ICE503 DSP-Homework#10

1. The convolution of discrete-time system with an impulse response h[n] is given by:

$$\mathrm{y[n]} = \sum_{k=-\infty}^{\infty} h[k]x[n-k],$$

   derive the z-transforms of transfer function $\mathrm{Y(z)} = \mathrm{H(z)X(z)}$ step by step.

2. A causal linear time-invariant system has the system function

$$H(z) = \frac{(1-1.5z^{-1}-z^{-2})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})}$$

   (a) Write the difference equation that characterizes the system with $x[n]$ and $y[n]$.

   (b) Plot the pole-zero diagram and indicate the region of convergence for the system function.

3. Matlab Simulation

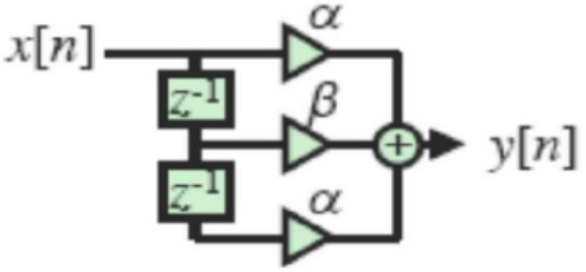
   Separate the following information in frequency.

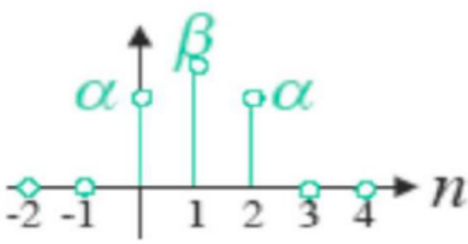
$$x[n] = Acos(\omega 1n) + Bcos(\omega 2n)$$

   with construct $H(e^{j\omega})$

$$H(e^{j\omega}) = \begin{cases} |H(e^{j\omega_1})| & \sim & 1, \\ |H(e^{j\omega_2})| & \sim & 0, \end{cases}$$

   Where $\omega_1 = 0.1$ and $\omega_2 = 0.4$. Consideer a 3 pt FIR filters with $h[n] = \{\alpha \ \ \beta \ \ \alpha\}$. Sketch the frequency response and compare the output signal with input signals.





1. The convolution of discrete-time system with an impulse response h[n] is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k],$$

derive the z-transforms of transfer function $Y(z) = H(z)X(z)$ step by step.

Given $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

$$Y(z) = \sum_{n=-\alpha}^{\alpha} y(n) z^{-n}$$

$$= \sum_{n=-\alpha}^{\alpha} \left(\sum_{k=-\alpha}^{\alpha} h(k)\, x(n-k) \right) z^{-n}$$

$$= \sum_{n=-\alpha}^{\alpha} \sum_{k=-\alpha}^{\alpha} h(k)\, x(n-k) z^{-n}$$

$$= \sum_{k=-\alpha}^{\alpha} h(k) \sum_{n=-\alpha}^{\alpha} x(n-k) z^{-n}$$

$$= \sum_{k=-\alpha}^{\alpha} h(k) \sum_{m=-\alpha}^{\alpha} x(m) z^{-(m+k)} \quad \text{Put } m = n-k$$

$$= \sum_{k=-\alpha}^{\alpha} h(k) \sum_{m=-\alpha}^{\alpha} x(m) z^{-m} \times z^{-k} \quad \Rightarrow n = m+k$$

$$= \sum_{k=-\alpha}^{\alpha} h(k) z^{-k} \left(\sum_{m=-\alpha}^{\alpha} x(m) z^{-m} \right)$$

$$= \sum_{k=-\alpha}^{\alpha} h(k) z^{-k}\, X(z)$$

$$= X(z) \left(\sum_{k=-\alpha}^{\alpha} h(k) z^{-k} \right)$$

$$= X(z) H(z)$$

$$= H(z) X(z)$$ ——— proved

2. A causal linear time-invariant system has the system function

$$H(z) = \frac{(1 - 1.5z^{-1} - z^{-2})(1 + 0.9z^{-1})}{(1 - z^{-1})(1 + 0.7jz^{-1})(1 - 0.7jz^{-1})}$$

(a) Write the difference equation that characterizes the system with $x[n]$ and $y[n]$.

(b) Plot the pole-zero diagram and indicate the region of convergence for the system function.

