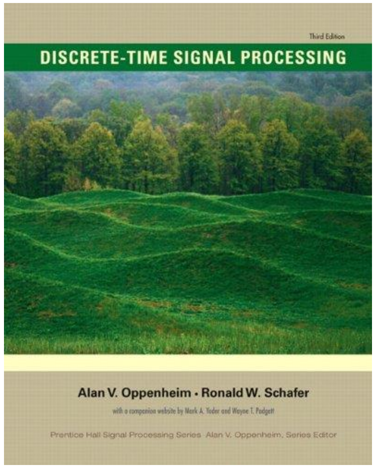


Lecture 03:

Discrete Time Systems (Part II)

Outlines

- 1. Discrete-time systems
- 2. Convolution
- ✓ 3. Linear Constant-Coefficient Difference Equations (LCCDEs)
- ✓ 4. Correlation

3. Linear Constant-Coefficient Difference Equation (LCCDE)

- General spec. of DT, LSI, finite-dim sys:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- defined by $\{d_k\},\{p_k\}$
- order = $\max(N,M)$

- Rearrange for $y[n]$ in causal form:

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

- WLOG, always have $d_0 = 1$

Solving LCCDEs

- "Total solution"

$$\underbrace{y[n] = y_c[n]}_{} + \underbrace{y_p[n]}_{}$$

Complementary Solution

satisfies $\sum_{k=0}^{N} d_k y[n-k] = 0$

Particular Solution

for given forcing function

$x[n]$

Complementary Solution

- General form of unforced oscillation i.e. system's 'natural modes'
- Assume y_c has form $y_c[n] = \lambda^n$

$$\Rightarrow \sum_{k=0}^{N} d_k \lambda^{n-k} = 0$$

$$\Rightarrow \lambda^{n-N}\left(d_0\lambda^N + d_1\lambda^{N-1} + \ldots + d_{N-1}\lambda + d_N\right) = 0$$

$$\Rightarrow \sum_{k=0}^{N} d_k \lambda^{N-k} = 0$$

Characteristic polynomial of system - depends only on $\{d_k\}$

Complementary Solution

- $\sum_{k=0}^{N} d_k \lambda^{N-k} = 0$ factors into roots λ_i, i.e.

  $$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots = 0$$

- Each/any λ_i satisfies eqn.
- Thus, complementary solution:

  $$y_c[n] = \alpha_1 \lambda_1^{\,n} + \alpha_2 \lambda_2^{\,n} + \alpha_3 \lambda_3^{\,n} + \ldots$$

  Any linear combination will work

  → α_is are free to match initial conditions

Complementary Solution

- Repeated roots in chr. poly:

$$(\lambda - \lambda_1)^L (\lambda - \lambda_2) \ldots = 0$$

$$\Rightarrow y_c[n] = \alpha_1 \lambda_1{}^n + \alpha_2 n \lambda_1{}^n + \alpha_3 n^2 \lambda_1{}^n + \ldots + \alpha_L n^{L-1} \lambda_1{}^n + \ldots$$

- Complex λ_is → sinusoidal $y_c[n] = \alpha_i \lambda_i{}^n$

Particular Solution

- Recall: Total solution $y[n] = y_c[n] + \boxed{y_p[n]}$
- Particular solution reflects input
- 'Modes' usually decay away for large n leaving just $y_p[n]$
- Assume 'form' of $x[n]$, scaled by β:
  e.g. $x[n]$ constant $\rightarrow y_p[n] = \beta$
  $x[n] = \lambda_0^n \rightarrow y_p[n] = \beta \cdot \lambda_0^n \quad (\lambda_0 \notin \lambda_i)$
  or $= \beta\, n^L \lambda_0^n \quad (\lambda_0 \in \lambda_i)$

LCCDE example

$$y[n] + y[n-1] - 6y[n-2] = x[n]$$

- Need input: $x[n] = 8\mu[n]$
- Need initial conditions:
  $y[-1] = 1,\ y[-2] = -1$

