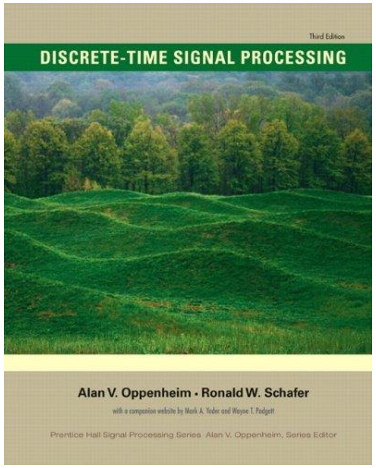


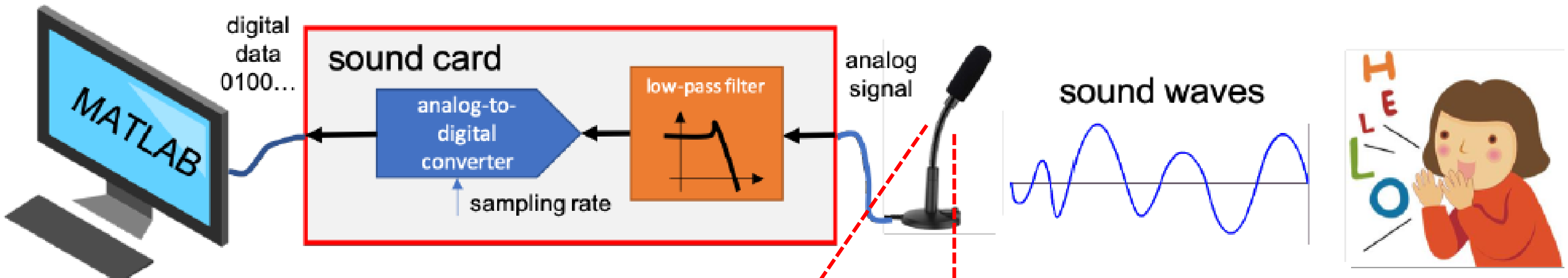
Lecture 01:
Discrete Time Signals

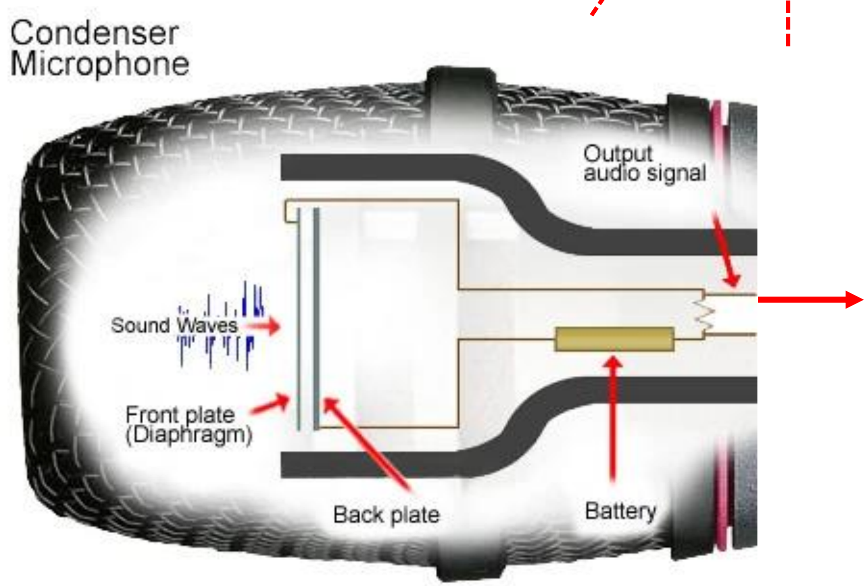
Outlines

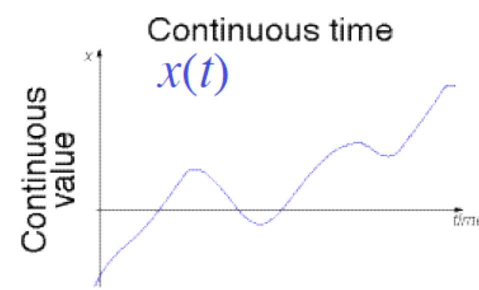
1. Digital Signal Processing
2. Operations on signals
3. Classes of sequences

1. What is signal?

Information-bearing function





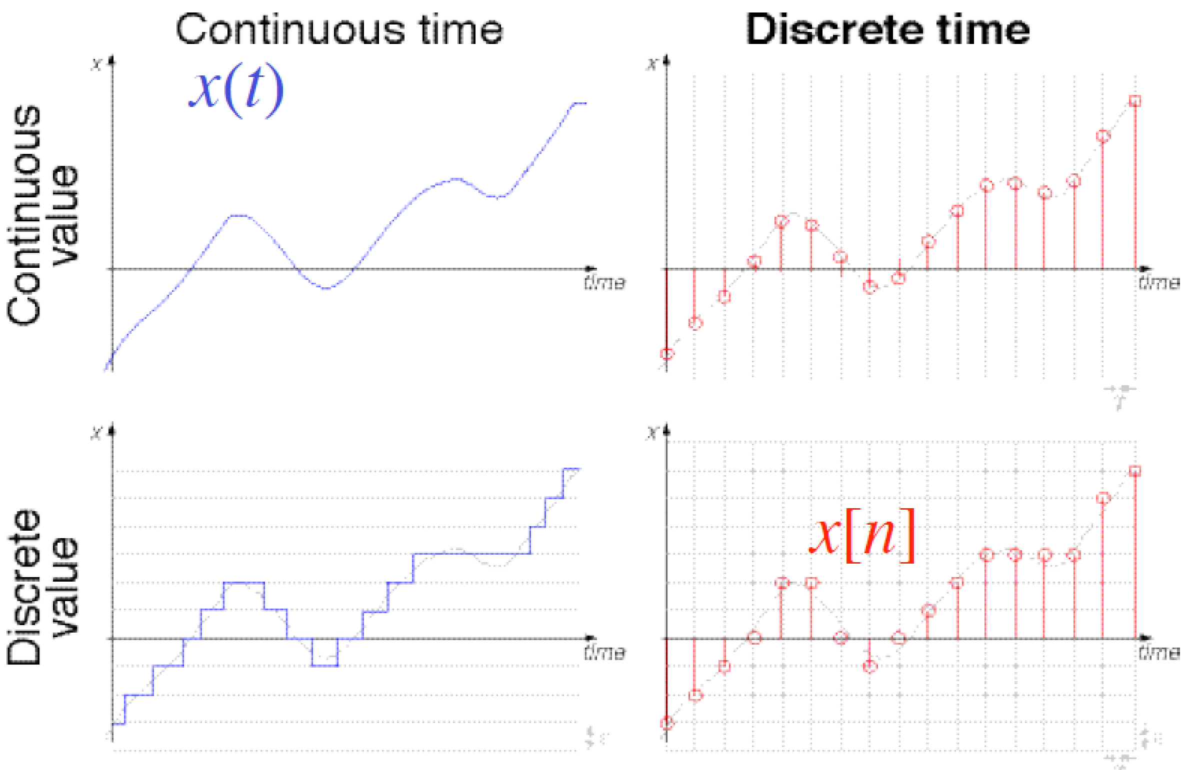


Signal processing

- Modify a signal to extract/enhance/ rearrange the information
- Origin in analog electronics e.g. radar
- Examples…
  - Noise reduction
  - Data compression
  - Representation for recognition/ classification…

