Discrete-Time Signal Processing

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1 The Discrete-Time Fourier Transform

1.1 Definition

The discrete-time Fourier transform (DTFT) maps an aperiodic discrete-time signal $x[n]$ to the frequency-domain function $X(e^{j\Omega}) = \mathcal{F}\left\{x[n]\right\}$. Likewise, we write $x[n] = \mathcal{F}^{-1}\left\{X(e^{j\Omega})\right\}$.

The DTFT pair $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\Omega})$ satisfies:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega \quad &\text{(synthesis equation)}\\
X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \quad &\text{(analysis equation)}
\end{aligned}
$$

Remarks:

- Note that
$$e^{j(\Omega+2\pi)n} = e^{j\Omega n}e^{j2\pi n} = e^{j\Omega n}$$
for any integer n. So, in the synthesis equation, $\int_{2\pi}$ represents integration over any interval $[a, a+2\pi)$ of length 2π. On the other hand, $x[n]$ is assumed to be aperiodic, so the summation in the analysis equation is over all n.

- Thus, the DTFT $X(e^{j\Omega})$ is always periodic with period 2π.

- The frequency response of a DT LTI system with unit impulse response $h[n]$
$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$
is precisely the DTFT of $h[n]$, so the response $y[n]$ that corresponds to the input $x[n] = e^{j\Omega n}$ is given by
$$y[n] = H(e^{j\Omega})e^{j\Omega n}.$$

Convergence. If $x[n]$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

then $X(e^{j\Omega})$ is well-defined (i.e., finite) for all $\Omega \in \mathbb{R}$. Thus, there is a unique $X(e^{j\Omega})$ for each absolutely summable $x[n]$.

Inversion. If $X(e^{j\Omega})$ is the DTFT of an absolutely summable DT signal $x[n]$, then

$$\begin{aligned}
\frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega &= \frac{1}{2\pi}\int_{2\pi}\left(\sum_{m=-\infty}^{\infty} x[m]e^{-j\Omega m}\right)e^{j\Omega n}d\Omega \\
&= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} x[m]\underbrace{\int_{2\pi} e^{-j\Omega(n-m)}d\Omega}_{2\pi\delta[n-m]} \\
&= \sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \\
&= x[n].
\end{aligned}$$

Thus, there is a one-to-one correspondence between an absolutely summable $x[n]$ and its DTFT $X(e^{j\Omega})$.

Periodic Signals. For a periodic DT signal, $x[n] = x[n+N]$, let a_k be the discrete-time Fourier series coefficients

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n},$$

where $\Omega_0 = 2\pi/N$. Then, the DTFT consists is a sum of Dirac delta functions:

$$X(e^{j\Omega}) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\Omega - k\Omega_0).$$

1.2 DTFT Examples

There are a few important DTFT pairs that are used regularly.

Example 1. Consider the *discrete-time rectangular pulse*

$$x[n] = \sum_{k=0}^{M} \delta[n-k].$$

The DTFT of this signal is given by

$$
\begin{aligned}
X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\
&= \sum_{k=0}^{M} e^{-j\Omega n} \\
&= \frac{e^{-j\Omega(M+1)} - 1}{e^{-j\Omega} - 1} \\
&= \frac{(e^{-j\Omega(M+1)/2} - e^{j\Omega(M+1)/2}) e^{-j\Omega(M+1)/2}}{(e^{-j\Omega/2} - e^{j\Omega/2}) e^{-j\Omega/2}} \\
&= \frac{\sin(\Omega(M+1)/2)}{\sin(\Omega/2)} e^{-j\Omega M/2}.
\end{aligned}
$$

The first term is the discrete-time analog of the sinc function and the second term is the phase shift associated with the fact that the pulse is not centered about $n = 0$. If M is even, we can advance the pule by $M/2$ samples to remove this term.

Example 2. An *ideal discrete-time low-pass filter* passing $|\Omega| < \Omega_c \leq \pi$ has the DTFT frequency-response

$$
H(e^{j\Omega}) = \begin{cases} 1 & \text{if } |\Omega| < \Omega_c \\ 0 & \text{if } \Omega_c < |\Omega| \leq \pi. \end{cases}
$$

Thus, the inverse DTFT implies that

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi} \int_{2\pi} H(e^{j\Omega}) e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi} \int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} d\Omega \\
&= \frac{1}{2\pi j n} \left[e^{j\Omega n} \right]_{-\Omega_c}^{\Omega_c} \\
&= \frac{1}{\pi n} \sin(\Omega_c n).
\end{aligned}
$$

This impulse response is not absolutely summable due to the discontinuity in $H(e^{j\Omega})$. To understand the DTFT in this case, let $H_M(e^{j\Omega})$ be the frequency response of the truncated impulse response

$$
h_M[n] = \begin{cases} h[n] & \text{if } |n| \leq M \\ 0 & \text{if } |n| > M. \end{cases}
$$

Due to the Gibb's phenomenon, $H_M(e^{j\Omega})$ does not converge uniformly to $H(e^{j\Omega})$. But, since $\sum_{n=-\infty}^{\infty} |h[n]|^2 < \infty$, the frequency response instead converges in the mean-square sense

$$
\lim_{M \to \infty} \int_{-\pi}^{\pi} \left| H(e^{j\Omega}) - H_M(e^{j\Omega}) \right|^2 d\Omega = 0.
$$