LCCDE example

- Complementary solution:

$$y[n] + y[n-1] - 6y[n-2] = 0; \quad y[n] = \lambda^n$$

$$\Rightarrow \lambda^{n-2}\left(\lambda^2 + \lambda - 6\right) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda - 2) = 0 \rightarrow \text{roots } \lambda_1 = -3, \lambda_2 = 2$$

$$\Rightarrow y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n$$

- α_1, α_2 are unknown at this point

LCCDE example

- Particular solution:
- Input $x[n]$ is constant $= 8\mu[n]$

  assume $y_p[n] = \beta$, substitute in:

  $y[n] + y[n-1] - 6y[n-2] = x[n]$ ('large' n)

  $\Rightarrow \beta + \beta - 6\beta = 8\mu[n]$

  $\Rightarrow -4\beta = 8 \Rightarrow \beta = -2$

LCCDE example

- Total solution $y[n] = y_c[n] + y_p[n]$
$$= \alpha_1(-3)^n + \alpha_2(2)^n + \beta$$
- Solve for unknown α_is by substituting *initial conditions* into DE at $n = 0, 1, \ldots$
$$y[n] + y[n-1] - 6y[n-2] = x[n]$$
from ICs
- $\underline{n = 0}$ $y[0] + y[-1] - 6y[-2] = x[0]$
$$\Rightarrow \alpha_1 + \alpha_2 + \beta + 1 + 6 = 8$$
$$\Rightarrow \alpha_1 + \alpha_2 = 3$$

LCCDE example

- $\underline{n = 1}$ $\quad y[1] + y[0] - 6y[-1] = x[1]$
  $\Rightarrow \alpha_1(-3) + \alpha_2(2) + \beta + \alpha_1 + \alpha_2 + \beta - 6 = 8$
  $\Rightarrow -2\alpha_1 + 3\alpha_2 = 18$
- solve: $\alpha_1 = -1.8$, $\alpha_2 = 4.8$
- Hence, system output:
  $y[n] = -1.8(-3)^n + 4.8(2)^n - 2 \quad n \geq 0$
- *Don't* find α_is by solving with ICs at $n = -1,-2$ *(ICs may not reflect natural modes; Mitra3 ex 2.37-8 (4.22-3) is* ***wrong****)*

LCCDE solving summary

- Difference Equation (DE):
  $Ay[n] + By[n\text{-}1] + \ldots = Cx[n] + Dx[n\text{-}1] + \ldots$
  Initial Conditions (ICs): $y[-1] = \ldots$
- DE RHS = 0 with $y[n]=\lambda^n \rightarrow$ roots $\{\lambda_i\}$
  gives complementary soln $y_c[n] = \sum \alpha_i \lambda_i^{\,n}$
- Particular soln: $y_p[n] \sim x[n]$
  solve for $\beta\lambda_0^{\,n}$ "at large n"
- α_is by substituting DE at $n = 0, 1, \ldots$
  ICs for $y[-1], y[-2]$; $y_t=y_c+y_p$ for $y[0], y[1]$

LCCDEs: zero input/zero state

- Alternative approach to solving LCCDEs is to solve two subproblems:
  - $y_{zi}[n]$, response with zero input (just ICs)
  - $y_{zs}[n]$, response with zero state (just $x[n]$)
- Because of linearity, $y[n] = y_{zi}[n] + y_{zs}[n]$
- Both subproblems are 'fully realized'
- But, have to solve for α_is twice (then sum them)

Impulse response of LCCDEs

- Impulse response: $\delta[n] \longrightarrow \boxed{\text{LCCDE}} \longrightarrow h[n]$

  i.e. solve with $x[n] = \delta[n] \rightarrow y[n] = h[n]$ (zero ICs)