(a) $H(z) = \dfrac{(1-1.5z^{-1} - z^{-2})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})}$

$(1+0.7j\,z^{-1})(1-0.7j\,z^{-1})$
$= (1)^2 + (0.7z^{-1})^2$
$= 1 + 0.49z^{-2}$

$(1-z^{-1})(1+0.49z^{-2})$
$= 1 - z^{-1} + 0.49z^{-2} - 0.49z^{-3}$

$(1-1.5z^{-1} - z^{-2})(1+0.9z^{-1})$
$= 1 - 1.5z^{-1} - z^{-2} + 0.9z^{-1} - 1.35z^{-2} - 0.9z^{-3}$
$= 1 - 0.6z^{-1} - 2.35z^{-2} - 0.9z^{-3}$

Hence,

$H(z) = \dfrac{1 - 0.6z^{-1} - 2.35z^{-2} - 0.9z^{-3}}{1 - z^{-1} + 0.49z^{-2} - 0.49z^{-3}} = \dfrac{Y(z)}{X(z)}$

$\Rightarrow X(z)(1 - 0.6z^{-1} - 2.35z^{-2} - 0.9z^{-3})$
$= Y(z)(1 - z^{-1} + 0.49z^{-2} - 0.49z^{-3})$

$\Rightarrow X(z) - 0.6z^{-1}X(z) - 2.35X(z)z^{-2} - 0.9z^{-3}X(z)$
$= Y(z) - z^{-1}Y(z) + 0.49z^{-2}Y(z) - 0.49z^{-3}Y(z)$

Take inverse z-transform of above, we get,

$x(t) - 0.6x(t-1) - 2.35x(t-2) - 0.9x(t-3)$
$= y(t) - y(t-1) + 0.49y(t-2) - 0.49y(t-3)$

which is the difference equation.

(b) Zeros: $(1-1.5z^{-1} - z^{-2})(1+0.9z^{-1}) = 0$

$\therefore\ 1 - 1.5z^{-1} - z^{-2} = 0 \qquad 1 + 0.9z^{-1} = 0$

$\Rightarrow z^2 - 1.5z - 1 = 0 \qquad \Rightarrow z + 0.9 = 0$

$\Rightarrow z = \dfrac{1.5 \pm \sqrt{1.5^2 + 4}}{2} \qquad \Rightarrow z = -0.9$

$= \dfrac{1.5 \pm \sqrt{2.25 + 4}}{2}$

$= \dfrac{1.5 \pm \sqrt{6.25}}{2}$

$= \dfrac{1.5 \pm 2.5}{2}$

$= \frac{4}{2}, -\frac{1}{2}$

$= 2, -0.5$

Hence, zeros at $(-0.9, 0)$, $(-0.5, 0)$, $(2, 0)$.

Poles:

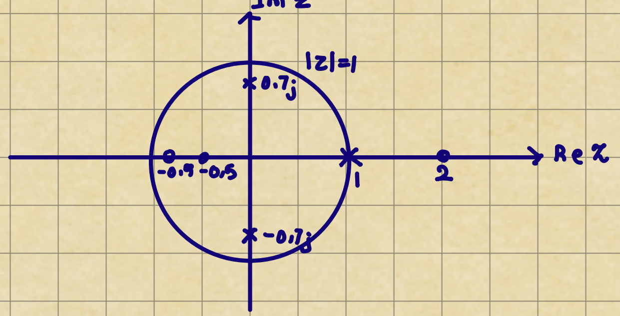
$(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1}) = 0$

$\Rightarrow 1 - z^{-1} = 0 \qquad 1 + 0.7j\,z^{-1} = 0 \qquad 1 - 0.7jz^{-1} = 0$

$\Rightarrow z - 1 = 0 \qquad z + 0.7j = 0 \qquad z - 0.7j = 0$

$\Rightarrow z = 1 \qquad z = -0.7j \qquad z = 0.7j$

$\therefore$ Poles at $(1, 0)$, $(0, -0.7j)$, $(0, 0.7j)$.



- The ROC cannot include poles
- ROC extends outward beyond outermost pole.

3. Matlab Simulation

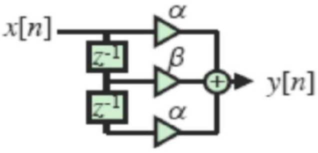
Separate the following information in frequency.