Continuous and digital-time signals

- Continuous-time signals = analog signals
- Discrete-time signals = discrete values at discrete time
- Digital signals = both time and amplitude are discrete



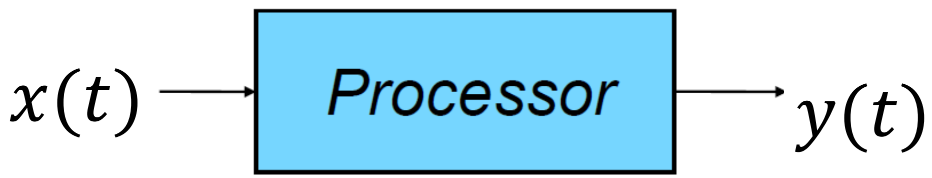
Notations and Examples

- Notations
  - Continuous-time (CT) signal $x(t)$: independent variable t takes continuous values
  - Discrete-time (DT) signal $x[n]$: independent variable n takes only integer values
  - Note: $x(t)$ is used to denote both the "signal" and "the signal value at time t"

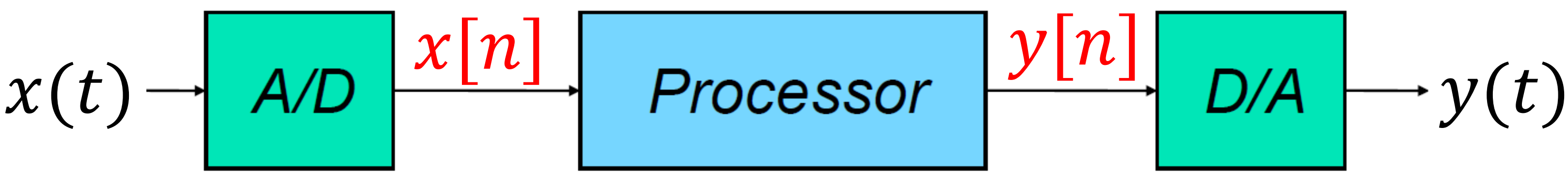
- Examples
  - Electrical signals: Voltages and currents in a circuit.
  - Acoustic signals: Audio and speech signals.
  - Biological signals: ECG, EEG, medical images.
  - Financial signals: Dow Jones indices.

DSP vs. analog SP

- Conventional signal processing:

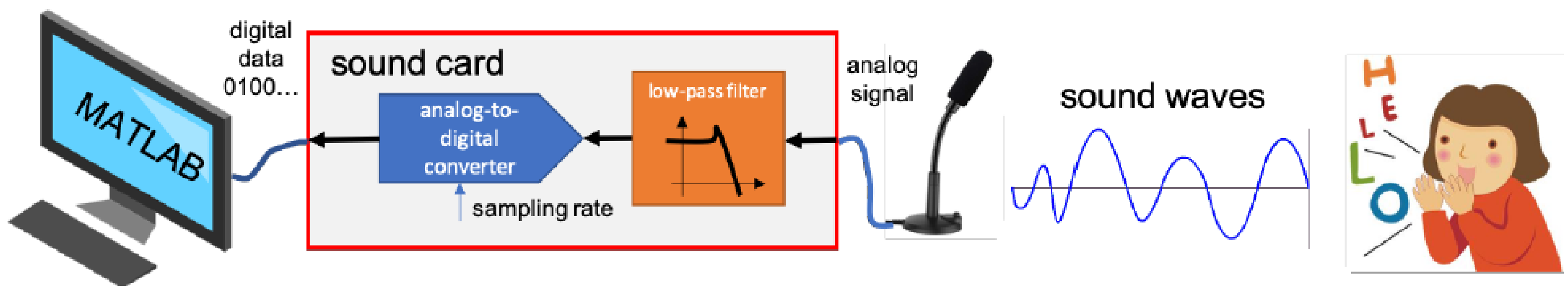


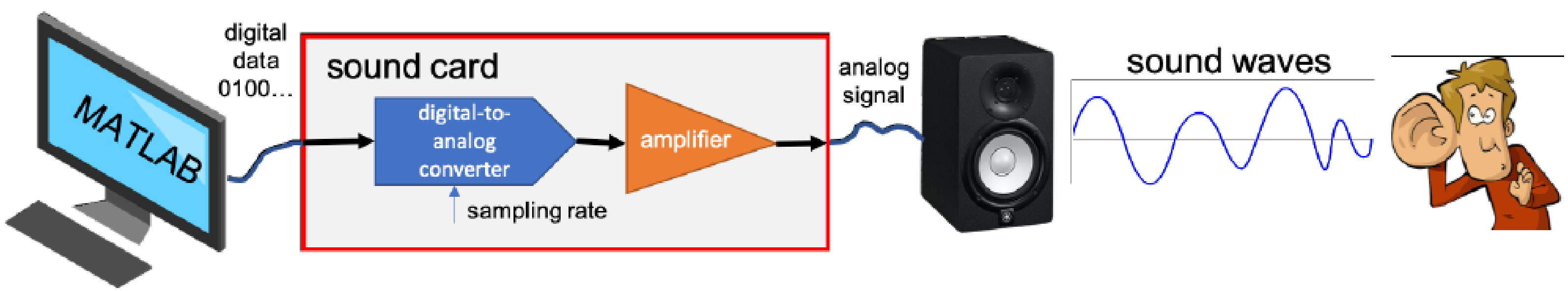
- Digital SP system:



DSP: An Example

Digital processing of an analog signal





Digital processing of an analog signal

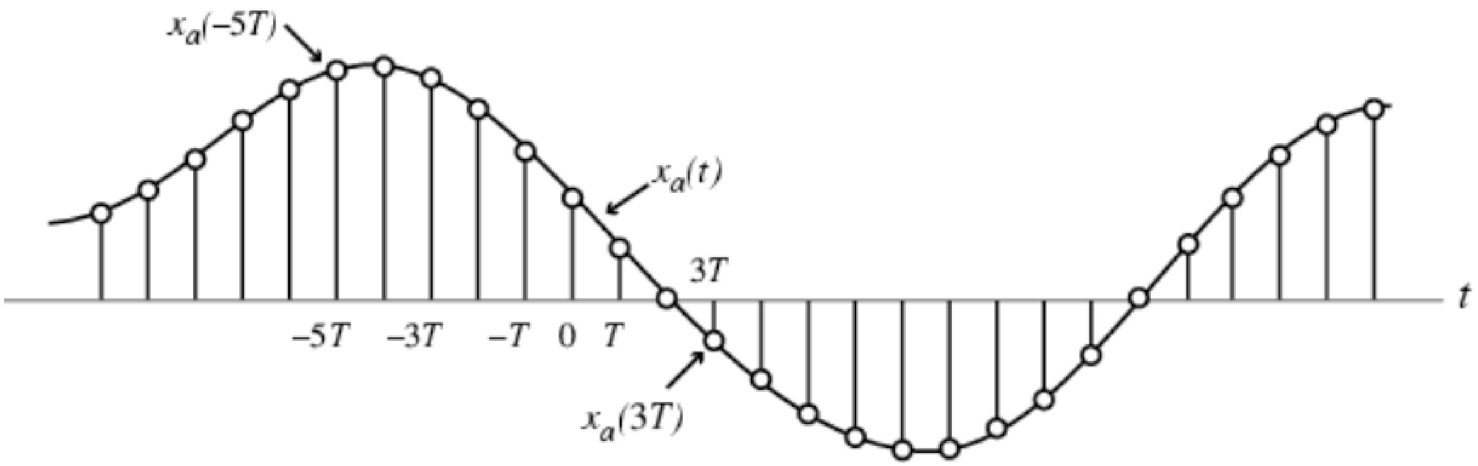
- Historical review:
  - 17th century: several numerical methods were developed to solve physical problem involving continuous variables and functions
  - 1950s: availability of large digital computers
  - 1960s: Researchers began to consider digital signal processing as a separate filed of itself.

Digital vs. Analog

- Advantages:
  - Many …
  - A digital circuit is less sensitive to tolerances of component values and is fairly independent of temperature, aging, and most other external parameters.
  - A digital circuit can be reproduced easily and does not require any adjustment.
  - It is amenable to full integration, and with the recent advantages in VLSI circuits, it is possible to integrate highly complex digital signal processing system on a single chip.
- Disadvantages:
  - System complexity & power consumption: the need for additional pre- and post-prcocessing devises such as A/D and D/A converters
  - Limited range of frequencies available for processing

2. Operations on signals

- Discrete time signal often obtained by sampling a continuous-time signal



- Sequence $\{x[n]\} = x_a(nT)$, $n = \ldots -1, 0, 1, 2 \ldots$
- T= samp. period; $1/T$= samp. frequency

Sequences

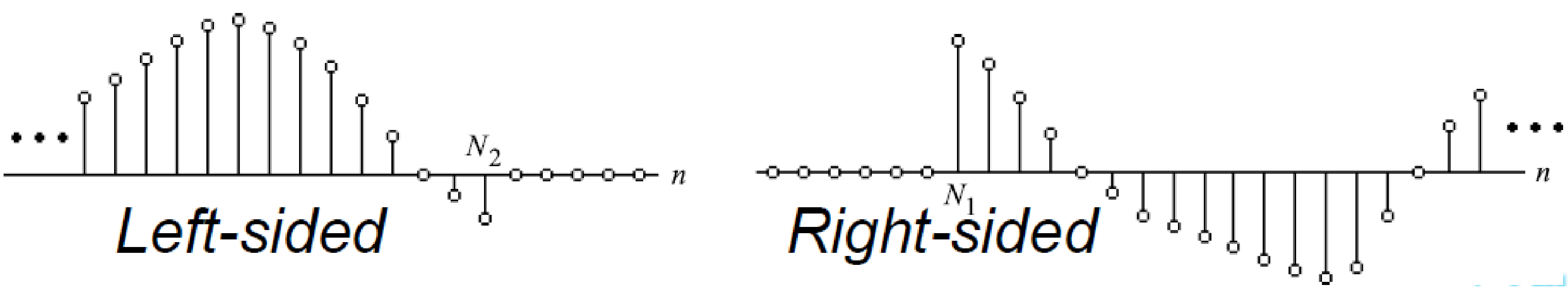
- Can write a sequence by listing values:

$$\{x[n]\} = \{\ldots, -0.2, \underset{\uparrow}{2.2}, 1.1, 0.2, -3.7, 2.9, \ldots\}$$

  - Arrow indicates where n=0
  - Thus, $x[-1] = -0.2,\ x[0] = 2.2,\ x[1] = 1.1,$

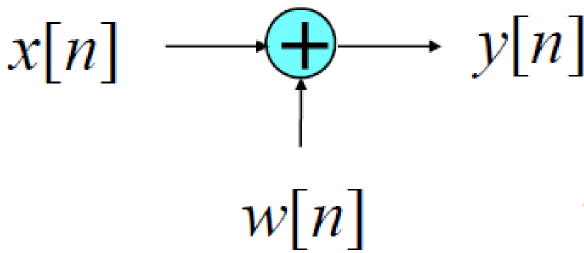
Left- and right-sided

- $x[n]$ may be defined only for certain n:
  - $N_1 \le n \le N_2$: Finite length (length = …)
  - $N_1 \le n$: Right-sided (Causal if $N_1 \ge 0$)
  - $n \le N_2$: Left-sided (Anticausal)
- Can always extend with zero-padding



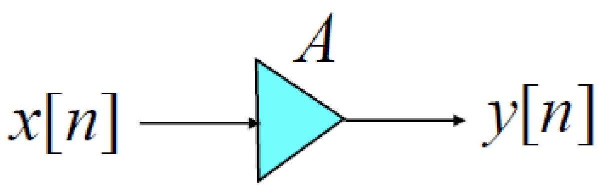
Sequence Operations

- **Addition** operation:
  - Adder



$$y[n] = x[n] + w[n]$$

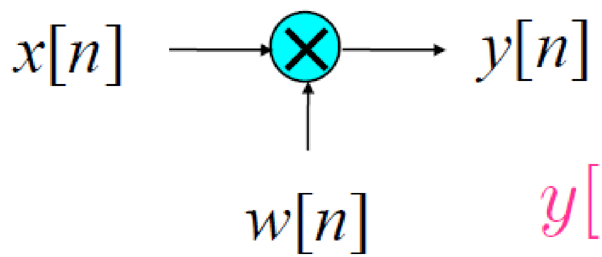
- **Multiplication** operation
  - Multiplier



$$y[n] = A \times x[n]$$

More operations

- **Product (modulation)** operation:
  - Modulator



$$y[n] = x[n] \times w[n]$$

- E.g. Windowing:
  Multiplying an infinite-length sequence by a finite-length window sequence to extract a region

