bmatrix}$$

c.

$$\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$$

18. Proof (by mathematical induction: Let the property $P(n)$ be the sentence

$$\mathbf{A}^n \text{ is symmetric.} \qquad \leftarrow P(n)$$

We will prove that $P(n)$ is true for all integers $n \geq 1$.

Show that $P(1)$ is true: $P(1)$ is true because by assumption $\mathbf{A}$ is a symmetric matrix.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose

$$\mathbf{A}^k \text{ is symmetric.} \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\mathbf{A}^{k+1} \text{ is symmetric.} \qquad \leftarrow P(k+1)$$

Let $\mathbf{A}^k = (b_{ij})$. Then for all $i, j = 1, 2, \ldots, m$,

$$\begin{aligned}
\text{the } ij\text{th entry of } \mathbf{A}^{k+1} &= \text{the } ij\text{th entry of } \mathbf{A}\mathbf{A}^k && \text{by definition of matrix power} \\
&= \sum_{r=1}^{m} a_{ir}b_{rj} && \text{by definition of matrix multiplication} \\
&= \sum_{r=1}^{m} a_{ri}b_{jr} && \text{because } A \text{is symmetric by hypothesis and } A^k \text{is symmetric by inductive hypothesis} \\
&= \sum_{r=1}^{m} b_{jr}a_{ri} && \text{because multiplication of real numbers is commutative} \\
&= \textit{the } ji\text{th entry of } \mathbf{A}^k\mathbf{A} && \text{by definition of matrix multiplication} \\
&= \textit{the } ji\text{th entry of } \mathbf{A}\mathbf{A}^k && \text{by exercise 17} \\
&= \textit{the } ji\text{th entry of } \mathbf{A}^{k+1} && \text{by definition of matrix power.}
\end{aligned}$$

Therefore, $\mathbf{A}^{k+1}$ is symmetric *[as was to be shown]*.

21. Proof (by mathematical induction: Let the property $P(n)$ be the sentence

> All the entries along the main diagonal of $\mathbf{A}^n$ are equal to each other and all the entries off the main diagonal are also equal to each other. ← $P(n)$

We will prove that $P(n)$ is true for all integers $n \geq 1$.

Show that $P(1)$ is true: $P(1)$ is true because

$$\mathbf{A}^1 = \mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which is the adjacency matrix for K_3, and all the entries along the main diagonal of $\mathbf{A}$ are 0 *[because K_3 has no loops]* and all the entries off the main diagonal are 1 *[because each pair of vertices is connected by exactly one edge]*.

Show that for all integers $m \geq 1$, if $P(m)$ is true then $P(m+1)$ is true: Let m be any integer with $m \geq 1$, and suppose

> All the entries along the main diagonal of $\mathbf{A}^m$ are equal to each other and all the entries off the main diagonal are also equal to each other. ← $P(m)$ inductive hypothesis

We must show that

> All the entries along the main diagonal of $\mathbf{A}^{m+1}$ are equal to each other and all the entries off the main diagonal are also equal to each other. ← $P(m+1)$

By inductive hypothesis,

$$\mathbf{A}^m = \begin{bmatrix} b & c & c \\ c & b & c \\ c & c & b \end{bmatrix} \quad \text{for some integers } b \text{ and } c.$$

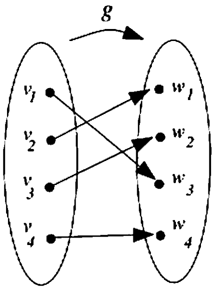
It follows that

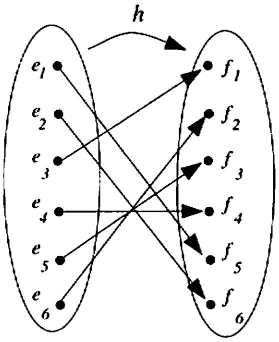
$$\mathbf{A}^{m+1} = \mathbf{A}\mathbf{A}^m = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} b & c & c \\ c & b & c \\ c & c & b \end{bmatrix} = \begin{bmatrix} 2c & b+c & b+c \\ b+c & 2c & b+c \\ b+c & b+c & 2c \end{bmatrix}$$

As can be seen, all the entries of $\mathbf{A}^{m+1}$ along the main diagonal are equal to each other and all the entries off the main diagonal are equal to each other. So the property is true for $n = m+1$.

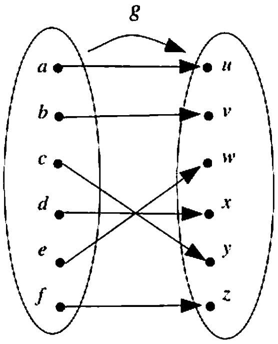
Section 10.4

3. The graphs are isomorphic. One way to define to isomorphism is as follows.

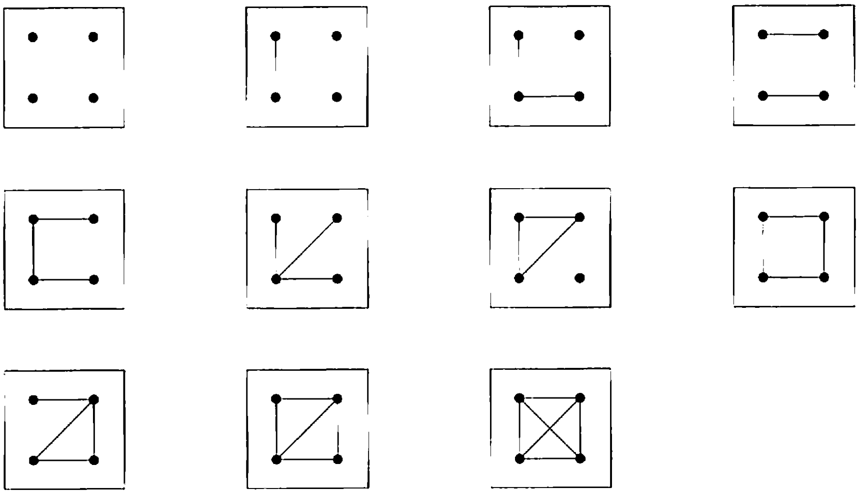




9. The graphs are isomorphic. One way to define to isomorphism is as follows.

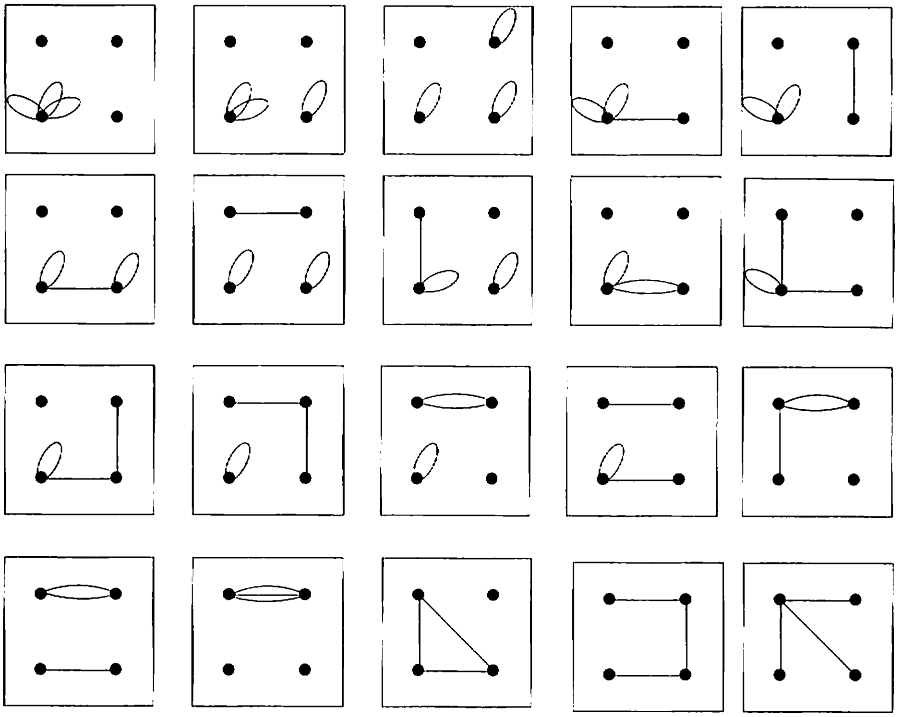


15. Of all nonisomorphic simple graphs with four vertices, there is one with 0 edges, one with 1 edge, two with 2 edges, three with 3 edges, two with 4 edges, one with 5 edges, and one with 6 edges. These eleven graphs are shown below.



18. There are three nonisomorphic graphs with four vertices and three edges in which all 3 edges are loops, five in which 2 edges are loops and 1 is not a loop, six in which 1 edge is a loop and

2 edges are not loops, and six in which none of the 3 edges is a loop. These twenty graphs are shown below.



24. Proof:

Suppose G and G' are isomorphic graphs and suppose G has a simple circuit C of length k, where k is a nonnegative integer. By definition of graph isomorphism, there are one-to-one correspondences $g\colon V(G) \to V(G')$ and $h\colon E(G) \to E(G')$ that preserve the edge-endpoint functions in the sense that for all v in $V(G)$ and e in $E(G)$, v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$.

Let C be $v_0e_1v_1e_2\ldots e_kv_k(=v_0)$, and let C' be $g(v_0)h(e_1)g(v_1)h(e_2)\ldots h(e_k)g(v_k)(=g(v_0))$. By the same reasoning as in the solution to exercise 23 in Appendix B, C' is a circuit of length k in G'.

Suppose C' is not a simple circuit. Then C' has a repeated vertex, say $g(v_i) = g(v_j)$ for some $i, j = 0, 1, 2, \ldots, k-1$ with $i \neq j$. But since g is a one-to-one correspondence this implies that $v_i = v_j$, which is impossible because C is a simple circuit. Hence the supposition is false, and so we conclude that C' is a simple circuit. Therefore G' has a simple circuit of length k.

27. Proof:

Suppose G and G' are isomorphic graphs and suppose G is connected. By definition of graph isomorphism, there are one-to-one correspondences $g\colon V(G) \to V(G')$ and $h\colon E(G) \to E(G')$ that preserve the edge-endpoint functions in the sense that for all v in $V(G)$ and e in $E(G)$, v is an endpoint of $e \Leftrightarrow g(v)$ is an endpoint of $h(e)$. Suppose w and x are any two vertices of G'. Then $u = g^{-1}(w)$ and $v = g^{-1}(x)$ are distinct vertices in G (because g is a one-to-one correspondence). Since G is connected, there is a walk in G connecting u and v. Say this walk is $ue_1v_1e_2v_2\ldots e_nv$. Because g and h preserve the edge-endpoint functions, $w = g(u)h(e_1)g(v_1)h(e_2)g(v_2)\ldots h(e_n)g(v) = x$ is a walk in G' connecting w and x.

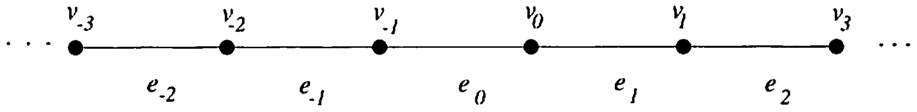
30. Suppose that G and G' are isomorphic via one-to-one correspondences $g\colon V(G) \to V(G')$ and $h\colon E(G) \to E(G')$, where g and h preserve the edge-endpoint functions. Now w_6 has degree

one in G', and so by the argument given in Example 10.4.4, w_6 must correspond to one of the vertices of degree one in G: either $g(v_1) = w_6$ or $g(v_6) = w_6$. Similarly, since w_5 has degree three in G', w_5 must correspond to one of the vertices of degree three in G: either $g(v_3) = w_5$ or $g(v_4) = w_5$. Because g and h preserve the edge-endpoint functions, edge f_6 with endpoints w_5 and w_6 must correspond to an edge in G with endpoints v_1 and v_3, or v_1 and v_4, or v_6 and v_3, or v_6 and v_4. But this contradicts the fact that none of these pairs of vertices are connected by edges in G. Hence the supposition is false, and G and G' are not isomorphic.

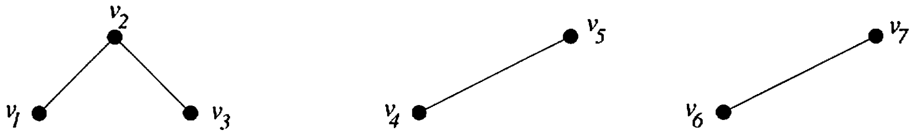
Section 10.5

3. By Theorem 10.5.2, a tree with n vertices (where $n \geq 1$) has $n - 1$ edges, and so by Theorem 10.1.1, its total degree is twice the number of edges, or $2(n - 1) = 2n - 2$.

6. Define an infinite graph G as follows: $V(G) = \{v_i \mid i \in \mathbf{Z}\} = \{\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\}$, $E(G) = \{e_i \mid i \in \mathbf{Z}\} = \{\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots\}$, and the edge-endpoint function is defined by the rule $f(e_i) = \{v_{i-1}, v_i\}$ for all $i \in \mathbf{Z}$. Then G is circuit-free, but each vertex has degree two. G is illustrated below.



15. One circuit-free graph with seven vertices and four edges is shown below.

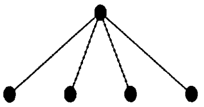


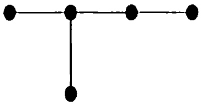
18. Any tree with five vertices has four edges. By Theorem 10.1.1, the total degree of such a graph is eight, not ten. Hence there is no tree with five vertices and total degree ten.

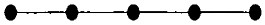
21. Any tree with ten vertices has nine edges. By Theorem 10.1.1, the total degree of such a tree is 18, not 24. Hence there is no such graph.

24. Yes. Given any two vertices u and w of G', then u and w are vertices of G neither equal to v. Since G is connected, there is a walk in G from u to w, and so by Lemma 10.2.1, there is a path in G from u to w. This path does not include edge e or vertex v because a path does not have a repeated edge, and e is the unique edge incident on v. *[If a path from u to w leads into v, then it must do so via e. But then it cannot emerge from v to continue on to w because no edge other than e is incident on v.]* Thus this path is a path in G'. It follows that any two vertices of G' are connected by a walk in G', and so G' is connected.

30. A tree with five vertices must have four edges and, therefore, a total degree of 8. Since at least two vertices have degree 1 and no vertex has degree greater than 4, the possible degrees of the five vertices are as follows: 1,1,1,1,4; 1,1,1,2,3; and 1,1,2,2,2. The corresponding trees are shown below.

