On the Problem of Distribution in Globular Star Clusters.

By H. C. Plummer, M.A. (Plate 8.)

1. Among the most remarkable objects in the sky must be reckoned the globular clusters. They suggest a number of interesting problems, the most elementary and fundamental of which is concerned with the statistical arrangement of the stars they contain. Simple counts form the basis of this study, and these can be made very easily when we possess good photographs. The close aggregation of the stars, however, demands high resolving power of the telescope, and the technical difficulty in photographing these objects successfully is very considerable. Hence the structure of stellar clusters has only received serious study in recent years. The first important investigation of the subject was made by Professor E. C. Pickering * about twenty years ago. He arrived at these two main conclusions:—

(1) The law of distribution is essentially the same for different clusters.

(2) The bright stars and the faint stars of a cluster obey the same law.

In search of the law of distribution of the stars in space he tested the forms $1 - r^2$ and $(1 - r)^n$, making the calculation for the latter with the values $n = 1$, 2, 3, and 4. The results were not very satisfactory. The same form of law was applied independently to the cluster M 13 by Mr. W. E. Plummer,† who gave the numerical results up to $n = 12$. The agreement with the observed numbers was not very close, nor could it be expected with a purely empirical law of this kind. The investigation merely served to show, since a high value of n was indicated, that the concentration of the stars in space increases very rapidly towards the centre of the cluster.

More recently Dr. v. Zeipel ‡ has made an important contribution in a study of the cluster M 3. He has shown how the law of distribution in space may be deduced numerically from the observed law of distribution in the projection as we see it. Further, he has compared § the distribution so determined for the clusters M 13 and ω Centauri with the densities to be expected in a spherical mass of gas in isothermal equilibrium. The result is to represent the central parts of the cluster satisfactorily, while in the outer parts the cluster is less dense than this theory requires.

It is proposed in this note to pursue the search for a physical basis on which the distribution of stars in clusters may be

* *Harvard Annals*, xxvi.
† *Mon. Not.*, lxv. p. 810.
‡ *Annales de l'Obs. de Paris*, xxv.
§ *C. R.*, cxliv. p. 361.

established. And first we examine the statistical relations between the density of the stars in space and the numbers obtained by different methods of counting.

2. Let R be the radius of the cluster, which for the present we may consider finite. Let three rectangular axes Ox, Oy, Oz be taken through the centre, Oz coinciding with the line of sight. We are concerned with three functions of the distance r from the centre, $\phi(r)$, $f(r)$, and $F(r)$. Of these $\phi(r)dv$ denotes the number of stars in a small volume dv; $f(r)da$ denotes the number of stars in a cylinder of small section da, the axis being at a distance r from the centre; and $F(r)dr$ denotes the number of stars lying between the planes $x=r$ and $x=r+dr$. Then

$$F(r)=\iint\phi(\rho)dydz$$

where

$$\rho^2=r^2+y^2+z^2, \quad R^2>r^2+y^2+z^2.$$

The boundary being given by the inequality, the integral becomes

$$F(r)=\int_r^R 2\pi\rho \, . \, \phi(\rho)d\rho \quad . \quad . \quad . \quad (1)$$

whence

$$\phi(r)=-F'(r)/2\pi r \quad . \quad . \quad . \quad . \quad (2)$$

Now $F(r)$ is the function which can be most easily determined from a photograph, namely, by simple counts of the stars in narrow parallel strips. Such counts can be directly compared with the result of applying an assumed law of distribution ϕ, or this law can be determined numerically by mechanical differentiation of F. The conditions necessary for complete accuracy are (1) that the counts shall cover the whole extent of the cluster, and (2) that the stars belonging to the background (or foreground) of the sky shall be excluded. If $\Sigma(r)$ is the number counted between the centre and the plane $x=r$, we have

$$F(r)=\Sigma'(r), \quad \Sigma(r)=\int_0^r F(r)dr.$$

3. It has been the more general practice to determine the function f which represents the distribution of the stars in the projection perpendicular to the line of sight. This is

$$f(r)=2\int_0^{\sqrt{(R^2-r^2)}}\phi(\rho)dz$$
$$=2\int_r^R\frac{\rho\phi(\rho)}{\sqrt{(\rho^2-r^2)}}d\rho \quad . \quad . \quad . \quad . \quad (3)$$

since $\rho^2=r^2+z^2$. Integrating by parts, since $\phi(R)=0$, we have

$$f(r)=-2\int_r^R\sqrt{(\rho^2-r^2)}\phi'(\rho)d\rho,$$

and therefore

$$f'(r) = 2r\int_r^R \frac{\phi'(\rho)}{\sqrt{(\rho^2 - r^2)}}d\rho$$
$$= 2r\int_0^{\sqrt{(R^2-r^2)}} \frac{\phi'(\rho)}{\rho}dz.$$

