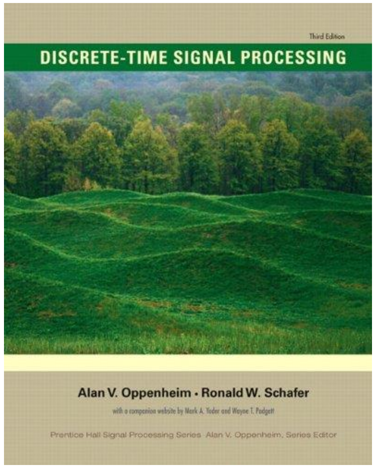


Lecture 04:
Fourier Domain

Outlines

1. The Fourier domain

2. Discrete-Time Fourier Transform (DTFT)

Joseph Fourier

(21 March 1768 – 16 May 1830)

Fourier claimed that any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable.

- French mathematician and physicist
- Initiate the investigation of Fourier series
- The Fourier transform and Fourier's Law are also named in his honor

1. The Fourier Transform

- Basic observation (continuous time):
  A periodic signal can be decomposed into sinusoids at integer multiples of the fundamental frequency
- i.e. if $\tilde{x}(t) = \tilde{x}(t+T)$
  we can approach $\tilde{x}$ with

Harmonics of the fundamental frequency

$$\tilde{x}(t) \approx \frac{a_0}{2} + \sum_{k=1}^{M} \left[a_k \cos\left(\boxed{\frac{2\pi k}{T}} t \right) + b_k \sin\left(\boxed{\frac{2\pi k}{T}} t \right) \right]$$

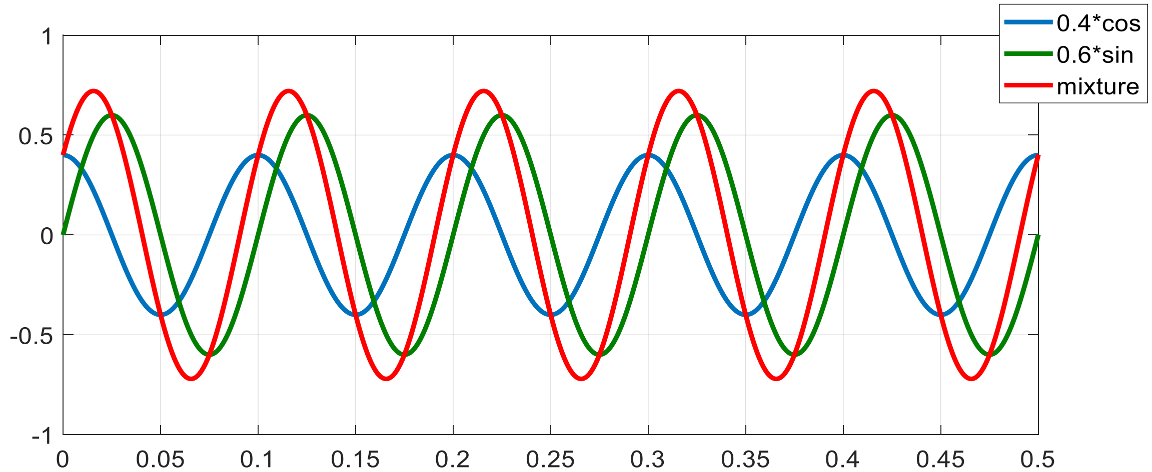
Fourier Coefficient

- It is also possible to use a sine (or cosine) wave function plus phase change to represent the addition of sine and cosine waves

$$a_k \cos\left(\frac{2\pi k}{T}t\right) + b_k \sin\left(\frac{2\pi k}{T}t\right) = c_k \cos\left(\frac{2\pi k}{T}t + \phi_k\right)$$

$$c_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \arctan\left(\frac{b_k}{a_k}\right)$$



- We can approximate $\tilde{x}(t)$ with

$$\tilde{x}(t) \approx \sum_{k=0}^{M} c_k \cos\left(\frac{2\pi k}{T}t + \phi_k\right)$$