- With $x[n] = \delta[n]$, 'form' of $y_p[n] = \beta\delta[n]$

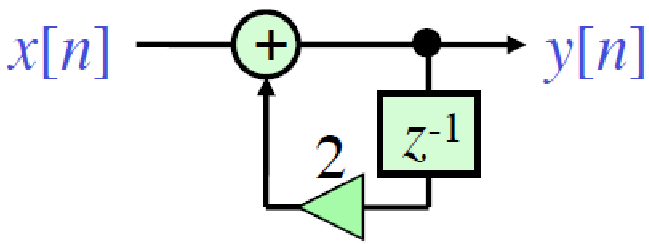
  $\rightarrow$ solve $y[n]$ for $n = 0, 1, 2\ldots$ to find α_is

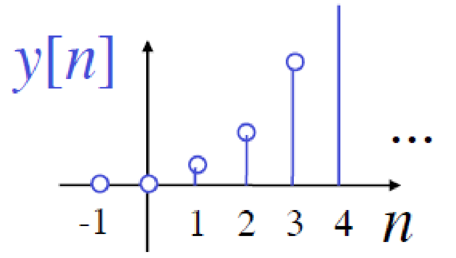
LCCDE IR example

- e.g. $y[n] + y[n-1] - 6y[n-2] = x[n]$
  (from before); $x[n] = \delta[n]$; $y[n] = 0$ for $n<0$
- $y_c[n] = \alpha_1(-3)^n + \alpha_2(2)^n$ $\quad y_p[n] = \beta\delta[n]$
- $\underline{n = 0:}$ $y[0] + \cancel{y[-1]} - 6\cancel{y[-2]} = \cancel{x[0]}$ 1
  $\Rightarrow \alpha_1 + \alpha_2 + \beta = 1$
- $\underline{n = 1:}$ $\alpha_1(-3) + \alpha_2(2) + 1 = 0$
- $\underline{n = 2:}$ $\alpha_1(9) + \alpha_2(4) - 1 - 6 = 0$
  $\Rightarrow \alpha_1 = 0.6,\ \alpha_2 = 0.4,\ \beta = 0$
- thus $h[n] = 0.6(-3)^n + 0.4(2)^n$ $\quad n \geq 0$
  Infinite length

System property: Stability

- Certain systems can be unstable e.g.





Output grows without limit in some conditions

Stability

- Several definitions for stability; we use **Bounded-input, bounded-output (BIBO) stable**
- For *every* bounded input $|x[n]| < B_x \quad \forall n$ output is also subject to a finite bound,

$$|y[n]| < B_y \quad \forall n$$

Stability example

- MA filter: $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right|$$

$$\leq \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]|$$

$$\leq \frac{1}{M} M \cdot B_x \leq B_y$$ → **BIBO Stable**

Stability & LCCDEs

- LCCDE output is of form:

$$y[n] = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n + \ldots + \beta \lambda_0^n + \ldots$$

- αs and βs depend on input & ICs, *but* to be bounded for any input we need $|\lambda| < 1$

4. Correlation

- Correlation ~ identifies similarity between sequences:

Cross correlation of x against y

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$$

"lag"

- Note: $r_{yx}[\ell] = \sum_{n=-\infty}^{\infty} y[n]x[n-\ell]$

call $m = n - \ell$

$$= \sum_{m=-\infty}^{\infty} y[m+\ell]x[m] = r_{xy}[-\ell]$$

Correlation and convolution

- Correlation: $$r_{xy}[n] = \sum_{k=-\infty}^{\infty} x[k]y[k-n]$$
- Convolution: $$x[n] \circledast y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$
- Hence: $r_{xy}[n] = x[n] \circledast y[-n]$

Correlation may be calculated by convolving with time-reversed sequence

Autocorrelation

- Autocorrelation (AC) is correlation of signal with itself:

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x[n-\ell] = r_{xx}[-\ell]$$

- Note: $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \varepsilon_x$ **Energy** of sequence $x[n]$