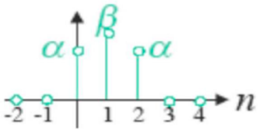
$$x[n] = A\cos(\omega 1 n) + B\cos(\omega 2 n)$$

with construct $H(e^{j\omega})$

$$H(e^{j\omega}) = \begin{cases} |H(e^{j\omega_1})| & \sim \quad 1, \\ |H(e^{j\omega_2})| & \sim \quad 0, \end{cases}$$

Where $\omega_1 = 0.1$ and $\omega_2 = 0.4$. Consideer a 3 pt FIR filters with $h[n] = \{\alpha \;\; \beta \;\; \alpha\}$. Sketch the frequency response and compare the output signal with input signals.





$x(n) = A\cos(\omega 1 n) + B\cos(\omega 2 n)$

$y(n) = \alpha x(n) + \beta x(n-1) + \alpha x(n-2)$

Take z transform,

$Y(z) = \alpha X(z) + \beta X(z) z^{-1} + \alpha X(z) z^{-2}$

Divide by $X(z)$,

$H(z) = \alpha + \beta z^{-1} + \alpha z^{-2}$

$\Rightarrow H(z) = \alpha(1 + z^{-2}) + \beta z^{-1}$

Let $z = e^{j\omega}$, we have,

$H(e^{j\omega}) = \alpha(1 + e^{-2j\omega}) + \beta e^{-j\omega}$

Given conditions are

$H(e^{j\omega_1}) = \alpha(1 + e^{-2j\omega_1}) + \beta e^{-j\omega_1} = 1 \quad$ — ①

$H(e^{j\omega_2}) = \alpha(1 + e^{-2j\omega_2}) + \beta e^{-j\omega_2} = 0 \quad$ — ②

To find α, β.

from (2),

$\beta e^{-j\omega_2} = -\alpha(1 + e^{-j2\omega_2})$

$\times e^{j\omega_2} \Rightarrow \beta = -\alpha(e^{j\omega_2} + e^{-j\omega_2})$

From (1),

$\alpha(1 + e^{-j2\omega_1}) - \alpha(e^{j\omega_2} + e^{-j\omega_2}) = 1$

$\Rightarrow \alpha(1 + e^{-j2\omega_1} - e^{j\omega_2} - e^{-j\omega_2}) = 1$

$$\Rightarrow \alpha = \frac{1}{1 + e^{-j2\omega_1} - (e^{j\omega_2} + e^{-j\omega_2})}$$

$$\beta = \frac{-(e^{j\omega_2} + e^{-j\omega_2})}{(1 + e^{-j2\omega_1}) - (e^{j\omega_2} + e^{-j\omega_2})}$$

ICE503 Homework-11

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Q. 2

Design of Filter

Consider a given input signal $x(n)$ and output $y(n)$. From the given symmetric filter diagram, we can write

$$y(n) = \alpha x(n) + \beta x(n-1) + \alpha x(n-1)$$

Taking z-transform and manipulating, we have the transfer function as

$$H(z) = \alpha\left(1 + z^{-2}\right) + \beta z^{-1}$$

The filter response function is written by substituting $z = e^{j\omega}$ as

$$H\left(e^{j\omega}\right) = \alpha\left(1 + e^{-j2\omega}\right) + \beta e^{-j\omega} \tag{1}$$

The filter response is given as:

$$H\left(e^{j\omega}\right) = \begin{cases} 1, & \omega = \omega_1 \\ 0, & \omega = \omega_2 \end{cases}$$

Hence by substitution in the filter response function (1) we have two equations:

$$H\left(e^{j\omega}\right) = \alpha\left(1 + e^{-j2\omega_1}\right) + \beta e^{-j\omega_1} = 1$$
$$H\left(e^{j\omega_2}\right) = \alpha\left(1 + e^{-j2\omega_2}\right) + \beta e^{-j\omega_2} = 0$$

Solving for α and β, we have the relations

$$\alpha = \frac{1}{\left(1 + e^{-j2\omega_1}\right) - \left(e^{j\omega_1} + e^{-j\omega_1}\right)}$$
$$\beta = \frac{-\left(e^{j\omega_2} + e^{-j\omega_2}\right)}{\left(1 + e^{-j2\omega_1}\right) - \left(e^{j\omega_2} + e^{-j\omega_2}\right)}$$

Hence, the 3-point filter response is given by $\mathbf{h} = [\alpha, \beta, \alpha]$. The plot below shows the given response frequencies $\omega_1 = 0.1$ and $\omega_2 = 0.2$.