Harmonics of the fundamental frequency

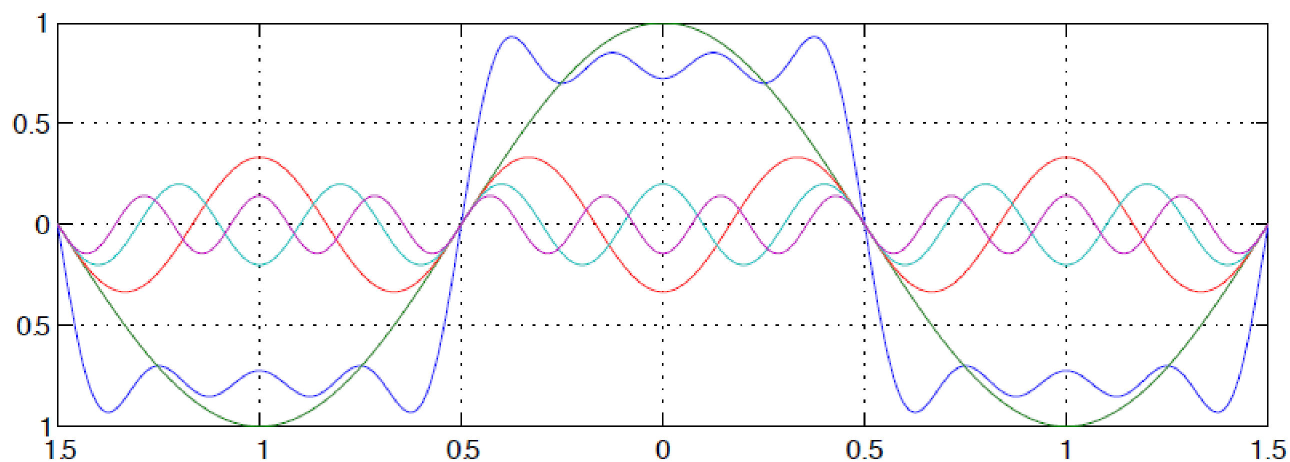
$$\sum_{k=0}^{M} c_k \cos\left(\frac{2\pi k}{T} t + \phi_k\right)$$

Fourier Series

- For a square wave,

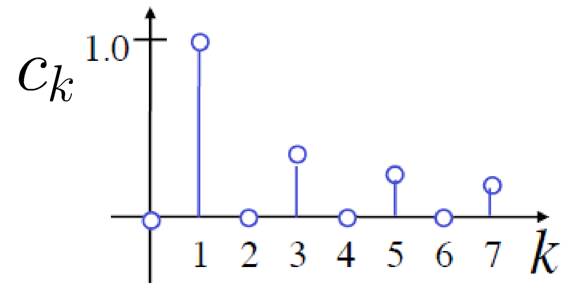
$$\phi_k = 0; \qquad c_k = \begin{cases} (-1)^{\frac{k-1}{2}} \frac{1}{k} & k = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases}$$

i.e. $x(t) = \cos\left(\frac{2\pi}{T} t\right) - \frac{1}{3}\cos\left(\frac{2\pi}{T} 3t\right) + \frac{1}{5}\cos\left(\frac{2\pi}{T} 5t\right) - \ldots$

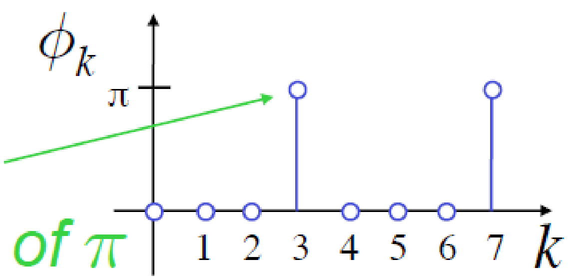


Fourier domain

- x is equivalently described by its Fourier Series parameters:



$$c_k = (-1)^{\frac{k-1}{2}} \frac{1}{k} \quad k = 1, 3, 5, \ldots$$



Negative a_k is equivalent to phase of π

- Complex form: $\tilde{x}(t) \approx \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$

$$\tilde{x}(t) \approx \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$$

Fourier analysis

- How to find $\{|c_k|\}, \{\arg\{c_k\}\}$?
  Inner product with
  (conjugate) complex sinusoids:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi k}{T}t} dt$$

An Example

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

- Consider $x(t) = \cos\left(l\frac{2\pi}{T}t\right)$

  .. so c_k should $= 0$ except $k = \pm l$

- Then

$$c_k = \frac{1}{T}\left(\int x(t)\cos\tfrac{2\pi kt}{T}\,dt - j\int x(t)\sin\tfrac{2\pi kt}{T}\,dt\right)$$

$0 \because$ *even·odd*

$$= \frac{1}{T}\left(\int \cos\tfrac{2\pi lt}{T}\cos\tfrac{2\pi kt}{T}\,dt - j\int \cancel{\cos\tfrac{2\pi lt}{T}\sin\tfrac{2\pi kt}{T}}\,dt\right)$$

An Example

- Works $\because$ if k, l are positive integers,

(say $T=2\pi$)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(kt)\cdot\cos(lt)\,dt=\begin{cases}1 & k=l\\ 0 & k\neq l\end{cases}$$