If we now integrate along the line $x = r_1$, $z = 0$, we obtain

$$\int_0^{\sqrt{(R^2-r_1^2)}} \frac{f'(r)}{r}dy = 2\int_0^{\sqrt{(R^2-r_1^2)}} dy \int_0^{\sqrt{(R^2-r^2)}} \frac{\phi'(\rho)}{\rho}dz$$

where $y^2 = r^2 - r_1^2$. Hence

$$\int_{r_1}^R \frac{f'(r)}{\sqrt{(r^2 - r_1^2)}}dr = \pi\int_{r_1}^R \frac{\phi'(\rho)}{\rho} \cdot \rho d\rho$$
$$= -\pi\phi(r_1) \quad . \quad . \quad . \quad . \quad (4)$$

or, writing ρ for r_1, and integrating by parts,

$$\phi(\rho) = \frac{1}{\pi}\int_\rho^R \sqrt{(r^2 - \rho^2)}\frac{d}{dr}\left[\frac{f'(r)}{r}\right]dr \quad . \quad . \quad . \quad (5)$$

if we may assume $f'(R) = 0$.

If the totals are counted up to successive distances r from the centre, and these are represented by $\sigma(r)$,

$$f(r) = \sigma'(r)/2\pi r.$$

The formulæ (4) and (5), for which we have given a simple, direct proof, were deduced by v. Zeipel from a theorem by Abel. It would appear that the application of this method of determining the function ϕ must be rather laborious.

4. If we suppose that a globular cluster is the outcome of a primitive spherical nebula, it is reasonable to expect that the density of matter at different distances from the centre will be approximately preserved. It is natural, therefore, to look for guidance to possible conditions of equilibrium in a spherical mass of gas. Let ϕ, p, t be the density, pressure, and temperature at the distance r from the centre of the nebula. For convective equilibrium in a perfect gas we have

$$p = at\phi = b\phi^\gamma$$

where a, b are constants and γ is the ratio of the specific heats. Also, if m is the mass within the radius r,

$$\frac{dp}{dr} = -cm\phi/r^2 \ , \ \frac{dm}{dr} = 4\pi r^2\phi \, .$$

Eliminating p and m,

$$\frac{d}{dr}\left(\frac{br^2}{\phi} \cdot \frac{d\phi^\gamma}{dr}\right) + 4\pi c r^2 \phi = 0 \quad . \quad . \quad . \quad (6)$$

or, choosing the units to give the results in the simplest form, since we are not concerned with absolute values,

$$\frac{d}{dr}\left(r^2 \cdot \frac{d\phi^{\gamma-1}}{dr}\right) + r^2\phi = 0 \quad . \quad . \quad . \quad (7)$$

If we change the independent variable to $u = 1/r$, this takes the well-known form

$$\frac{d^2\phi^{\gamma-1}}{du^2} + \frac{\phi}{u^4} = 0 \quad . \quad . \quad . \quad . \quad (8)$$

On the other hand, we get the corresponding equations for isothermal equilibrium by putting $\gamma = 1$ in (6) and obtain similarly

$$\frac{d}{dr}\left(r^2 \cdot \frac{d \,.\, \log \phi}{dr}\right) + r^2\phi = 0 \quad . \quad . \quad . \quad (9)$$

$$\frac{d^2(\log \phi)}{du^2} + \frac{\phi}{u^4} = 0 \,.$$

5. These equations have received much study.* The appropriate solutions cannot in general be expressed mathematically in finite terms, but the more interesting cases have been worked out in a numerical form. The corresponding distributions either have a finite boundary, *e.g.* when $\gamma = 1 \cdot 4$ (diatomic gases), or they extend to infinity, as in the isothermal state given by (9). There are, however, two special cases which are known to be susceptible of solution in a simple form. The first solution was discovered by Ritter † for the case of $\gamma = 2$, namely,

$$\phi(r) = \sin r/r \quad . \quad . \quad . \quad . \quad (10)$$

and the second by Schuster ‡ for $\gamma = 1 \cdot 2$, namely,

$$\phi(r) = (1 + r^2)^{-\frac{5}{2}} \quad . \quad . \quad . \quad . \quad (11)$$

except for a numerical factor which we can omit. The values of γ for simple gases are near $1\frac{2}{3}$ (*e.g.* mercury), $1 \cdot 4$ (*e.g.* hydrogen), and $1\frac{1}{3}$ (*e.g.* chlorine), in accordance with the kinetic theory. But values less than $1 \cdot 2$ have been found for compound gases (*e.g.* chloroform), while greater values than 2 have been deduced by Amagat § for carbon dioxide under great pressures at moderate temperatures, though no values so great seem to have been determined by direct experiment. On the whole we may well be content to employ, at least in the first instance, that indication of

* *Cf.* Lord Kelvin, *Proc. R.S.E.*, xxviii. p. 259; Emden, *Gaskugeln* (Leipzig, 1907).