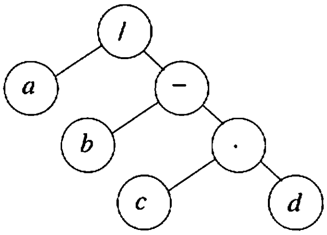






Section 10.6

3. *b.*



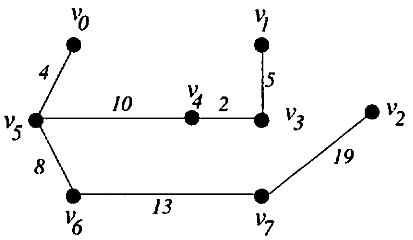
12. There is no tree with the given properties because any full binary tree with eight internal vertices has nine terminal vertices, not seven.

15. There is no tree with the given properties because a full binary tree with five internal vertices has $2 \cdot 5 + 1$ or eleven vertices in all, not nine.

18. There is no full binary tree with sixteen vertices because a full binary tree has $2k + 1$ vertices, where k is the number of internal vertices, and $16 \neq 2k + 1$ for any integer k.

Section 10.7

6. Minimum spanning tree:



Order of adding the edges: $\{v_3, v_4\}$, $\{v_0, v_5\}$, $\{v_1, v_3\}$, $\{v_5, v_6\}$, $\{v_4, v_5\}$, $\{v_6, v_7\}$, $\{v_2, v_7\}$

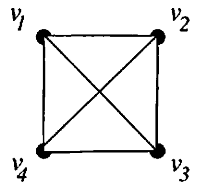
15.

Step	$newpageV(T)$	$E(T)$	F	$L(a)$	$L(b)$	$L(c)$	$L(d)$	$L(e)$	$L(f)$	$L(g)$
0	$\{a\}$	$\emptyset$	$\{a\}$	0	∞	∞	∞	∞	∞	∞
1	$\{a\}$	$\emptyset$	$\{b, e, g\}$	0	**3***	∞	∞	3	∞	4
2	$\{a, b\}$	$\{\{a, b\}\}$	$\{c, e, g\}$	0	3	10	∞	**3**	∞	4
3	$\{a, b, e\}$	$\{\{a, b\}, \{a, e\}\}$	$\{c, d, f, g\}$	0	3	10	14	3	7	**4**
4	$\{a, b, e, g\}$	$\{\{a, b\}, \{a, e\}, \{a, g\}\}$	$\{c, d, f\}$	0	3	10	14	3	**5**	4
5	$\{a, b, e, g, f\}$	$\{\{a, b\}, \{a, e\}, \{a, g\}, \{g, f\}\}$								

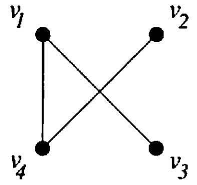
*At this point, vertex e could have been chosen instead of vertex b.

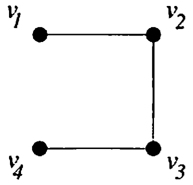
18. *a.* If there were two distinct paths from one vertex of a tree to another, they (or pieces of them) could be patched together to obtain a circuit. But a tree cannot have a circuit.

21. *b.* Counterexample: Let G be the following simple graph.



Then G has the spanning trees shown below.





These trees have no edge in common.

27. Proof: Suppose not. That is, suppose there exists a connected, weighted graph G with n vertices and an edge e that (1) has larger weight than any other edge of G, (2) is in a circuit C of G, and (3) is in a minimum spanning tree T for G. Let the endpoints of e be vertices v and w, and let H be the graph obtained from T by removing e. In other words, $V(H) = V(T)$ and $E(H) = E(T) - \{e\}$. Then H is a circuit-free subgraph of T that contains all the vertices of G but only $n - 2$ edges, too few to be a tree. Now H consists of two components, one, say H_v, containing v and the other, say H_w, containing w. Let e' be a "bridge" from H_w to H_v. That is, as shown in the solution to exercise 39 in Section 10.2, there is a trail in G from v to w that does not include e, and so we may let e' be the edge in the trail that immediately precedes the first vertex in the trail that is in H_v. Let T' be the graph obtained from H by adding e'. More precisely, $V(T') = V(H)$ and $E(T') = E(H) \cup \{e\}$. Then T' is connected, contains every vertex of G (as does T), and has $n - 1$ edges (the same as T). Hence, by Theorem 10.5.4, T' is a spanning tree for G. Now

$$w(T') = w(T) - w(e) + w(e') = w(T) - (w(e) - w(e')) < w(T)$$

because $w(e) > w(e')$. Thus T' is a spanning tree of smaller weight than a minimum spanning tree for G, which is a contradiction. Hence the supposition is false, and the given statement is true.

30. Proof: Suppose that G is a connected, weighted graph with n vertices and that T is the output graph produced when G is input to Algorithm 10.7.4. Clearly T is a subgraph of G and T is connected because no edge is removed from T as T is being constructed if its removal would disconnect T. Also T is circuit-free because if T had a circuit then the circuit would contain edges $e_1, e_2, \ldots, e_k$ of maximal weight. At some point during execution of the algorithm, each of these edges would be examined (since all edges are examined eventually). Let e_i be the first such edge to be examined. When examined, e_i must be removed because deletion of an edge from a circuit does not disconnect a graph and at the time e_i is examined no other edge of the circuit would have been removed. But this contradicts the supposition that e_i was one of the edges in the output graph T. Thus T is circuit-free. Furthermore, T contains every vertex of G since only edges, not vertices, are removed from G in the construction of T. Hence T is a spanning tree for G.

Next we show that T has minimum weight. Let T_1 be any minimum spanning tree for G such that the number of edges T_1 and T have in common is a maximum. If $T = T_1$, we are done. So suppose $T \neq T_1$. Then there is an edge e of T that is not in T_1. *[Since trees T and*

T_1 *both have the same vertex set, if they differ at all, they must have different, but same-size, edge sets.]* Now adding e to T_1 produces a graph with a unique circuit (exercise 19). Let e' be an edge of this circuit such that e' is not in T. *[Such an edge must exist because* T *is a tree and hence circuit-free.]* Let T_2 be the graph obtained from T_1 by removing e' and adding e. Note that T_2 has n vertices and $n-1$ edges and that T_2 is connected *[since, by Lemma 10.5.3, the subgraph obtained by removing an edge from a circuit in a connected graph is connected].* Consequently, T_2 is a spanning tree for G. In addition,

$$w(T_2) = w(T_1) - w(e') + w(e).$$

Now $w(e) \le w(e')$ because at the stage in Algorithm 10.7.4 when e' was removed, e could have been removed, and it would have been removed if $w(e) > w(e')$. Thus

$$w(T_2) \;=\; w(T_1) - \underbrace{w(e') + w(e)}_{\ge 0} \;\le\; w(T_1).$$

But T_1 is minimum spanning tree for G, and thus, since T_2 is a spanning tree with weight less than or equal to the weight of T_1, T_2 is also a minimum spanning tree for G.

Finally note that by construction, T_2 has one more edge in common with T than T_1 does, which contradicts the choice of T_1 as a minimum spanning tree for G with a maximum number of edges in common with T. Thus the supposition that $T \ne T_1$ is false, and hence T itself is a minimum spanning tree for G.

Review Guide: Chapter 10

Definitions: How are the following terms defined?

- graph, edge-endpoint function *(p. 626)*
- loop in a graph, parallel edges, adjacent edges, isolated vertex, edge incident on an endpoint *(p. 626)*
- directed graph *(p. 629)*
- simple graph *(p. 632)*
- complete graph on n vertices *(p. 633)*
- complete bipartite graph on (m, n) vertices *(p. 633)*
- subgraph *(p. 634)*
- degree of a vertex in a graph, total degree of a graph *(p. 635)*
- walk, trail, path, closed walk, circuit, simple circuit *(p. 644)*
- connected vertices, connected graph *(p. 646)*
- connected component of a graph *(p. 647)*
- Euler circuit in a graph *(p. 648)*
- Euler trail in a graph *(p. 652)*
- Hamiltonian circuit in a graph *(p. 654)*
- adjacency matrix of a directed (or undirected) graph *(p. 662)*
- symmetric matrix *(p. 664)*
- $n \times n$ identity matrix *(p. 669)*
- powers of a matrix *(p. 670)*
- isomorphic graphs *(p. 676)*
- isomorphic invariant for graphs *(p. 679)*
- circuit-free graph *(p. 683)*
- tree, forest, trivial tree *(p. 683)*
- parse tree, syntactic derivation tree *(p. 684)*
- terminal vertex (or leaf), internal vertex (or branch vertex) *(p. 688)*
- rooted tree, level of a vertex in a rooted tree, height of a rooted tree *(p. 694)*
- parents, children, siblings, descendants, and ancestors in a rooted tree *(p. 694)*
- binary tree, full binary tree, subtree *(p. 696)*
- spanning tree *(p. 702)*
- weighted graph, minimum spanning tree *(p. 704)*

Graphs

- How can you use a graph as a model to help solve a problem? *(p. 631)*
- What does the handshake theorem say? In other words, how is the total degree of a graph related to the number of edges of the graph? *(p. 636)*
- How can you use the handshake theorem to determine whether graphs with specified properties exist? *(pp. 636-638)*
- If an edge is removed from a circuit in a graph, does the graph remain connected? *(p. 647, 690)*
- A graph has an Euler circuit if, and only if, it satisfies what two conditions? *(p. 652)*
- A graph has a Hamiltonian circuit if, and only if, it satisfies what four conditions? *(p. 655)*
- What is the traveling salesman problem? *(p. 656)*
- How do you find the adjacency matrix of a directed (or undirected) graph? How do you find the graph that corresponds to a given adjacency matrix? *(p. 663)*

- How can you determine the connected components of a graph by examining the adjacency matrix of the graph? *(p. 666)*
- How do you multiply two matrices? *(p. 666)*
- How do you use matrix multiplication to compute the number of walks from one vertex to another in a graph? *(p. 672)*
- How do you show that two graphs are isomorphic? *(p. 677)*
- What are some invariants for graph isomorphisms? *(p. 679)*
- How do you establish that two simple graphs are isomorphic? *(p. 680)*

Trees

- How do you show that a saturated carbon molecule with k carbon atoms has $2k + 2$ hydrogen atoms? *(p. 686 and exercise 4 in Section 10.5)*
- If a tree has more than one vertex, how many vertices of degree 1 does it have? Why? *(p. 687)*
- If a tree has n vertices, how many edges does it have? Why? *(p. 688)*
- If a connected graph has n vertices, what additional property guarantees that it will be a tree? Why? *(p. 692)*
- How can you represent an algebraic expression using a binary tree? *(p. 696)*
- Given a full binary tree, what is the relation among the number of its internal vertices, terminal vertices, and total number of vertices? *(p. 697)*
- Given a binary tree, what is the relation between the number of its terminal vertices and its height? *(p. 698)*
- What is the relation between the number of edges in two different spanning trees for a graph? *(p. 702)*
- How does Kruskal's algorithm work? *(p. 704)*
- How do you know that Kruskal's algorithm produces a minimum spanning tree? *(p. 706)*
- How does Prim's algorithm work? *(p. 707)*
- How do you know that Prim's algorithm produces a minimum spanning tree? *(p. 708)*
- How does Dijkstra's shortest path algorithm work? *(p. 711)*
- How do you know that Dijkstra's shortest path algorithm produces a shortest path? *(p. 713)*

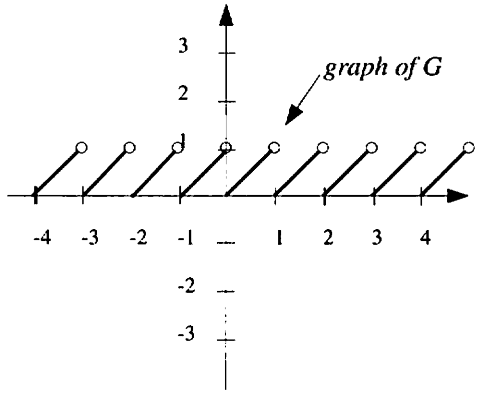
Chapter 11: Analysis of Algorithm Efficiency

The focus of Chapter 11 is the analysis of algorithm efficiency in Sections 11.3 and 11.5. The chapter opens with a brief review of the properties of function graphs that are especially important for understanding O-, Ω-, and Θ-notations, which are introduced in Section 11.2. For simplicity, the examples in Section 11.2 are restricted to polynomial and rational functions. Section 11.3 introduces the analysis of algorithm efficiency with examples that include sequential search, insertion sort, selection sort (in the exercises), and polynomial evaluation (in the exercises). Section 11.4 discusses the properties of logarithms that are particularly important in the analysis of algorithms and other areas of computer science, and Section 11.5 applies the properties to analyze algorithms whose orders involve logarithmic functions. Examples in Section 11.5 include binary search and merge sort.

Section 11.1

9.

x	*G(x)*
0	*0*
1/2	*1/2*
3/4	*3/4*
1	*0*
1 1/2	*1/2*
1 3/4	*3/4*
2	*0*



18. *b.* When $x < 0$, k is increasing.

Proof:

Suppose $x_1 < x_2 < 0$. Multiplying both sides of this inequality by -1 gives $-x_1 > -x_2$, and adding x_1x_2 to both sides gives $x_1x_2 - x_1 > x_1x_2 - x_2$. Now, since x_1 and x_2 are both negative, x_1x_2 is positive, and hence

$$\frac{x_1x_2 - x_1}{x_1x_2} > \frac{x_1x_2 - x_2}{x_1x_2}.$$

Simplifying the two fractions gives

$$\frac{x_2 - 1}{x_2} > \frac{x_1 - 1}{x_1}$$

and so $k(x_1) < k(x_2)$ by definition of k.

21. *b.* Proof by contradiction:

Suppose that g is not increasing. Then there exist real numbers x_1 and x_2 such that $0 < x_1 < x_2$ and $g(x_1) \geq g(x_2)$. By definition of g,

$$x_1^{\frac{m}{n}} \geq x_2^{\frac{m}{n}}.$$

Applying part (a) to this inequality gives

$$\left(x_1^{\frac{m}{n}}\right)^n \geq \left(x_2^{\frac{m}{n}}\right)^n.$$

By the laws of exponents, $x_1{}^{\frac{m}{n}\cdot n} = x_1{}^m$ and $x_2{}^{\frac{m}{n}\cdot n} = x_2{}^m$, and so

$$x_1{}^m \geq x_2{}^m.$$

But, by part (a), $x_1{}^m < x_2{}^m$, and so we have reached a contradiction. Hence the supposition is false, and thus g is increasing.

