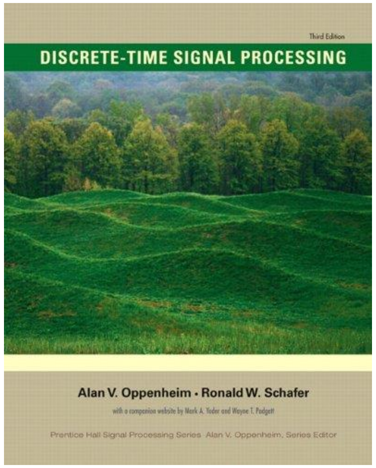


Lecture 06:

Sampling of Continuous-Time Signals

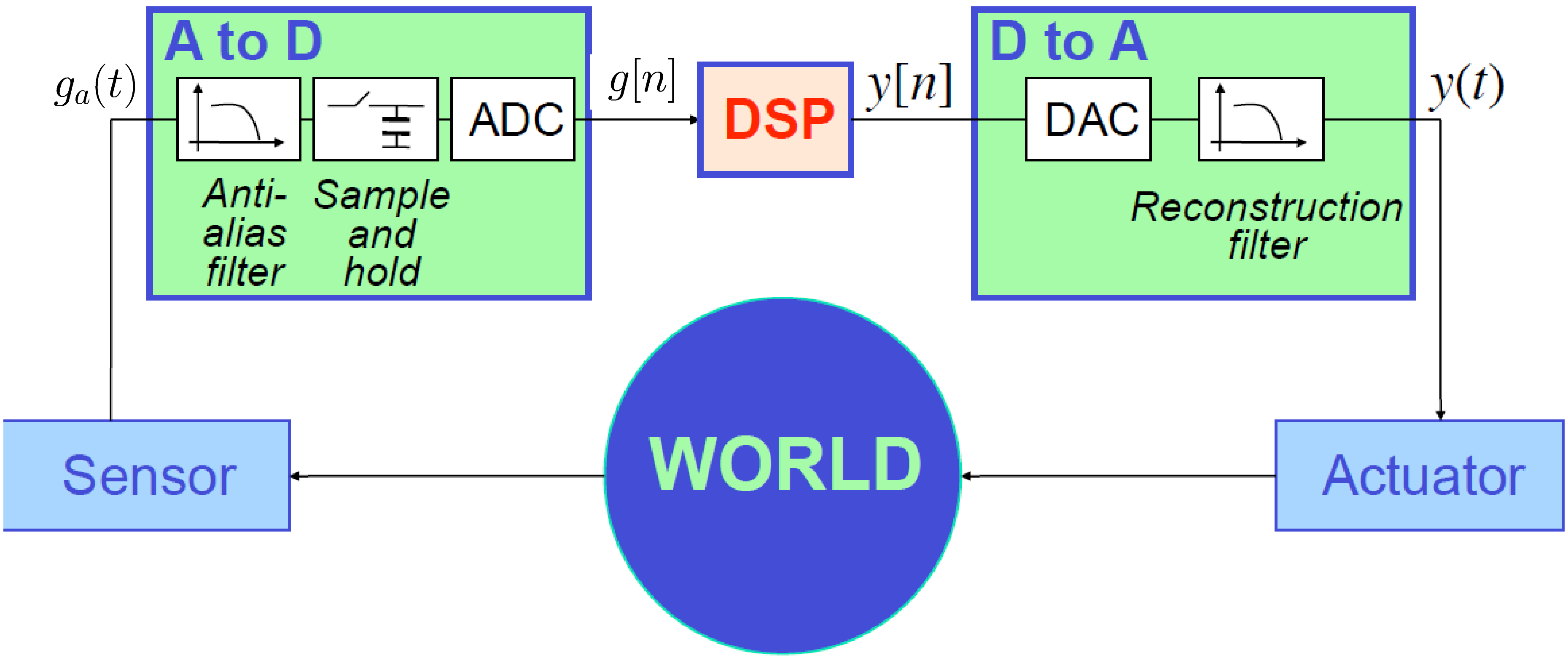
Outline

1. Sampling and Reconstruction

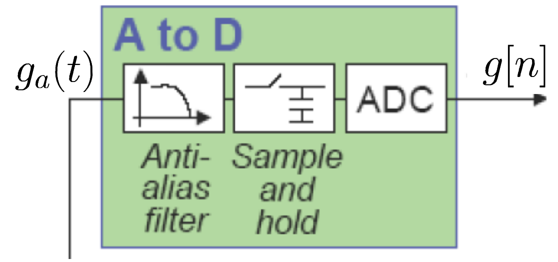
2. Quantization

1. Sampling & Reconstruction

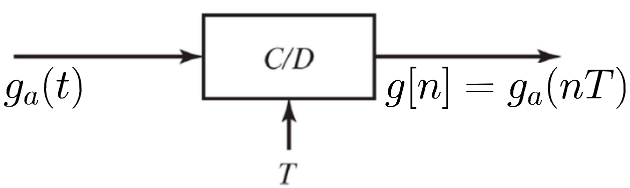
- DSP must interact with an analog world:



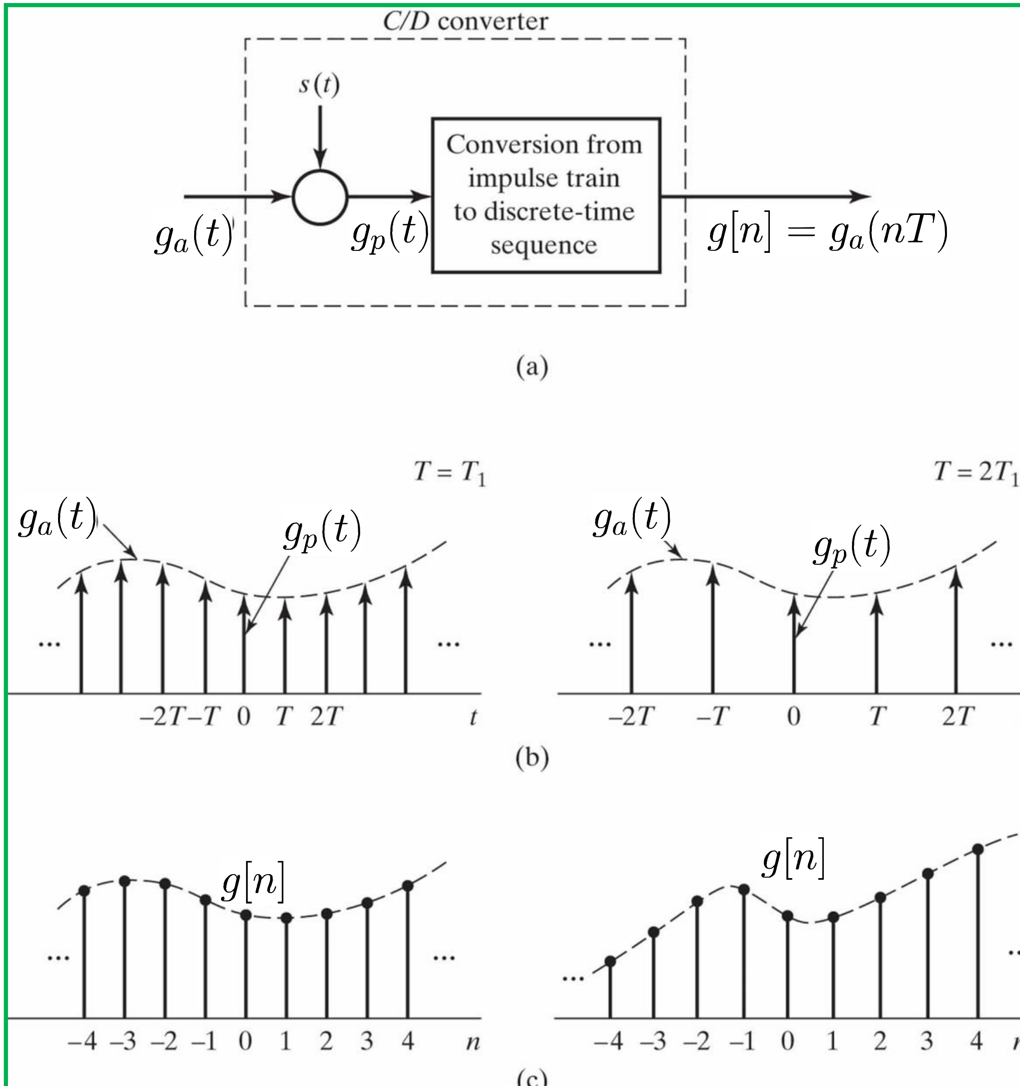
Sampling



$g_a(t)$
A to D
ADC
$g[n]$
Anti-alias filter
Sample and hold



$g_a(t)$
C/D
$g[n] = g_a(nT)$
T



C/D converter
s(t)
Conversion from impulse train to discrete-time sequence
$g_a(t)$
$g_p(t)$
$g[n] = g_a(nT)$
(a)
$T = T_1$
$T = 2T_1$
$g_a(t)$
$g_p(t)$
–2T –T 0 T 2T t
(b)
$g[n]$
–4 –3 –2 –1 0 1 2 3 4 n
(c)

Sampling : Frequency Domain

- Sampling: CT signal → DT signal by recording values at 'sampling instants':

Discrete $$g[n] = g_a(nT)$$ Continuous

Sampling period T

→ samp.freq. $\Omega_{samp} = 2\pi/T$ *rad/sec*

- What is the relationship of the spectra?

Ω in rad/**second**

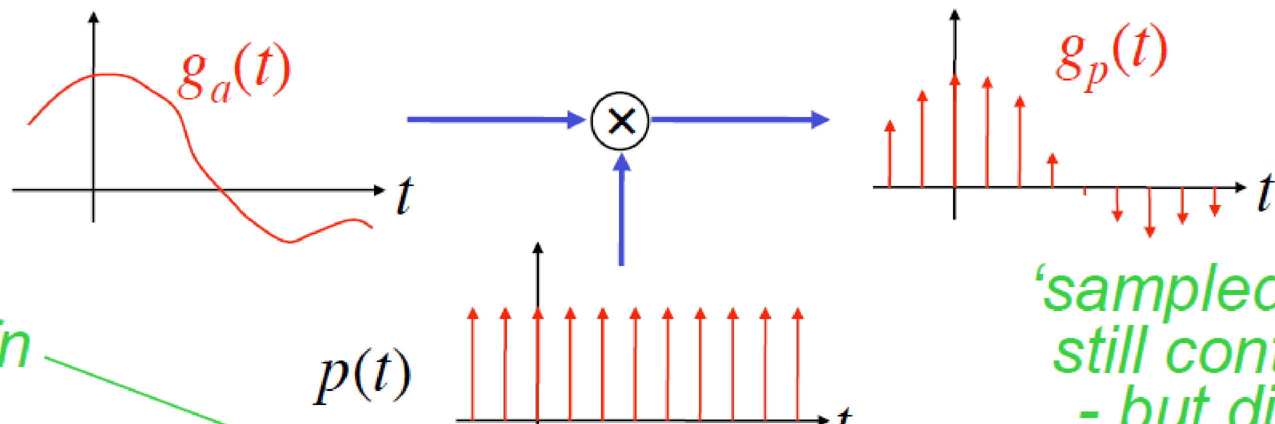
- i.e. relate $$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} \, dt$$ CTFT

and $$G(e^{j\omega}) = \sum_{-\infty}^{\infty} g[n] e^{-j\omega n}$$ DTFT

ω in rad/**sample**

Sampling

- DT signals have same 'content' as CT signals gated by an impulse train:



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- $g_p(t) = g_a(t) \cdot p(t)$ is a CT signal with the same information as DT sequence $g[n]$

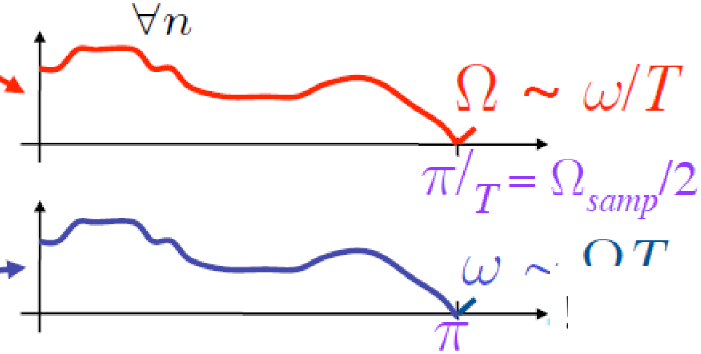
Spectra of Sampled Signals

- Given CT $g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT) \cdot \delta(t - nT)$
- CTFT Spectrum *by linearity*

$$G_p(j\Omega) = \mathcal{F}\{g_p(t)\} = \sum_{\forall n} g_a(nT)\mathcal{F}\{\delta(t-nT)\}$$

$$\Rightarrow G_p(j\Omega) = \sum_{\forall n} g_a(nT)e^{-j\Omega nT}$$

- Compare to DTFT $G(e^{j\omega}) = \sum_{\forall n} g[n]e^{-j\omega n}$
- i.e. $G(e^{j\omega}) = G_p(j\Omega)|_{\Omega T = \omega}$