Date: December 7, 2024

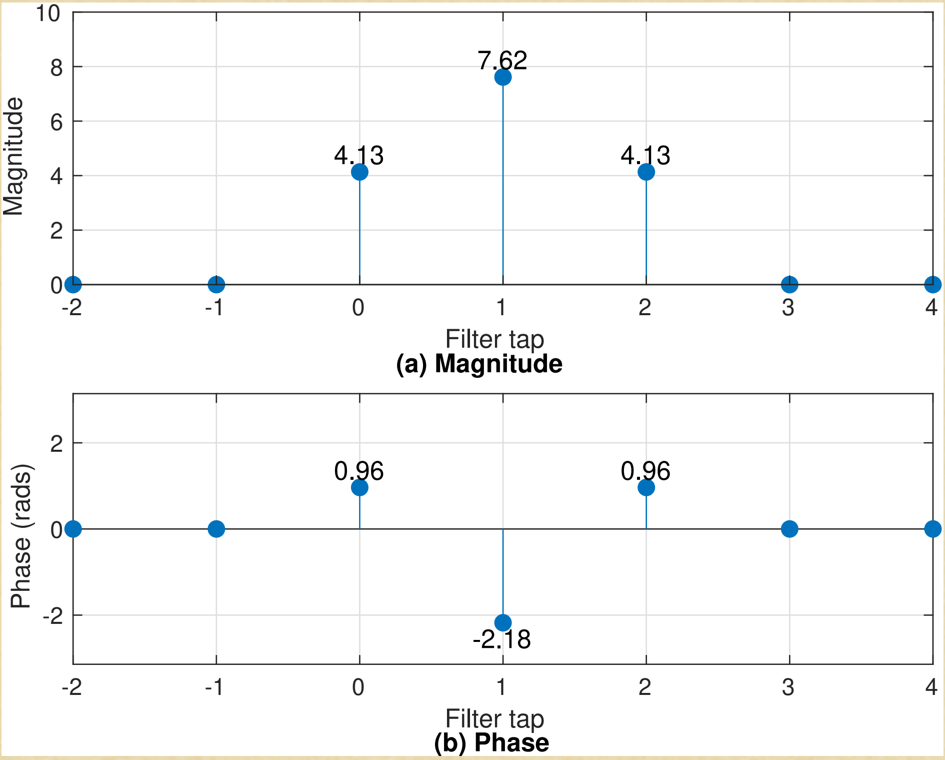

Fig. 1: Plot of filter coefficients.