† Wiedemann's *Annalen*, xi. p. 338.

‡ *B.A. Report*, 1883, p. 428.

§ *C.R.*, cxxi. p. 863.

theory which appears in the simplest form. This is expressed by (11), and we have

$$\phi(r) = \mathrm{N}(1+r^2)^{-\frac{5}{2}},\ \mathrm{F}(r) = \tfrac{2}{3}\pi\mathrm{N}(1+r^2)^{-\frac{3}{2}},\ \Sigma(r) = \tfrac{2}{3}\pi\mathrm{N}r(1+r^2)^{-\frac{1}{2}} \quad (12)$$

$$f(r) = 2\mathrm{N}\int_r^\infty \frac{\rho d\rho}{(\rho^2 - r^2)^{\frac{1}{2}}(1+\rho^2)^{\frac{5}{2}}} = \frac{2\mathrm{N}}{(1+r^2)^2}\int_0^\infty \frac{dq}{(1+q^2)^{\frac{5}{2}}}$$

where

$$q^2(1+r^2) = \rho^2 - r^2.$$

The definite integral is equal to $\frac{2}{3}$, and we have

$$f(r) = \tfrac{4}{3}\mathrm{N}(1+r^2)^{-2}, \quad \sigma(r) = \tfrac{4}{3}\pi\mathrm{N}r^2(1+r^2)^{-1} \quad . \quad (13)$$

6. As our first example we will take the counts of the ω Centauri cluster made by Professor S. I. Bailey.* The results of a comparison with the formula (12) are shown in the following Table I. :—

TABLE I.

ω Centauri.

To r=	S.	N.	W.	E.	Means.	Sums.	C.	O−C.	ΔC.	Δ(O−C).
1·5	777	781	681	672	728	728	727	+ 1	727	+ 1
3·0	689	673	637	632	658	1386	1371	+15	644	+14
4·5	524	538	470	506	510	1896	1887	+ 9	516	− 6
6·0	392	348	335	476	388	2284	2277	+ 7	390	− 2
7·5	248	249	280	336	278	2562	2564	− 2	287	− 9
9·0	197	156	201	253	202	2764	2773	− 9	209	− 7
10·5	132	140	146	185	151	2915	2927	−12	154	− 3
12·0	128	119	110	141	125	3040	3042	− 2	115	+10
13·5	92	64	78	104	85	3125	3129	− 4	87	− 2
15·0	75	67	63	83	72	3197	3196	+ 1	67	+ 5
∞							3540			
								+33		+30
								−29		−29

Here the four columns following the first contain the numbers of stars in complete strips, the edges of which are at the given distances from the centre of the cluster. The means are given in column 6; incidentally, the use of means is easily seen to reduce the effect of an error in the assumed position of the centre. Column 7 contains the derived sums from the centre to the distances given, the last being half the total number for the cluster. These are compared in the next two columns with the formula

$$\Sigma(r) = 3540r/\sqrt{(1+r^2)}$$

* *Astronomy and Astro-Physics*, xii. p. 689; *Harvard Annals*, xxvi. p. 213.

the unit of r being chosen at 7′·14. If the counts were extended to infinity we should thus expect to find 7080 stars, as against 6389 counted. In the last column but one the computed sums are again split up according to the successive strips, and the last column gives the residuals when the comparison is made with the means in column 6. When the divergence of the separate numbers in columns 2–5 is considered, it is clear that the counts are at least as well represented by the formula we have chosen as they are consistent with the fundamental hypothesis that we are dealing with a truly spherical distribution.

One qualification must be made. We have made no allowance for the average density of the sky apart from the cluster. It is exceedingly difficult to do this with any certainty. According to our present ideas the cluster has no finite boundary. Hence an estimate based on counts in the vicinity will be too high, because we shall include stars which belong to the cluster. On the other hand, counts made at a considerable distance may not represent the density of the actual region. For the present it is enough to say that the neglect of this correction can scarcely invalidate the excellent agreement which has been found between the counts and the formula.

7. Before going further in testing the law indicated by (11), it will be well to examine the suitability of the other simple law, represented by (10). This gives

$$F(r) = \int_r^R \sin\rho \, . \, d\rho = 1 + \cos r$$

$$\Sigma(r) = \int_0^r F(r) dr = r + \sin r$$

when a constant factor, $2\pi N$, is ignored. Here the distribution has a finite boundary, $R = \pi$. If we assume that the counts extend to this boundary and are complete within it, we have the results shown in Table II. when we multiply $\Sigma(r)$ by 1018 to give the proper total.