Time shifting

- **Time-shifting** operation: $y[n] = x[n-N]$
  where *N* is an integer
- If *N* > 0, it is **delaying** operation
  - Unit delay $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n]$

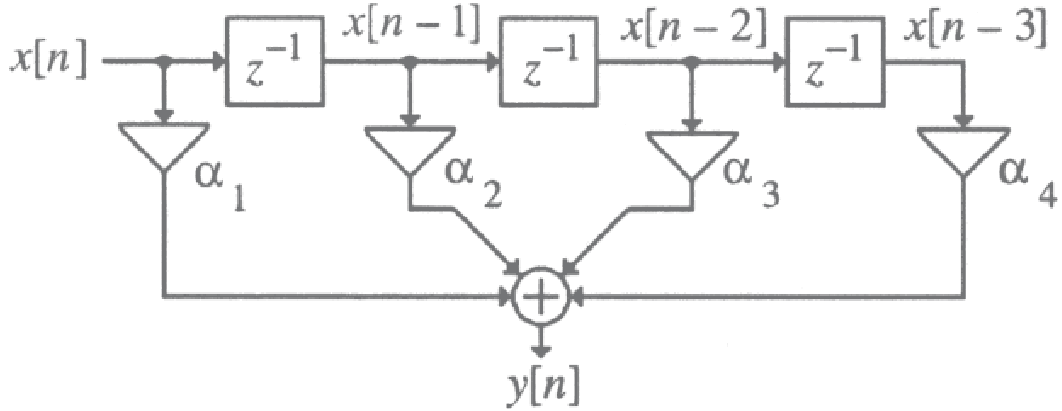
$$y[n] = x[n-1]$$

- If *N* < 0, it is an **advance** operation
  - Unit advance

$$x[n] \longrightarrow \boxed{z} \longrightarrow y[n] \qquad y[n] = x[n+1]$$

Combination of basic operations

- Example



$$y[n] = \alpha_1 x[n] \;+\; \alpha_2 x[n-1]$$
$$+\; \alpha_3 x[n-2] \;+\; \alpha_4 x[n-3]$$

Up- and down-sampling

- Certain operations change the effective sampling rate of sequences by adding or removing samples
- Up-sampling = adding more samples
  = interpolation
- Down-sampling = discarding samples
  = decimation

Down-sampling

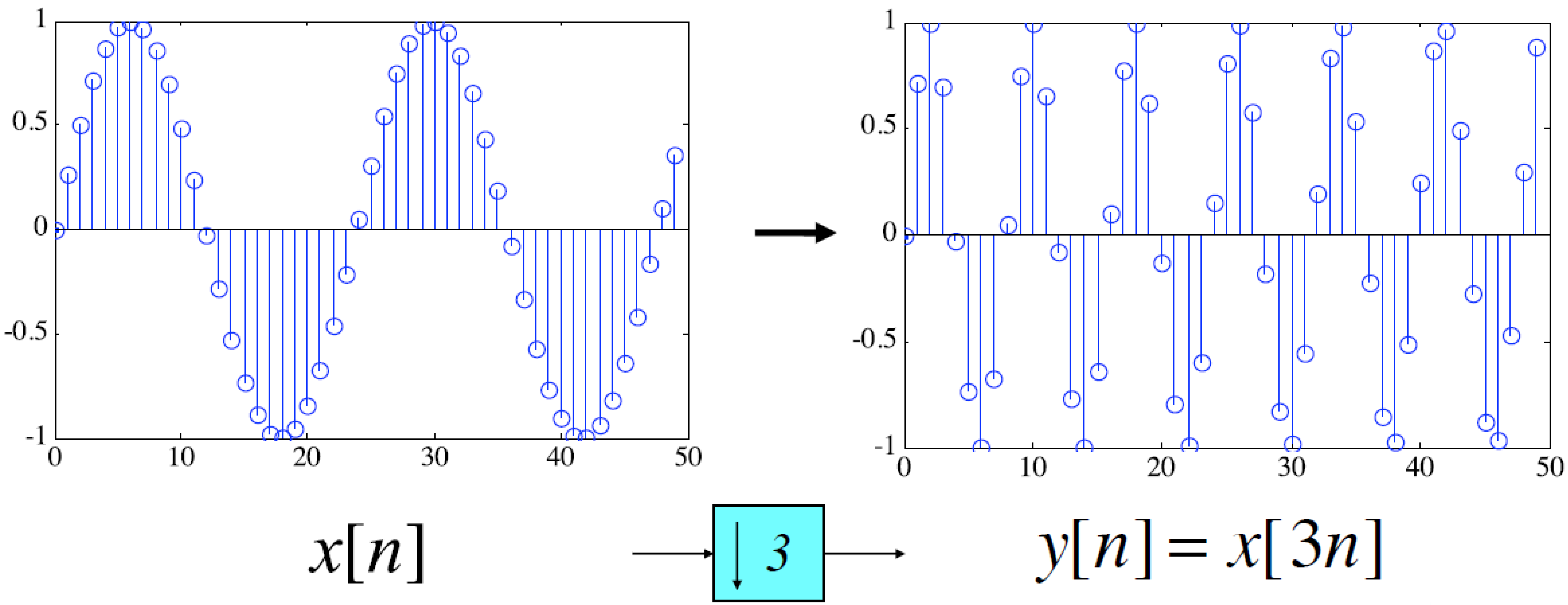
- In **down-sampling** by an integer factor $M > 1$, every M-th sample of the input sequence is kept and M - 1 in-between samples are removed:

$$x_d[n] = x[nM]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

Down-sampling

- An example of down-sampling



$x[n]$

$y[n] = x[3n]$

Up-sampling

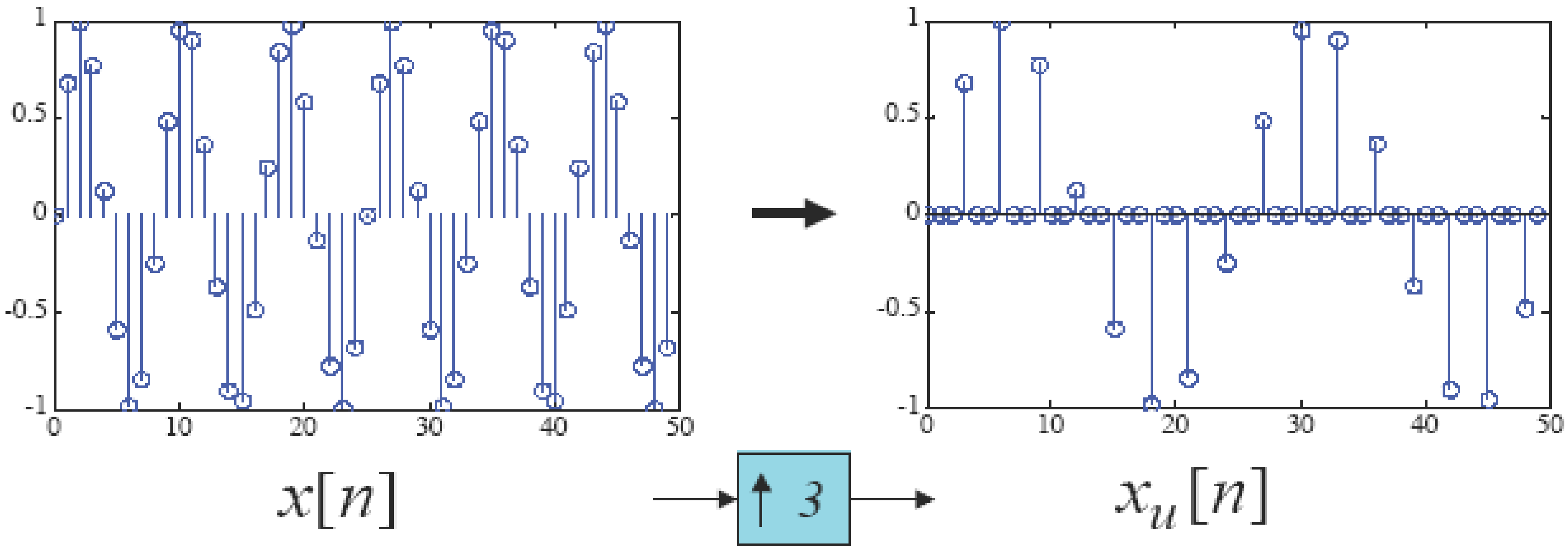
- Up-sampling is the converse of down-sampling: L-1 zero values are inserted between each pair of original values.

$$x_u[n] = \begin{cases} x[n/L] & n = 0, \pm L, 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

Up-sampling

- An example of up-sampling



not inverse of downsampling!

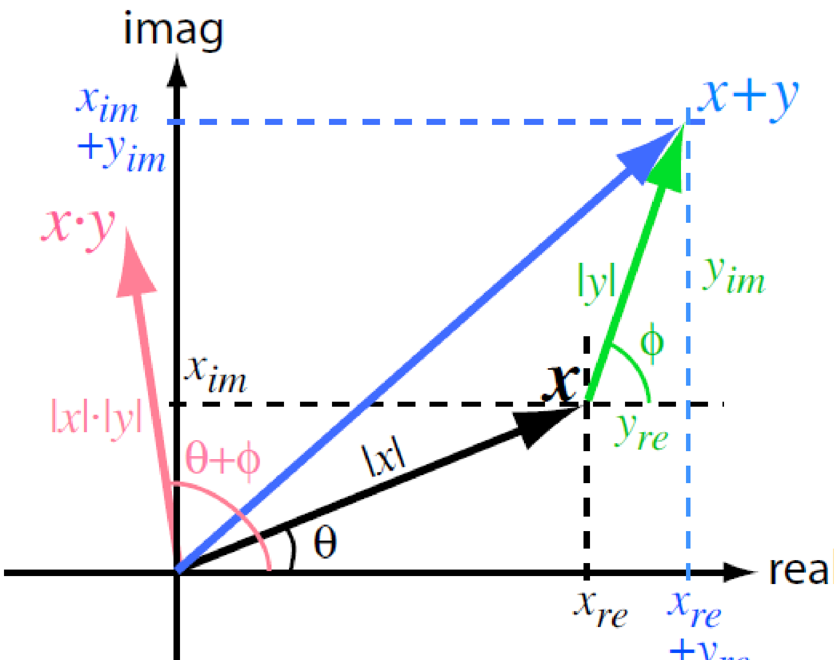
Complex numbers

- .. a mathematical convenience that lead to simple expressions
- A second "imaginary" dimension ($j = \sqrt{-1}$) is added to all values.
- **Rectangular** form: $x = x_{\mathrm{Re}} + j x_{\mathrm{Im}}$

  where **magnitude** $|x| = \sqrt{x_{\mathrm{R}}^2 + x_{\mathrm{Im}}^2}$

  and **phase** $\theta = \tan^{-1}\left(\frac{x_{\mathrm{Im}}}{x_{\mathrm{Re}}}\right)$
- **Polar form:** $x = |x| e^{j\theta} = |x| \cos\theta + j|x| \sin\theta$

Complex math

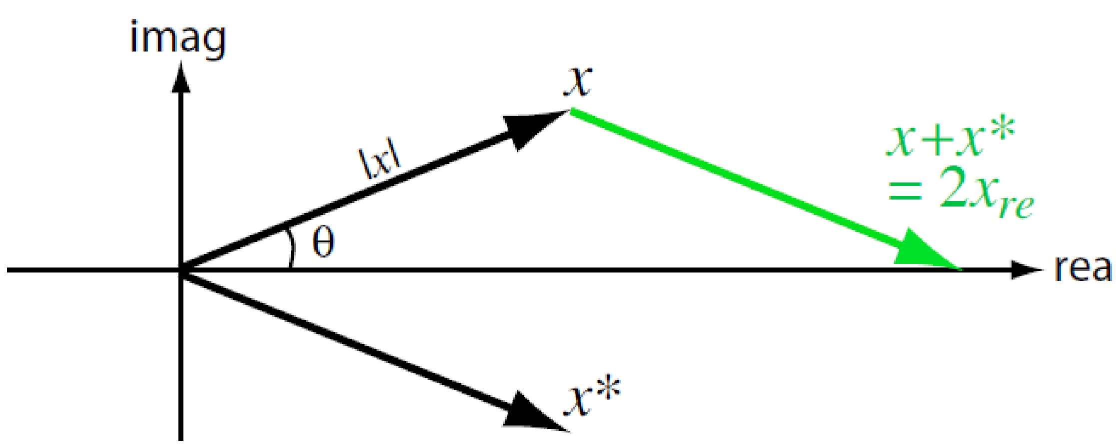


- When **adding**, real and imaginary parts add: $(a+jb) + (c+jd)$
  $= (a+c) + j(b+d)$
- When **multiplying**, magnitudes multiply and phases add:
  $re^{j\theta} \cdot se^{j\varphi} = rse^{j(\theta+\varphi)}$
- Phases modulo 2π

Complex conjugate

- Flips imaginary part / negates phase:
  Conjugate $x^* = x_{re} - j \cdot x_{im} = |x|\, e^{j(-\theta)}$

- Useful in resolving to real quantities:
  $x + x^* = x_{re} + j \cdot x_{im} + x_{re} - j \cdot x_{im} = 2x_{re}$
  $x \cdot x^* = |x|\, e^{j(\theta)}\, |x|\, e^{j(-\theta)} = |x|^2$