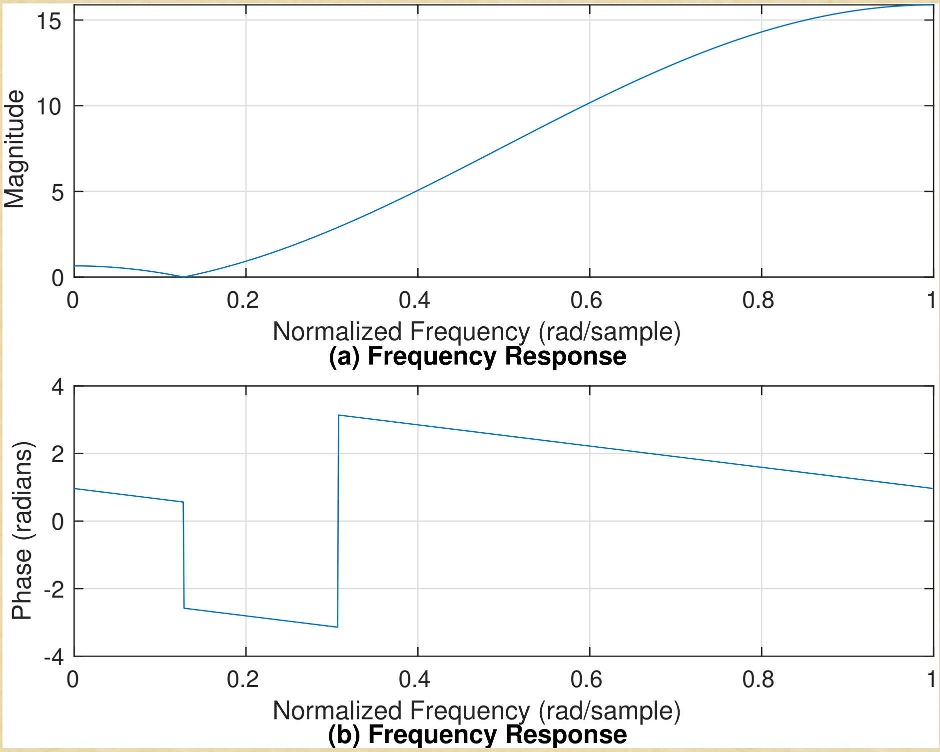

Fig. 2: Plot of filter response for the 3-point filter whose coefficients are shown in Fig. 1 above.

Comparison of Input-Output Response

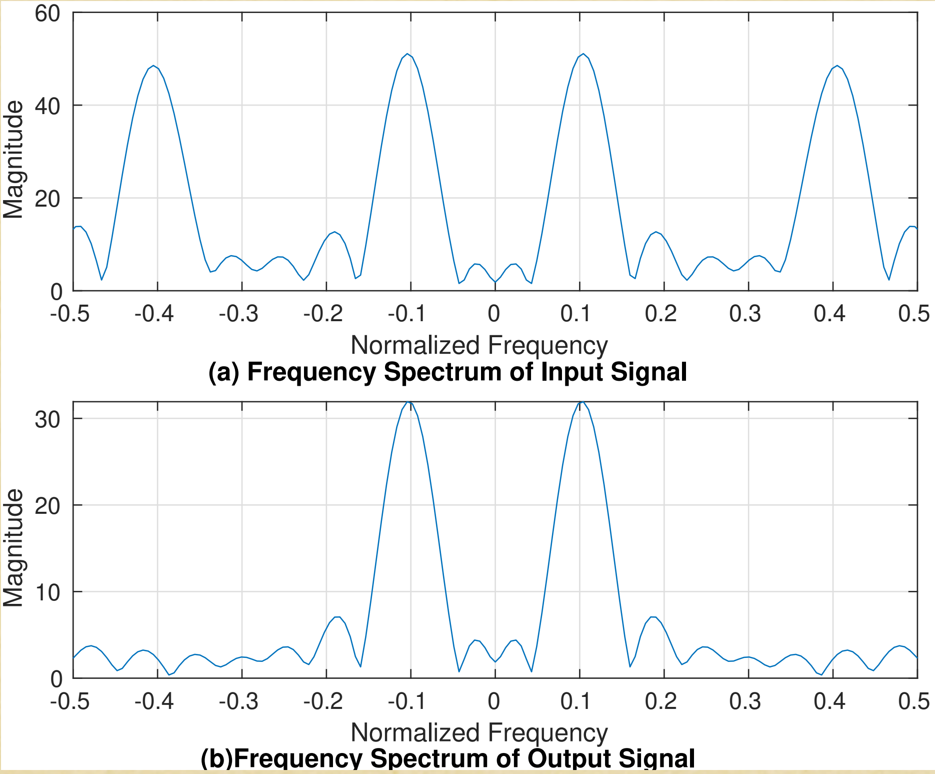

Fig. 3: Plot of input and output signal.