Section 11.2

3. *a. Formal version of negation*: $f(x)$ is not $\Theta(g(x))$ if, and only if, $\forall$ positive real numbers k, A, and B, $\exists$ a real number $x > k$ such that either $|f(x)| < A\,|g(x)|$ or $|f(x)| > B\,|g(x)|$.

   b. Informal version of negation: $f(x)$ is not $O(g(x))$ if, and only if, no matter what positive real numbers k, A, and B might be chosen, it is possible to find a real number x greater than k with the property that either $|f(x)| < A\,|g(x)|$ or $|f(x)| > B\,|g(x)|$.

9. Let $A = 1/2$, $B = 3$, and $k = 33$. Then by substitution, $A|x^2| \leq |3x^2 - 80x + 7| \leq B|x^2|$ for all $x > k$, and hence by definition of Θ-notation, $3x^2 - 80x + 7$ is $\Theta(x^2)$.

15. *a.* Proof (by mathematical induction): Let the property $P(n)$ be the sentence

$$\text{If } x \text{ is any real number with } x > 1, \text{ then } x^n > 1. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: We must show that if x is any real number with $x > 1$, then $x^1 > 1$. But this is true because $x^1 = x$. So $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$\text{If } x \text{ is any real number with } x > 1, \text{ then } x^k > 1. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$\text{If } x \text{ is any real number with } x > 1, \text{ then } x^{k+1} > 1. \qquad \leftarrow P(k+1)$$

So suppose x is any real number with $x > 1$. By inductive hypothesis, $x^k > 1$, and multiplying both sides by the positive number x gives $x \cdot x^k > x \cdot 1$, or, equivalently, $x^{k+1} > x$. But $x > 1$, and so, by transitivity of order, $x^{k+1} > 1$ *[as was to be shown]*.

b. Proof:

Suppose x is any real number with $x > 1$ and m and n are integers with $m < n$. Then $n - m$ is an integer with $n - m \geq 1$, and so, by part (a), $x^{n-m} > 1$. Multiplying both sides by x^m gives $x^m \cdot x^{n-m} > x^m \cdot 1$, and so, by the laws of exponents, $x^n > x^m$ *[as was to be shown]*.

21. *a.* For any real number $x > 1$,

$$\begin{aligned} |\lfloor\sqrt{x}\rfloor| &= \lfloor\sqrt{x}\rfloor && \text{because since } x > 1 > 0, \text{ then } \lfloor\sqrt{x}\rfloor > 0 \\ &\leq \sqrt{x} && \text{because } \lfloor r\rfloor \leq r \text{ for all real numbers } r \\ &= |\sqrt{x}| && \text{because } \sqrt{x} \geq 0. \end{aligned}$$

Therefore, by transitivity of equality and order, $|\lfloor\sqrt{x}\rfloor| \leq |\sqrt{x}|$.

b. Suppose x is any real number with $x > 1$. By definition of floor,

$$\lfloor\sqrt{x}\rfloor \leq \sqrt{x} < \lfloor\sqrt{x}\rfloor + 1.$$

Now

$$\begin{array}{rcll} \lfloor\sqrt{x}\rfloor + 1 & \leq & 2\lfloor\sqrt{x}\rfloor & \\ \Leftrightarrow 1 & \leq & \lfloor\sqrt{x}\rfloor & \text{by subtracting } \lfloor\sqrt{x}\rfloor \text{ from both sides} \\ \Leftrightarrow 1 & \leq & \sqrt{x} & \text{by definition of floor} \\ \Leftrightarrow 1 & \leq & x & \text{by squaring both sides (okay because } x \text{ is positive),} \end{array}$$

and the last inequality is true because we are assuming that $x > 1$. Thus,

$$\sqrt{x} < \lfloor\sqrt{x}\rfloor + 1 \quad \text{and} \quad \lfloor\sqrt{x}\rfloor + 1 \leq 2\lfloor\sqrt{x}\rfloor,$$

and so, by the transitivity of order (Appendix A, T18),

$$\sqrt{x} \leq 2\lfloor\sqrt{x}\rfloor.$$

Dividing both sides by 2 gives

$$\frac{1}{2}\sqrt{x} \leq \lfloor\sqrt{x}\rfloor.$$

Finally, because all quantities are positive, we conclude that

$$\frac{1}{2}\left|\sqrt{x}\right| \leq \left|\lfloor\sqrt{x}\rfloor\right|.$$

c. Let $A = \frac{1}{2}$ and $a = 1$. Then by substitution,

$$A\left|\sqrt{x}\right| \leq \left|\lfloor\sqrt{x}\rfloor\right| \quad \text{for all } x > a,$$

and hence by definition of Ω-notation, $\lfloor\sqrt{x}\rfloor$ is $\Omega(\sqrt{x})$.

Let $B = 1$ and $b = 1$. Then

$$\left|\lfloor\sqrt{x}\rfloor\right| \leq B\left|\sqrt{x}\right|$$

for all real numbers $x > b$, and so by definition of O-notation, $\lfloor\sqrt{x}\rfloor$ is $O(\sqrt{x})$.

d. By part (c) and Theorem 11.2.1(1), we can immediately conclude that $\lfloor\sqrt{x}\rfloor$ is $\Theta(\sqrt{x})$.

24. *a.* For all real numbers $x > 1$,

$$\begin{array}{rcll} \left|\frac{1}{4}x^5 - 50x^3 + 3x + 12\right| & \leq & \left|\frac{1}{4}x^5\right| + \left|50x^3\right| + \left|3x\right| + \left|12\right| & \text{by the triangle inequality} \\ & = & \frac{1}{4}x^5 + 50x^3 + 3x + 12 & \text{because } \frac{1}{4}x^5,\ 50^3,\ 3x, \\ & & & \text{and 12 are positive} \\ & \leq & \frac{1}{4}x^5 + 50x^5 + 3x^5 + 12x^5 & \text{because } x^3 < x^5,\ x < x^5, \\ & & & \text{and } 1 < x^5 \text{ for } x > 1 \\ & \leq & 66x^5 & \text{because } \frac{1}{4} + 50 + 3 + 12 < 66 \\ & = & 66\left|x^5\right| & \text{because } x^5 \text{is positive.} \end{array}$$

Therefore, by transitivity of equality and order, $\left|\frac{1}{4}x^5 - 50x^3 + 3x + 12\right| \leq 66\left|x^5\right|$.

b. Let $B = 66$ and $b = 1$. Then by substitution, $\left|\frac{1}{4}x^5 - 50x^3 + 3x + 12\right| \leq B\left|x^2\right|$ for all $x > b$. Hence by definition of O-notation,

$$\frac{1}{4}x^5 - 50x^3 + 3x + 12 \text{ is } O(x^2).$$

27. Proof:

Suppose a_0, a_1, $a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$; and let

$$d = 2\left(\frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}\right).$$

Let a be greater than or equal to the maximum of d and 1. Then if $x > a$

$$\begin{aligned}
& x && \geq 2\left(\frac{|a_0| + |a_1| + |a_2| + \cdots + |a_3| + |a_{n-1}|}{|a_n|}\right) \\
\Rightarrow\quad & \frac{1}{2}|a_n|\, x && \geq |a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}| \\
& && \quad \text{by multiplying both sides by } \tfrac{1}{2}|a_n| \\
\Rightarrow\quad & (1 - \tfrac{1}{2})\, |a_n| x && \geq |a_0| \cdot \frac{1}{x^{n-1}} + |a_1| \cdot \frac{1}{x^{n-2}} + |a_2| \cdot \frac{1}{x^{n-3}} + \cdots + |a_{n-2}| \cdot \frac{1}{x} + |a_{n-1}| \cdot 1 \\
& && \quad \text{because by exercise 15, when } x > 1 \text{ and } m \geq 1, \\
& && \quad \text{then } x^m > 1, \text{ and so } 1 > \tfrac{1}{x^m} \\
\Rightarrow\quad & |a_n|\, x^n - \frac{|a_n|}{2} x^n && \geq |a_0| + |a_1|\, x + |a_2|\, x^2 + \cdots + |a_{n-2}|\, x^{n-2} + |a_{n-1}|\, x^{n-1} \\
& && \quad \text{by multiplying both sides by } x^{n-1}.
\end{aligned}$$

Subtracting all terms on the right-hand side from both sides and adding the second term on the left-hand side to both sides gives

$$|a_n|\, x^n - |a_{n-1}|\, x^{n-1} - |a_{n-2}|\, x^{n-2} - \cdots - |a_2|\, x^2 - |a_1|\, x - |a_0| \geq \frac{|a_n|}{2} x^n. \qquad (*)$$

Now, by the triangle inequality, for all real numbers r and s,

$$|r| = |(r+s) + (-s)| \leq |r+s| + |-s| = |r+s| + |s|,$$

and thus,

$$|r| - |s| \leq |r+s|.$$

It follows by repeated application of this result that, when $x > 1$,

$$\begin{aligned}
& |a_n|\, x^n - |a_{n-1}|\, x^{n-1} - |a_{n-2}|\, x^{n-2} - \cdots - |a_2|\, x^2 - |a_1|\, x - |a_0| \\
& \quad = |a_n x^n| - |a_{n-1} x^{n-1}| - |a_{n-2} x^{n-2}| - \cdots - |a_2 x^2| - |a_1 x| - |a_0| \\
& \quad \leq |a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0|. \qquad (**)
\end{aligned}$$

Using the transitive property of order to combine (*) and (**) gives that

$$\frac{|a_n|}{2} x^n \leq |a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0|.$$

Let $A = \dfrac{|a_n|}{2}$ and let a be as defined above. Then

$Ax^n \leq |a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0|$ for all real numbers $x > a$.

It follows by definition of Ω-notation that

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 \text{ is } \Omega(x^n).$$

30. Let $a = 2\left(\dfrac{50+3+12}{1/4}\right) = 520$, and let $A = \dfrac{1}{2}\cdot\dfrac{1}{4} = \dfrac{1}{8}$. If $x > 520$, then

$$
\begin{array}{rrcl}
 & x & \geq & 2\left(\dfrac{50+3+12}{1/4}\right) \\
\Rightarrow & \dfrac{1}{2}\cdot\dfrac{1}{4}x & \geq & 50+3+12 \\
 & & & \text{by multiplying both sides by } \tfrac{1}{2}\cdot\tfrac{1}{4} \\
\Rightarrow & (1-\dfrac{1}{2})\cdot\dfrac{1}{4}x & \geq & 50\dfrac{1}{x}+3\cdot\dfrac{1}{x^3}+12\cdot\dfrac{1}{x^4} \\
 & & & \text{because } 1-\tfrac{1}{2} = \tfrac{1}{2}\text{, and, since } x > 520 > 1, \\
 & & & \text{then } 1 > \tfrac{1}{x},\ 1 > \tfrac{1}{x^3} \text{ and } 1 > \tfrac{1}{x^4} \\
\Rightarrow & \dfrac{1}{4}x^5 - \dfrac{1}{2}\cdot\dfrac{1}{4}x^5 & \geq & 50x^3+3x+12 \\
 & & & \text{by multiplying both sides by } x^4 \\
\Rightarrow & \dfrac{1}{4}x^5 - 50x^3 - 3x - 12 & \geq & \dfrac{1}{2}\cdot\dfrac{1}{4}x^5 \\
 & & & \text{by subtracting } 50x^3\text{+ } 3x + 12 \text{ from and} \\
 & & & \text{adding } \tfrac{1}{2}\cdot\tfrac{1}{4}x^5 \text{ to both sides} \\
\Rightarrow & \dfrac{1}{4}x^5 - 50x^3 + 3x + 12 & \geq & \dfrac{1}{2}\cdot\dfrac{1}{4}x^5 \\
 & & & \text{because } 3x + 12 > \text{-}3x \text{ - } 12 \text{ since } x > 0 \\
\Rightarrow & \left|\dfrac{1}{4}x^5 - 50x^3 + 3x + 12\right| & \geq & \dfrac{1}{8}\left|x^5\right| \\
 & & & \text{because both sides are nonnegative.}
\end{array}
$$

Thus for all real numbers $x > a$, $\left|\dfrac{1}{4}x^5 - 50x^3 + 3x + 12\right| \geq A\left|x^5\right|$.Hence, by definition of Ω-notation, we conclude that $\dfrac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Omega(x^5)$.

33. By exercise 24, $\dfrac{1}{4}x^5 - 50x^3 + 3x + 12$ is $O(x^5)$ and, by exercise 31, $\dfrac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Omega(x^5)$. Thus, by Theorem 11.2.1(1), $\dfrac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Theta(x^5)$.

