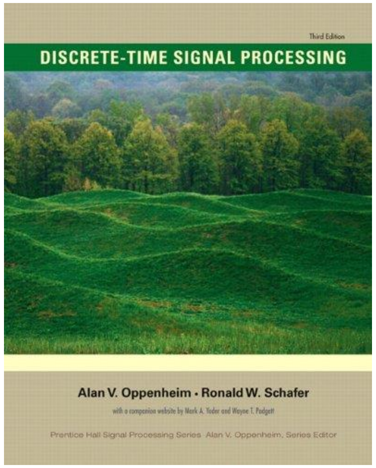


Lecture 14:

Filter Design: FIR

Outlines

- Windowed Impulse Response
- Window Shapes
- Design by Iterative Optimization

1. FIR Filter Design

- FIR filters
  - no poles (just zeros)
  - no precedent in analog filter design
- Approaches
  - windowing ideal impulse response
  - iterative (computer-aided) design

Least Integral-Squared Error

- Given desired FR $H_d(e^{j\omega})$, what is the **best** finite $h_t[n]$ to approximate it?

best in what sense?

- Can try to minimize Integral Squared Error (ISE) of frequency responses:

$$\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d\left(e^{j\omega}\right) - H_t\left(e^{j\omega}\right)\right|^2 d\omega$$

$= \mathrm{DTFT}\{h_t[n]\}$

Least Integral-Squared Error

- Ideal IR is $h_d[n] = \mathrm{IDTFT}\{H_d(e^{j\omega})\}$, (usually infinite-extent)
- By Parseval, ISE $\phi = \sum_{n=-\infty}^{\infty} \left| h_d[n] - h_t[n] \right|^2$
- But: $h_t[n]$ only exists for n = -$M..M$,

$$\Rightarrow \phi = \underbrace{\sum_{n=-M}^{M} \left| h_d[n] - h_t[n] \right|^2}_{\text{minimized by making } h_t[n] = h_d[n],\ -M \le n \le M} + \underbrace{\sum_{n<-M,\, n>M} \left| h_d[n] \right|^2}_{\text{not altered by } h_t[n]}$$

minimized by making
$h_t[n] = h_d[n]$, $-M \le n \le M$

not altered by $h_t[n]$

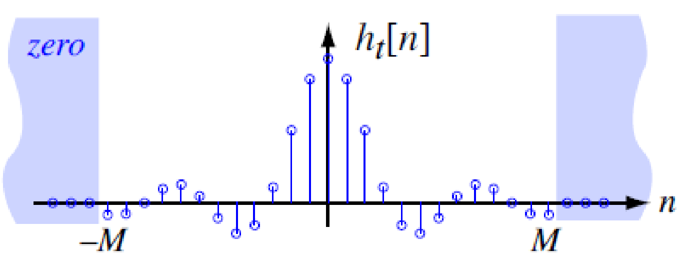
Least Integral-Squared Error

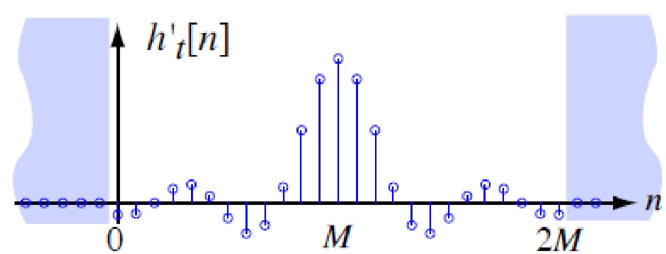
- Thus, minimum mean-squared error approximation in $2M+1$ point FIR is truncated IDTFT:

$$h_t[n] = \begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

- Make causal by delaying by M points

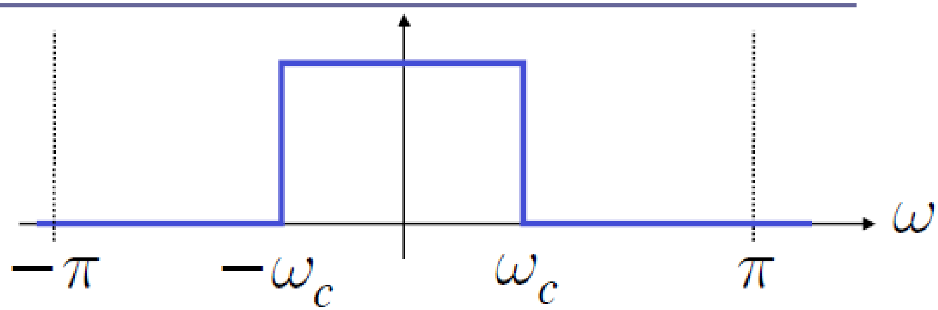
$\rightarrow h'_t[n] = 0$ for $n < 0$





Approximating Ideal Filters

- From topic 6, ideal lowpass has:

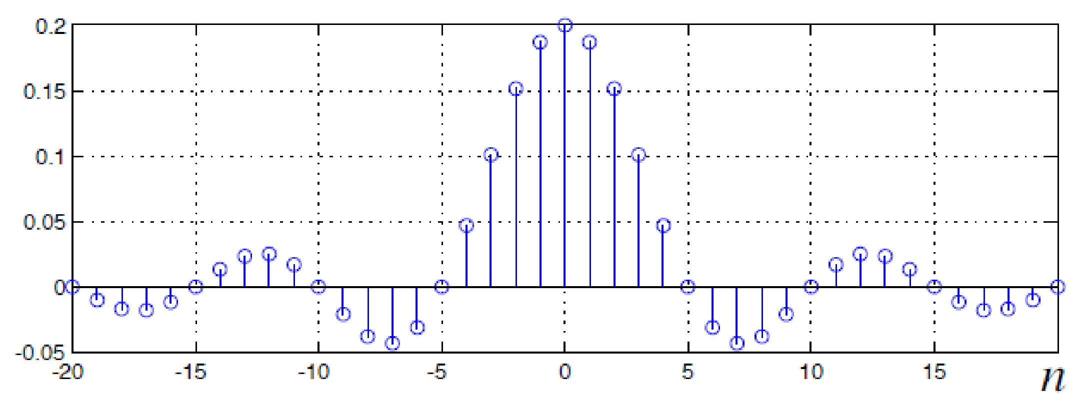


$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

and:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$$

(doubly infinite)

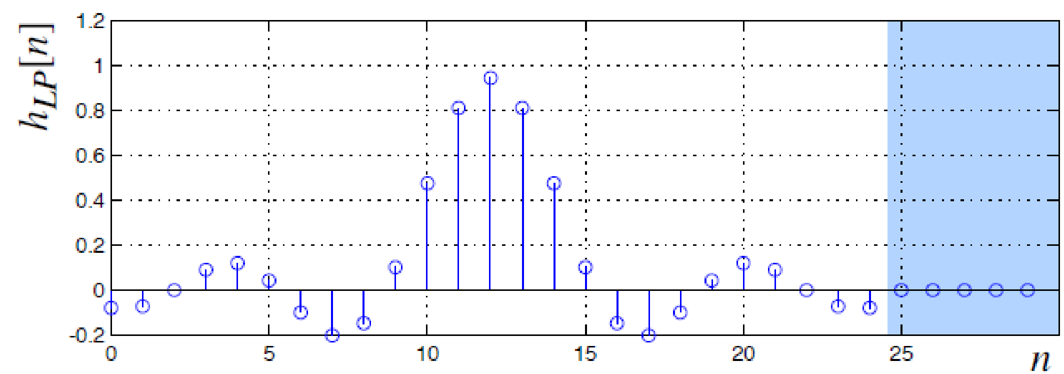


Approximating Ideal Filters

- Thus, minimum ISE causal approximation to an ideal lowpass

$$\hat{h}_{LP}[n] = \begin{cases} \frac{\sin \omega_c (n-M)}{\pi (n-M)} & 0 \leq n \leq 2M \\ 0 & \text{otherwise} \end{cases}$$