The input signal comprises frequency components at $\omega_1 = 0.1$ and $\omega_2 = 0.4$. However, the low pass filter designed retains the lower frequency $\omega_1 = 0.1$ and eliminates the higher frequency $\omega_2 = 0.4$. This is evident from the plots above.

```
% HW11
% Q02

% ---------- reset -----------------
close all;
clear all;
clc;


% ---------- configs ----------------
w1 = 0.1;
w2 = 0.4;

% ------ generate the filter --------
[a b] = get_coeff_filter(w1, w2);  % generate the filter coeffs.


h = [a b a]; % actual 3 point filter coefficients

% ----------- generate the input signals ---------
n = 0: 99; % time index
A = 1; % amplitudes
B = 1;

x = A * cos(w1*n) + B * cos(w2*n); % input signal

% ----------- output = filter(input) -------------
y = filter(h, 1, x);

% ----------- Analyze the filter -----------------


% Compute frequency response of the filter
[H, w] = freqz(h, 1, 1024);

figure(1)
subplot(2,1,1)

% Plot the magnitude response
plot(w/pi, abs(H));
xlabel('Normalized Frequency (rad/sample)');
ylabel('Magnitude');
title('(a) Frequency Response','Units', 'normalized', 'Position',
 [0.5, -0.35, 0]);
grid on;

% Plot the phase response (optional)
subplot(2,1,2);
plot(w/pi, angle(H));
xlabel('Normalized Frequency (rad/sample)');
ylabel('Phase (radians)');
title('Phase Response');
```

```
title('(b) Frequency Response','Units', 'normalized', 'Position',
 [0.5, -0.35, 0]);
grid on;


saveas(gca, 'hw11_2a.eps', 'epsc');


% --------- Analyze I/O signals ---------

% Fourier Transform of Input Signal
N_fft = 1024;
X = fft(x, N_fft); % Compute the FFT of input signal
X = fftshift(X);
frequencies = 0: N_fft-1; % Define the frequency axis (normalized)
frequencies = frequencies/N_fft - 0.5;

frequencies = frequencies * 2*pi;

% Fourier Transform of Output Signal
Y = fft(y, N_fft); % Compute the FFT of output signal
Y = fftshift(Y);

% Magnitudes of the FFTs
magnitude_X = abs(X); % Magnitude of FFT of input signal
magnitude_Y = abs(Y); % Magnitude of FFT of output signal

% Plot the frequency spectrum
figure;

% Input signal spectrum
subplot(2, 1, 1);
plot(frequencies, magnitude_X);
title('(a) Frequency Spectrum of Input
 Signal','Units', 'normalized', 'Position', [0.5, -0.35, 0]);

xlabel('Normalized Frequency');
ylabel('Magnitude');
grid on;
xlim([-0.5,0.5])
xticks(-0.5:0.1:0.5);

% Output signal spectrum
subplot(2, 1, 2);
plot(frequencies, magnitude_Y);
title('(b)Frequency Spectrum of Output
 Signal', 'Units', 'normalized', 'Position', [0.5, -0.35, 0]);
xlabel('Normalized Frequency');
ylabel('Magnitude');
grid on;
xlim([-0.5,0.5])
xticks(-0.5:0.1:0.5);
```

```
saveas(gca, 'hw11_2b.eps', 'epsc');



% ----- plot filter coeff ---------
h = [0 0 a b a 0 0];
n = [-2 -1 0 1 2 3 4];
h_mag = abs(h);
h_ph = angle(h);
figure(3)

subplot(2,1,1)
stem(n, h_mag, 'filled')
for i = 1:length(h)
    if h_mag(i) <= 1e-6
        continue
    end
    text(n(i), h_mag(i), sprintf(' %.2f',
 h_mag(i)), 'VerticalAlignment', 'bottom', 'HorizontalAlignment', 'center');
end
xlim([-2, 4])
ylim([0,10])
xlabel('Filter tap')
ylabel('Magnitude')
grid on
title('(a) Magnitude ','Units', 'normalized', 'Position', [0.5, -0.35,
 0]);


subplot(2,1,2)
stem(n, h_ph, 'filled')
for i = 1:length(h)
    if abs(h_mag(i)) <= 1e-6
        continue
    end
    if h_ph(i) >= 0
        text(n(i), h_ph(i), sprintf(' %.2f',
 h_ph(i)), 'VerticalAlignment', 'bottom', 'HorizontalAlignment', 'center');
    else
        text(n(i), h_ph(i), sprintf(' %.2f',
 h_ph(i)), 'VerticalAlignment', 'top', 'HorizontalAlignment', 'center');
    end
end
xlim([-2, 4])
ylim([-pi,pi])
xlabel('Filter tap')
ylabel('Phase (rads)')
grid on
title('(b) Phase','Units', 'normalized', 'Position', [0.5, -0.35, 0]);


saveas(gca, 'hw11_2c.eps', 'epsc');

% ---------------- functions -------------
```

```
function [a, b] = get_coeff_filter(w1, w2)
    term1 = 1 + exp(-1j * 2* w1);
    term2 = exp(1j*w2) + exp(-1j*w2);

    a = 1/(term1 - term2);
    b = -term2/(term1 - term2);


end
```