36. Note that

$$
\begin{aligned}
\frac{x(x-1)}{2} + 3x &= \frac{x^2-x}{2} + \frac{6}{2}x \\
&= \frac{1}{2}x^2 + \frac{5}{2}x \qquad \text{by algebra,}
\end{aligned}
$$

and so, by the theorem on polynomial orders, $\dfrac{x(x-1)}{2} + 3x$ is $\Theta(x^2)$

39. Note that

$$
\begin{aligned}
2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right) &= 2n - 2 + \frac{n^2}{2} + \frac{n}{2} + 2(n^2-n) \\
&= \frac{5}{2}n^2 + \frac{1}{2}n - 2 \qquad \text{by algebra,}
\end{aligned}
$$

and so, by the theorem on polynomial orders,

$$2(n-1) + \frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right) \text{ is } \Theta(n^2).$$

45. Note that

$$\begin{aligned}
\sum_{k=1}^{n}(k+3) &= \sum_{k=1}^{n} k + \sum_{k=1}^{n} 3 && \text{by Theorem 5.1.1}\\
&= \frac{n(n+1)}{2} + \underbrace{(3+3+\cdots+3)}_{n \text{ terms}} && \text{by Theorem 5.2.2}\\
&= \frac{1}{2}n^2 + \frac{1}{2}n + 3n && \text{by definition of multiplication}\\
&= \frac{1}{2}n^2 + \frac{7}{2}n && \text{by algebra,}
\end{aligned}$$

and so, by the theorem on polynomial orders,

$$\sum_{k=1}^{n}(k+3) \text{ is } \Theta(n^2).$$

48. *a.* Proof: Suppose $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$. Then

$$\begin{aligned}
&\lim_{x\to\infty}\left|\frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{a_nx^n}\right|\\
&= \lim_{x\to\infty}\left(1 + \left|\frac{a_{n-1}}{a_n}\right|\frac{1}{x} + \cdots + \left|\frac{a_2}{a_n}\right|\frac{1}{x^{n-2}} + \left|\frac{a_1}{a_n}\right|\frac{1}{x^{n-1}} + \left|\frac{a_0}{a_n}\right|\frac{1}{x^n}\right)\\
&= \lim_{x\to\infty} 1 + \left|\frac{a_{n-1}}{a_n}\right| \lim_{x\to\infty}\left(\frac{1}{x}\right) + \cdots + \left|\frac{a_2}{a_n}\right| \lim_{x\to\infty}\left(\frac{1}{x^{n-2}}\right) + \left|\frac{a_1}{a_n}\right| \lim_{x\to\infty}\left(\frac{1}{x^{n-1}}\right) + \left|\frac{a_0}{a_n}\right| \lim_{x\to\infty}\left(\frac{1}{x^n}\right)\\
&= 1
\end{aligned}$$

because $\lim_{x\to\infty}\left(\frac{1}{x^k}\right) = 0$ for all integers $k \geq 1$.

b. Proof:

Suppose $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$. By part (a) and the definition of limit, we can make the following statement: For all positive real numbers ε, there exists a real number M (which we may take to be positive) such that

$$1 - \varepsilon < \left|\frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{a_nx^n}\right| < 1 + \varepsilon \quad \text{for all real numbers } x > M.$$

Let $\varepsilon = 1/2$. Then there exists a real number M_0 such that

$$1 - \frac{1}{2} < \left|\frac{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0}{a_nx^n}\right| < 1 + \frac{1}{2} \quad \text{for all real numbers } x > M_0.$$

Equivalently, for all real numbers $x > M_0$,

$$\frac{1}{2}|a_n||x^n| < \left|a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\right| < \frac{3}{2}|a_n||x^n|.$$

Let $A = \frac{1}{2}|a_n|$, $B = \frac{3}{2}|a_n|$, and $k = M_0$. Then

$$A|x^n| \leq \left|a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\right| \leq B|x^n| \quad \text{for all real numbers } x > k,$$

and so, by definition of Θ-notation,

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \text{ is } \Theta(n^n).$$

57. *b.* Proof (by mathematical induction): Let the property $P(n)$ be the inequality

$$\frac{1}{2}n^{3/2} \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: We must show that $\frac{1}{2}\cdot 1^{3/2} \le \sqrt{1}$. But the left-hand side of the inequality is $1/2$ and the right-hand side is 1, and $1/2 < 1$. So $P(1)$ is true.

Show that for all integers $k \ge 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \ge 1$, and suppose that

$$\frac{1}{2}k^{3/2} \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{k}. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis}\end{array}$$

We must show that

$$\frac{1}{2}(k+1)^{3/2} \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{k+1}. \qquad \leftarrow P(k+1)$$

By adding $\sqrt{k+1}$ to both sides of the inductive hypothesis, we have

$$\frac{1}{2}k^{3/2}+\sqrt{k+1} \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{k}+\sqrt{k+1}.$$

Thus, by the transitivity of order, it suffices to show that

$$\frac{1}{2}(k+1)^{3/2} \le \frac{1}{2}k^{3/2}+\sqrt{k+1}.$$

Now when $k \ge 1$,

$$k^2 \ge k^2-1 = (k-1)(k+1).$$

Divide both sides by $k(k-1)$ to obtain

$$\frac{k}{k-1} \ge \frac{k+1}{k}.$$

But $\frac{k+1}{k} \ge 1$, and any number greater than or equal to 1 is greater than or equal to its own square root. Thus

$$\frac{k}{k-1} \ge \frac{k+1}{k} \ge \sqrt{\frac{k+1}{k}} = \frac{\sqrt{k+1}}{\sqrt{k}}.$$

Hence

$$k\sqrt{k} \ge (k-1)\sqrt{k+1} = (k+1-2)\sqrt{k+1} = (k+1)^{3/2}-2\sqrt{k+1}.$$

Multiplying the extreme-left and extreme-right sides of the inequality by $1/2$ gives

$$\frac{1}{2}k^{3/2} \ge \frac{1}{2}(k+1)^{3/2}-\sqrt{k+1}, \quad \text{or, equivalently,} \quad \frac{1}{2}(k+1)^{3/2} \le \frac{1}{2}k^{3/2}+\sqrt{k+1}.$$

[This is what was to be shown].

60. Proof: Suppose $f(x)$ and $g(x)$ are $o(h(x)$ and a and b are any real numbers. Then by properties of limits,

$$\lim_{x\to\infty}\frac{af(x)+bg(x)}{h(x)} = a\lim_{x\to\infty}\frac{f(x)}{h(x)}+b\lim_{x\to\infty}\frac{g(x)}{h(x)} = a\cdot 0+b\cdot 0 = 0.$$

So $af(x)+bg(x)$ is $o(h(x))$.

Section 11.3

3. *a.* When the input size is increased from m to $2m$, the number of operations increases from cm^3 to $c(2m)^3 = 8cm^3$.

   b. By part (a), the number of operations increases by a factor of $\dfrac{8cm^3}{cm^3} = 8$.

   c. When the input size is increased by a factor of 10 (from m to $10m$), the number of operations increases by a factor of $\dfrac{c(10m^3)}{cm^3} = \dfrac{1000cm^3}{cm^3} = 1000$.

12. *a.* For each iteration of the inner loop there is one comparison. The number of iterations of the inner loop can be deduced from the following table, which shows the values of k and i for which the inner loop is executed.

k	1				2				$\cdots$	$n-2$		$n-1$
i	2	3	$\cdots$	n	3	4	$\cdots$	n	$\cdots$	$n-1$	n	n
	$\underbrace{\qquad\qquad}_{n-1}$				$\underbrace{\qquad\qquad}_{n-2}$					$\underbrace{\qquad}_{2}$		$\underbrace{\quad}_{1}$

Therefore, by Theorem 5.2.2, the number of iterations of the inner loop is

$$(n-1)+(n-2)+\cdots+2+1 = \frac{n(n-1)}{2}.$$

It follows that the total number of elementary operations that must be performed when the algorithm is executed is

$$1\cdot(\frac{n(n-1)}{2}) = \frac{1}{2}n^2 - \frac{1}{2}n.$$

By the theorem on polynomial orders, $\frac{1}{2}n^2 - \frac{1}{2}n$ is $\Theta(n^2)$, and so the algorithm segment has order n^2.

15. *a.* There are three multiplications for each iteration of the inner loop, and there is one additional addition for each iteration of the outer loop. The number of iterations of the inner loop can be deduced from the following table, which shows the values of i and j for which the inner loop is executed.

i	1				2				$\cdots$	$n-2$		$n-1$
j	2	3	$\cdots$	n	3	4	$\cdots$	n	$\cdots$	$n-1$	n	n
	$\underbrace{\qquad\qquad}_{n-1}$				$\underbrace{\qquad\qquad}_{n-2}$					$\underbrace{\qquad}_{2}$		$\underbrace{\quad}_{1}$

Hence, by Theorem 5.2.2, the total number of iterations of the inner loop is

$$(n-1)+(n-2)+\cdots+2+1 = \frac{n(n-1)}{2}.$$

Because three multiplications are performed for each iteration of the inner loop, the number of operations that are performed when the inner loop is executed is

$$3\cdot\frac{n(n-1)}{2} = \frac{3}{2}(n^2-n) = \frac{3}{2}n^2 - \frac{3}{2}n.$$

Now an additional operation is performed each time the outer loop is executed, and because the outer loop is executed n times, this gives an additional n operations. Therefore, the total number of operations is

$$(\frac{3}{2}n^2 - \frac{3}{2}n) + n = \frac{3}{2}n^2 - \frac{1}{2}n.$$

b. By the theorem on polynomial orders, $\frac{3}{2}n^2 - \frac{1}{2}n$ is $\Theta(n^2)$, and so the algorithm segment has order n^2.

18. *a.* There is one multiplication for each iteration of the inner loop. If n is odd, the number of iterations of the inner loop can be deduced from the following table,which shows the values of i and j for which the inner loop is executed.

i	1				2				$\cdots$	$n-2$				$n-1$				n			
$\lfloor\frac{i+1}{2}\rfloor$	1				1				$\cdots$	$\frac{n-1}{2}$				$\frac{n-1}{2}$				$\frac{n+1}{2}$			
j	1	2	$\cdots$	n	1	2	$\cdots$	n	$\cdots$	$\frac{n-1}{2}$	$\frac{n-1}{2}+1$	$\cdots$	n	$\frac{n-1}{2}$	$\frac{n-1}{2}+1$	$\cdots$	n	$\frac{n+1}{2}$	$\frac{n+1}{2}+1$	$\cdots$	n
	$\underbrace{\qquad\qquad}_{n}$				$\underbrace{\qquad\qquad}_{n}$					$\underbrace{\qquad\qquad}_{n-\frac{n-1}{2}+1=\frac{n+3}{2}}$				$\underbrace{\qquad\qquad}_{n-\frac{n-1}{2}+1=\frac{n+3}{2}}$				$\underbrace{\qquad\qquad}_{n-\frac{n+1}{2}+1=\frac{n+1}{2}}$			

Thus the number of iterations of the inner loop is

$$\begin{aligned}
n+n+(n-1)+(n-1)+\cdots+\frac{n+3}{2}+\frac{n+3}{2}+\frac{n+1}{2} \\
&= 2\left(n+(n-1)+\cdots+\frac{n+3}{2}\right)+\frac{n+1}{2} \\
&= 2\cdot\left(\sum_{k=1}^{n}k-\sum_{k=1}^{(n+1)/2}k\right)+\frac{n+1}{2} && \text{because } \tfrac{n+3}{2} - 1 = \tfrac{n+1}{2} \\
&= n(n+1)-\frac{n+1}{2}\left(\frac{n+1}{2}+1\right)+\frac{n+1}{2} && \text{by Theorem 5.2.2} \\
&= \frac{4n(n+1)}{4}-\frac{(n+1)^2}{4} \\
&= \frac{4n^2+4n-n^2-2n-1}{4} \\
&= \frac{3n^2+2n-1}{4} \\
&= \frac{3}{4}n^2+\frac{1}{2}n-\frac{1}{4}.
\end{aligned}$$

By similar reasoning, if n is even, then the number of iterations of the inner loop is

$$\begin{aligned}
n+n+(n-1)+(n-1)+\cdots+\frac{n+2}{2}+\frac{n+2}{2} \\
&= 2\left(n+(n-1)+\cdots+\frac{n}{2}\right) \\
&= 2\cdot\left(\sum_{k=1}^{n}k-\sum_{k=1}^{n/2}k\right) && \text{because } \tfrac{n+2}{2} - 1 = \tfrac{n}{2} \\
&= n(n+1)-\frac{n}{2}\left(\frac{n}{2}+1\right) && \text{by Theorem 5.2.2} \\
&= \frac{4n(n+1)}{4}-\frac{n^2}{4}-\frac{2n}{4} \\
&= \frac{4n^2+4n-n^2-2n}{4} \\
&= \frac{3n^2+2n}{4} \\
&= \frac{3}{4}n^2+\frac{1}{2}n.
\end{aligned}$$

Because one operation is performed for each iteration of the inner loop, the answer is that

$$1\cdot\left(\frac{3}{4}n^2+n-\frac{3}{4}\right)=\frac{3}{4}n^2+3n-\frac{11}{4}$$

elementary operations are performed when n is odd and

$$1 \cdot \left(\frac{3}{4}n^2 + n - 1\right) = \frac{3}{4}n^2 + 6n$$

elementary operations are performed when n is even.

b. By the theorem on polynomial orders,$\frac{3}{4}n^2 + 3n - \frac{11}{4}$ is $\Theta(n^2)$ and $\frac{3}{4}n^2 + 6n$ is also $\Theta(n^2)$ and so this algorithm segment has order n^2.

21.

	$a[1]$	$a[2]$	$a[3]$	$a[4]$	$a[5]$
initial order	7	3	6	9	5
result of step k = 2	3	7	6	9	5
result of step k = 3	3	6	7	9	5
result of step k = 4	3	6	7	9	5
result of step k = 5	3	5	6	7	9

27. *a.*

$$\begin{aligned} E_1 &= 0 \\ E_2 &= E_1 + 2 + 1 = 3 \\ E_3 &= E_2 + 3 + 1 = 3 + 4 \\ E_4 &= E_3 + 4 + 1 = 3 + 4 + 5 \\ E_5 &= E_4 + 5 + 1 = 3 + 4 + 5 + 6 \\ &\vdots \end{aligned}$$

$$\text{Guess: } E_n = 3 + 4 + 5 + \cdots + (n+1) = [1 + 2 + 3 + 4 + 5 + \cdots + (n+1)] - (1+2)$$
$$= \frac{(n+1)(n+2)}{2} - 3 = \frac{n^2 + 3n + 2 - 6}{2} = \frac{n^2 + 3n - 4}{2}$$

b. Proof (by mathematical induction): Let $E_1, E_2, E_3; \ldots$. be a sequence that satisfies the recurrence relation $E_k = E_{k-1} + k + 1$ for all integers $k \geq 2$, with initial condition $E_1 = 0$, and let the property $P(n)$ be the equation

$$E_n = \frac{n^2 + 3n - 4}{2}. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: The left-hand side of $P(1)$ is E_1, which equals 0 by definition of $E_1, E_2, E_3, \ldots$, and the right-hand side of $P(1)$ is

$$\frac{1^2 + 3 \cdot 1 - 4}{2} = \frac{1 + 3 - 4}{2} = 0$$

also. So $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$E_k = \frac{k^2 + 3k - 4}{2}. \qquad \leftarrow \begin{array}{l} P(k) \\ \text{inductive hypothesis} \end{array}$$

We must show that

$$E_{k+1} = \frac{(k+1)^2 + 3(k+1) - 4}{2}. \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ equals

$$\begin{aligned} E_{k+1} &= E_k + (k+1) + 1 && \text{by definition of } E_1, E_2, E_3, \ldots \\ &= \frac{k^2+3k-4}{2} + (k+1) + 1 && \text{by inductive hypothesis} \\ &= \frac{k^2+3k-4+2k+4}{2} \\ &= \frac{k^2+5k}{2}. \end{aligned}$$

And the right-hand side of $P(k+1)$ equals

$$\frac{(k+1)^2+3(k+1)-4}{2} = \frac{k^2+2k+1+3k+3-4}{2} = \frac{k^2+5k}{2}$$

also, and so $P(k+1)$ is true *[as was to be shown]*.