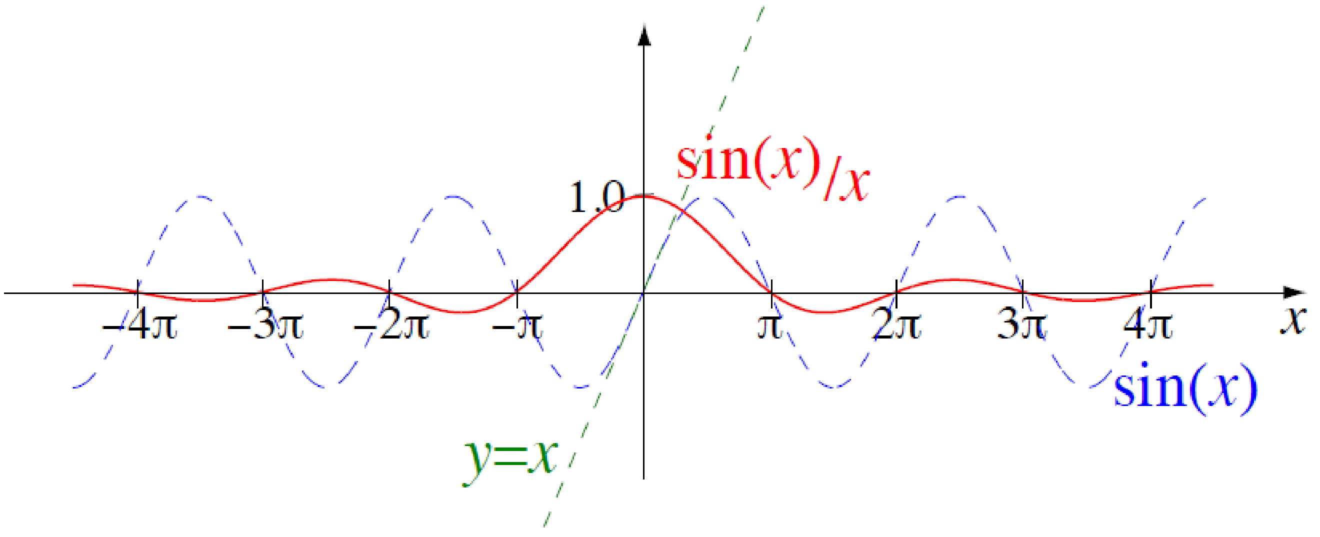
$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\cos(k+l)t+\cos(k-l)t\,dt$$

$$=\frac{1}{4\pi}\left[\frac{\sin(k+l)t}{k+l}+\frac{\sin(k-l)t}{k-l}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2}\left(\operatorname{sinc}\pi(k+l)+\operatorname{sinc}\pi(k-l)\right)$$

sinc

- $\operatorname{sinc} x \triangleq \dfrac{\sin x}{x}$



- $= 1$ when $x = 0$

  $= 0$ when $x = r\cdot\pi,\ r \neq 0,\ r = \pm 1, \pm 2, \pm 3, \ldots$

$$\tilde{x}(t) \approx \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$$

Fourier analysis

$$\hat{c}_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\frac{2\pi k}{T}t} dt \quad ; \text{call } \tau = \frac{2\pi}{T}t$$

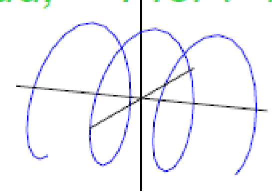
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\left(\frac{T}{2\pi}\tau\right) e^{-jk\tau} d\tau$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_l c_l e^{jl\tau}\right) e^{-jk\tau} d\tau$$

$$= \sum_l c_l \boxed{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(l-k)\tau} d\tau\right)}$$

$$= c_k$$

Integral of $(l\text{-}k)$ complete cycles of a complex sinusoid; = 0 for $l \neq k$ $\because$ real (cos) part is complete cycles, imag (sin) part is odd; = 1 for $l = k$ $\because \int 1 d\tau$



sinc again

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-l)}\,dw = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-l)}}{j(n-l)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\pi(n-l)} - e^{-j\pi(n-l)}}{j(n-l)}\right)$$

$$= \frac{1}{2\pi}\left(\frac{2j\sin\pi(n-l)}{j(n-l)}\right) = \operatorname{sinc}\pi(n-l)$$