Causal shift



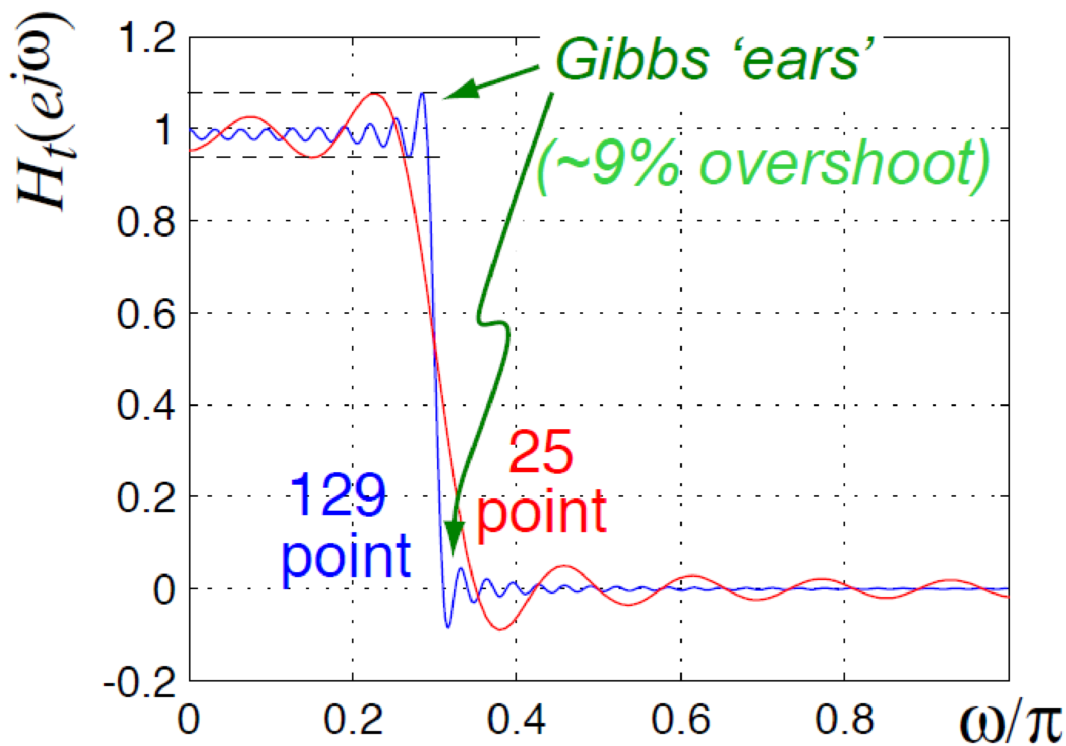
2M+1 points

Gibbs Phenomenon

- Truncated ideal filters have *Gibbs' Ears*:

Increasing filter length
→ narrower ears
(reduces ISE)
but *height the same*

→ ***not*** *optimal by*
minimax criterion

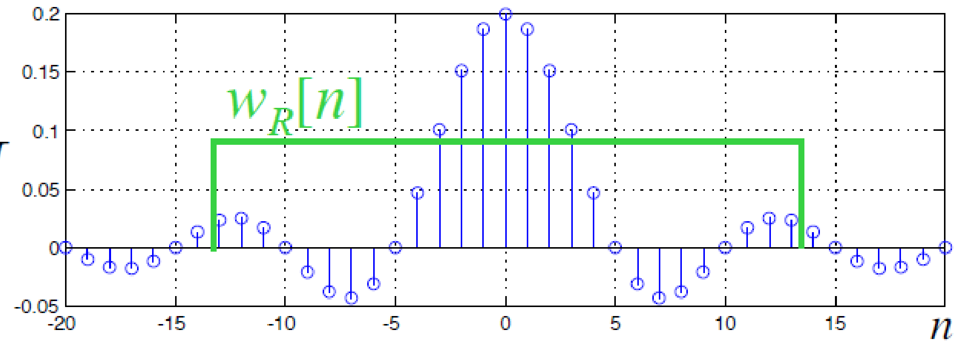


Where Gibbs comes from

- Truncation of $h_d[n]$ to $2M+1$ points is multiplication by a rectangular window:

$$h_t[n] = h_d[n] \cdot w_R[n]$$

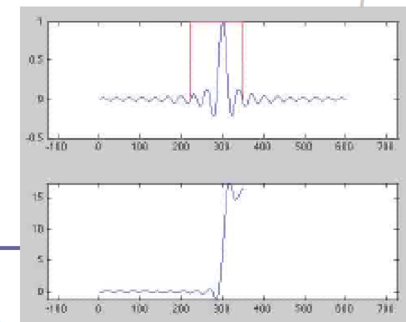
$$w_R[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases}$$



- Multiplication in time domain is convolution in frequency domain:

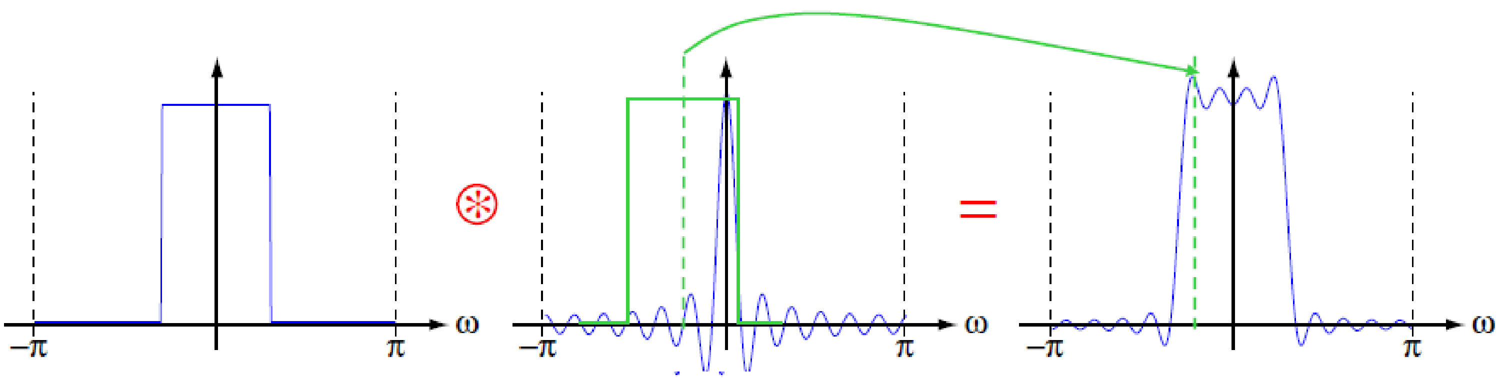
$$g[n] \cdot h[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta$$

convanim.mp4



Where Gibbs comes from

- Thus, FR of truncated response is convolution of ideal FR and FR of rectangular window (pd.sinc):