Spectra of Sampled Signals

- Also, note that $p(t) = \sum_{\forall n} \delta(t - nT)$

  is periodic, thus has Fourier Series:

$$p(t) = \boxed{\frac{1}{T}} \sum_{k=-\infty}^{\infty} e^{j\left(\frac{2\pi}{T}\right)kt}$$

$$\because c_k = \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-j2\pi kt/T}\, dt = \frac{1}{T}$$

- But $F\left\{e^{j\Omega_0 t} x(t)\right\} = X\left(j(\Omega - \Omega_0)\right)$ *shift in frequency*

  so $\boxed{G_p(j\Omega) = \frac{1}{T} \sum_{\forall k} \underline{G_a\left(j(\Omega - k\Omega_{samp})\right)}}$

- scaled sum of replicas of $G_a(j\Omega)$
shifted by multiples of sampling frequency Ω_{samp}

CT and DT Spectra

- So: $G\left(e^{j\omega}\right) = G_p(j\Omega)\Big|_{\Omega T=\omega} = \frac{1}{T}\sum_{\forall k} G_a\left(j\left(\frac{\omega}{T} - k\frac{2\pi}{T}\right)\right)$

  or DTFT $G\left(e^{jT\Omega}\right) = \frac{1}{T}\sum_{\forall k} G_a\left(j(\Omega - k\Omega_T)\right)$ CTFT

$g_a(t) \leftrightarrow G_a(j\Omega)$

×

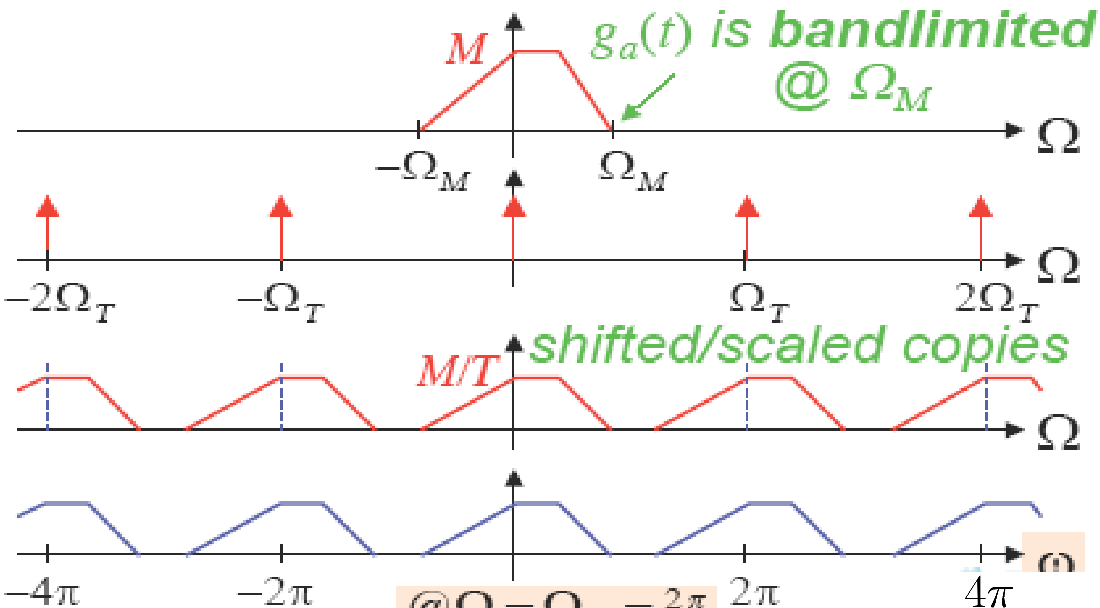
$p(t) \leftrightarrow P(j\Omega)$

=

$g_p(t) \leftrightarrow G_p(j\Omega)$

↓

$g[n] \leftrightarrow G\left(e^{j\omega}\right)$

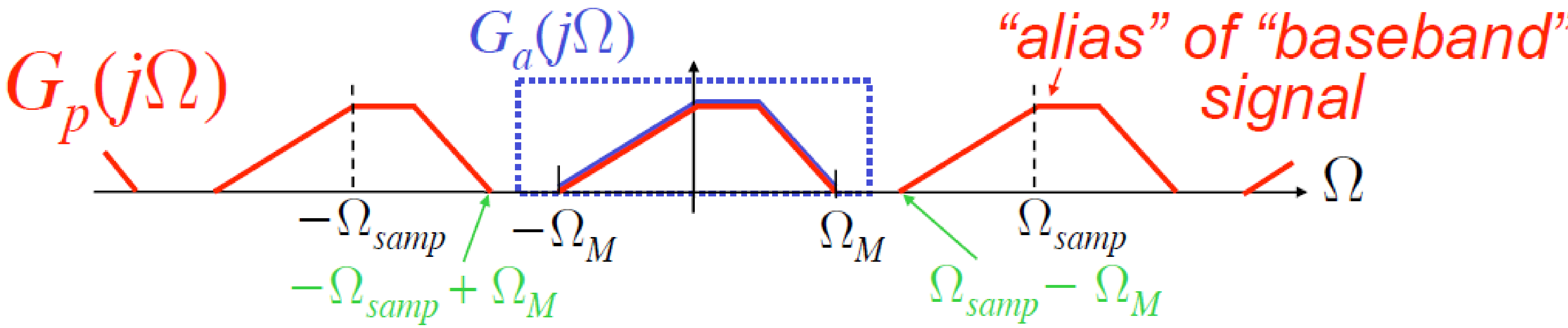


$@\Omega = \Omega_T = \frac{2\pi}{T}$

$\Rightarrow \omega = \frac{\Omega}{T} = 2\pi$

Avoid Aliasing

- Sampled analog signal has spectrum:



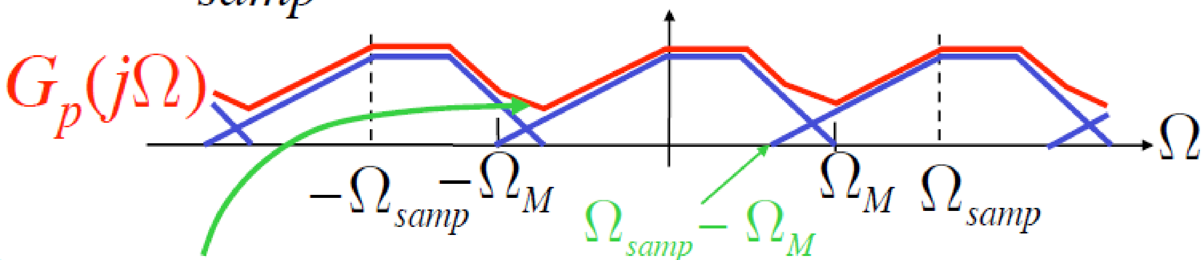
- $g_a(t)$ is bandlimited to $\pm\,\Omega_M$ rad/sec
- When sampling frequency Ω_{samp} is large…

→ no overlap between aliases

→ can recover $g_a(t)$ from $g_p(t)$ by low-pass filtering

The Nyquist Limit

- If bandlimit Ω_M is too large, or sampling rate Ω_{samp} is too small, aliases will overlap:



- Spectral effect cannot be filtered out
  → cannot recover $g_a(t)$

Sampling theorum

- Avoid by: $\Omega_{samp} - \Omega_M \geq \Omega_M \Rightarrow \Omega_{samp} \geq 2\Omega_M$
- i.e. bandlimit $g_a(t)$ at $\leq \dfrac{\Omega_{samp}}{2}$

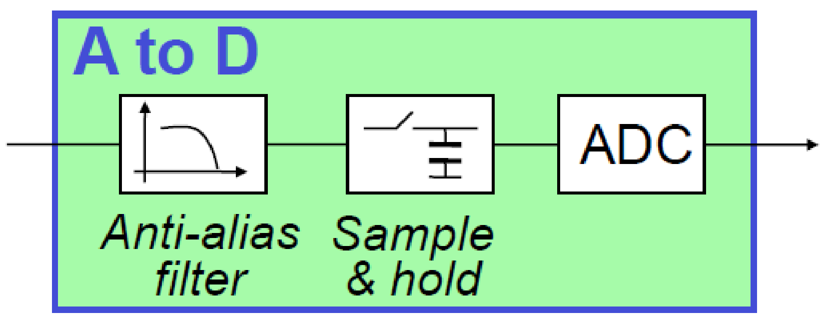
Nyquist frequency

Anti-Alias Filter

- To understand speech, need ~ 3.4 kHz
  → 8 kHz sampling rate (i.e. up to 4 kHz)
- Limit of hearing ~20 kHz
  → 44.1 kHz sampling rate for CDs

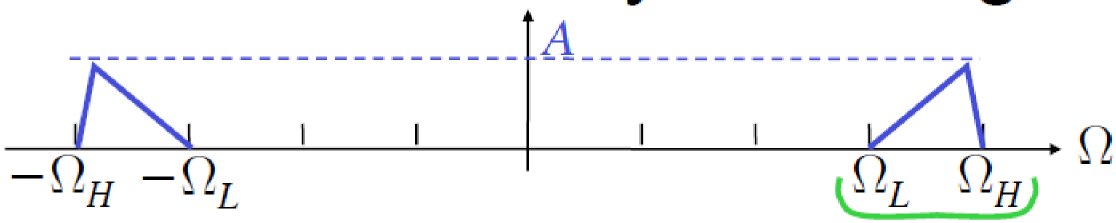
'space' for filter rolloff

- Must remove energy above Nyquist with LPF before sampling: "Anti-alias" filter



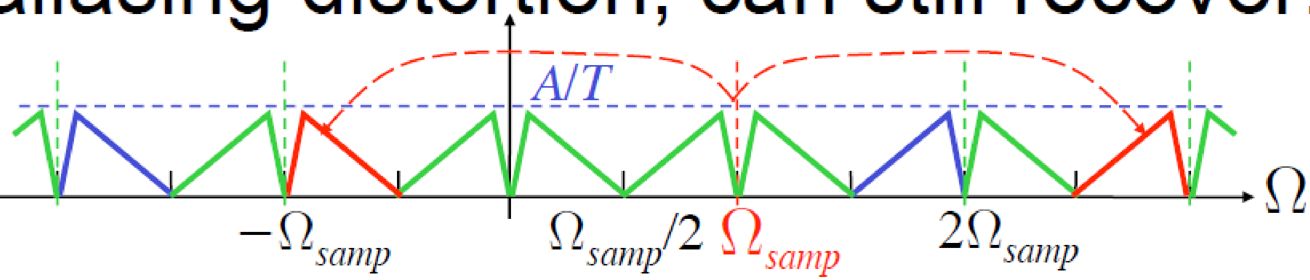
Sampling Bandpass Signals

- Signal is not always in 'baseband' around $\Omega = 0$ ... may be at higher Ω:



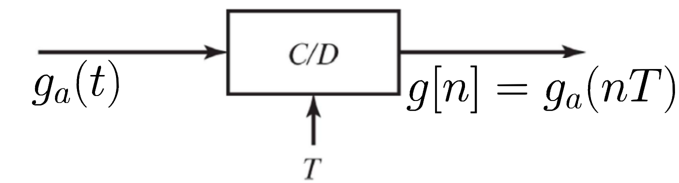
Bandwidth $\Delta\Omega = \Omega_H - \Omega_L$

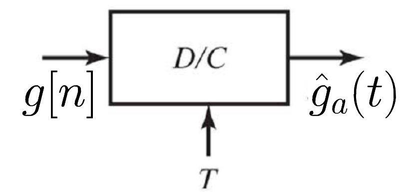
- If aliases from sampling don't overlap, no aliasing distortion; can still recover:

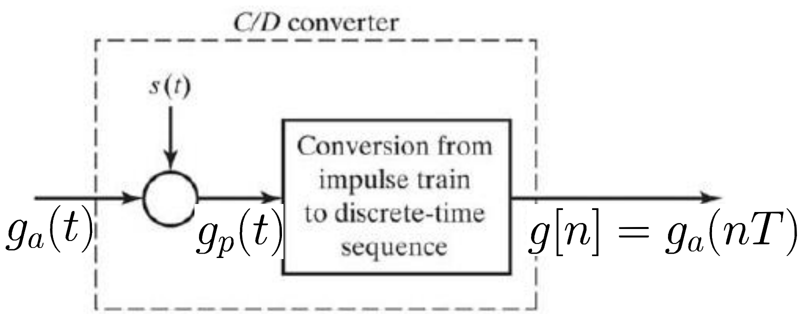


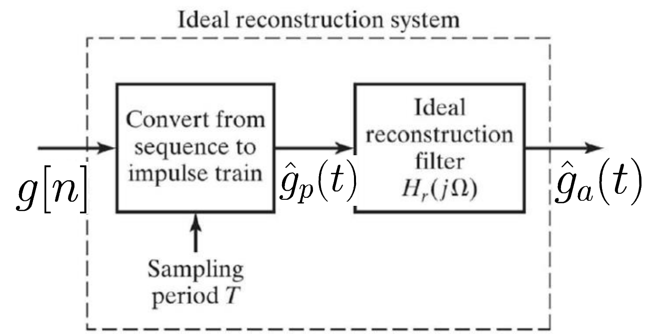
- Basic limit: $\Omega_{samp}/2 \geq$ bandwidth $\Delta\Omega$

Reconstruction (1/4)

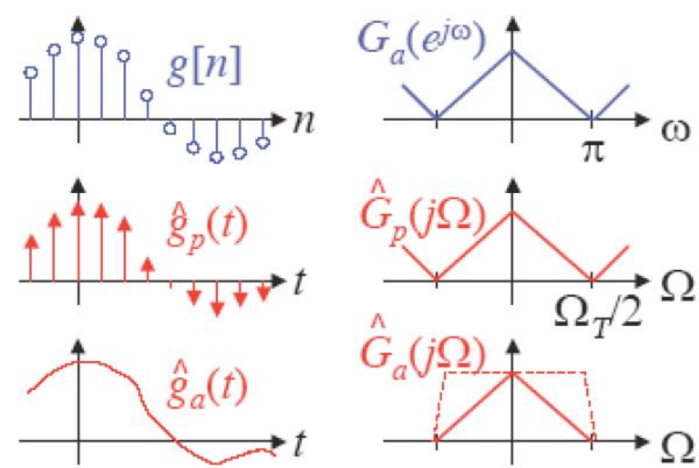






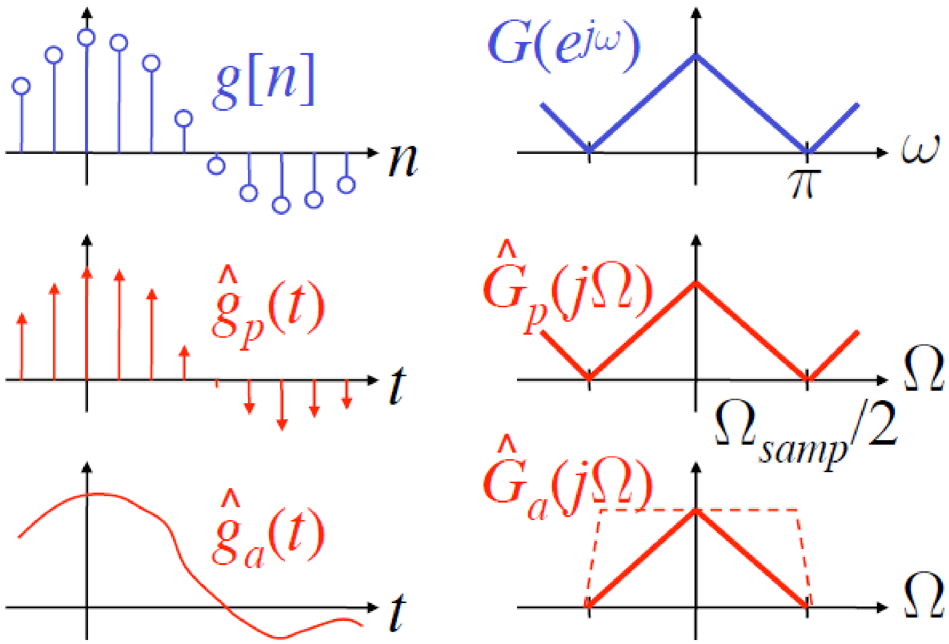


- To turn $g[n]$ back to $\hat{g}_a(t)$:



Reconstruction (2/4)

- To turn $g[n]$ back to $\hat{g}_a(t)$:
  - make a continuous impulse train $\hat{g}_p(t)$
  - lowpass filter to extract baseband $\rightarrow \hat{g}_a(t)$

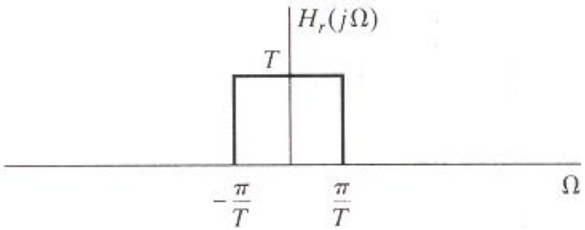


- Ideal reconstruction filter is *brickwall*
  - i.e. sinc - not realizable (especially analog!)
  - use something with finite transition band…

Reconstruction (3/4)

- Time axis is normalized by T
- Frequency axis is normalized by fs = 1/T
- Reconstruction filter

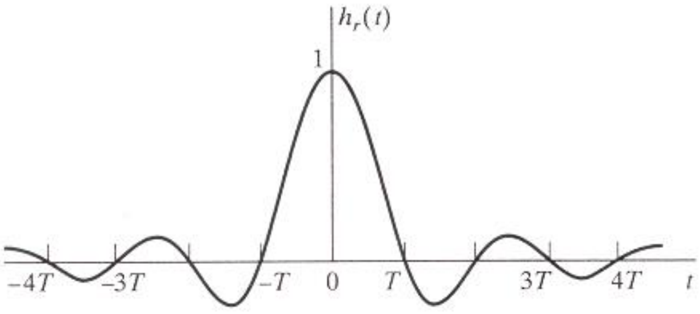
$$h_r(t) = \frac{\sin(\pi t/T)}{(\pi t/T)}$$