- Same as $\int \cos$ $\because$ imag $j\sin$ part cancels

Fourier Analysis

- Thus, $$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

  because complex sinusoids $e^{-j\frac{2\pi k}{T}t}$ pick out the corresponding sinusoidal components linearly combined in

$$x(t) = \sum_{k=-M}^{M} c_k e^{j\frac{2\pi k}{T}t}$$

Fourier Transform

- Fourier series for periodic signals extends naturally to Fourier Transform for any (CT) signal (not just periodic):

$$X(j\Omega) = \int_{\infty}^{\infty} x(t)e^{-j\Omega t}\,dt$$

Fourier Transform (FT)

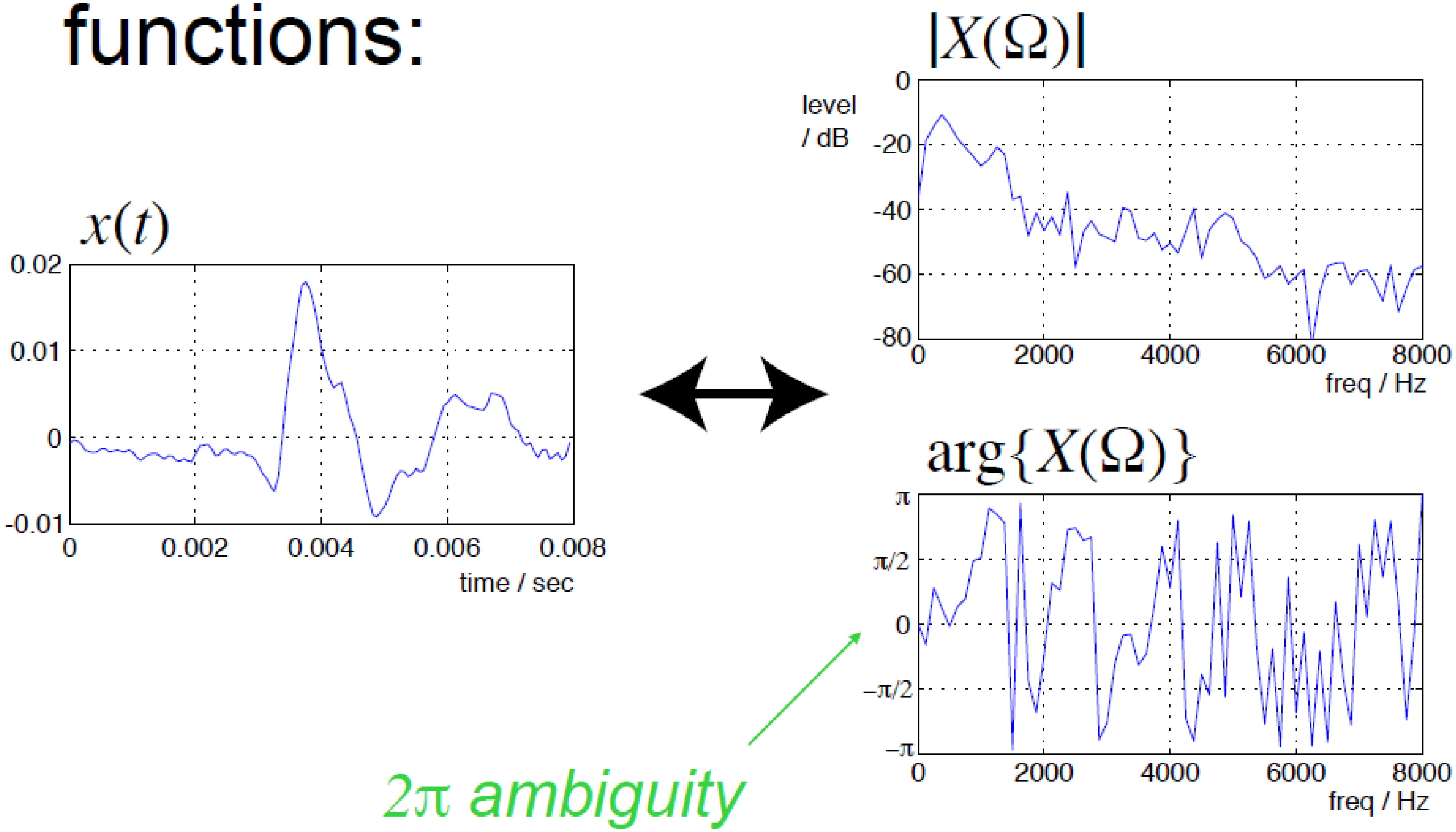
$$x(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

Inverse Fourier Transform (IFT)

- Discrete index k → continuous freq. Ω

Fourier Transform

- Mapping between two continuous functions:



Fourier Transform of a sine

- Assume $x(t) = e^{j\Omega_0 t}$

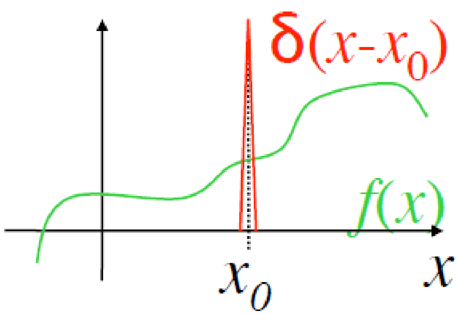
  Now, since $x(t) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$

  ...we know $X(\Omega) = 2\pi\delta(\Omega - \Omega_0)$

  ...where $\delta(x)$ is the Dirac delta function

  (continuous time) i.e.