33. As i goes from $k+1$ to 5 through $5-(k+1)+1 = 5-k$ values (where k goes from 1 to 4), the number of comparisons is

$$(5-1)+(5-2)+(5-3)+(5-4) = 4+3+2+1 = 10.$$

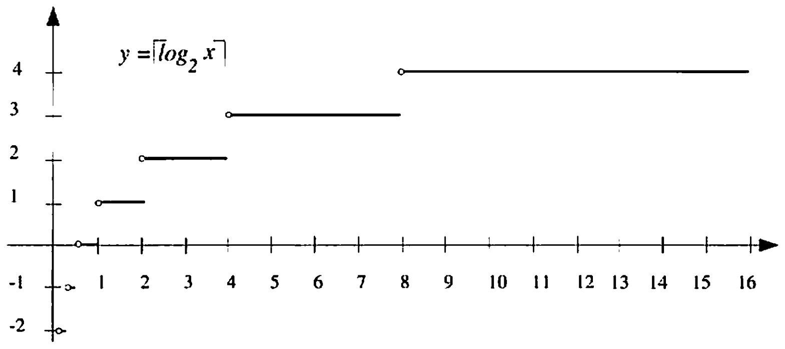
39. By the result of exercise 38, $s_n = \frac{1}{2}n^2 + \frac{3}{2}n$, which is $\Theta(n^2)$ by the theorem on polynomial orders.

42. There are two operations (one addition and one multiplication) per iteration of the loop, and there are n iterations of the loop. Therefore, $t_n = 2n$.

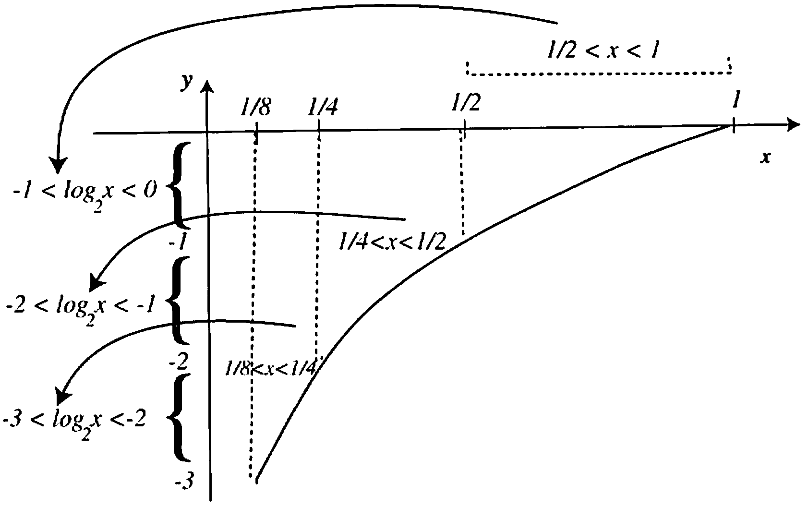
Section 11.4

6.

x	$\lceil log_2 x \rceil$
$1 < x \leq 2$	1
$2 < x \leq 4$	2
$4 < x \leq 8$	3
$1/2 < x \leq 0$	0
$1/4 < x \leq 1/8$	-1
$1/8 < x \leq 1/4$	-2



12. When $\frac{1}{2} < x < 1$, then $-1 < \log_2 x < 0$.
When $\frac{1}{4} < x < \frac{1}{2}$, then $-2 < \log_2 x < -1$.
When $\frac{1}{8} < x < \frac{1}{4}$, then $-3 < \log_2 x < -2$.
And so forth.



24. Proof (by strong mathematical induction): Let $c_1, c_2, c_3, \ldots$ be a sequence that satisfies the recurrence relation

$$c_k = 2c_{\lfloor k/2 \rfloor} + k \text{ for all integers } k \geq 2, \text{ with initial condition } c_1 = 0,$$

and let the property $P(n)$ be the inequality

$$c_n \leq n \log_2 n. \qquad \leftarrow P(n)$$

Show that P(1) is true: For $n = 1$ the inequality states that $c_1 \leq 1 \cdot \log_2 1 = 1 \cdot 0 = 0$, which is true because $c_1 = 0$. So $P(1)$ is true.

Show that if $k \geq 1$ and $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is true: Let k be any integer with $k \geq 1$, and suppose that

$$c_i \leq i \log_2 i \text{ for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \begin{array}{l}\text{inductive}\\ \text{hypothesis}\end{array}$$

We must show that

$$c_{k+1} \leq (k+1)\log_2(k+1).$$

First note that because k is greater than 1 and by definition of floor,

$$1 \leq \left\lfloor \frac{k+1}{2} \right\rfloor \leq \frac{k+1}{2}.$$

Also, because k is an integer with $k \geq 1$, we have

$$1 \leq k \Rightarrow k+1 \leq k+k \Rightarrow k+1 \leq 2k \Rightarrow \frac{k+1}{2} \leq k.$$

Thus, by the transitive property of order,

$$\left\lfloor \frac{k+1}{2} \right\rfloor \leq k.$$

Then

$$\begin{aligned}
c_{k+1} &= 2c_{\lfloor (k+1)/2 \rfloor} + (k+1) \\
&\qquad\qquad \text{by definition of } c_1, c_2, c_3, \ldots \\
&\le 2\left(\left\lfloor \frac{k+1}{2} \right\rfloor \log_2 \left\lfloor \frac{k+1}{2} \right\rfloor\right) + (k+1) \\
&\qquad\qquad \text{by inductive hypothesis because } \left\lfloor \tfrac{k+1}{2} \right\rfloor \le k \\
&\le (k+1)\log_2\left(\frac{k+1}{2}\right) + (k+1) \\
&\qquad\qquad \text{since } 1 \le \left\lfloor \tfrac{k+1}{2} \right\rfloor \le \tfrac{k+1}{2}\text{, we have by property (11.4.1)} \\
&\qquad\qquad \text{that } \log_2 \left\lfloor \tfrac{k+1}{2} \right\rfloor \le \log_2\left(\tfrac{k+1}{2}\right) \\
&= (k+1)\left[\log_2(k+1) - \log_2 2\right] + (k+1) \\
&\qquad\qquad \text{by Theorem 7.2.1(b)} \\
&= (k+1)\left[\log_2(k+1) - 1\right] + (k+1) \\
&\qquad\qquad \text{because } \log_2 2 = 1 \\
&= (k+1)\log_2(k+1) \\
&\qquad\qquad \text{by algebra,}
\end{aligned}$$

Therefore, by transitivity of equality and order, $c_{k+1} \le (k+1)\log_2(k+1)$ *[as was to be shown].*

33. For all integers $n > 0$,

$$2^n \le 2^{n+1} \le 2 \cdot 2^n.$$

Thus, let $A = 1$, $B = 2$, and $k = 0$. Then

$$A \cdot 2^n \le 2^{n+1} \le B \cdot 2^n \quad \text{for all integers } n > k,$$

and so, by definition of Θ-notation, 2^{n+1} is $\Theta(2^n)$.

36. By factoring out a 4 and using the formula for the sum of a geometric sequence (Theorem 5.2.3), we have that for all integers $n > 1$,

$$\begin{aligned}
4 + 4^2 + 4^3 + \cdots + 4^n &= 4(1 + 4 + 4^2 + \cdots + 4^{n-1}) \\
&= 4\left(\frac{4^{(n-1)+1} - 1}{4 - 1}\right) \\
&= \frac{4}{3}(4^n - 1) \\
&= \frac{4}{3} \cdot 4^n - \frac{4}{3} \\
&\le \frac{4}{3} \cdot 4^n.
\end{aligned}$$

Moreover, because

$$4 + 4^2 + 4^3 + \cdots + 4^{n-1} \ge 0, \quad \text{then} \quad 4^n \le 4 + 4^2 + 4^3 + \cdots + 4^{n-1} + 4^n.$$

So let $A = 1$, $B = 4/3$, and $k = 1$. Then, because all quantities are positive,

$$A \cdot |4^n| \le \left|4 + 4^2 + 4^3 + \cdots + 4^n\right| \le B \cdot |4^n| \quad \text{for all integers } n > k,$$

and thus, by definition of Θ-notation, $4 + 4^2 + 4^3 + \cdots + 4^n$ is $\Theta(4^n)$.

42. $1 + \frac{1}{2} = \frac{3}{2}$, $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{50}{24} = \frac{25}{12}$, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$

45. *a.* Proof: If n is any positive integer, then $\log_2 n$ is defined and by definition of floor,

$$\lfloor \log_2 n \rfloor \ \le\ \log_2 n \ <\ \lfloor \log_2 n \rfloor + 1.$$

If, in addition, n is greater than 2, then since the logarithmic function with base 2 is increasing

$$\log_2 n \ >\ \log_2 2 \ =\ 1.$$

Thus, by definition of floor,

$$1 \ \le\ \lfloor \log_2 n \rfloor .$$

Adding $\lfloor \log_2 n \rfloor$ to both sides of this inequality gives

$$\lfloor \log_2 n \rfloor + 1 \ \le\ 2 \lfloor \log_2 n \rfloor .$$

Hence, by the transitive property of order (T18 in Appendix A),

$$\log_2 n \ \le\ 2 \lfloor \log_2 n \rfloor ,$$

and dividing both sides by 2 gives

$$\frac{1}{2} \log_2 n \ \le\ \lfloor \log_2 n \rfloor .$$

Let $A = 1/2$, $B = 1$, and $k = 2$. Then

$$A \log_2 n \ \le\ \lfloor \log_2 n \rfloor \ \le\ B \log_2 n \ \text{ for all integers } n \ge k,$$

and, because $\log_2 n$ is positive for $n > 2$, we may write

$$A\, |\log_2 n| \ \le\ |\lfloor \log_2 n \rfloor| \ \le\ B\, |\log_2 n| \ \text{ for all integers } n \ge k.$$

Therefore, by definition of Θ-notation, $\lfloor \log_2 n \rfloor$ is $\Theta(\log_2 n)$.

b. Proof: If n is any positive real number, then $\log_2 n$ is defined and by definition of floor,

$$\lfloor \log_2 n \rfloor \le \log_2 n.$$

If, in addition, n is greater than 2, then, as in part (a),

$$\log_2 n \ <\ \lfloor \log_2 n \rfloor + 1 \quad \text{and} \quad \lfloor \log_2 n \rfloor + 1 \ \le\ 2 \log_2 n.$$

Hence, because $\log_2 n$ is positive for $n > 2$, we may write

$$|\log_2 n| \le |\lfloor \log_2 n \rfloor + 1| \le 2\, |\log_2 n| .$$

Let $A = 1$, $B = 2$ and $k = 2$. Then

$$A\, |\log_2 n| \ \le\ |\lfloor \log_2 n \rfloor + 1| \ \le\ B\, |\log_2 n| \ \text{ for all integers } n \ge k.$$

Therefore, by definition of Θ-notation, $\lfloor \log_2 n \rfloor + 1$ is $\Theta(\log_2 n)$.

48. Proof:

Suppose n is a variable that takes positive integer values. Then whenever $n \ge 2$,

$$2^n \ =\ \underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ factors}} \ \le\ \underbrace{2 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n}_{n \text{ factors}} \ \le\ 2n!.$$

Let $B = 2$ and $b = 2$. Since 2^n and $n!$ are positive for all n,

$$|2^n| \ \le\ B|n!| \ \text{ for all integers } n \ge b.$$

Hence by definition of O-notation 2^n is $O(n!)$.

51. *a.* Let n be any positive integer. Then for any real number x *[because $u < 2^u$ for all real numbers u]*,

$$\frac{x}{n} < 2^{\frac{x}{n}} \Rightarrow x < n2^{\frac{x}{n}} \Rightarrow x^n < (n2^{\frac{x}{n}})^n = n^n \cdot 2^x.$$

So $x^n < n^n 2^x$.

b. Let x be any positive real number and let n be any positive integer. Then

$$x^n = |x^n| \quad \text{and} \quad n^n 2^x = 2^x\,|n^n|,$$

and thus the result of part (a) may be written as

$$|x^n| \le 2^x\,|n^n|.$$

Let $B = 2^x$ and $b = 0$. Then $|x^n| \le B\,|n^n|$ for all integers $n > b$, and so by definition of O-notation x^n is $O(n^n)$.

Section 11.5

6. *a.*

index	0			
bot	1	1	1	1
top	10	4	1	0
mid		5	2	1

b.

index	0		8
bot	1	6	
top	10		
mid		5	8

12.

n	424	141	47	15	5	1	0

15. If $n \ge 3$, then

$$\begin{array}{rcll} & b_n & = & 1 + \lfloor \log_3 n \rfloor & \text{by the result of exercise 14} \\ \Rightarrow & b_n & \le & 1 + \log_3 n & \text{because } \lfloor \log_3 n \rfloor \le \log_3 n \text{ by definition of floor} \\ \Rightarrow & b_n & \le & \log_3 n + \log_3 n & \text{because if } n \ge 3 \text{ then } \log_3 n \ge 1 \\ \Rightarrow & b_n & \le & 2\log_3 n & \text{by algebra.} \end{array}$$

Furthermore, because $\log_3 n \ge 0$ for $n > 2$, we may write

$$|\log_3 n| < |\lfloor \log_3 n \rfloor + 1| \le 2\,|\log_3 n|.$$

Let $A = 1$, $B = 2$, and $k = 2$. Then all quantities are positive, and so

$$A\,|\log_3 n| < |\lfloor \log_3 n \rfloor + 1| \le B\,|\log_3 n| \quad \text{for all integers } n > k.$$

Hence by definition of Θ-notation, $b_n = 1 + \lfloor \log_3 n \rfloor$ is $\Theta(\log_3 n)$, and thus the algorithm segment has order $\log_3 n$.