Correlation maxima

- Note: $r_{xx}[\ell] \le r_{xx}[0] \Rightarrow \left|\dfrac{r_{xx}[\ell]}{r_{xx}[0]}\right| \le 1$
- Similarly: $r_{xy}[\ell] \le \sqrt{\varepsilon_x \varepsilon_y} \Rightarrow \dfrac{r_{xy}[\ell]}{\sqrt{r_{xx}[0] r_{yy}[0]}} \le 1$
- From geometry,

$$\langle \mathbf{xy} \rangle = \sum_i x_i y_i = \underbrace{|\mathbf{x}|}_{\sqrt{\sum x_i^2}} |\mathbf{y}| \cos\theta$$

angle between x and y

- when $\mathbf{x}//\mathbf{y}$, $\cos\theta = 1$, else $\cos\theta < 1$

AC of a periodic sequence

- Sequence of period N: $\tilde{x}[n] = \tilde{x}[n+N]$
- Calculate AC over a finite window:

$$r_{\tilde{x}\tilde{x}}[\ell] = \frac{1}{2M+1}\sum_{n=-M}^{M}\tilde{x}[n]\tilde{x}[n-\ell]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{x}[n-\ell] \qquad \text{if } M >> N$$

AC of a periodic sequence

$$r_{\tilde{x}\tilde{x}}[0] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}^2[n] = P_{\tilde{x}}$$

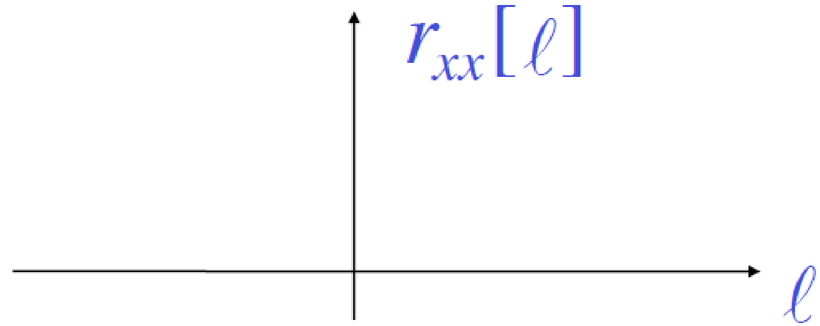
Average energy per sample or **Power** of x

$$r_{\tilde{x}\tilde{x}}[\ell + N] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}[n]\tilde{x}[n-\ell-N] = r_{\tilde{x}\tilde{x}}[\ell]$$

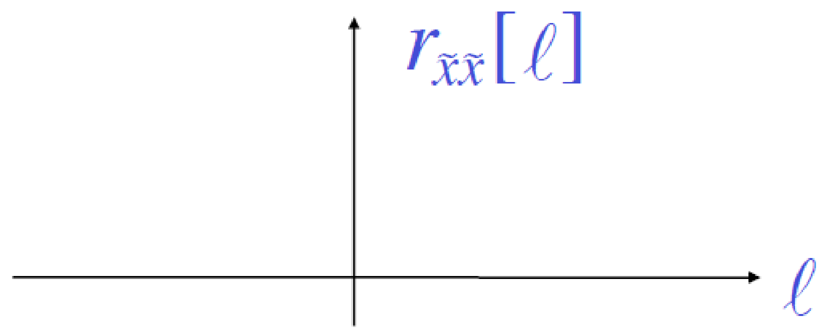
- i.e AC of periodic sequence is periodic