  $\int \delta(x - x_0) f(x) dx = f(x_0)$



- $\rightarrow \; x(t) = Ae^{j\Omega_0 t} \leftrightarrow X(\Omega) = A\delta(\Omega - \Omega_0)$

Fourier Transforms

	Time	*Frequency*
Fourier Series (FS)	Continuous periodic $\tilde{x}(t)$	Discrete infinite c_k
Fourier Transform (FT)	Continuous infinite $x(t)$	Continuous infinite $X(\Omega)$
Discrete-Time FT (DTFT)	Discrete infinite $x[n]$	Continuous periodic $X(e^{j\omega})$
Discrete FT (DFT)	Discrete finite/pdc $\tilde{x}[n]$	Discrete finite/pdc $X[k]$

2. Discrete Time FT (DTFT)

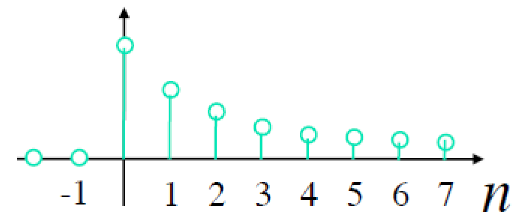
- FT defined for discrete sequences:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad \text{DTFT}$$

- Summation (not integral)
- Discrete (normalized) frequency variable ω
- Argument is $e^{j\omega}$, not $j\omega$

DTFT

- e.g. $x[n] = \alpha^n \cdot \mu[n],\ |\alpha| < 1$



$$\Rightarrow \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n}$$

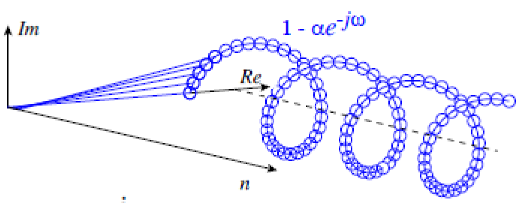
$$= \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n$$

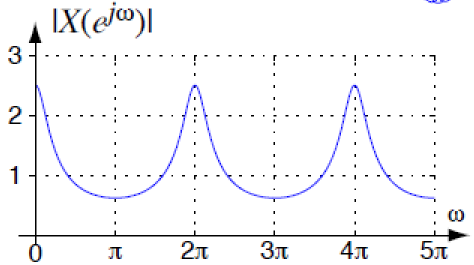
$$= \frac{1}{1-\alpha e^{-j\omega}}$$

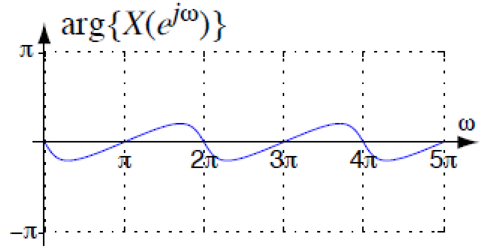
$$S = \sum_{n=0}^{\infty} c^n \Rightarrow cS = \sum_{n=1}^{\infty} c^n$$