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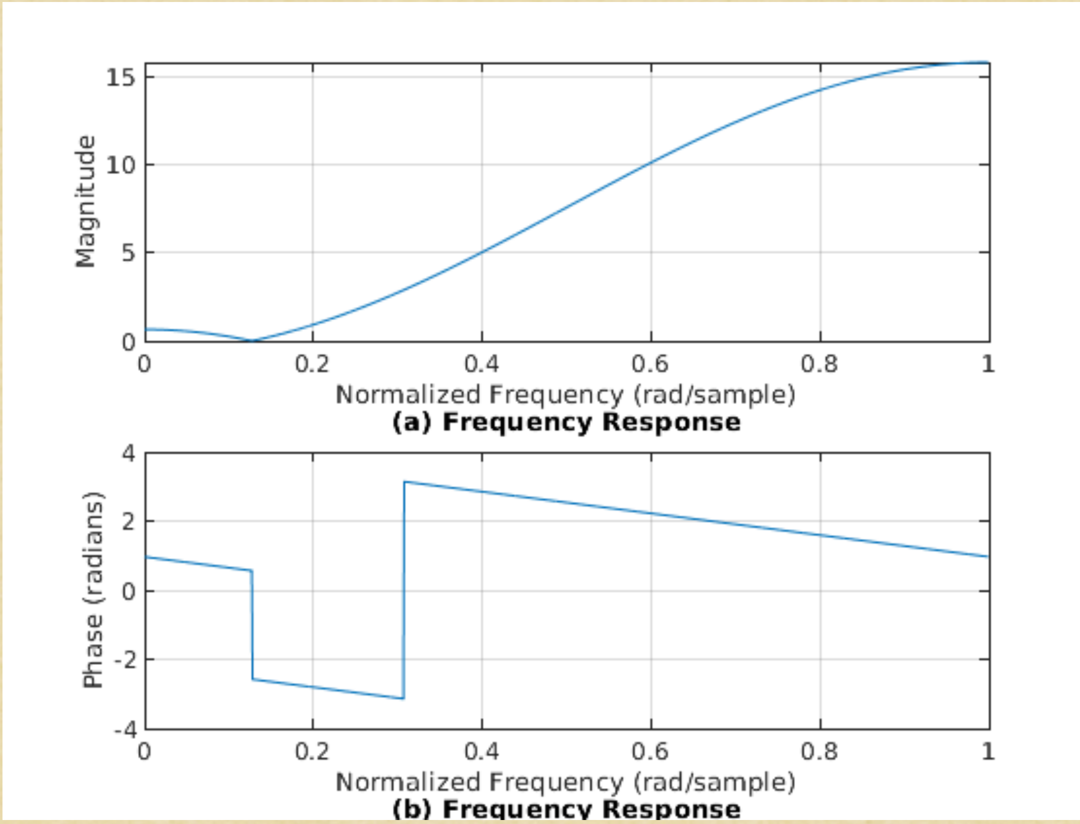
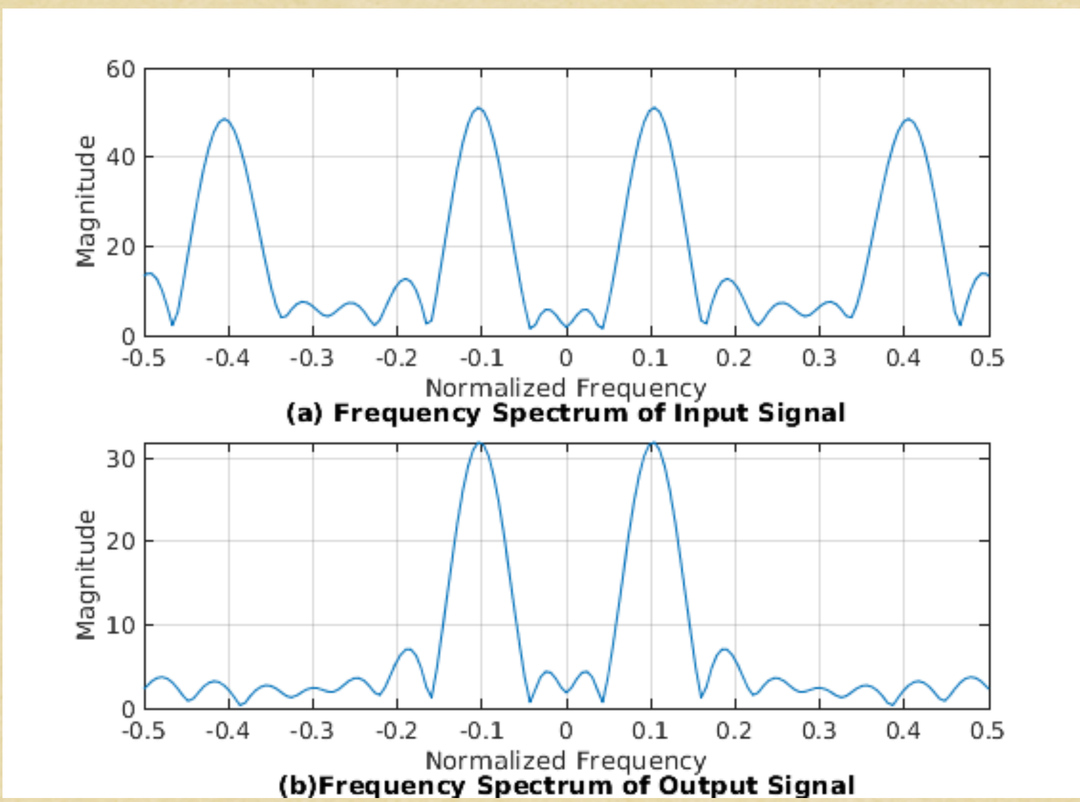