Table II.

r.		Sums.	C.	O – C.	Means.	ΔC.	Δ(O – C).
1′·5 =	18°	728	634	+ 94	728	634	+ 94
3·0	36	1386	1238	+ 148	658	604	+ 54
4·5	54	1896	1784	+ 112	510	546	− 36
6·0	72	2284	2248	+ 36	388	464	− 76
7·5	90	2562	2617	− 55	278	369	− 91
9·0	108	2764	2887	− 123	202	270	− 68
10·5	126	2915	3062	− 147	151	175	− 24
12·0	144	3040	3158	− 118	125	96	+ 29
13·5	162	3125	3193	− 68	85	35	+ 50
15·0	180	3197	3199	− 2	72	6	+ 66
				+ 390			+ 293
				− 513			− 295

Here the first two columns contain the distance from the centre in actual angular measure and in the angular unit appropriate to the formula. Columns 3 and 6, which are identical with columns 7 and 6 in Table I., give the totals in strips up to these distances, and their differences. Column 4 gives the computed totals, with the differences in column 7. The results of the comparison are given in columns 5 and 8. There can be no doubt that the formula fails to represent the counts. It may be that a better appearance might be obtained by altering the scale value in r and adjusting the density by a different factor. Also, as before, the correction for general sky density has been omitted. But it does not appear that a good agreement could be obtained by such means. The evidence suggests strongly that in comparison with the law $(1+r^2)^{-\frac{5}{2}}$ the law $\sin r/r$ may be set aside for our purpose.

8. The curves which show the number of stars in the projection at different distances from the centre can be made directly comparable by reducing the central density to unity and choosing an appropriate scale value in distance. This has been done by Professor Pickering for the clusters ω Centauri, 47 Tucanæ (bright and faint stars separately), and M 13 (for which the material was derived from Professor Scheiner's measures). The actual densities (central or mean) and scale values cannot be insisted on as material, since the former is a function of the optical power and time of exposure, and the latter depends on the distance of the cluster. In fact, the counts are never complete and the parallaxes are unknown. Through the points which he had thus reduced to a common system Professor Pickering drew a smooth curve to represent the general law. This curve we now compare with the law represented by (13),