18. Suppose an array of length k is input to the **while** loop and the loop is iterated one time. The elements of the array can be matched with the integers from 1 to k with $m = \left\lceil \dfrac{k+1}{2} \right\rceil$, as shown below:

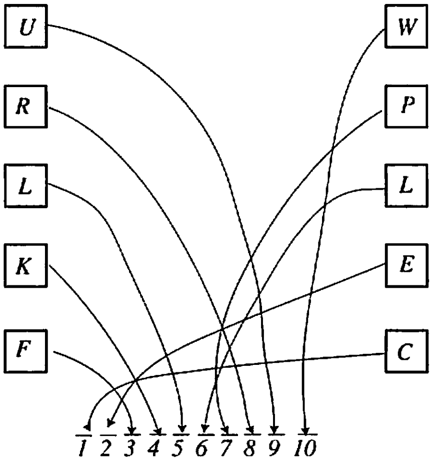
$$\begin{array}{ccccccccc} \multicolumn{4}{c}{\overbrace{\hspace{7cm}}^{\text{left subarray}}} & & \multicolumn{4}{c}{\overbrace{\hspace{7cm}}^{\text{right subarray}}} \\ a[bot] & a[bot+1] & \ldots & a[mid-1] & a[mid] & a[mid+1] & \ldots & a[top-1] & a[top] \\ \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ 1 & 2 & & m-1 & m & m+1 & & k-1 & k \end{array}$$

Case 1 (k is even): In this case $m = \left\lceil \frac{k+1}{2} \right\rceil = \left\lceil \frac{k}{2} + \frac{1}{2} \right\rceil = \frac{k}{2} + 1$, and so the number of elements in the left subarray equals $m - 1 = (\frac{k}{2} + 1) - 1 = \frac{k}{2} = \left\lfloor \frac{k}{2} \right\rfloor$. The number of elements in the right subarray equals $k - (m+1) - 1 = k - m = k - (\frac{k}{2} + 1) = \frac{k}{2} - 1 < \left\lfloor \frac{k}{2} \right\rfloor$. Hence both subarrays (and thus the new input array) have length at most $\left\lfloor \frac{k}{2} \right\rfloor$.

Case 2 (k is odd): In this case $m = \left\lceil \frac{k+1}{2} \right\rceil = \frac{k+1}{2}$, and so the number of elements in the left subarray equals $m - 1 = \frac{k+1}{2} - 1 = \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor$. The number of elements in the right subarray equals $k - m = k - \frac{k+1}{2} = \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor$ also. Hence both subarrays (and thus the new input array) have length $\left\lfloor \frac{k}{2} \right\rfloor$.

The arguments in cases 1 and 2 show that the length of the new input array to the next iteration of the **while** loop has length at most $\lfloor k/2 \rfloor$.

21.



24. *a*. Refer to Figure 11.5.3. Observe that when k is odd, the subarray $a[mid{+}1], a[mid{+}2], \ldots a[top]$ has length

$$k - \left(\frac{k+1}{2} + 1\right) + 1 = \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor.$$

And when k is even, the subarray $a[mid + 1], a[mid + 2], \ldots a[top]$ has length

$$k - \left(\frac{k}{2} + 1\right) + 1 = \frac{k}{2} = \left\lfloor \frac{k}{2} \right\rfloor.$$

So in either case the subarray has length $\lfloor k/2 \rfloor$.

Review Guide: Chapter 11

Definitions: How are the following terms defined?

- real-valued function of a real variable *(p. 717)*
- graph of a real-valued function of a real variable *(p. 717)*
- power function with exponent a *(p. 718)*
- floor function *(p. 719)*
- multiple of a real-valued function of a real variable *(p. 721)*
- increasing function *(p. 722)*
- decreasing function *(p. 722)*
- $f(x)$ is $\Omega(g(x)$, where f and g are real-valued functions of a real variable defined on the same set of nonnegative real numbers *(p. 727)*
- $f(x)$ is $O(g(x)$, where f and g are real-valued functions of a real variable defined on the same set of nonnegative real numbers *(p. 727)*
- $f(x)$ is $\Theta(g(x)$, where f and g are real-valued functions of a real variable defined on the same set of nonnegative real numbers *(p. 727)*
- algorithm A is $\Theta(g(n)$ (or A has order $g(n)$) *(p. 741)*
- algorithm A is $\Omega(g(n)$ (or A has a best case order $g(n)$) *(p. 741)*
- algorithm A is $O(g(n)$ (or A has a worst case order $g(n)$) *(p. 741)*
- polynomial time algorithms, NP class, NP-complete problems, the P vs. NP problem, tractable and intractable problems *(pp. 775-776)*

Polynomial and Rational Functions and Their Orders

- What is the graph of the floor function? *(pp. 719-720)*
- What is the difference between the graph of a function defined on an interval of real numbers and the graph of a function defined on a set of integers? *(p. 720)*
- How do you graph a multiple of a real-valued function of a real variable? *(p. 721)*
- How do you prove that a function is increasing (decreasing)? *(p. 723)*
- What are some properties of O-, Ω-, and Θ-notation? Can you prove them? *(p. 728)*
- If $x > 1$, what is the relationship between x^r and x^s, where r and s are rational numbers and $r < s$? *(p. 729)*
- Given a polynomial, how do you use the definition of Θ-notation to show that the polynomial has order x^n, where n is the degree of the polynomial? *(pp. 730-732)*
- What is the theorem on polynomial orders? *(p. 733)*
- What is an order for the sum of the first n integers? *(p. 735)*
- What is an order for a function that is a ratio of rational power functions? *(p. 736)*

Efficiency of Algorithms

- How do you compute the order of an algorithm segment that contains a loop? a nested loop? *(pp. 742-744)*
- How do you find the number of times a loop will iterate when an algorithm segment is executed? *(p. 743)*
- How do you use the theorem on polynomial orders to help find the order of an algorithm segment? *(p. 744)*
- What is the sequential search algorithm? How do you compute its worst case order? its average case order? *(pp. 739-740)*
- What is the insertion sort algorithm? How do you compute its best and worst case orders? *(pp. 740, 744-746)*

Logarithmic and Exponential Orders

- What do the graphs of logarithmic and exponential functions look like? *(pp. 751-752)*
- What can you say about the base 2 logarithm of a number that is between two consecutive powers of 2? *(p. 753)*
- How do you compute the number of bits needed to represent a positive integer in binary notation? *(p. 755)*
- How are logarithms used to solve recurrence relations? *(pp. 755-757)*
- If $b > 1$, what can you say about the relation among $\log_b x$, x^r, and $x \log_b x$? *(p. 758)*
- If $b > 1$ and $c > 1$, how are orders of $\log_b x$ and $\log_c x$ related? *(p. 760)*
- What is an order for a harmonic sum? *(pp. 760-762)*
- What is a divide-and-conquer algorithm? *(p. 765)*
- What is the binary search algorithm? *(pp. 765-767)*
- What is the worst case order for the binary search algorithm, and how do you find it? *(pp. 768-772)*
- What is the merge sort algorithm? *(pp. 772-775)*
- What is the worst case order for the merge sort algorithm, and how do you find it? *(p. 775)*

Chapter 12: Regular Expressions and Finite-State Automata

This chapter opens with some historical background about the connections between computers and formal languages. Section 12.1 focuses on regular expressions and emphasizes their utility for pattern matching, whether for compilers or for general text processing.

Section 12.2 introduces the concept of finite-state automaton. In one sense, it is a natural sequel to the discussions of digital logic circuits in Section 2.4 and Boolean functions in Section 7.1, with the next-state function of an automaton governing the operation of sequential circuit in much the same way that a Boolean function governs the operation of a combinatorial circuit. The section also provides practice in finding a finite-state automaton that corresponds to a regular expression and shows how to write a program to implement a finite-state automaton. Both abilities are useful for computer programming. The section ends with a statement and partial proof of Kleene's theorem, which describes the exact nature of the relationship between finite-state automata and regular languages.

The equivalence and simplification of finite-state automata, discussed in Section 12.3, provides an additional application for the concept of equivalence relation, introduced in Section 8.3. Note the parallel between the simplification of digital logic circuits discussed in Section 2.4 and the simplification of finite-state automata developed in this section. Both kinds of simplification have obvious practical use.

Section 12.1

3. *b.* $L = \{11*, 11/, 12*, 12/, 21*, 21/, 22*, 22/\}$

   $11* = 1*1 = 1$, $11/ = 1/1 = 1$, $12* = 1*2 = 2$, $12/ = 1/2 = 0.5$, $21* = 2*1 = 2$, $21/ = 2/1 = 2$, $22* = 2*2 = 4$, $22/ = 2/2 = 1$

6. L_1L_2 is the set of strings of 0's and 1's that both start and end with a 0.

   $L_1 \cup L_2$ is the set of strings of 0's and 1's that start with a 0 or end with a 0 (or both).

   $(L_1 \cup L_2)^*$ is the set of strings of 0's and 1's that start with a 0 or end with a 0 (or both) or that contain 00.

9. $(((x \mid (y(z^*)))^*)((yx) \mid (((yz)^*)z)))$

12. $xy(x^*y)^* \mid (yx \mid y)y^*$

15. $L((a \mid b)c) = L(a \mid b)L(c) = (L(a) \cup L(b))L(c) = (\{a\} \cup \{b\})\{c\} = \{a, b\}\{c\} = \{ac, bc\}$

18. $x, yxxy, xx, xyxxy, xyxxyyxxy, \ldots$

21. The language consists of the set of all strings of x's and y's that start with xy or yy followed by any string of x's and y's.

24. The string 120 does not belong to the language defined by $(01^*2)^*$ because it does not start with 0. However, 01202 does belong to the language because 012 and 02 are both defined by 01^*2 and the language is closed under concatenation.

27. $x \mid y^* \mid y^*(xyy^*)(\epsilon \mid x)$

30. Note that for any regular expression x, $(x^*)^*$ defines the set of all strings obtained by concatenating a finite number of a finite number of concatenations of copies of x. But any such string can equally well be obtained simply by concatenating a finite number of copies of x, and thus $(x^*)^* = x^*$. Hence the given languages are the same: $L((rs)^*) = L(((rs)^*)^*)$.

33. $[a - z]\{3\}[a - z]^*ly$

Section 12.2

3. *a.* U_0, U_1, U_2, U_3 *b.* a, b *c.* U_0 *d.* U_3

e.

			input	
			a	b
state	→	U_0	U_2	U_1
		U_1	U_2	U_3
		U_2	U_2	U_2
	◎	U_3	U_3	U_3

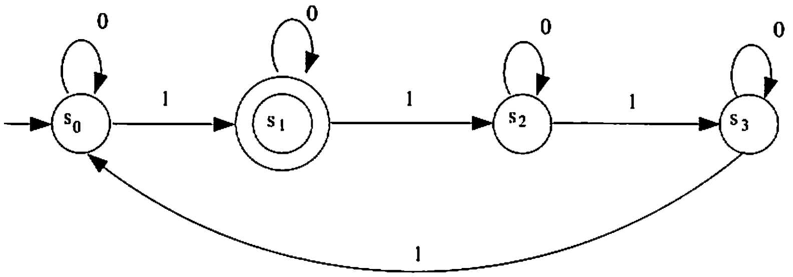
6. *a.* s_0, s_1, s_2, s_3 *b.* 0, 1 *c.* s_0 *d.* s_0

e.

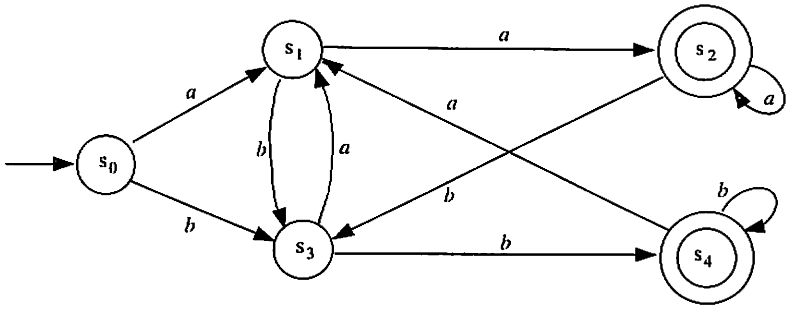
			input	
			0	1
state	→ ◎	s_0	s_0	s_1
		s_1	s_1	s_2
		s_2	s_2	s_3
		s_3	s_3	s_0

9. *a.* s_0, s_1, s_2, s_3 *b.* 0, 1 *c.* s_0 *d.* s_1

e.

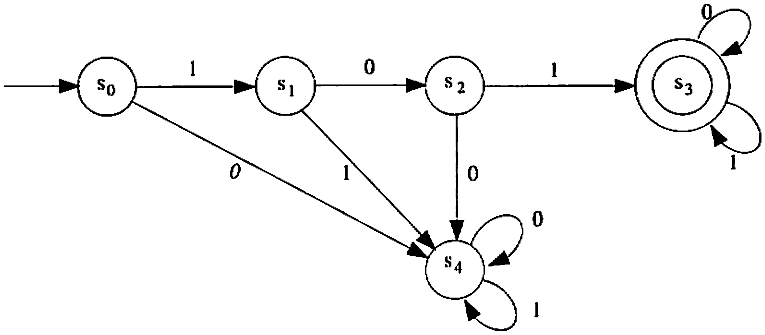


21. *a.*



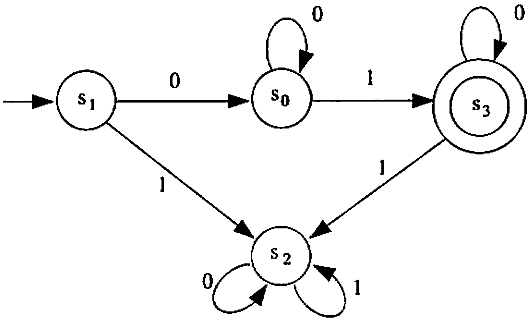
b. $(a|b)^*(aa|bb)$

24. *a.*



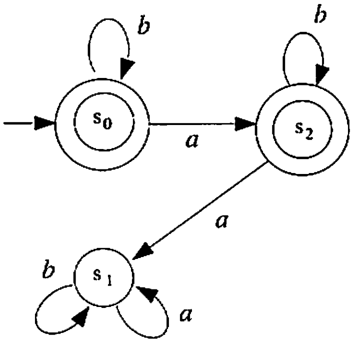
b. 101(0|1)*

27. *a.*

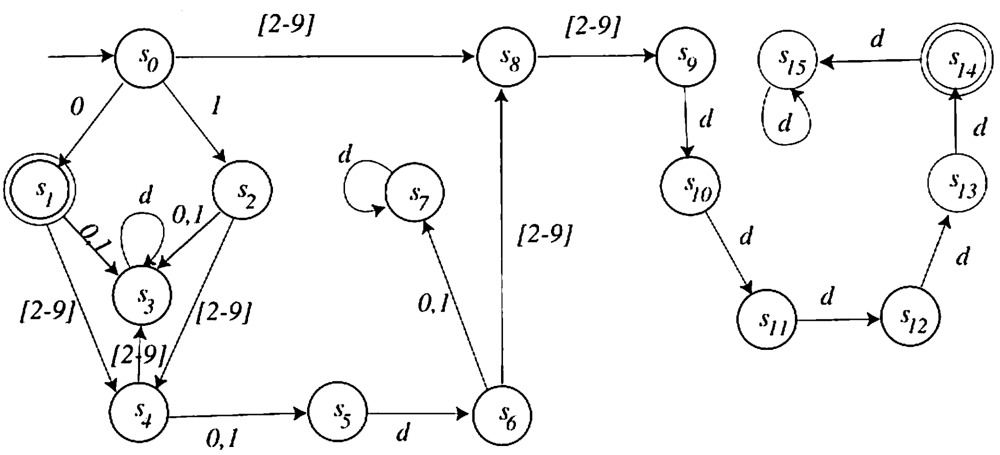


b. 00*10* (or using the $^+$ notation: 0^+10^*)

30.