What correlations look like

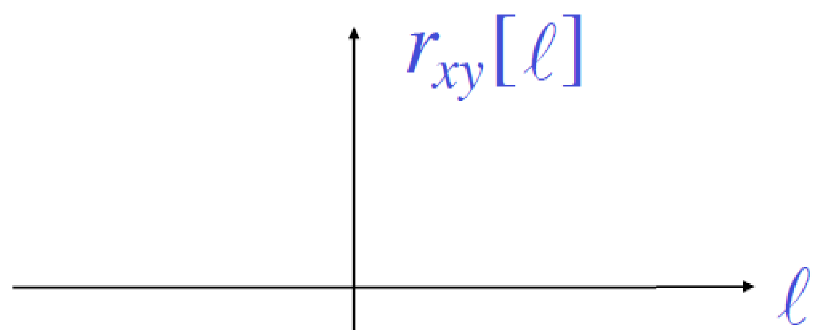
- AC of any $x[n]$



- AC of periodic

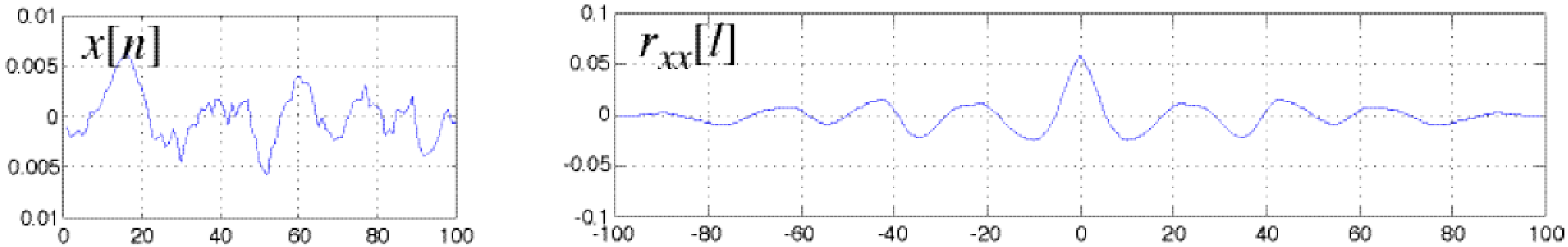


- Cross correlation

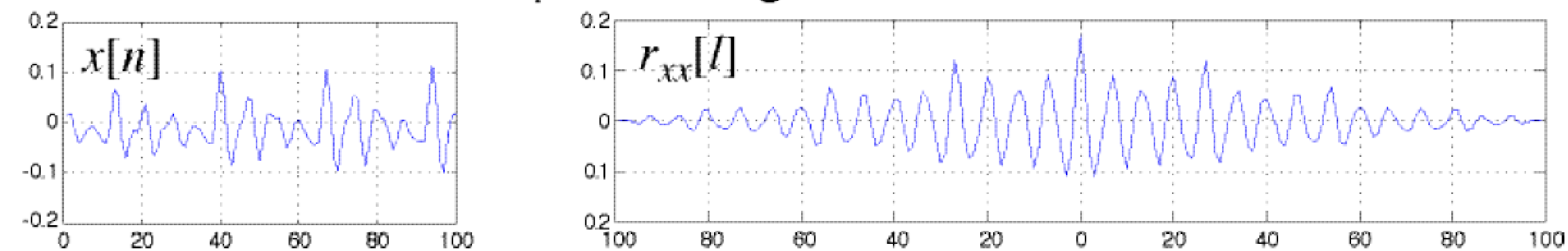


What correlation looks like

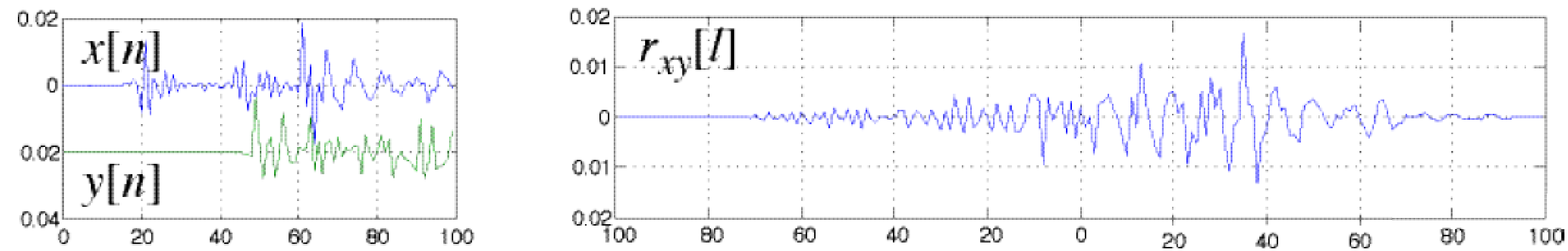