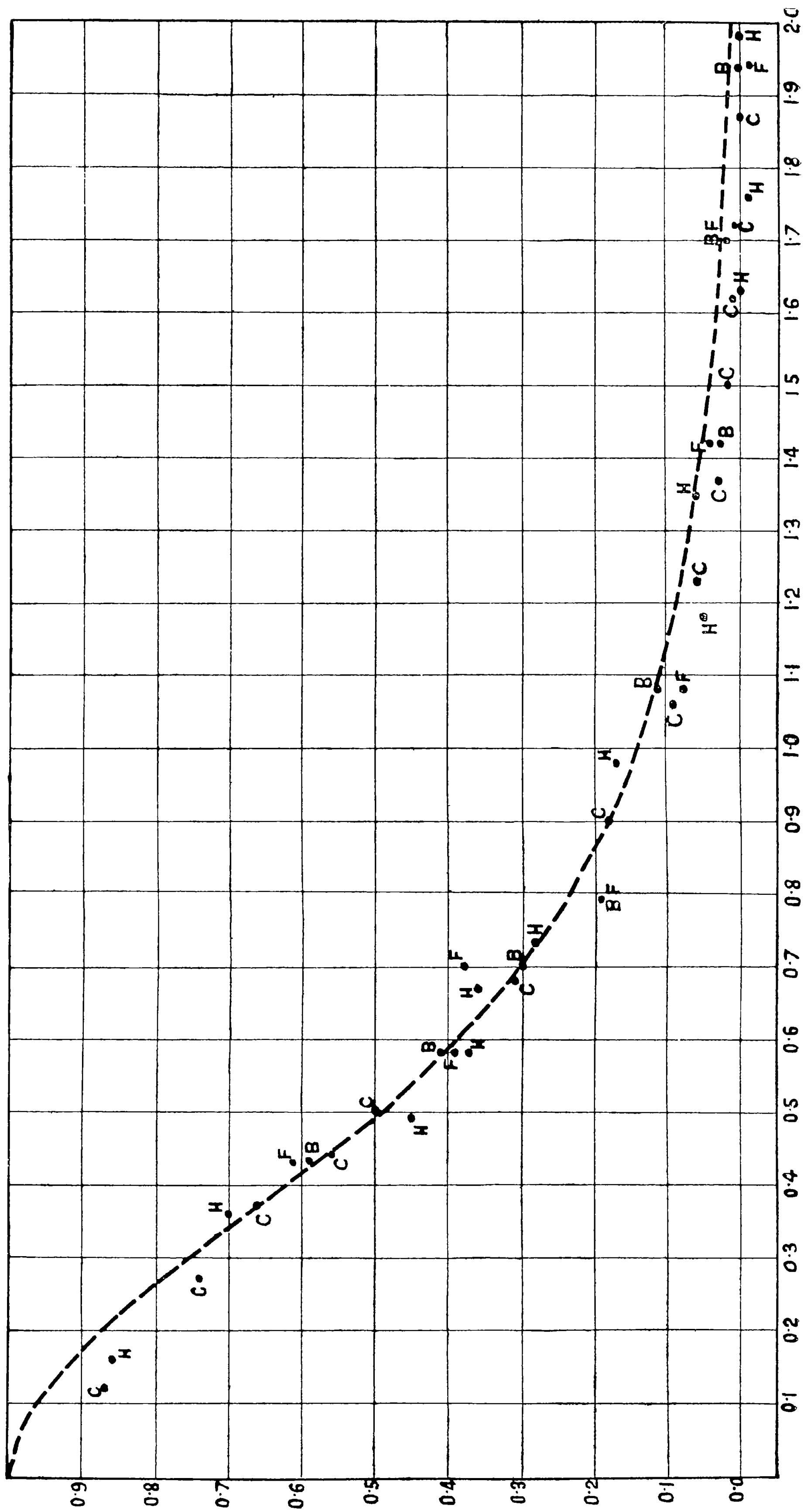
$$y=(1+r^2)^{-2}$$

but making $r=1{\cdot}29\,x$. The result is shown in Table III.

Table III.

x.	Curve.	Formula.	F − C.	x.	Curve.	Formula.	F − C.
0·0	1·000	1·000	·000	1·0	0·132	0·141	+ ·009
0·1	0·900	0·968	+ ·068	1·1	·098	·110	+ ·012
0·2	·800	·879	+ ·079	1·2	·071	·087	+ ·016
0·3	·700	·756	+ ·056	1·3	·050	·069	+ ·019
0·4	·600	·624	+ ·024	1·4	·034	·055	+ ·021
0·5	·500	·499	− ·001	1·5	·022	·044	+ ·022
0·6	·402	·391	− ·011	1·6	·013	·036	+ ·023
0·7	·311	·303	− ·008	1·7	·007	·030	+ ·023
0·8	·235	·235	·000	1·8	·003	·024	+ ·021
0·9	·176	·181	+ ·005	1·9	·001	·020	+ ·019
1·0	·132	·141	+ ·009	2·0	·000	·017	+ ·017

CURVE OF APPARENT DENSITY IN CLUSTERS.



C = ω Centauri, B = 47 Tucanæ (bright stars), F = 47 Tucanæ (faint stars), H = M 13.

It will be noticed that the values given by the formula are on the average ·020 in excess of the values given by the curve. This is due, it may fairly be contended, to a too liberal allowance being made for the general sky average in the determination of the Harvard curve. That the stars of a cluster die out at a particular distance is confessedly an arbitrary assumption. It is right that the law should be compared not simply with the interpolated curve but also with the actual points on which that curve is based. This is done in the figure (Plate 8), in which the points are reproduced from the diagram given by Professor Pickering. We see that the agreement of the points with the curve now proposed is good, and that it would be even better if the points were all raised by the amount ·020.

9. We have thus found that the evidence drawn from the Harvard material lends support to the idea that the distribution of stars in clusters resembles the distribution of density in a gravitating sphere of a particular kind of gas to which Schuster's solution applies. It would seem, at all events, that this simple form of law deserves to be borne in mind when further statistical results become available. At present the material for discussion is scanty. But we have v. Zeipel's measures of the cluster M 3, to which reference has already been made, and the examination of this case will be found useful, since it will warn us not to attribute excessive generality or precision to the preceding results. Of the three plates used in forming the complete catalogue, the one with the longest exposure (2^{h}) alone is used in making the counts by strips parallel to the axis of y. The centre of the cluster is assumed at $x = +2'\cdot 1$, and means of the numbers found on opposite sides of this centre are used. It is noted (p. 17) that "une trentaine d'étoiles" in the centre of the cluster could not be included among the measures owing to the density of the plate; on this account we add 15 stars to the central strip. A further correction is made by subtracting 5 stars for each strip 1′ wide to allow for the average density of the sky apart from the cluster. We now have the numbers in Table IV.