48. Let d represent the character class $[0-9]$.



51. Proof (by contradiction):

Suppose there were a finite-state automaton A that accepts L. Consider all strings of the form a^i for some integer $i \geq 0$.

Since the set of all such strings is infinite and the number of states of A is finite, by the pigeonhole principle at least two of these strings, say a^p and a^q with $p < q$, must send A to the same state, say s, when input to A starting in its initial state. (The strings of the given form are the pigeons, the states are the pigeonholes, and each string is associated with the state to which A goes when the string is input to A starting in its initial state.)

Because A accepts L, A accepts $a^q b^q$ but A does not accept $a^p b^q$.

Now since $a^q b^q$ is accepted by A, A goes to an accepting state if, starting from the initial state, first a^q is input to it (sending it to state s) and then b^q is input to it. But A also goes to state s after a^p is input to it. Hence, inputting b^q to A after inputting a^p also sends A to an accepting state. In other words, A accepts $a^p b^q$.

Thus $a^p b^q$ is accepted by A and yet it is not accepted by A, which is a contradiction. Hence the supposition is false: there is no finite-state automaton that accepts L.

54. *a.* Proof:

Suppose A is a finite-state automaton with input alphabet $\sum$, and suppose $L(A)$ is the language accepted by A.

Define a new automaton A' as follows: Both the states and the input symbols of A' are the same as the states and input symbols of A. The only difference between A and A' is that each accepting state of A is a non-accepting state of A', and each non-accepting state of A is an accepting state of A'.

It follows that each string in $\sum^*$ that is accepted by A is not accepted by A', and each string in $\sum^*$ that is not accepted by A is accepted by A'. Thus $L(A') = (L(A))^c$.

b. Proof:

Let A_1 and A_2 be finite-state automata, and let $L(A_1)$ and $L(A_2)$ be the languages accepted by A_1 and A_2, respectively.

By part (a), there exist automata A_1' and A_2' such that $L(A_1') = (L(A_1))^c$ and $L(A_2') = (L(A_2))^c$.

Hence, by Kleene's theorem (part 1), there are regular expressions r_1 and r_2 that define $(L(A_1))^c$ and $(L(A_2))^c$, respectively. So we may write $(L(A_1))^c = L(r_1)$ and $(L(A_2))^c = L(r_2)$.

Now by definition of regular expression, $r_1 \mid r_2$ is a regular expression, and, by definition of the language defined by a regular expression, $L(r_1 \mid r_2) = L(r_1) \cup L(r_2)$.

Thus, by substitution and De Morgan's law, $L(r_1 \mid r_2) = (L(A_1))^c \cup (L(A_2))^c = (L(A_1) \cap L(A_2))^c$, and so, by Kleene's theorem (part(2)), there is a finite-state automaton, say A, that accepts $(L(A_1) \cap L(A_2))^c$.

It follows from part (a) that there is a finite-state automaton, A', that accepts $((L(A_1) \cap L(A_2))^c)^c$. But, by the double complement law for sets, $((L(A_1) \cap L(A_2))^c)^c = L(A_1) \cap L(A_2)$.

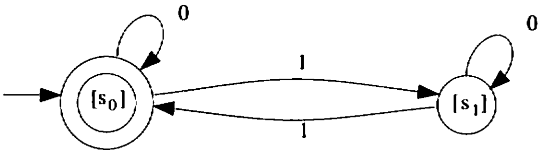
So there is a finite-state automaton, A', that accepts $L(A_1) \cap L(A_2)$, and hence, by Kleene's theorem and the definition of regular language, $L(A_1) \cap L(A_2)$ is a regular language.

Section 12.3

3. *a.* 0-equivalence classes: $\{s_1, s_3\}$, $\{s_0, s_2\}$

   1-equivalence classes: $\{s_1, s_3\}$, $\{s_0, s_2\}$

   b. transition diagram for $\bar{A}$:



6. *a.* 0-equivalence classes: $\{s_0, s_1, s_3, s_4, s_5\}$, $\{s_2, s_6\}$

   1-equivalence classes: $\{s_0, s_4, s_5\}$, $\{s_1, s_3\}$, $\{s_2\}$, $\{s_6\}$

   2-equivalence classes: $\{s_0, s_4\}$, $\{s_5\}$, $\{s_1\}$, $\{s_3\}$, $\{s_2\}$, $\{s_6\}$

   3-equivalence classes: $\{s_0\}$, $\{s_4\}$, $\{s_5\}$, $\{s_1\}$, $\{s_3\}$, $\{s_2\}$, $\{s_6\}$

   b. The transition diagram for $\overline{A}$ is the same as the one given for A except that the states are denoted $[s_0], [s_1], [s_2], [s_3], [s_4], [s_5], [s_6]$.

15. Proof:

Suppose k is an integer such that $k \geq 1$ and C_k is a k-equivalence class. We must show that there is a $k-1$ equivalence class, C_{k-1}, such that $C_k \subseteq C_{k-1}$.

By property (12.3.3), the $(k-1)$-equivalence classes partition the set of all states of A in to a union of mutually disjoint subsets.

Let s be any state in C_k. Then s is in *some* $(k-1)$-equivalence class; call it C_{k-1}.

Let t be any other state in C_k. *[We will show that $t \in C_{k-1}$ also.]* Then $t \; R_k \; s$, and so for all input strings of length k, $N^*(t, w)$ is an accepting state $\Leftrightarrow N^*(s, w)$ is an accepting state.

Since $k-1 < k$, it follows that for all input strings of length $k-1$, $N^*(t, w)$ is an accepting state $\Leftrightarrow N^*(s, w)$ is an accepting state.

Consequently, $t \; R_{k-1} \; s$, and so t and s are in the same $(k-1)$-equivalence class.

But $s \in C_{k-1}$. Hence $t \in C_{k-1}$ also. We, therefore, conclude that $C_k \subseteq C_{k-1}$.

18. Proof:

Suppose A is an automaton and C is a $*$-equivalence class of states of A.

By Theorem 12.3.2, there is an integer $K \geq 0$ such that C is a K-equivalence class of A. Suppose C contains both an accepting state s and a nonaccepting state t of A.

Since both s and t are in the same K-equivalence class, s is K-equivalent to t (by exercises 36 and 37 of Section 8.3), and so by exercise 17, s is 0-equivalent to t.

But this is impossible because there are only two 0-equivalence classes, the set of all accepting states and the set of all nonaccepting states, and these two sets are disjoint.

Hence the supposition that C contains both an accepting and a nonaccepting state is false: C consists entirely of accepting states or entirely of nonaccepting states.

Review Guide: Chapter 12

Definitions: How are the following terms defined?

- alphabet, string over an alphabet, formal language over an alphabet *(p. 781)*
- Σ^n, Σ^* (the Kleene closure of Σ), and Σ^+ (the positive closure of Σ), where Σ is an alphabet *(p. 781)*
- concatenation of x and y, where x and y are strings *(p. 783)*
- concatenation of L and L', where L and L' are languages *(p. 783)*
- union of L and L', where L and L' are languages *(p. 783)*
- Kleene closure of L , where L is a language *(p. 783)*
- regular expression over an alphabet *(p. 783)*
- language defined by a regular expression *(p. 784)*
- character class *(p. 787)*
- finite-state automaton, next-state function *(p. 793)*
- language accepted by a finite-state automaton *(p. 795)*
- eventual-state function for a finite-state automaton *(p. 797)*
- regular language *(p. 804)*
- $*$-equivalence of states in a finite-state automaton *(p. 809)*
- k-equivalence of states in a finite-state automaton *(p. 810)*
- quotient automaton *(p. 814)*
- equivalent automata *(p. 816)*

Regular Expressions

- What is the order of precedence for the operations in a regular expression? *(p. 784)*
- How do you find the language defined by a regular expression? *(p. 785)*
- Given a language, how do you find a regular expression that defines the language? *(p. 786)*
- What are some practical uses of regular expressions? *(pp. 787-789)*

Finite-State Automata

- How do you construct an annotated next-state table for a finite-state automaton given the transition diagram for the automaton? *(p. 794)*
- How do you construct a transition diagram for a finite-state automaton given its next-state table? *(pp. 794-795)*
- How do you find the state to which a finite-state automaton goes if the characters of a string are input to it? *(p. 796)*
- How do you find the language accepted by a finite-state automaton? *(p. 796)*
- Given a simple formal language, how do you construct a finite-state automaton to accept the language? *(p. 798)*
- How can you use software to simulate the action of a finite-state automaton? *(pp. 799-801)*
- What do the two parts of Kleene's theorem say about the relation between the language accepted by a finite-state automaton and the language defined by a regular expression? *(pp. 799. 803)*
- How can the pigeonhole principle be used to show that a language is not regular? *(p. 804)*
- How do you find the k-equivalence classes for a finite-state automaton? *(p. 811)*
- How do you find the $*$-equivalence classes for a finite-state automaton? *(p. 812)*
- How do you construct the quotient automaton for a finite-state automaton? *(pp. 814-815)*
- What is the relation between the language accepted by a finite-state automaton and the language accepted by the corresponding quotient automaton? *(p. 814)*

Conventions for Mathematical Writing

1. When introducing a new variable into a discussion, the convention is to place the new variable to the left of the equal sign and the expression that defines it to the right. This convention is identical to the one used in computer programming. For example, in a computer program, if a and b have previously been defined, and you want to assign the value of $a+b$ to a new variable s, you would write something like

$$s := a + b.$$

Similarly, in a mathematical proof, if a and b have previously been introduced into a discussion, and you want to let s be their sum,

instead of writing "Let $a+b=s$," you should write, "Let $s=a+b$."

2. It is considered good mathematical writing to avoid starting a sentence with a variable. That is one reason that mathematical writing frequently uses words and phrases such as Then, Thus, So, Therefore, It follows that, Hence, etc. For example, in a proof that any sum of even integers is even, instead of writing,

By definition of even, $m = 2a$ and $n = 2b$ for some integers a and b.

$$m + n = 2a + 2b\ldots$$

write

By definition of even, $m = 2a$ and $n = 2b$ for some integers a and b.
Then

$$m + n = 2a + 2b\ldots$$

The fact that $m+n = 2a+2b$ is a consequence of the facts that $m = 2a$ and $n = 2b$. Including the word "Then" in your proof alerts your reader to this reasoning.

3. Standard mathematical writing avoids repeating the left-hand side in a sequence of equations in which the left-hand side remains constant. For example, if $n = 5q+4$, instead of writing

$$\begin{aligned} n^2 &= (5q+4)^2 \\ n^2 &= 25q^2+40q+16 \\ n^2 &= 25q^2+40q+15+1 \\ n^2 &= 5(5q^2+8q+3)+1 \end{aligned}$$

all the n^2 except the first are omitted and each subsequent equal sign is read as "which equals," as shown below:

$$\begin{aligned} n^2 &= (5q+4)^2 \\ &= 25q^2+40q+16 \\ &= 25q^2+40q+15+1 \\ &= 5(5q^2+8q+3)+1 \end{aligned}$$

4. Respecting the equal sign is one of the most important mathematical conventions. An equal sign should only be used between quantities that are equal, not as a substitute for words like "is," "means that," "if and only if," $\Leftrightarrow$, or "is equivalent to." For example, if $a = 4$ and $b = 12$, students occasionally write:

$$a \mid b = 4 \mid 12 \text{ since } 12 = 4 \cdot 3.$$

But if this were read out loud, it would be, "a divides b equals 4 divides 12 since 12 equals 4 times 3," which makes no sense. A correct version would be

$$a \mid b \Leftrightarrow 4 \mid 12, \text{ which is true because } 12 = 4 \cdot 3$$

or

$$a \mid b \text{ because } 4 \mid 12 \text{ since } 12 = 4 \cdot 3.$$

5. It is unnecessary, and even risky, to place full statements of definitions and theorems inside the bodies of proofs. The reason is that the variables used to express them can become confused with variables that are part of the proof. So instead of including the statement of the definition of divisibility, for example, just write, "by definition of divisibility." Similarly, instead of including the statement of, say, Theorem 8.4.3, just write, "by Theorem 8.4.3." For instance, to prove that a sum of any even integer plus any odd integer is odd, someone might write the following:

> Suppose m is any even integer and n is any odd integer.
> For an integer to be even means that it equals $2k$ for some integer k, and
> for an integer to be odd, means that it equals $2k + 1$ for some integer k.
> Thus $m + n = 2k + (2k + 1) = 4k + 1$...

The problem is that although the letter k appears in the statements of the definitions in the text, it refers to a different quantity in each one. However, when the statements are combined together in the proof, the letter k can have only one interpretation. The result is that the argument in the "proof" only applies to an even integer and the next successive odd integer, not to *any* even integer and *any* odd integer.

Tips for Success with Proofs and Disproofs

Make sure your proofs are genuinely convincing. Express yourself carefully and completely – but concisely! Write in complete sentences, but don't use an unnecessary number of words.

Disproof by Counterexample

- To disprove a universal statement, give a counterexample.
- Write the word "Counterexample" at the beginning of a counterexample.
- Write counterexamples in complete sentences.
- Give values of the variables that you believe show the property is false.
- Include the computations that prove beyond any doubt that these values really do make the property false.

All Proofs

- Write the word "Proof" at the beginning of a proof.
- Write proofs in complete sentences.
- Start each sentence with a capital letter and finish with a period.

Direct Proof

- Begin each direct proof with the word "Suppose."
- In the "Suppose" sentence:
  - Introduce a variable or variables (indicating the general set they belong to - e.g., integers, real numbers etc.), and
  - Include the hypothesis that the variables satisfy.
- Identify the conclusion that you will need to show in order to complete the proof.
- Reason carefully from the "suppose" to the "conclusion to be shown."
- Include the little words (like "Then," "Thus," "So," "It follows that") that make your reasoning clear.
- Give a reason to support each assertion you make in your proof.

Proof by Contradiction

- Begin each proof by contradiction by writing "Suppose not. That is, suppose...," and continue this sentence by carefully writing the negation of the statement to be proved.
- After you have written the "suppose," you need to show that this supposition leads logically to a contradiction.
- Once you have derived a contradiction, you can conclude that the think you supposed is false. Since you supposed that the given statement was false, you now know that the given statement is true.