$H_d(e^{j\omega})$

$\mathrm{DTFT}\{w_R[n]\}$

periodic sinc...

$$\frac{\sin Nx}{\sin x}$$

$H_t(e^{j\omega})$

Where Gibbs comes from

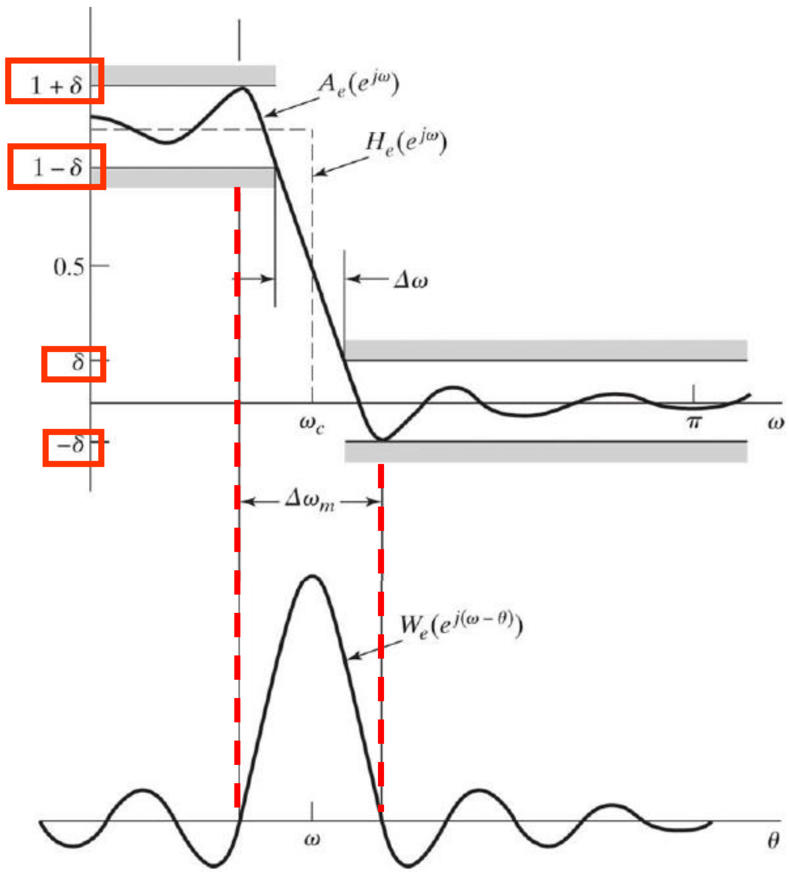
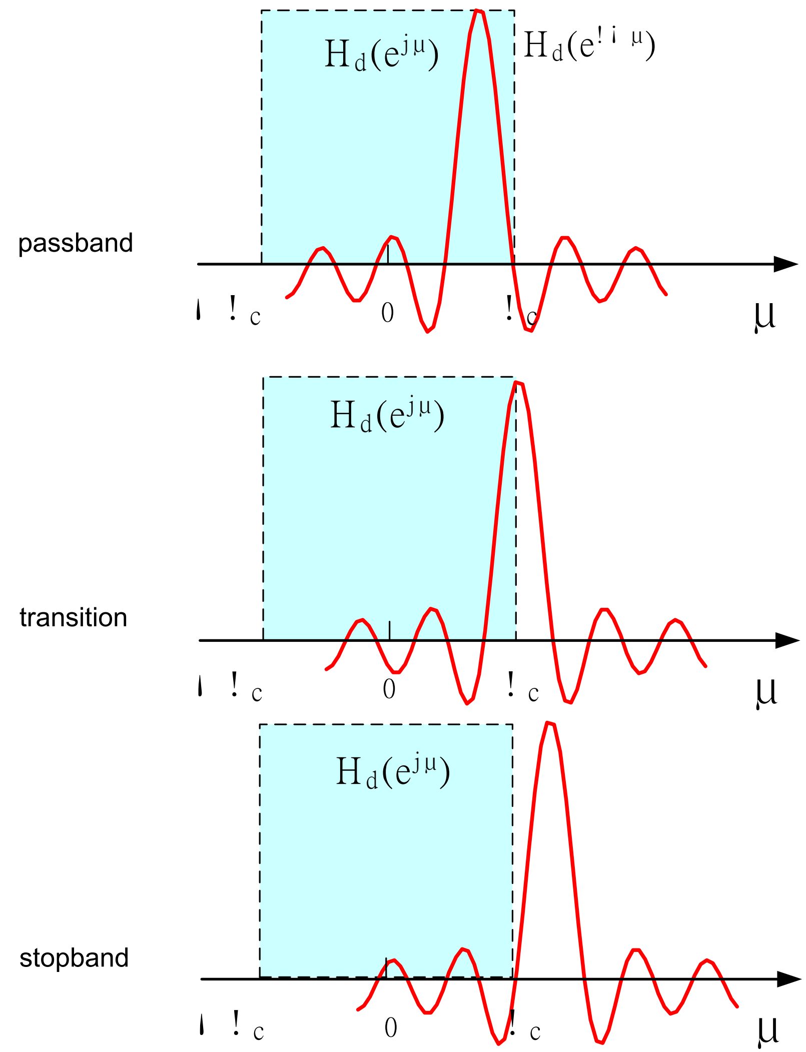