3. Classes of sequences

- Useful to define broad categories…
  - Finite/infinite (extent in n)
  - Real/complex:

    $$x[n] = x_{re}[n] + j \cdot x_{im}[n]$$

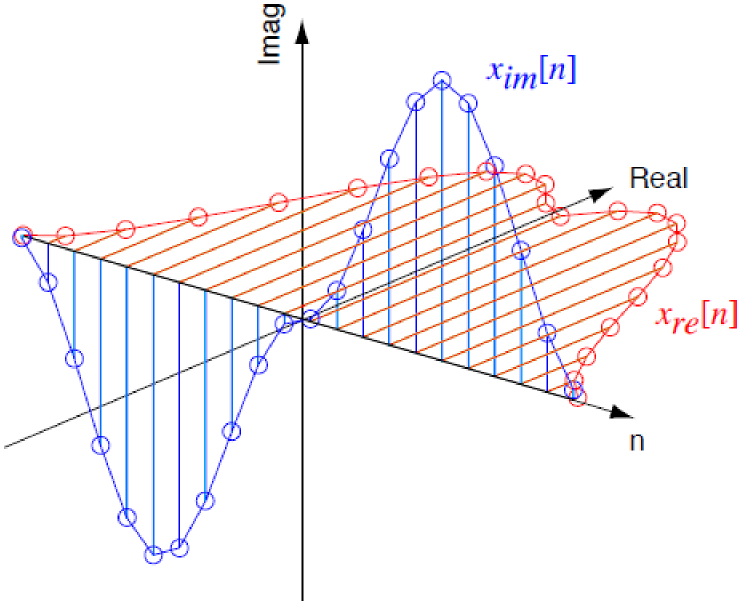
Classification by symmetry

- Conjugate symmetric sequence:

if $x[n] = x_{re}[n] + j \cdot x_{im}[n]$

then $x_{cs}[n] = x_{cs}{}^*[-n]$

$= x_{re}[-n] - j \cdot x_{im}[-n]$



- Conjugate antisymmetric:

$x_{ca}[n] = -x_{ca}{}^*[-n] = -x_{re}[-n] + j \cdot x_{im}[-n]$

Conjugate symmetric decomposition

- Any sequence can be expressed as conjugate symmetric (CS) / antisymmetric (CA) parts:

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where:

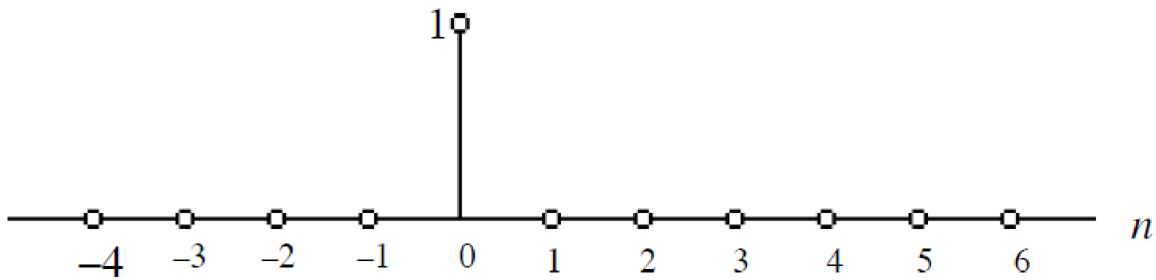
$$x_{cs}[n] = {}^1/_2(x[n] + x^*[-n]) \qquad = x_{cs}^*[-n]$$

$$x_{ca}[n] = {}^1/_2(x[n] - x^*[-n]) \qquad = -x_{ca}^*[-n]$$

- When signals are real,
CS → Even ($x_{re}[n] = x_{re}[-n]$), CA → Odd

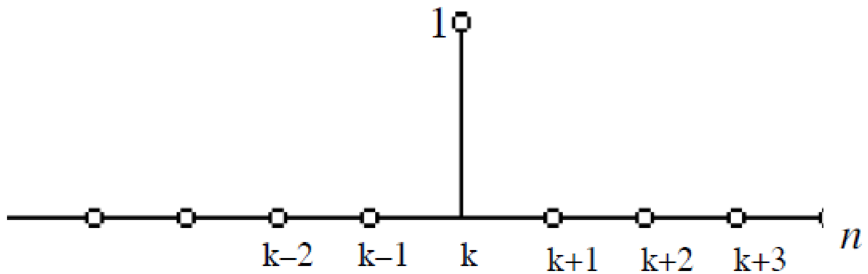
Basic sequences

- Unit sample sequence: $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- Shift in time:

  $\delta[n - k]$

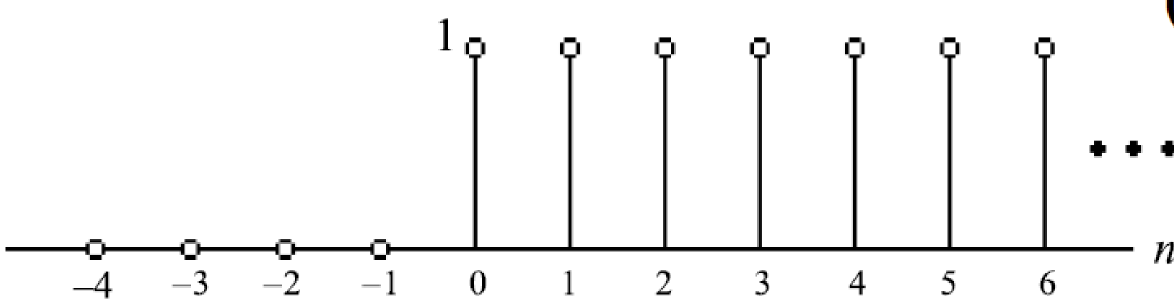


- Can express any sequence with δ:

$$\{\underset{\uparrow}{\alpha_0}, \alpha_1, \alpha_2 ..\} = \alpha_0 \delta[n] + \alpha_1 \delta[n\text{-}1] + \alpha_2 \delta[n\text{-}2] ..$$