Proof by Contraposition

- Look to see if the statement to be proved is a universal conditional statement.
- If so, you can prove it by writing a direct proof of its contrapositive.

Find-the-Mistake Problems

All of the following problems contain a mistake. Identify and correct each one.

1. **Section 2.2**: The negation of "$1 < a < 5$" is "$1 \geq a \geq 5$."
2. **Section 2.2**: "P only if Q" means "if Q then P."
3. **Section 3.2**
   (a) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x > 2$ then $x^2 \leq 4$."
   (b) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exist real numbers x such that if $x > 2$ then $x^2 \leq 4$."
   (c) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exists a real number x such that $x > 2$ and $x^2 < 4$."
4. **Section 3.2**: The contrapositive of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x \leq 2$ then $x^2 \leq 4$."
5. **Section 3.3**: Statement: $\exists$ a real number x such that $\forall$ real numbers y, $x + y = 0$. Proposed negation: $\forall$ real numbers x, if y is a real number then $x + y \neq 0$.
6. **Section 4.1**: A person is asked to prove that the square of any odd integer is odd. Toward the end of a proof the person writes: "Therefore $n^2 = 2k + 1$, which is the definition of odd."
7. **Section 4.1**: *Prove:* The square of any even integer is even.

   Beginning of proof: Suppose that r is any integer. Then if m is any even integer, $m = 2r$. . . .
8. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^n = (-1)^{2r} = 1$ because $2r$ is even....
9. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^{2r} = ((-1)^2)^r$....
10. **Section 4.3**: *Prove:* For all integers a and b, if a and b are divisible by 3 then $a+b$ is divisible by 3.

    Beginning of proof: Suppose that for all integers a and b, if a and b are divisible by 3 then $a + b$ is divisible by 3. By definition of divisibility, ...
11. **Section 4.3**: *Prove:* For all integers a, if 3 divides a, then 3 divides a^2.

    Beginning of proof: Suppose a is any integer such that 3 divides a. Then $a = 3k$ for any integer k....
12. **Section 4.3**: *Prove:* For all integers a, if $a = 3b + 1$ for some integer b, then $a^2 - 1$ is divisible by 3.

    Beginning of proof: Let a be any integer such that $a = 3b + 1$ for some integer b. We will prove that $a^2 - 1$ is divisible by 3. This means that $a^2 - 1 = 3q$ for some integer q. Then $(3b + 1)^2 - 1 = 3q$, and, since q is an integer, by definition of divisibility, $a^2 - 1$ is divisible by 3....

13. **Section 4.4**: *Prove:* For all integers a, $a^2 - 2$ is not divisible by 3.

    Beginning of proof: Suppose a is any integer. By the quotient-remainder theorem with divisor $d = 3$, there exist unique integers q and r such that $a = 3q + r$, where $0 < r \leq 3$....

14. **Section 4.6**: *Prove by contradiction:* The product of any irrational number and any rational number is irrational.

    Beginning of proof: Suppose not. That is, suppose the product of any irrational number and any rational number is rational....

15. **Section 4.6**: The negation of "n is not divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$" is "n is divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$."

16. **Section 5.2**: The equation $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ is true for $n = 1$ because $1 + 2 + 3 + \cdots + 1 = \dfrac{1(1+1)}{2}$ is true.

17. **Section 5.2**: The equation $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ is true for $n = 1$ because

$$1 = \frac{1(1+1)}{2} \Rightarrow 1 = \frac{2}{2} \Rightarrow 1 = 1.$$

18. **Section 5.2**: *Prove by mathematical induction:* For all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

    Beginning of proof: Let the property $P(n)$ be

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \text{ for all integers } n \geq 1....$$

19. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that there is an element x such that x is in A and x is in B.

20. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that for all x, x is in A and x is in B.

21. **Section 7.2**: To prove that F: $A \to B$ is one-to-one, assume that if $F(x_1) = F(x_2)$ then $x_1 = x_2$.

22. **Section 7.2**: To prove that F: $A \to B$ is one-to-one, we must show that for all x_1 and x_2 in A, $F(x_1) = F(x_2)$ and $x_1 = x_2$.

23. **Section 8.2**: Define a relation R on the set of all integers by $a\,R\,b$ if, and only if, $ab > 0$. To show that R is symmetric, assume that for all integers a and b, $a\,R\,b$. We will show that $b\,R\,a$.

Answers for the Find-the-Mistake Problems

All of the following problems contain a mistake. Identify and correct each one.

1. **Section 2.2**: The negation of "$1 < a < 5$" is "$1 \geq a \geq 5$."

   Answer: A statement of the form "$1 < a < 5$" is an *and* statement. Thus, by De Morgan's law, its negation is an *or* statement. The correct negation is $1 \geq a$ or $a \geq 5$.

2. **Section 2.2**: "P only if Q" means "if Q then P."

   Answer: "P only if Q" means that the only way P can occur is for Q to occur. This means that if Q does not occur, then P cannot occur, or, equivalently, "if P occurs then Q must have occurred," i.e., "if P then Q."

3. **Section 3.2**

   (a) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x > 2$ then $x^2 \leq 4$."

   (b) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exist real numbers x such that if $x > 2$ then $x^2 \leq 4$."

   (c) The negation of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "There exists a real number x such that $x > 2$ and $x^2 < 4$."

   Answer to a, b, and c: The negation of a "For all" statement is a "There exists" statement, the negation of "if p then q" is "p and not q," and the negation of "$x^2 > 4$" is "$x^2 \leq 4$." The correct negation in all three cases is "There exists a real number x such that $x > 2$ and $x^2 \leq 4$."

4. **Section 3.2**: The contrapositive of "For all real numbers x, if $x > 2$ then $x^2 > 4$" is "For all real numbers x, if $x \leq 2$ then $x^2 \leq 4$."

   Answer: The contrapositive of "if p then q" is "if not q then not p." In this case p is $x > 2$ and q is $x^2 > 4$. Thus the correct answer is "For all real numbers x, if $x^2 \leq 4$ then $x \leq 2$."

5. **Section 3.3**: Statement: $\exists$ a real number x such that $\forall$ real numbers y, $x + y = 0$. Proposed negation: $\forall$ real numbers x, if y is a real number then $x + y \neq 0$.

   Answer: The proposed negation began correctly with "$\forall$ real numbers x," but the continuation should be the existential statement " $\exists$ a real number y such that $x + y \neq 0$."

6. **Section 4.1**: A person is asked to prove that the square of any odd integer is odd. Toward the end of a proof the person writes: "Therefore $n^2 = 2k + 1$, which is the definition of odd."

   Answer: For an integer to be odd means that it equals 2 times some integer plus 1.So it is not correct to say that "$2k + 1$ *is* the definition of odd." The person should have written: "Therefore $n^2 = 2k + 1$, where k is an integer, and so n^2 is odd by definition of odd."

7. **Section 4.1**: *Prove:* The square of any even integer is even.

   Beginning of proof: Suppose that r is any integer. Then if m is any even integer, $m = 2r \ldots$.

   Answer: To prove that the square of any even integer is even, you must start by supposing you have a *[particular but arbitrarily chosen]* even integer. By using the definition of even, you can *deduce* what the even integer must look like, namely that it must equal $2 \cdot$(some integer). A correct proof would start with an even integer m and deduce the existence of an integer r such that $m = 2r$. This "proof" has it backwards.

8. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^n = (-1)^{2r} = 1$ because $2r$ is even....

   Answer: By claiming that $(-1)^{2r} = 1$, this "proof" assumes what is to be proved, namely that (-1) raised to an even power equals 1.

9. **Section 4.1**: *Prove directly from the definition of even:* For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r. By substitution, $(-1)^{2r} = ((-1)^2)^r$...

   Answer: The fact that $(-1)^{2r} = ((-1)^2)^r$ follows from a property of exponents; it is not true "by substitution." When you write "by substitution," you have to include the original variable in the equation that you write. Thus the following would be correct:
   Prove directly from the definition of even: For all even integers n, $(-1)^n = 1$.

   Beginning of proof: Suppose n is any even integer. Then $n = 2r$ for some integer r, and so

   $$\begin{aligned}(-1)^n &= (-1)^{2r} && \text{by substitution}\\ &= ((-1)^2)^r && \text{by a property of exponents...}\end{aligned}$$

10. **Section 4.3**: *Prove:* For all integers a and b, if a and b are divisible by 3 then $a+b$ is divisible by 3.

    Beginning of proof: Suppose that for all integers a and b, if a and b are divisible by 3 then $a + b$ is divisible by 3. By definition of divisibility, ...

    Answer: This proof begins by assuming exactly what is to be proved. If one assumes what is to be proved, there is nothing left to do!

11. **Section 4.3**: *Prove:* For all integers a, if 3 divides a, then 3 divides a^2.

    Beginning of proof: Suppose a is any integer such that 3 divides a. Then $a = 3k$ for any integer k....

    Answer: It is incorrect to say that "$a = 3k$ for any integer k" because k cannot be just "any" integer; in fact, the only integer that k can be is $k = a/3$. The correct thing to say is, "Then $a = 3k$ for some integer k."

12. **Section 4.3**: *Prove:* For all integers a, if $a = 3b+1$ for some integer b, then $a^2 - 1$ is divisible by 3.

    Beginning of proof: Let a be any integer such that $a = 3b + 1$ for some integer b. We will prove that $a^2 - 1$ is divisible by 3. This means that $a^2 - 1 = 3q$ for some integer q. Then $(3b + 1)^2 - 1 = 3q$, and, since q is an integer, by definition of divisibility, $a^2 - 1$ is divisible by 3....

    Answer: This "proof" assumes something equivalent to what is to be proved. After stating "We will prove that $a^2 - 1$ is divisible by 3" it is correct to state that ."This means that $a^2 - 1 = 3q$ for some integer q." However, the following sentence assumes that the integer q has been shown to exist, which is not the case.

13. **Section 4.4**: *Prove:* For all integers a, $a^2 - 2$ is not divisible by 3.

    Beginning of proof: Suppose a is any integer. By the quotient-remainder theorem with divisor $d = 3$, there exist unique integers q and r such that $a = 3q + r$, where $0 < r \leq 3$.

    Answer: The inequality is incorrect; it should be $0 \leq r < 3$.

14. **Section 4.6**: *Prove by contradiction:* The product of any irrational number and any rational number is irrational.

Beginning of proof: Suppose not. That is, suppose the product of any irrational number and any rational number is rational.

Answer: A proof by contradiction start with the negations of the statement to be proved. In this case, the statement to be proved is universal, and so its negation is existential. However, this proposed proof begins with a universal statement. A correct way to begin the proof is the following:

Beginning of proof: Suppose not. That is, suppose there exists an irrational number and a rational number whose product is rational.

15. **Section 4.6**: The negation of "n is not divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$" is "n is divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$."

    Answer: Consider negating the statement "He does not have any money." The negation is not "He does have any money," it is "He does have some money." Similarly, the negation of "n is not divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$" is not "n is divisible by any prime number greater than 1 and less than or equal to $\sqrt{n}$." It is "n is divisible by *some* prime number greater than 1 and less than or equal to $\sqrt{n}$," or "There exists a prime number greater than 1 and less than or equal to $\sqrt{n}$ that divides n."

16. **Section 5.2**: The equation $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ is true for $n=1$ because $1+2+3+\cdots+1=\dfrac{1(1+1)}{2}$ is true.

    Answer: When $n=1$, the expression $1+2+3+\cdots+n=1$; it does not equal $1+2+3+\cdots+1$.

17. **Section 5.2**: The equation $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ is true for $n=1$ because

$$1=\frac{1(1+1)}{2}\Rightarrow 1=\frac{2}{2}\Rightarrow 1=1.$$

    Answer: A false statement can imply a true conclusion. So deducing a true conclusion from a statement is not a valid way to prove that the statement is true.

18. **Section 5.2**: *Prove by mathematical induction:* For all integers $n \geq 1$,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

    Beginning of proof: Let the property $P(n)$ be

$$1+2+3+\cdots+n=\frac{n(n+1)}{2} \text{ for all integers } n \geq 1....$$

    Answer: The job of a proof by mathematical induction is to prove that a given property is true for all integers greater than or equal to a given integer. In this example, the property $P(n)$ is simply the equation

$$1+2+3+\cdots+n=\frac{n(n+1)}{2},$$

    and the proof by mathematical induction establishes that $P(n)$ is true for all integers $n \geq 1$. The mistake is including the words "for all integers $n \geq 1$" as part of $P(n)$ because these words make $P(n)$ identical with what is to be proved.

19. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that there is an element x such that x is in A and x is in B.

    Answer: This answer implies that for A to be a subset of B, it is enough for there to be a single element that is in both sets. But this is false. For instance, if $A = \{1, 2\}$ and $B = \{2, 3\}$, then 2 is in both A and B, but A is not a subset of B because 1 is in A and 1 is not in B. In fact, for A to be a subset of B means that for all x, ***if*** x is in A ***then*** x must be in B.

20. **Section 6.1**: Given sets A and B, to show that A is a subset of B, we must show that for all x, x is in A and x is in B.

    Answer: There are two problems with this answer. One is that it implies that A and B are identical sets, whereas for A to be a subset of B it is possible for B to contain elements that are not in A. In addition, because no domain is specified for x, it appears to say that everything in the universe is in both A and B, which is not the case for most sets A and B.

21. **Section 7.2**: To prove that F: $A \to B$ is one-to-one, assume that if $F(x_1) = F(x_2)$ then $x_1 = x_2$.

    Answer: Assuming that "if $F(x_1) = F(x_2)$ then $x_1 = x_2$" is essentially the same as assuming that F is one-to-one. In other words, it essentially assumes what needs to be proved.

22. **Section 7.2**: To prove that F: $A \to B$ is one-to-one, we must show that for all x_1 and x_2 in A, $F(x_1) = F(x_2)$ and $x_1 = x_2$.

    Answer: This statement implies that for all x_1 and x_2 in A, $x_1 = x_2$. In other words, it implies that there is only one element in A, which is very seldom the case.

23. **Section 8.2**: Define a relation R on the set of all integers by $a\,R\,b$ if, and only if, $ab > 0$. To show that R is symmetric, assume that for all integers a and b, $a\,R\,b$. We will show that $b\,R\,a$.

    Answer: The problem with these statements is that saying "assume that for all integers a and b, $a\,R\,b$" is equivalent to saying that every integer is related to every other integer by R. This is not the case. For instance, -1 is not related to 1 because $(-1) \cdot 1 = -1$ and $-1 \not> 0$.