TABLE IV.

r.	O.	$\Sigma(r)$.	Diff.	C.	Δ.	r.	O.	$\Sigma(r)$.	Diff.	C.	Δ.
0′·5	135	148	148	137	137	6′	549	534	16	533	11
1·0	253	263	115	251	114	7	559	539	5	541	8
1·5	337	345	82	337	86	8	573	548	9	545	4
2·0	388	393	48	397	60	9	582	552	4	549	4
2·5	432	435	42	438	41	10	592	557	5	551	2
3·0	461	461	26	468	30	15	630	570	13	557	6
4·0	501	496	35	502	34	20	664	579	9	559	2
5·0	528	518	22	522	20	25	691	581	2	560	1

Here the column headed O contains the means of counts to

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the given distances on both sides of the centre, the column headed $\Sigma(r)$ the corresponding numbers when the corrections just explained have been applied, and the next column the differences of the preceding, *i.e.* the number of stars belonging to the cluster in each strip. With these last two columns are to be compared the columns headed C and Δ respectively, in which the numbers calculated from the formula

$$\Sigma(r) = 562r(1 + r^2)^{-\frac{1}{2}}$$

with $2'$ as the unit of r, and their differences are given. The general agreement is fairly good, and if no other tests were available, there would be little reason to be dissatisfied with this representation of the distribution of the stars.

10. Dr. v. Zeipel has given (p. 28) the counts of the stars lying between circles drawn at regular distances around the centre of the cluster. These counts, corrected for the average density of the sky, determine $\sigma(r)$ for different values of r, and we must compare them with formula (13), which in this case we make

$$\sigma(r) = 1160r^2(1 + r^2)^{-1}$$

with $2'$ as the unit of r as before. This gives the numbers in the column C of Table V.

TABLE V.

r.	$\sigma(r)$.	Diff.	C.	Δ.	R.	r.	$\sigma(r)$.	Diff.	C.	Δ.	R.
0′·5	103	103	68	68	213	5′	959	59	1000	72	280
1·0	326	223	232	164	212	6	1009	50	1044	44	283
1·5	499	173	418	186	238	7	1045	36	1073	29	278
2·0	634	135	580	162	244	8	1075	30	1092	19	279
2·5	747	113	706	126	253	9	1091	16	1105	13	279
3·0	819	72	803	97	259	10	1115	24	1116	11	278
4·0	900	81	928	125	273	13	1144	29	1133	17	278

The comparison of the fifth column (headed Δ) with the third column, which contains the corrected counts in rings about the centre, reveals a want of agreement which can only be attributed to the inability of our formulæ to represent the true structure of the cluster. In particular, the central part of the cluster is much denser than our theory allows, and there is one simple test which shows this very clearly. If we compare the formulæ (12) and (13), we have simply

$$\{\Sigma(r)\}^2/\sigma(r) = \tfrac{1}{3}\pi N = \tfrac{1}{2}\Sigma(\infty).$$

It is true that in this case we have deduced $\Sigma(r)$ from one plate only, while $\sigma(r)$ contains a few stars in addition which occur on two plates of shorter exposure. But this will not invalidate the conclusion that the defined ratio should be constant for all values of r. This test is the more convenient because, unlike the previous comparisons, it is independent of the assumed unit to which r is referred. Accordingly, the ratio deduced from the numbers shown in Tables IV. and V. is given under R in Table V. The numbers show a systematic run which it would be impossible to alter greatly by any reasonable change in the assumed density of the stars not belonging to the cluster. The direction of the run proves unmistakably that the concentration of the cluster towards the centre is greater than is possible according to the law which is the subject of examination.

11. The same point is brought out very clearly in another way. We have used $2'$ as the unit of r and 1160 as the numerical factor in $\sigma(r)$. This latter is $\frac{4}{3}\pi N$ according to (13), whence $N = 277$. Thus

$$\phi(r) = 277(1 + r^2)^{-\frac{5}{2}}/(2')^3 = 34{\cdot}6(1 + r^2)^{-\frac{5}{2}}/(1')^3.$$

Now, Dr. v. Zeipel has calculated numerically by means of (5) the values of $\phi(r)$ corresponding to the numbers found for $f(r)$. These we can now compare with the above formula, and the result is shown in Table VI., where the numbers calculated from the formula are given under the heading C.

Table VI.

$4r$.	$\phi(r)$.	C.	$4r$.	$\phi(r)$.	C.	$4r$.	$\phi(r)$.	C.
1'	58·5	33·3	8'	4·83	6·11	19'	0·19	0·30
2	56·6	29·8	9	4·18	4·52	21	0·17	0·20
3	48·0	24·9	10	3·34	3·30	23	0·14	0·13
4	31·6	19·8	11	1·84	2·45	25	0·10	0·09
5	16·2	15·2	13	0·96	1·37	27	0·06	0·06
6	8·81	11·3	15	0·41	0·80	29	0·06	0·05
7	6·26	8·34	17	0·21	0·49	31	0·06	0·03

The deficiency in the calculated density of the cluster within 1' of the centre is very marked. Beyond this distance the agreement improves. As the numbers $\phi(r)$ themselves are not free from irregularities which become very evident when $r\phi(r)$ is tabulated and differenced, and which could with difficulty be reconciled with any ordinary function, the representation of the density in the outer parts of the cluster is fairly satisfactory, but the discrepancy in the central density is, at first sight, serious.