$$\Rightarrow S - cS = c^0 = 1$$

$$\Rightarrow S = \frac{1}{1-c} \quad (\ |c| < 1\ )$$





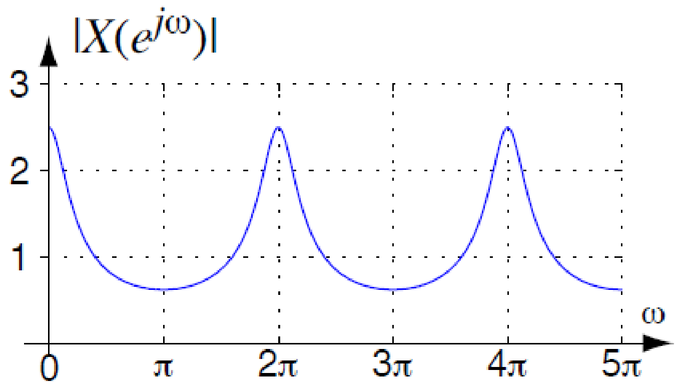


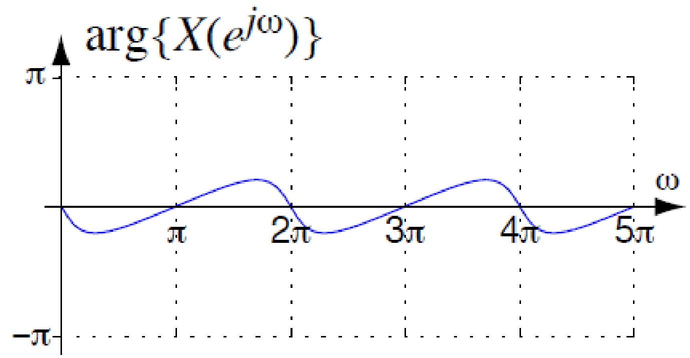
Periodicity of DFT

- $X(e^{j\omega})$ has periodicity 2π in ω :

$$X(e^{j(\omega+2\pi)}) = \sum x[n] e^{-j(\omega+2\pi)n}$$

$$= \sum x[n] e^{-j\omega n} e^{-j2\pi n} = X(e^{j\omega})$$





- Phase ambiguity of $e^{j\omega}$ makes it implicit

Inverse DTFT (IDTFT)

- Same basic "Fourier Synthesis" form:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega \quad \text{IDTFT}$$

- Note: continuous, periodic $X(e^{j\omega})$

  discrete, infinite $x[n]$ ...

- IDTFT is actually Fourier Series *analysis* (except for sign of ω)

IDTFT

- Verify by substituting in DTFT:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_l x[l]e^{-j\omega l}\right)e^{j\omega n}\,d\omega$$

$$= \sum_l x[l]\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-l)}\,d\omega$$

$= 0$ *unless* $n = l$ *i.e.* $= \delta[n\text{-}l]$

$$= \sum_l x[l]\,\mathrm{sinc}\pi(n-l) = x[n] \quad \checkmark$$

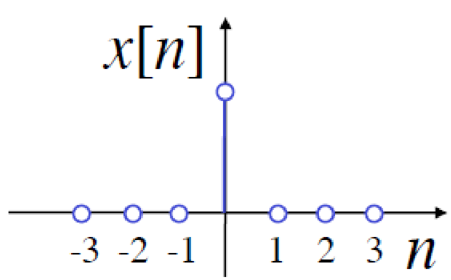
DTFTs of simple sequences

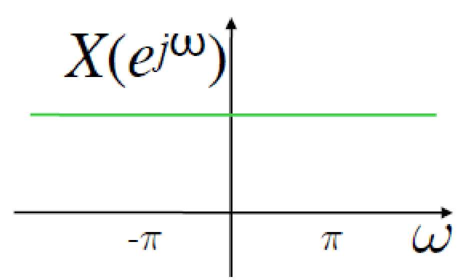
- $x[n] = \delta[n] \Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$= e^{-j\omega 0} = 1 \quad \text{(for all } \omega\text{)}$$

- i.e.

$x[n]$		$X(e^{j\omega})$
$\delta[n]$	$\leftrightarrow$	1





DTFTs of simple sequences

IDTFT

- $x[n] = e^{j\omega_0 n}$: $\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$

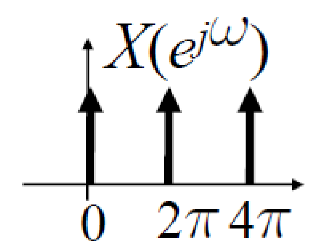
  $\Rightarrow X(e^{j\omega}) = 2\pi\cdot\delta(\omega-\omega_0)$ over $-\pi < \omega < \pi$

  but $X(e^{j\omega})$ must be periodic in $\omega \Rightarrow$

  $$e^{j\omega_0 n} \leftrightarrow \sum_k 2\pi\cdot\delta(\omega-\omega_0-2\pi k)$$

- If $\omega_0 = 0$ then $x[n] = 1 \;\forall\; n$

  so $$1 \leftrightarrow \sum_k 2\pi\cdot\delta(\omega-2\pi k)$$



DTFTs of simple sequences

- From before:

$$\alpha^n \mu[n] \leftrightarrow \frac{1}{1-\alpha e^{-j\omega}} \qquad (\,|\alpha| < 1)$$

- $\mu[n]$ tricky - not finite

$$\mu[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \underbrace{\sum_k \pi\delta(\omega + 2\pi k)}_{\text{DTFT of 1/2}}$$