Autocorrelation of generic signal



Autocorrelation of near-periodic signal

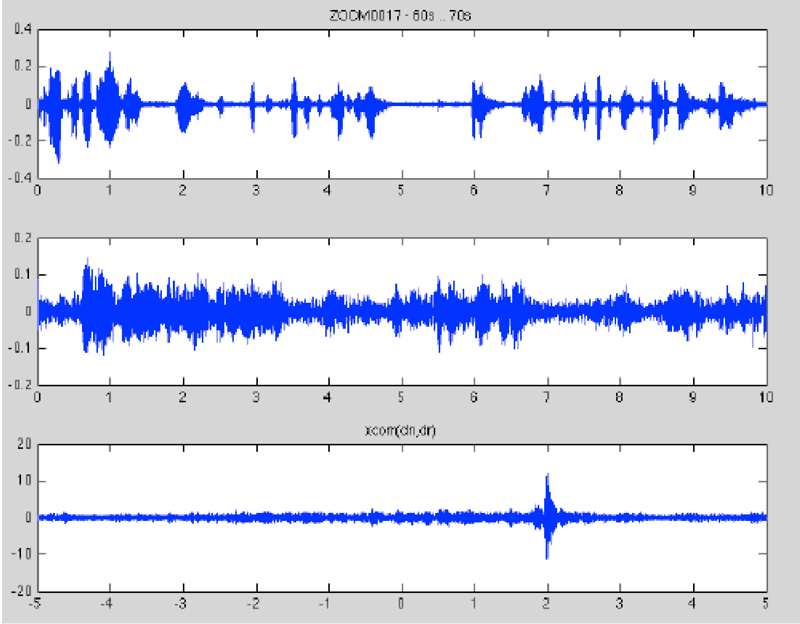


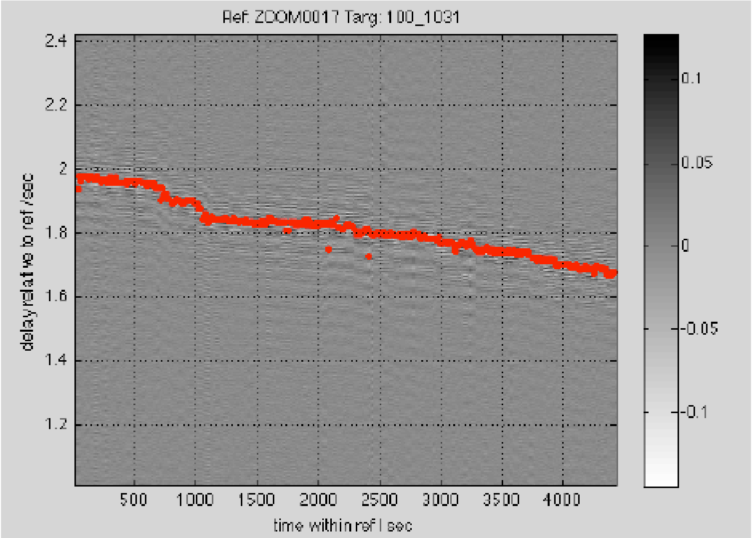
Cross-correlation



Correlation in action

- Close mic vs. video camera mic





- Short-time cross-correlation