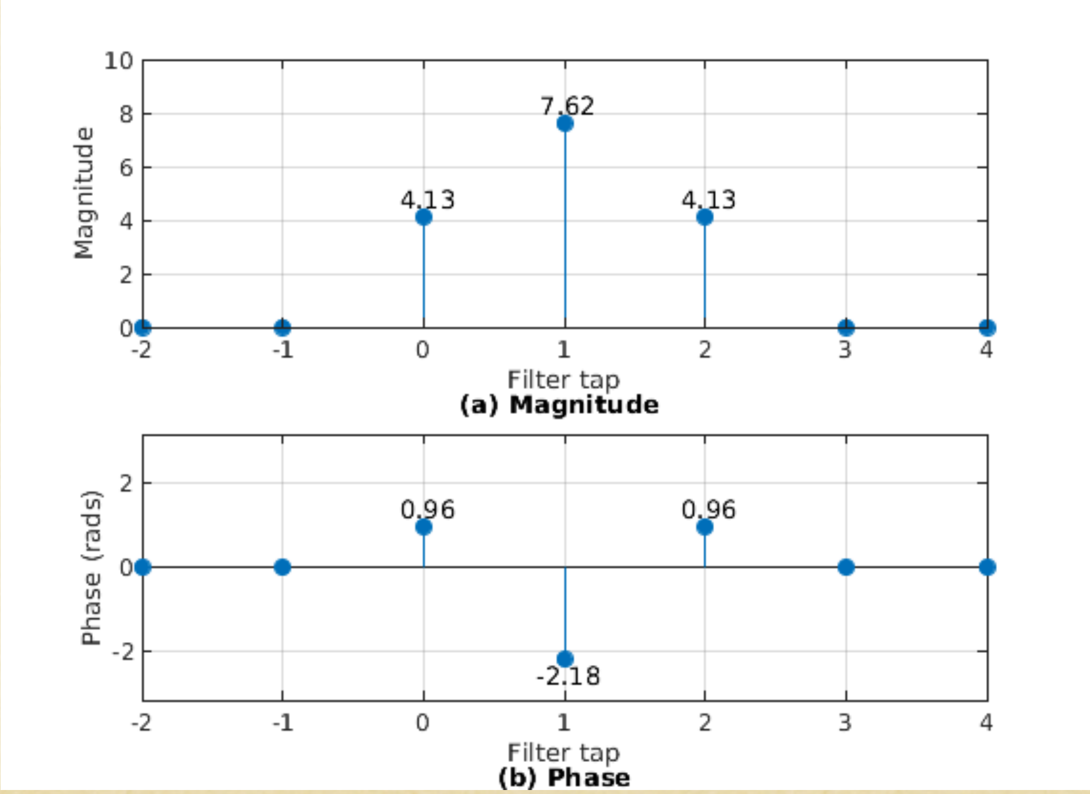
(a) Frequency Response

(b) Frequency Response



(a) Frequency Spectrum of Input Signal

(b)Frequency Spectrum of Output Signal



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