DTFT properties

- Linear:

$$\alpha g[n] + \beta h[n] \quad \leftrightarrow \quad \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$$

- Time shift:

$$g[n - n_0] \quad \leftrightarrow \quad e^{-j\omega n_0} G(e^{j\omega})$$

- Frequency shift:

$$e^{j\omega_0 n} g[n] \quad \leftrightarrow \quad G\left(e^{j(\omega-\omega_0)}\right)$$

'delay' in frequency

DTFT example

- $x[n] = \delta[n] + \alpha^n \mu[n\text{-}1] \quad \leftrightarrow \quad ?$

$$= \delta[n] + \alpha(\alpha^{n\text{-}1} \mu[n\text{-}1])$$

$$\Rightarrow X(e^{j\omega}) = 1 + \alpha\left(e^{-j\omega\cdot 1} \cdot \frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$= 1 + \frac{\alpha e^{-j\omega}}{1-\alpha e^{-j\omega}} = \frac{1-\alpha e^{-j\omega} + \alpha e^{-j\omega}}{1-\alpha e^{-j\omega}}$$

$$= \frac{1}{1-\alpha e^{-j\omega}} \qquad \Rightarrow x[n] = \alpha^n \mu[n]$$

DTFT symmetry

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- If $x[n] \quad \leftrightarrow \quad X(e^{j\omega})$ then...

$$x[-n] \quad \leftrightarrow \quad X(e^{-j\omega})$$ *from summation*

$$x^*[n] \quad \leftrightarrow \quad X^*(e^{-j\omega})$$ $(e^{-j\omega})^* = e^{j\omega}$

$$\mathrm{Re}\{x[n]\} \quad \leftrightarrow \quad X_{CS}(e^{j\omega}) = \frac{1}{2}\left[X\left(e^{j\omega}\right) + X^*\left(e^{-j\omega}\right)\right]$$

conjugate symmetry cancels Im *parts on IDTFT*

$$j\mathrm{Im}\{x[n]\} \leftrightarrow \quad X_{CA}(e^{j\omega}) = \frac{1}{2}\left[X\left(e^{j\omega}\right) - X^*\left(e^{-j\omega}\right)\right]$$

$$x_{cs}[n] \quad \leftrightarrow \quad \mathrm{Re}\{X(e^{j\omega})\}$$

$$x_{ca}[n] \quad \leftrightarrow \quad j\mathrm{Im}\{X(e^{j\omega})\}$$

DTFT of real $x[n]$

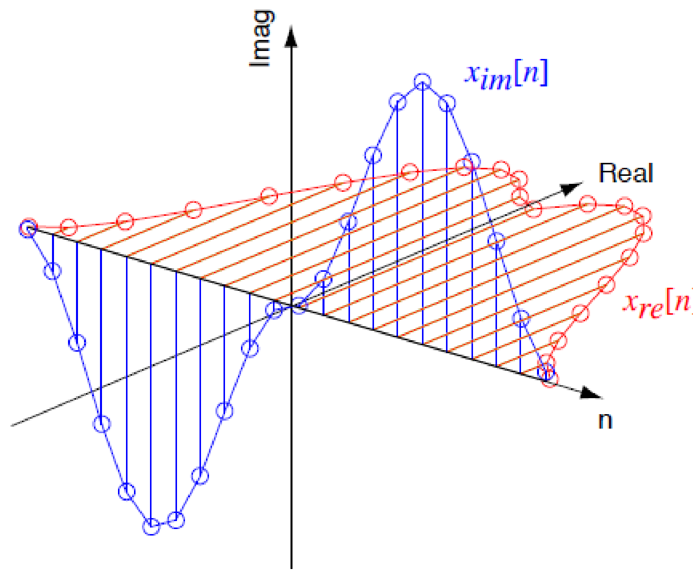
- When $x[n]$ is pure real, $\Rightarrow \underbrace{X(e^{j\omega}) = X^*(e^{-j\omega})}_{X_{CS}}$

$$x_{cs}[n] \equiv x_{ev}[n] = x_{ev}[-n] \quad \leftrightarrow \quad X_R(e^{j\omega}) = X_R(e^{-j\omega})$$

$$x_{ca}[n] \equiv x_{od}[n] = -x_{od}[-n] \quad \leftrightarrow \quad X_I(e^{j\omega}) = -X_I(e^{-j\omega})$$

$x[n]$ *real, even*

$\leftrightarrow X(e^{j\omega})$ *even, real*



DTFT and convolution

- Convolution: $x[n] = g[n] \circledast h[n]$

$$\Rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(g[n] \circledast h[n]\right) e^{-j\omega n}$$
$$= \sum_n \left(\sum_k g[k] h[n-k]\right) e^{-j\omega n}$$
$$= \sum_k \left(g[k] e^{-j\omega k} \sum_n h[n-k] e^{-j\omega(n-k)}\right)$$
$$= G(e^{j\omega}) \cdot H(e^{j\omega})$$