More basic sequences

- Unit step sequence: $\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



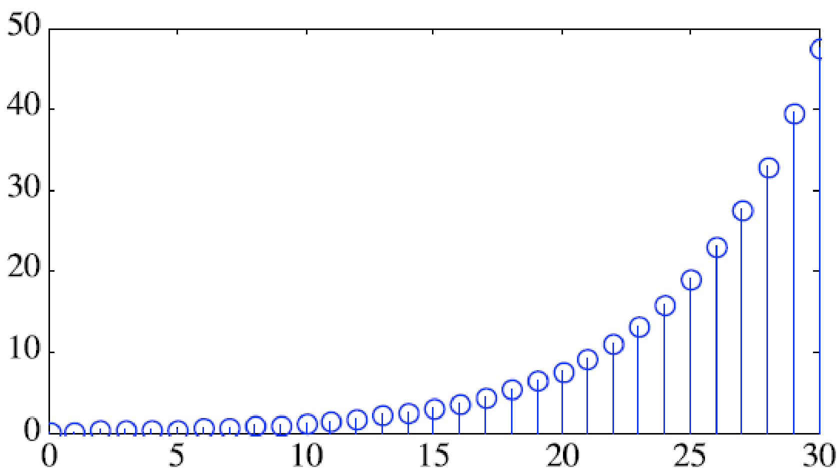
- Relate to unit sample:

$$\delta[n] = \mu[n] - \mu[n-1]$$

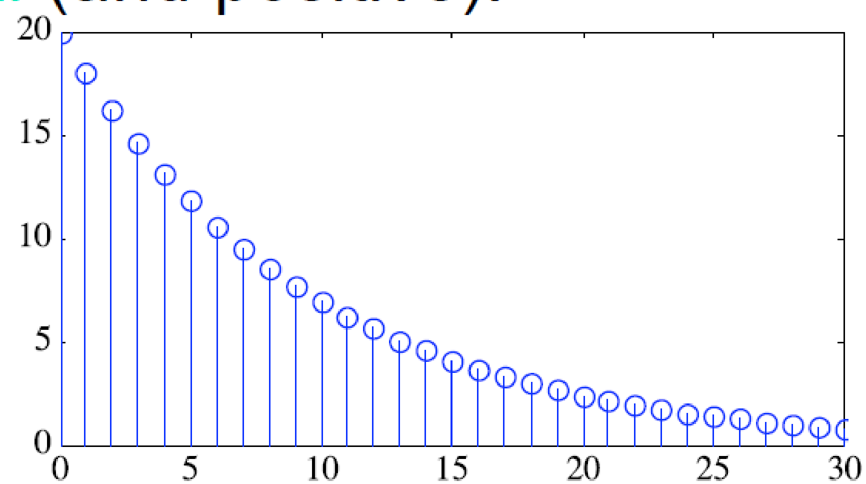
$$\mu[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Exponential sequences

- Exponential sequences are *eigenfunctions* of LTI systems
- General form: $x[n] = A \cdot \alpha^n$
  - If A and α are *real* (and positive):



$|\alpha| > 1$



$|\alpha| < 1$

Complex exponentials

$$x[n] = A \cdot \alpha^n$$

- Constants A, α can be complex :

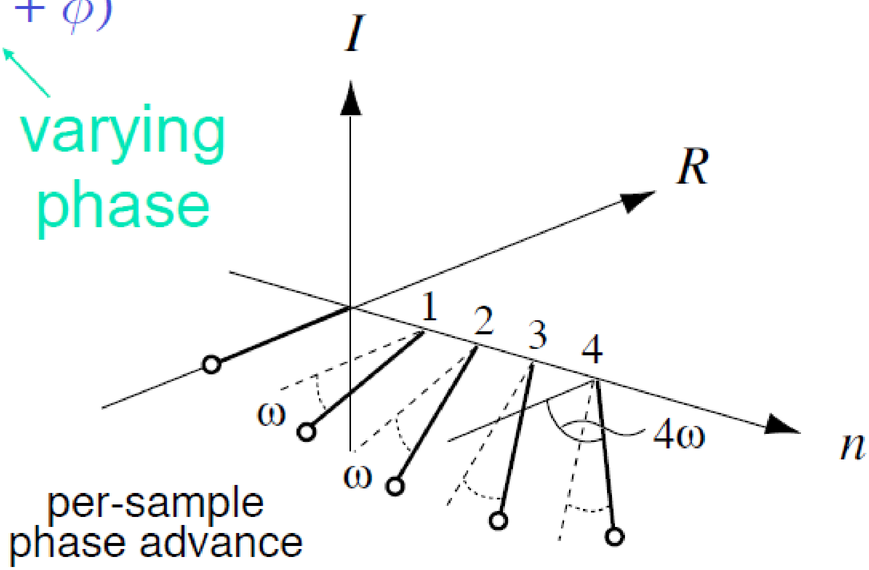
$A = |A| e^{j\phi}$; $\alpha = e^{(\sigma + j\omega)}$

$\rightarrow x[n] = |A|\, e^{\sigma n}\, e^{j(\omega n + \phi)}$

scale

varying magnitude

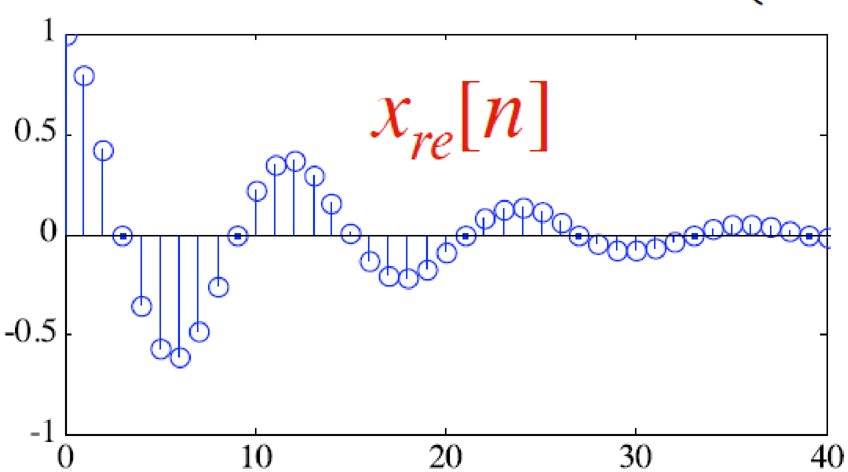
varying phase

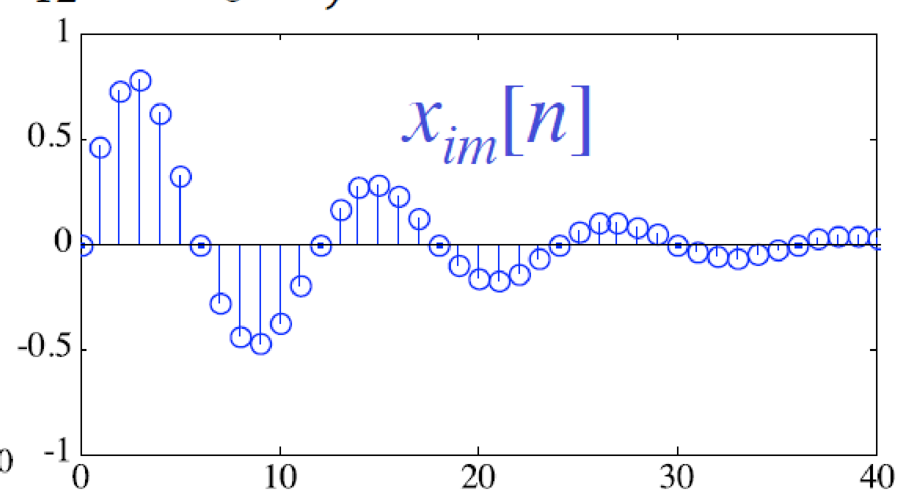


Complex exponentials

- Complex exponential sequence can 'project down' onto real & imaginary axes to give sinusoidal sequences

$$x[n] = \exp\left\{\left(-\tfrac{1}{12} + j\tfrac{\pi}{6}\right)n\right\} \qquad e^{j\theta} = \cos\theta + j\sin\theta$$