This difficulty, however, by no means compels us to abandon the general physical basis on which we have proceeded. It has been mentioned in § 1 that Dr. v. Zeipel found that the distribution in the central parts of this cluster is satisfactorily represented by the density of a sphere of gas in isothermal equilibrium. This law failed to correspond with the density observed in the outer parts, while the opposite state of things presents itself when we seek to apply a particular law of convective equilibrium. Hence we are led to combine both solutions and to conclude that the cluster is the outcome of a nebula in convective equilibrium outside a central core in isothermal equilibrium. There is nothing strange in such a distribution, for Ritter and others who have investigated the subject of equilibrium in gaseous spheres have contemplated the possibility of such isothermal-adiabatic phases. This combination will occur when convective currents transfer portions of the gas in the outer regions too quickly to allow an equilibrium of temperature to be established, while in the interior the greater density of the gas and the comparative freedom from disturbances make conduction operative and consequently the isothermal condition possible. Thus the facts obtained from the cluster M 3 can be explained without doing violence to the general line of thought which we have followed.

12. The contents of this paper are here summarised :—

§ 1. Introductory.

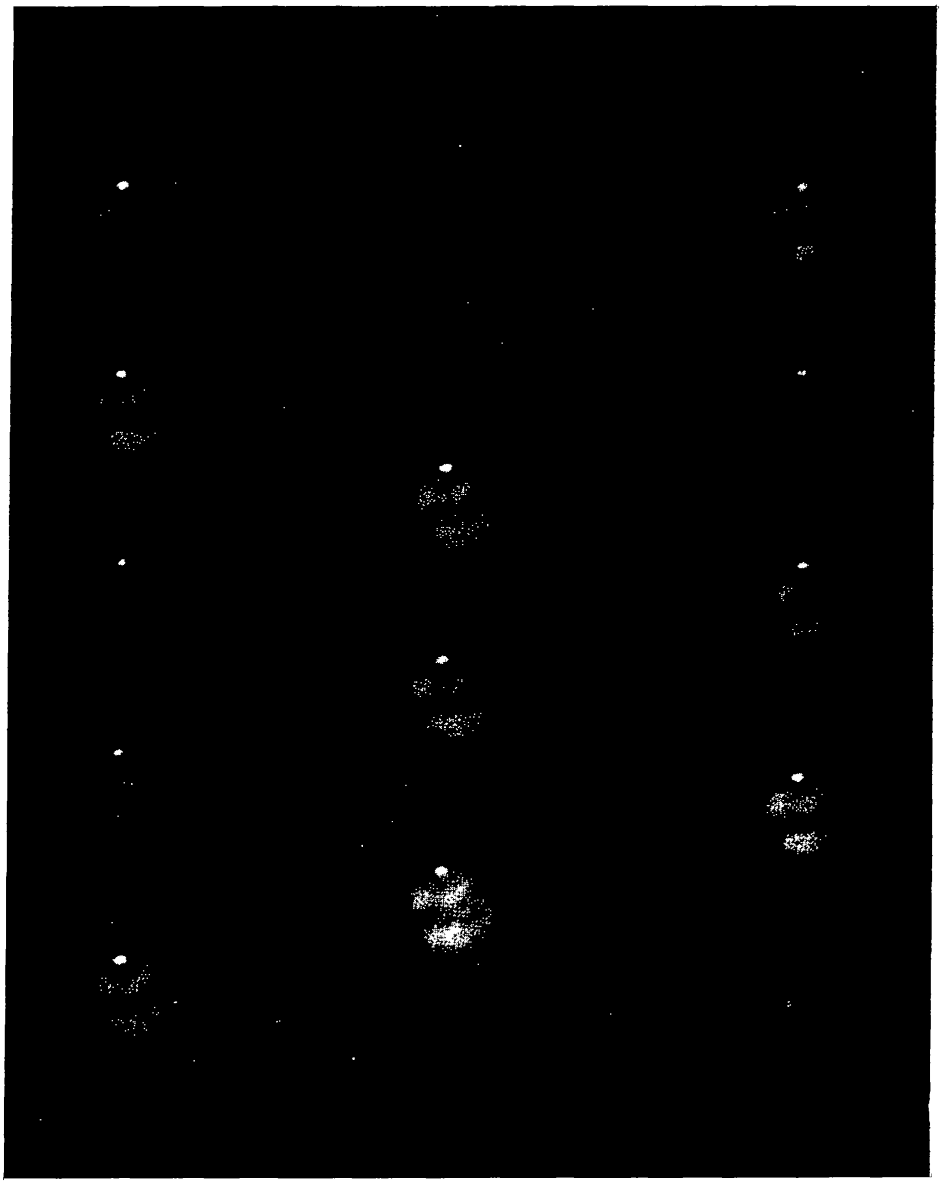
§§ 2, 3. Examination of the relations between the distribution of the stars of a cluster in space and in projection, and of the methods of making counts.