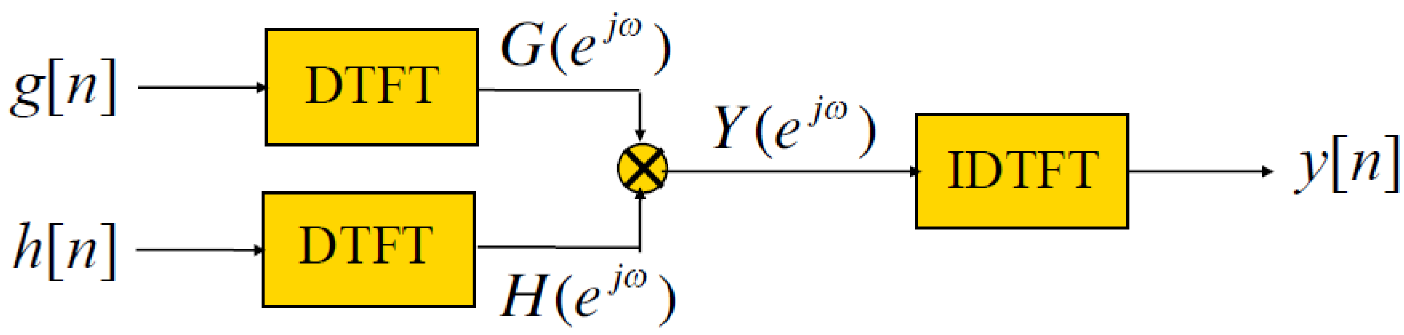
$$g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega}) H(e^{j\omega})$$

Convolution becomes multiplication

Convolution with DTFT

- Since $g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega})H(e^{j\omega})$

  we can calculate a convolution by:
  - finding DTFTs of $g, h \rightarrow G, H$
  - multiply them: $G \cdot H$
  - IDTFT of product is result, $g[n] \circledast h[n]$



DTFT convolution example

- $x[n] = \alpha^n \cdot \mu[n] \Rightarrow X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}}$
- $h[n] = \delta[n] - \alpha\delta[n-1]$

$$\Rightarrow H(e^{j\omega}) = 1 - \alpha\left(e^{-j\omega \cdot 1}\right) \cdot 1$$

- $y[n] = x[n] \circledast h[n]$

$$\Rightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$= \frac{1}{1-\alpha e^{-j\omega}} \cdot \left(1 - \alpha e^{-j\omega}\right) = 1$$

$\Rightarrow$ $y[n] = \delta[n]$ i.e. ...

DTFT modulation

- Modulation: $x[n] = g[n] \cdot h[n]$
  Could solve if $g[n]$ was just sinusoids...

$$X(e^{j\omega}) = \sum_{\forall n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) e^{j\theta n} d\theta \right) \cdot h[n] e^{-j\omega n}$$

write g[n] as IDTFT

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) \left[\sum_{\forall n} h[n] e^{-j(\omega-\theta)n} \right] d\theta$$

$$\Rightarrow \quad g[n] \cdot h[n] \;\leftrightarrow\; \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega-\theta)}) d\theta$$

Dual *of convolution in time*

Parseval's relation

- "Energy" in time and frequency domains are equal:

$$\sum_{\forall n} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})d\omega$$

- If $g = h$, then $g \cdot g^* = |g|^2$ = energy...

Energy density spectrum

- Energy of sequence $\varepsilon_g = \sum_{\forall n} |g[n]|^2$
- By Parseval $\varepsilon_g = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \, d\omega$
- Define Energy Density Spectrum (EDS)

$$S_{gg}(e^{j\omega}) = |G(e^{j\omega})|^2$$

EDS and autocorrelation

- Autocorrelation of $g[n]$:

$$r_{gg}[\ell] = \sum_{n=-\infty}^{\infty} g[n]g[n-\ell] = g[n] \circledast g[-n]$$

$$\Rightarrow \quad DTFT\{r_{gg}[\ell]\} = G(e^{j\omega})G(e^{-j\omega})$$

- If $g[n]$ is *real*, $G(e^{-j\omega}) = G^*(e^{j\omega})$, so

$$DTFT\{r_{gg}[\ell]\} = \left|G(e^{j\omega})\right|^2 = S_{gg}(e^{j\omega})$$

no phase info.

- Mag-sq of spectrum is DTFT of autoco