Illustration of type of approximation obtained at a discontinuity of ideal frequency response.



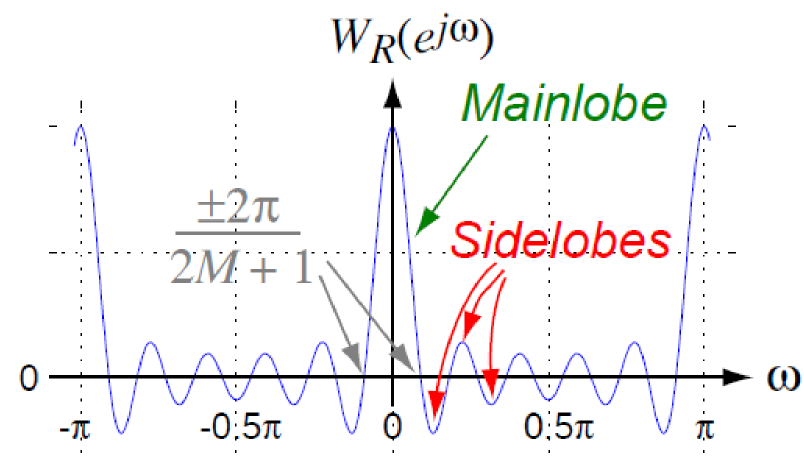
Where Gibbs comes from

- Rectangular window:

$$w_R[n] = \begin{cases} 1 & -M \le n \le M \\ 0 & \text{otherwise} \end{cases} \Rightarrow \begin{aligned} W_R(e^{j\omega}) &= \sum_{n=-M}^{M} e^{-j\omega n} \\ &= \frac{\sin(2M+1)\frac{\omega}{2}}{\sin\frac{\omega}{2}} \end{aligned}$$

- Mainlobe width ($\propto 1/L$) determines transition band
- Sidelobe *height* determines ripples

≈ *invariant with length*



"periodic sinc"

2. Window Shapes for Filters

- Windowing (infinite) ideal response
  → FIR filter: $h_t[n] = h_d[n] \cdot w[n]$
- Rectangular window has best ISE error
- Other “tapered windows” vary in:
  - mainlobe → transition band width
  - sidelobes → size of ripples near transition
- Variety of ‘classic’ windows...

Window Shapes for FIR Filters

- Rectangular:

$$w[n] = 1 \quad -M \le n \le M$$

- Hann:

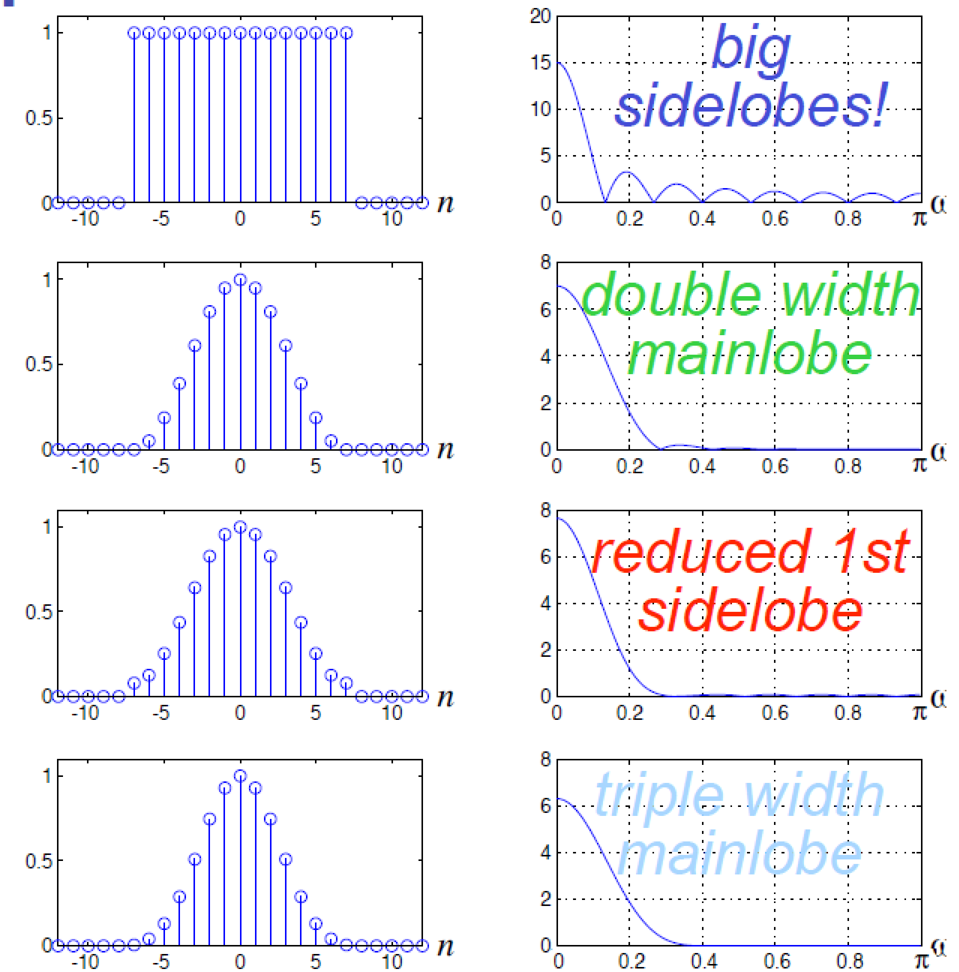
$$0.5 + 0.5\cos(2\pi \tfrac{n}{2M+1})$$

- Hamming:

$$0.54 + 0.46\cos(2\pi \tfrac{n}{2M+1})$$

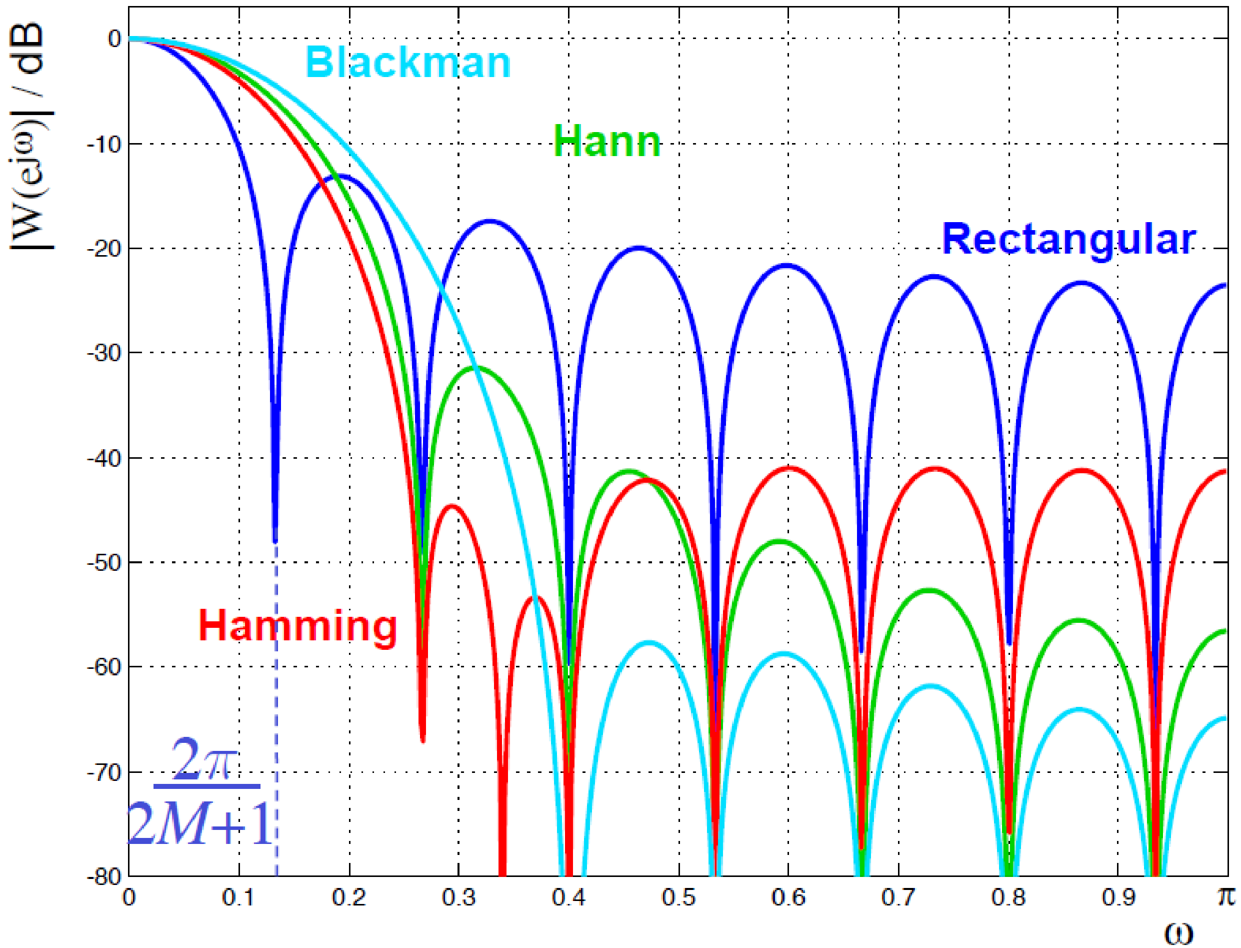
- Blackman:

$$0.42 + 0.5\cos(2\pi \tfrac{n}{2M+1}) + 0.08\cos(2\pi \tfrac{2n}{2M+1})$$



Window Shapes for FIR Filters

- Comparison on dB scale:



Adjustable Windows

- So far, discrete main-sidelobe tradeoffs..
- Kaiser window = parametric, continuous tradeoff:

$$w[n] = \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)} \qquad -M \le n \le M$$

modified zero-order Bessel function

- Empirically, for min. SB atten. of α dB:

$$\beta = \begin{cases} 0.11(\alpha - 8.7) & \alpha > 50 \\ 0.58(\alpha - 21)^{0.4} + 0.08(\alpha - 21) & 21 \le \alpha \le 50 \\ 0 & \alpha < 21 \end{cases}$$

required order

$$N = \frac{\alpha - 8}{2.3\Delta\omega}$$

transition width

Commonly used Windows

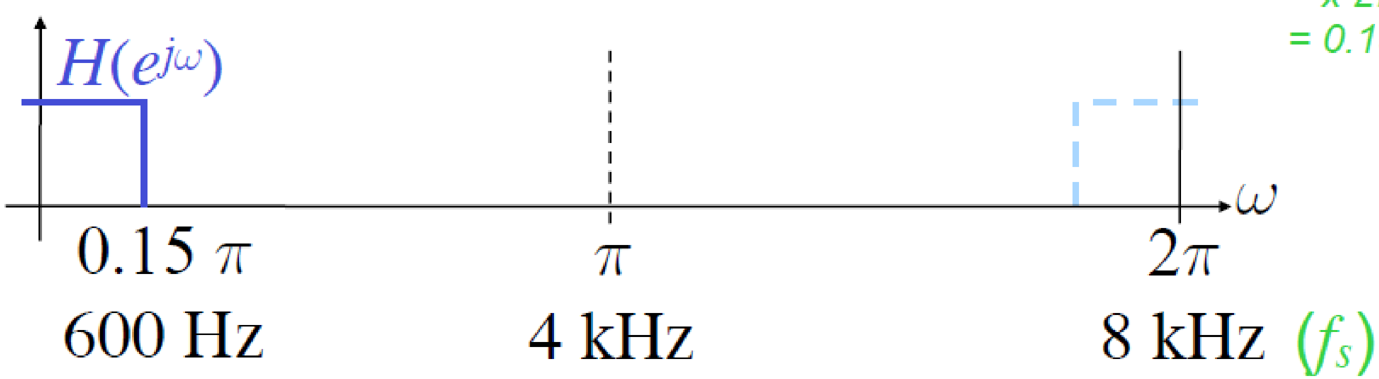
COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	-13	$4\pi/(M+1)$	-21	0	$1.81\pi/M$
Bartlett	-25	$8\pi/M$	-25	1.33	$2.37\pi/M$
Hann	-31	$8\pi/M$	-44	3.86	$5.01\pi/M$
Hamming	-41	$8\pi/M$	-53	4.86	$6.27\pi/M$
Blackman	-57	$12\pi/M$	-74	7.04	$9.19\pi/M$

Windowed Filter Example

- Design a 25 point FIR low-pass filter with a cutoff of 600 Hz (SR = 8 kHz)
- No specific transition/ripple req's
  → compromise: use Hamming window
- Convert the frequency to radians/ sample: $\omega_c = \frac{600}{8000} \times 2\pi = 0.15\pi$