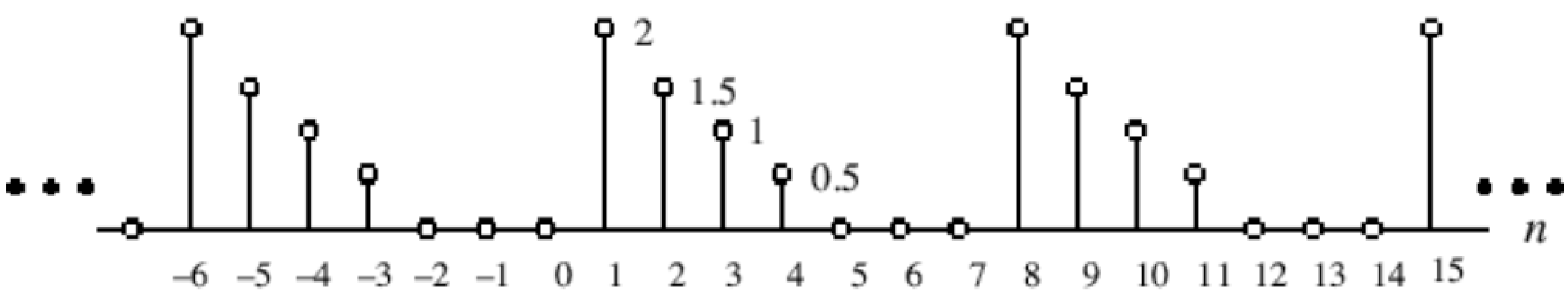


$x_{re}[n] = e^{-n/12}\cos(\pi n/6)$ $\quad x_{im}[n] = e^{-n/12}\sin(\pi n/6)$

Periodic sequences

- A sequence $\tilde{x}[n]$ satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$, is called a **periodic sequence** with a **period** N where N is a positive integer and k is any integer.

  Smallest value of N satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called the **fundamental period**



Periodic exponentials

- Sinusoidal sequence $A\cos(\omega_o n + \phi)$ and complex exponential sequence $B\exp(j\omega_o n)$ are periodic sequences of period N **only if** $\omega_o N = 2\pi r$ with N & r positive integers
- Smallest value of N satisfying $\omega_o N = 2\pi r$ is the **fundamental period** of the sequence
- $r = 1$ → one sinusoid cycle per N samples
  $r > 1$ → r cycles per N samples

Symmetry of periodic sequences

- An N-point finite-length sequence $x_f[n]$ defines a periodic sequence:

$$x[n] = x_f[\langle n \rangle_N]$$

"n modulo N" $\langle n \rangle_N = n + rN$

s.t. $0 \leq \langle n \rangle_N < N,\ r \in \mathbb{Z}$

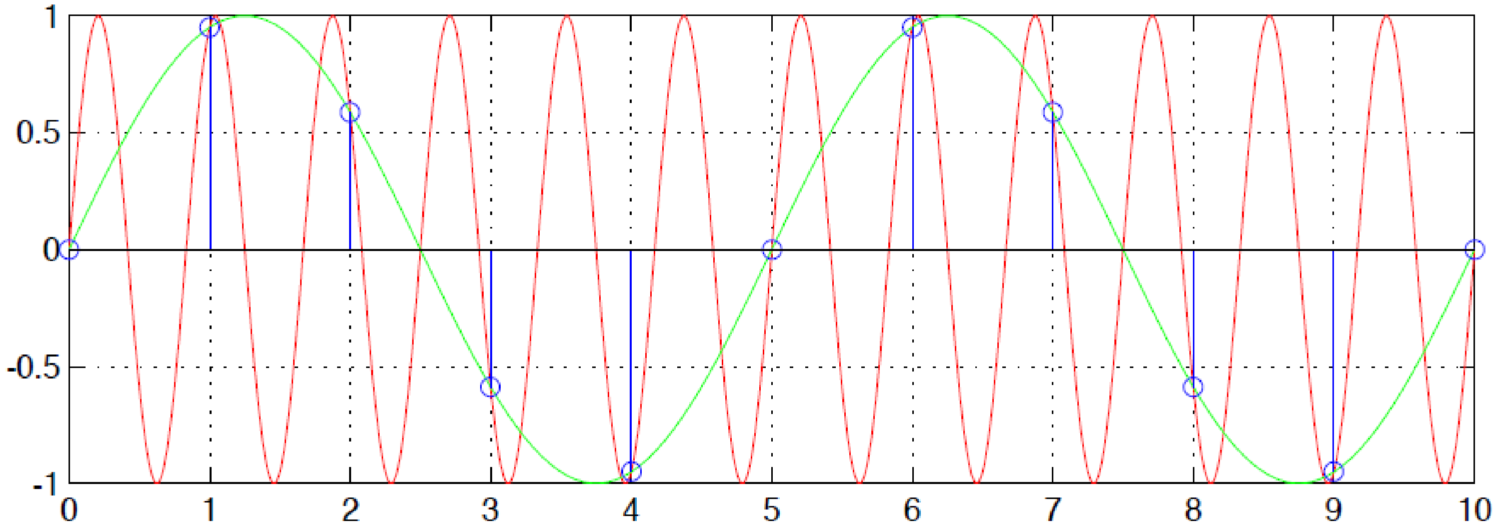
- Symmetry of $x_f[n]$ is not defined because $x_f[n]$ is undefined for $n < 0$
- Define Periodic Conjugate Symmetric:

$$x_{pcs}[n] = 1/2\left(x[n] + x^*[\langle -n \rangle_N]\right)$$

$$= 1/2\left(x_f[n] + x_f^*[N-n]\right) \quad 1 \leq n < N$$

Sampling sinusoids

- Sampling a sinusoid is *ambiguous*:



$$x_1[n] = \sin(\omega_0 n)$$

$$x_2[n] = \sin((\omega_0 + 2\pi r)n) = \sin(\omega_0 n) = x_1[n]$$

Aliasing

- E.g. for $\cos(\omega n)$, $\omega = 2\pi r \pm \omega_0$

  all (integer) r appear the same after sampling
- We say that a larger ω appears

  aliased to a lower frequency
- **Principal value** for discrete-time frequency: $0 \le \omega_0 \le \pi$

  (i.e. less than 1/2 cycle per sample)