(b)

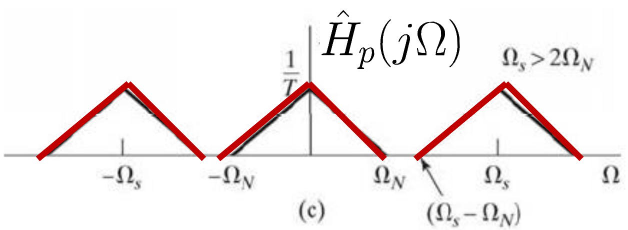
$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g(nT) h_r(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \pi((t-nT)/T)}{\pi(t-nT)/T}$$

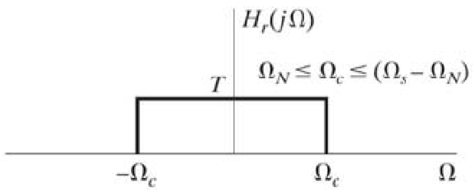


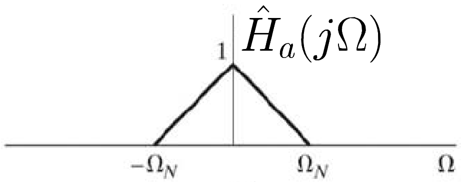
(c)

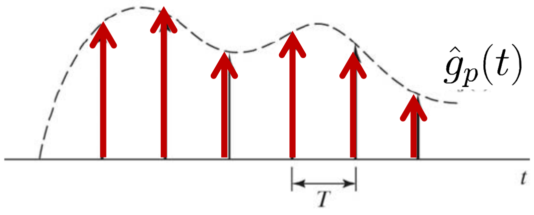
Reconstruction (4/4)



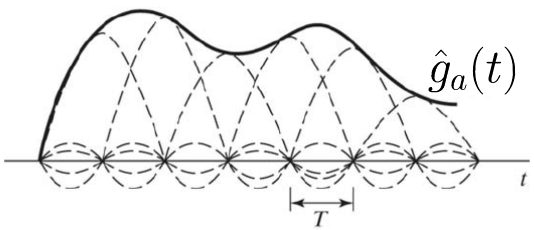
(c)







(b)



(c)

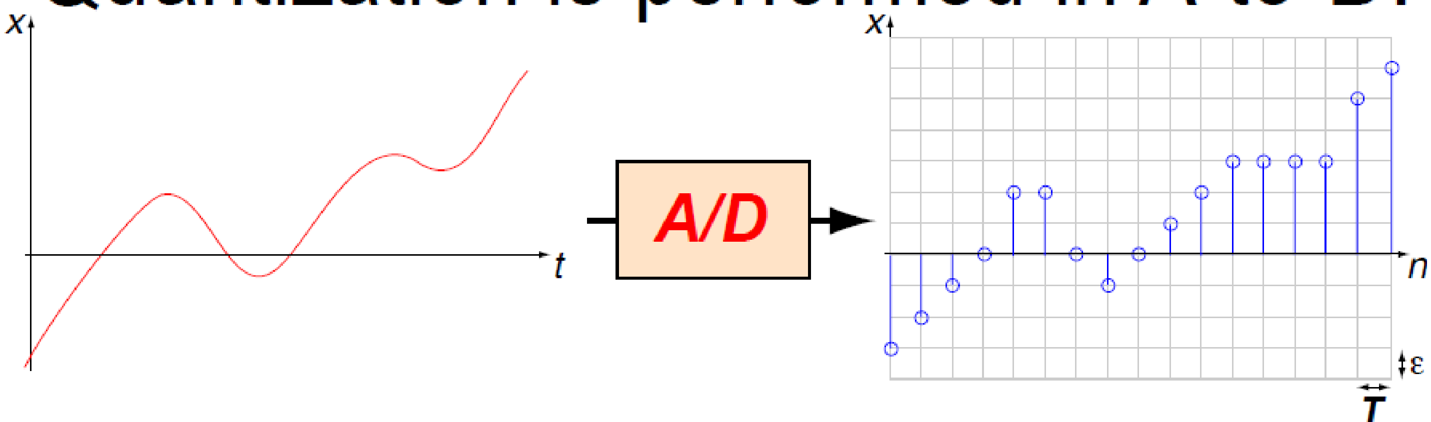
$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \pi \left((t - nT)/T\right)}{\pi (t - nT)/T}$$

2. Quantization

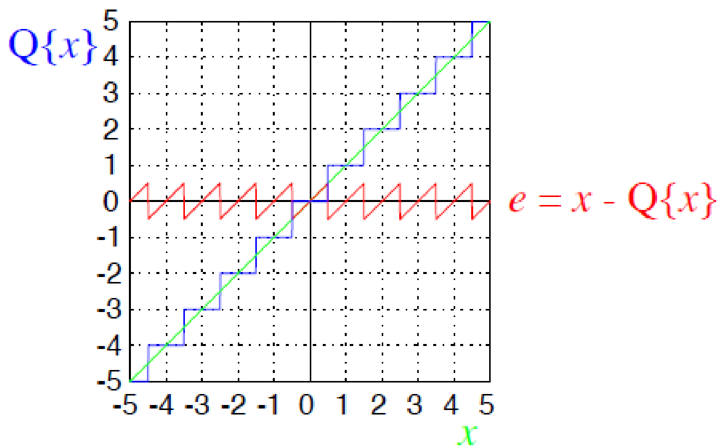
- Course so far has been about discrete-time i.e. quantization of time
- Computer representation of signals also quantizes level (e.g. 16 bit integer word)
- Level quantization introduces an error between ideal & actual signal → noise
- Resolution (# bits) affects data size → quantization critical for compression
  - smallest data ↔ coarsest quantization

Quantization

- Quantization is performed in A-to-D:



- Quantization has simple transfer curve:



Quantized signal

$$\hat{x} = Q\{x\}$$

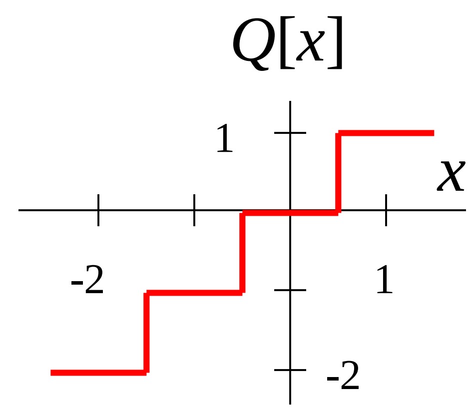
Quantization error

$$e = x - \hat{x}$$

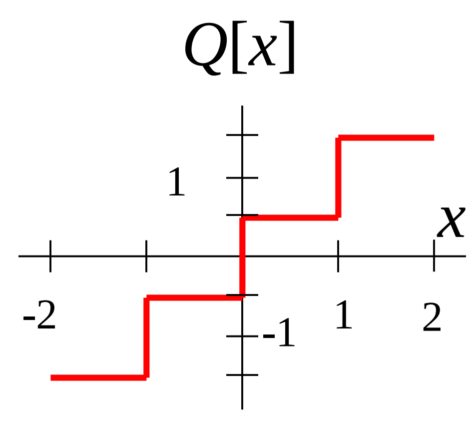
Uniform Amplitude Quantization

- Round to nearest integer (midtread)
  - Quantize amplitude to levels {-2, -1, 0, 1}
  - Step size Δ for linear region of operation
  - Represent levels by {00, 01, 10, 11} or {10, 11, 00, 01} …
  - Latter is two's complement representation
- Rounding with offset (midrise)
  - Quantize to levels {-3/2, -1/2, 1/2, 3/2}
  - Represent levels by {11, 10, 00, 01} …
  - Step size

$$\Delta = \frac{\frac{3}{2} - \left(-\frac{3}{2}\right)}{2^2 - 1} = \frac{3}{3} = 1$$



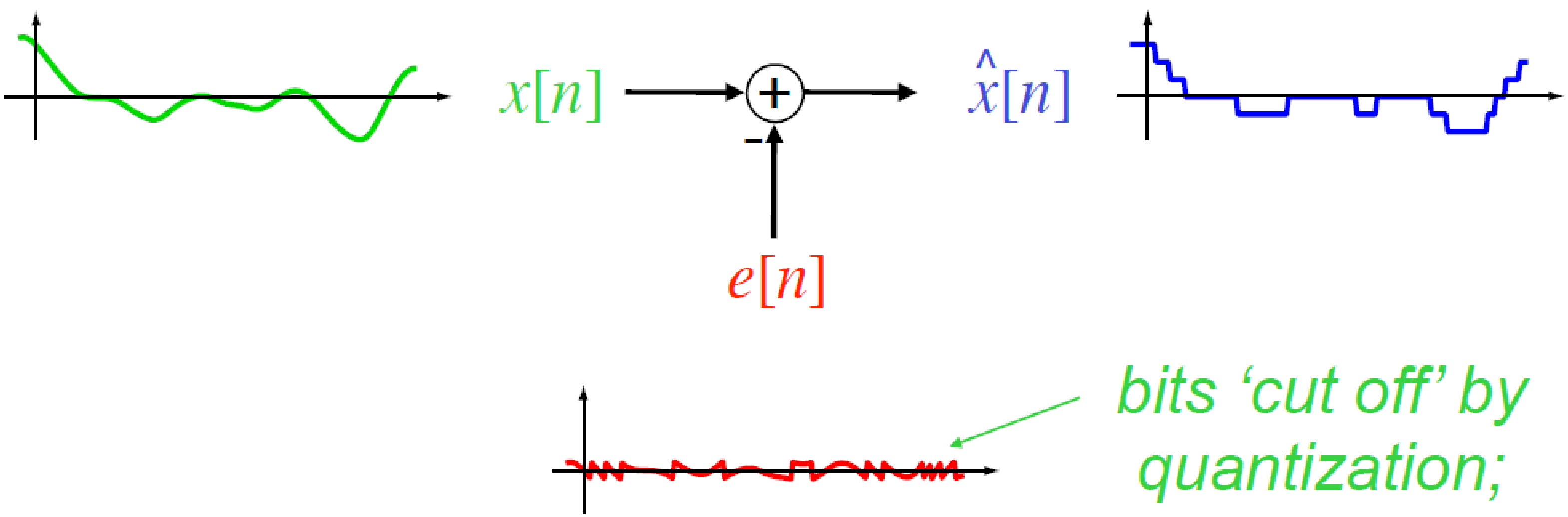
$$\Delta = \frac{1-(-2)}{2^2 - 1} = \frac{3}{3} = 1$$



Quantization Noise

- Can usually model quantization as additive white noise:

i.e. uncorrelated with self or signal x



bits 'cut off' by quantization; hard amplitude limit

Dynamic Range

- Signal-to-noise ratio in dB

$$\mathrm{SNR_{dB}} = 10 \log_{10} \frac{\text{Signal Power}}{\text{Noise Power}}$$
$$= 10 \log_{10} \text{Signal Power} -$$
$$10 \log_{10} \text{Noise Power}$$

- For linear systems, dynamic range equals SNR
- Lowpass anti-aliasing filter for audio CD format

  Ideal magnitude response of 0 dB over passband

  $A_{stopband}$ = 0 dB – Noise Power in dB = -98.08 dB

Why 10 $\log_{10}$?

For amplitude A,

$|A|_{\mathrm{dB}} = 20 \log_{10} |A|$

With power $P \propto |A|^2$,

$P_{\mathrm{dB}} = 10 \log_{10} |A|^2$

$P_{\mathrm{dB}} = 20 \log_{10} |A|$

Quantization SNR

- Common measure of noise is Signal-to-Noise ratio (SNR) in dB:

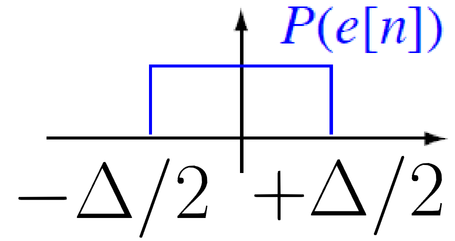
$$SNR = 10 \cdot \log_{10} \frac{\sigma_x^2}{\sigma_e^2} \text{ dB}$$

signal power

noise power

- When $|x|$ >> 1 LSB, quantization noise has ~ uniform distribution:

$$\Rightarrow \sigma_e^2 = \frac{\Delta^2}{12}$$



(quantizer step = Δ)