600 cyc/sec
/ 8000 samp/sec
x 2π rad/cyc
= 0.15π rad/samp



Windowed Filter Example

1. Get ideal filter impulse response:

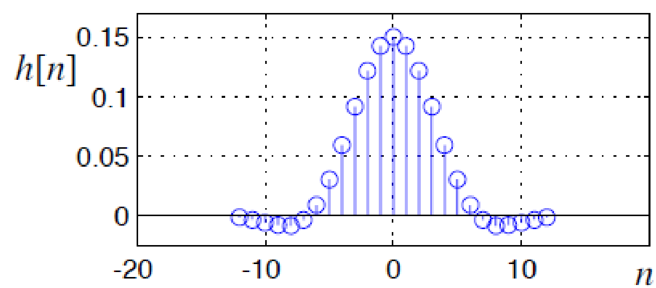
$$\omega_c = 0.15\pi \quad \Rightarrow h_d[n] = \frac{\sin 0.15\pi n}{\pi n}$$

2. Get window:

Hamming @ $N = 25 \rightarrow M = 12$ $(N = 2M+1)$

$$\Rightarrow w[n] = 0.54 + 0.46\cos\left(2\pi \tfrac{n}{25}\right) \quad -12 \le n \le 12$$

3. Apply window:



$$h[n] = h_d[n] \cdot w[n]$$

$$= \frac{\sin 0.15\pi n}{\pi n}\left(0.54 + 0.46\cos\frac{2\pi n}{25}\right) \quad -12 \le n \le 12$$

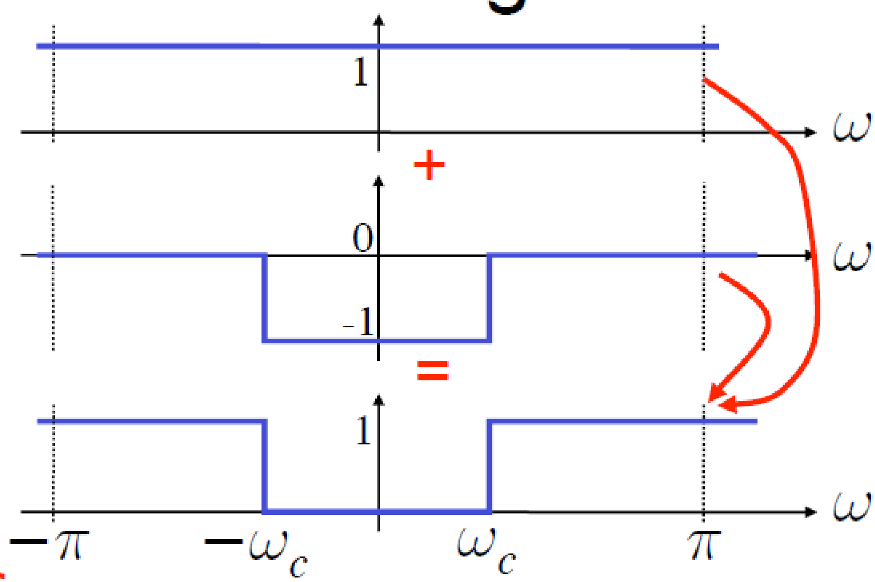
Freq. Resp. (FR) Arithmetic

- Ideal LPF has pure-real FR i.e.

$$\theta(\omega) = 0,\ H(e^{j\omega}) = |H(e^{j\omega})|$$

→ Can build piecewise-constant FRs by combining ideal responses, e.g. HPF:

wouldn't work if phases were nonzero!



$\delta[n]$ i.e. $H(e^{j\omega}) = 1$

+

$-h_{LP}[n]$ $H_{LP}(e^{j\omega}) = 1$ for $|\omega| < \omega_c$

=

$$h_{HP}[n] = \delta[n] - (\sin\omega_c n)/\pi n$$

Window Technique: “Overdesign”

- $\delta_p = \delta_s$
  - over design in either pass-band or stop-band

- non equal ripple in window design,
  - over design in part of the pass-band and stop-band

Note on IIR and FIR Design

passband edge freq. = 0.22
stopband edge freq. = 0.29
maximum passband gain = 0dB
minimum passband gain = -1dB
maximum stopband gain = -40dB

TABLE 7.3 ORDERS OF DESIGNED FILTERS.

Filter design	Order
Butterworth	18
Chebyshev I	8
Chebyshev II	8
Elliptic	5
Kaiser	63
Parks–McClellan	44

- number of multiplications (symmetric structure)
  - Kaiser: 32 (linear phase)
  - Parks-McClellan: 23 (linear phase)
  - Elliptic: 8
- IIR design requires fewer order or fewer taps (less complexity)
- FIR can achieve linear phase no stability concern