§§ 4, 5. The conditions of equilibrium of a spherical mass of gas with the two simplest forms of solution.

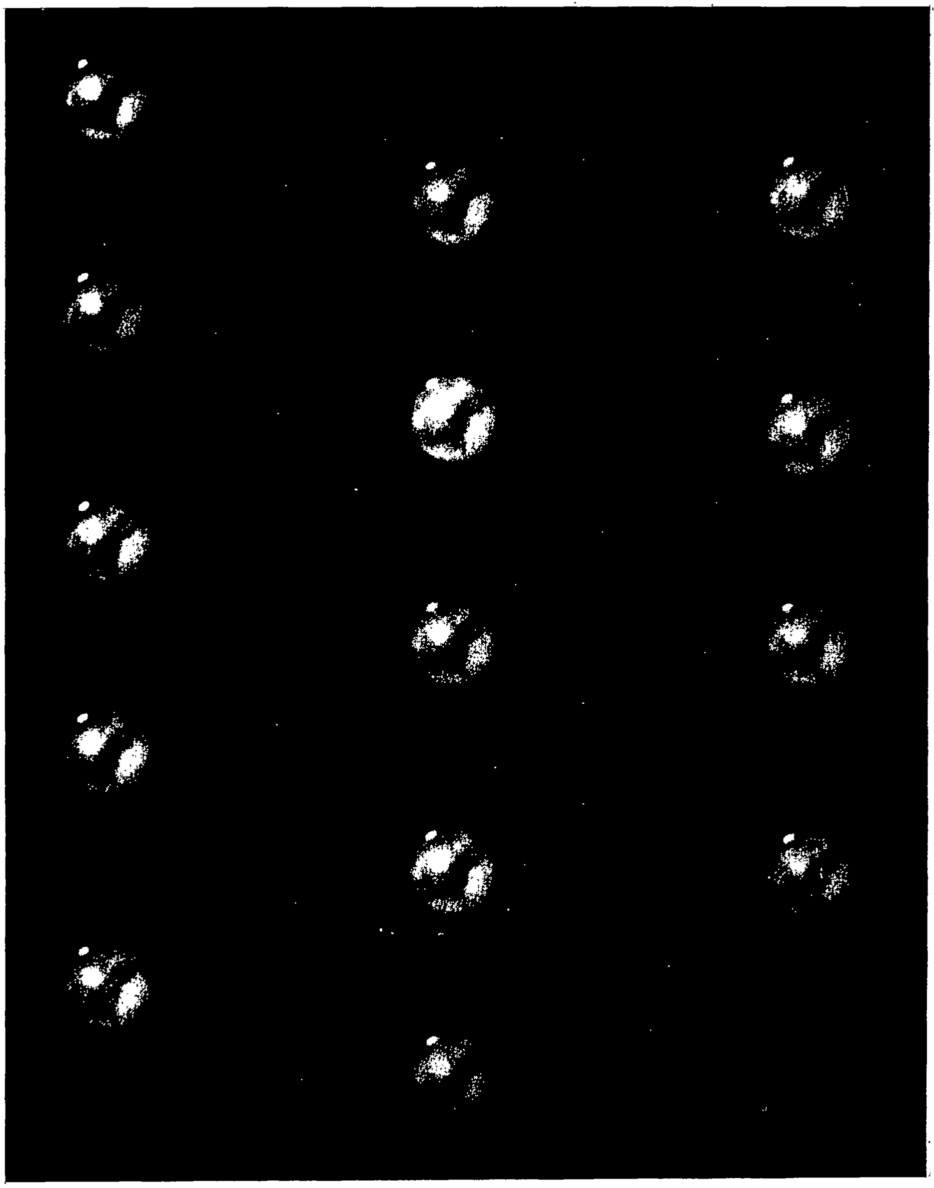
§§ 6, 7. The numbers counted in the ω Centauri cluster accord with one law $[N(1+r^2)^{-\frac{5}{2}}]$ and not with the other $[N \sin r/r]$.

§ 8. The same law represents well the apparent (projected) distribution found by Professor Pickering for the clusters ω Centauri, 47 Tucanæ (bright and faint stars), and M 13.

§§ 9–11. The cluster M 3 is compared with the same law and found to be marked by a higher degree of condensation towards the centre than can be reconciled with the law. It is suggested that this can be explained by supposing that the original nebular matrix possessed a central core in isothermal equilibrium, with an outer envelope in convective equilibrium.



MARS, 1909 SEPTEMBER 24. 16H. 55M. G.M.T.—E. E. BARNARD.



MARS, 1909 SEPTEMBER 28. 16H. 46M. G.M.T.—E. E. BARNARD.