Quantization Error (Noise) Analysis

- Quantizer step size

$$\Delta = \frac{2\,m_{\max}}{L-1} \approx \frac{2\,m_{\max}}{L}$$

- Quantization error

$$-\frac{\Delta}{2} \le q \le \frac{\Delta}{2}$$

q is sample of zero-mean random process *Q*

q is uniformly distributed

$$\sigma_Q^2 = E\left\{Q^2\right\} - \underbrace{\mu_Q^2}_{zero}$$

$$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{1}{3}\,m_{\max}^2\;2^{-2B}$$

- Input power: $P_{\text{average,m}}$

$$\text{SNR} = \frac{\text{Signal Power}}{\text{Noise Power}}$$

$$\text{SNR} = \frac{P_{\text{average,m}}}{\sigma_Q^2} = \left(\frac{3P_{\text{average,m}}}{m_{\max}^2}\right)2^{2B}$$

SNR exponential in *B*

Adding 1 bit increases SNR by factor of 4

Quantization Error (Noise) Analysis

- SNR in dB = constant + 6.02 dB/bit * *B*

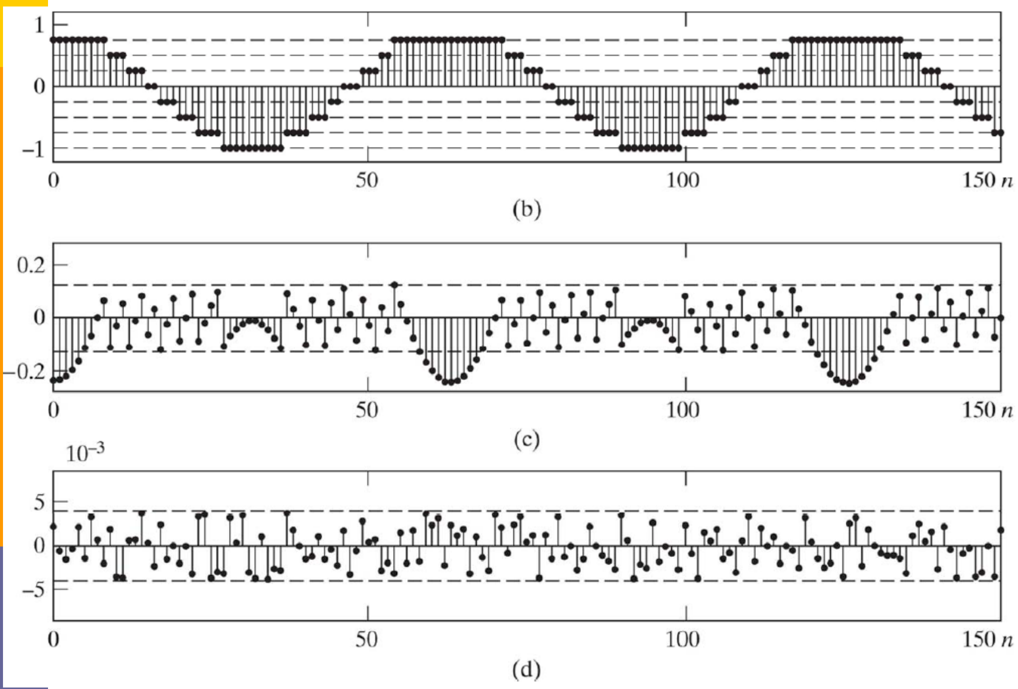
Loose upper bound

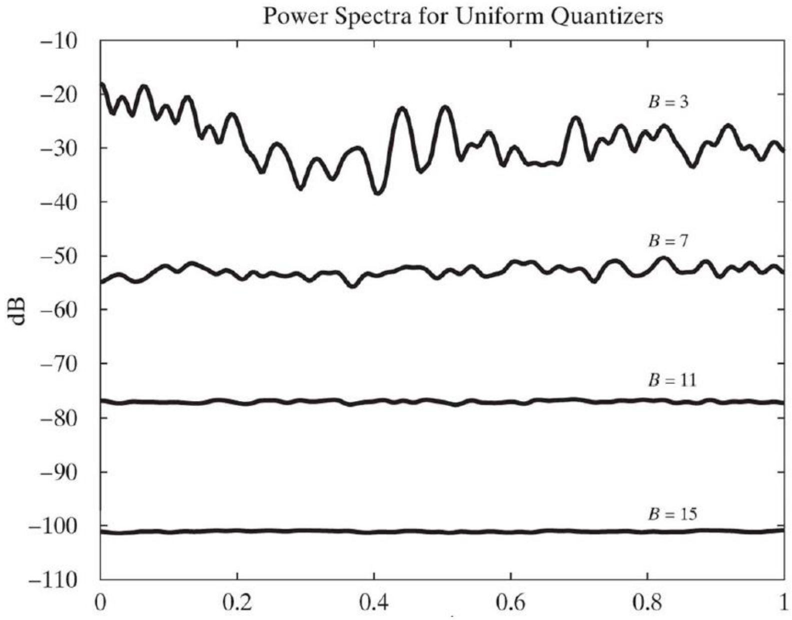
$$
\begin{aligned}
10 \log_{10} \mathrm{SNR} &= 10 \log_{10}\left(\left(\frac{3P_{average,m}}{m_{\max}^2}\right) 2^{2B}\right) \\
&= 10 \log_{10} 3 + 10 \log_{10}\left(P_{average,m}\right) - 20 \log_{10}\left(m_{\max}\right) + 20\,\mathrm{B} \log_{10}(2) \\
&= \underbrace{0.477 + 10 \log_{10}\left(P_{average,m}\right) - 20 \log_{10}\left(m_{\max}\right)} + 6.02\,B
\end{aligned}
$$

1.76 and 1.17 are common constants used in audio

- What is maximum number of bits of resolution for
  - Audio CD signal with SNR of 95 dB
  - TI TLV320AIC23B stereo codec used on TI DSP board
    - ADC 90 dB SNR (14.6 bits) and 80 dB THD (13 bits)
    - DAC has 100 dB SNR (16 bits) and 88 dB THD (14.3 bits)

PSD of Quantization Noise





Coefficient Quantization

- Quantization affects not just signal but filter constants too
  - .. depending on implementation
  - .. may have different resolution
- Some coefficients are very sensitive to small changes
  - e.g. poles near unit circle