1.3 Properties of the DTFT

The canonical properties of the DTFT are very similar to the canonical properties of the continuous-time Fourier transform (CTFS). For $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\Omega})$ and $y[n] \overset{\mathcal{F}}{\longleftrightarrow} Y(e^{j\Omega})$, we observe that:

1) Linearity: $ax[n] + by[n] \overset{\mathcal{F}}{\longleftrightarrow} aX(e^{j\Omega}) + bY(e^{j\Omega})$

2) Time shift: $x[n - n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\Omega n_0} X(e^{j\Omega})$

3) Frequency shift: $e^{j\Omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\Omega-\Omega_0)})$

4) Conjugation: $x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\Omega})$

5) Time flip: $x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{-j\Omega}\right)$

6) Convolution: $x[n] * y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\Omega})Y(e^{j\Omega})$

7) Multiplication: $x[n]y[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\Omega}) \circledast Y(e^{j\Omega}) \triangleq \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\Omega-\theta)})d\theta$

8) Symmetry:

$$x[n] \text{ real} \Longrightarrow Re\{X(e^{j\Omega})\} \text{ even, } Im\{X(e^{j\Omega})\} \text{ odd, } |X(e^{j\Omega})| \text{ even, } \angle X(e^{j\Omega}) \text{ odd}$$

$$x[n] \text{ real \& even} \Longrightarrow X(e^{j\Omega}) \text{ real \& even}$$

$$x[n] \text{ real \& odd} \Longrightarrow X(e^{j\Omega}) \text{ pure imaginary \& odd}$$

9) Parseval's relation:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\Omega})|^2 d\Omega$$

1.4 Frequency-Domain Characterization of DT LTI Systems

The time-domain characterization of DT LTI systems is given by

$$y[n] = x[n] * h[n],$$

where $h[n]$ is the unit impulse response of the system. Taking the DTFT of this equation gives the frequency-domain characterization of DT LTI systems,

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}).$$

The DTFT, $H(e^{j\Omega})$, of $h[n]$ is called the frequency response of the system.

1.5 Spectral Density

For an energy-type signal $x[n]$, the energy spectral density is defined to be

$$R_{xx}(e^{j\Omega}) \triangleq \left|X(e^{j\Omega})\right|^2.$$

This notation is used because $R_{xx}(e^{j\Omega})$ is the DTFT of the autocorrelation signal $r_{xx}[\ell]$. To see this, we first note that $r_{xx}[\ell] = x[n] * y[n]$ for $y[n] = x^*[-n]$. Then, the convolution theorem implies that

$$R_{xx}(e^{j\Omega}) = \mathcal{F}\left\{x[n]\right\}\mathcal{F}\left\{x^*[-n]\right\} = X(e^{j\Omega})X^*(e^{j\Omega}) = \left|X(e^{j\Omega})\right|^2.$$

For a power-type signal (e.g., a periodic signal), one considers a sequence (in M) of normalized energy spectral densities for the windowed signals $x_M[n] = x[n]w[n]$ with $w[n] = u[n+M] - u[n-M+1]$. Using this, the power spectral density is defined to be

$$\begin{aligned}
\overline{R}_{xx}(e^{j\Omega}) &\triangleq \lim_{M\to\infty} \frac{1}{2M+1}\left|X_M(e^{j\Omega})\right|^2 \\
&= \lim_{M\to\infty} \frac{1}{2M+1}\mathcal{F}\left\{x_M[n]\right\}\mathcal{F}\left\{x_M^*[-n]\right\} \\
&= \lim_{M\to\infty} \mathcal{F}\left\{\frac{1}{2M+1}\sum_{n=-M+\max(0,\ell)}^{M+\min(0,\ell)} x[n]x^*[n-\ell]\right\} \\
&= \mathcal{F}\left\{\lim_{M\to\infty}\frac{1}{2M+1}\sum_{n=-M+\max(0,\ell)}^{M+\min(0,\ell)} x[n]x^*[n-\ell]\right\} \\
&= \mathcal{F}\left\{\lim_{M\to\infty}\frac{1}{2M+1}\sum_{n=-M}^{M} x[n]x^*[n-\ell]\right\} \\
&= \mathcal{F}\left\{\overline{r}_{xx}[\ell]\right\},
\end{aligned}$$

where $\overline{r}_{xx}[\ell]$ is the time-average autocorrelation function of $x[n]$.

The operational meaning of these definitions is that the total energy (or power) signal energy contained in the frequency band $[\Omega_1, \Omega_2]$ is given by

$$E_{[\Omega_1,\Omega_2]} = \frac{1}{2\pi}\int_{\Omega_1}^{\Omega_2} R_{xx}(e^{j\Omega})d\Omega,$$

where the 2π scale factor is chosen so that

$$E_{[-\pi,\pi]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(e^{j\Omega})d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\Omega})\right|^2 d\Omega = \sum_{n=-\infty}^{\infty}|x[n]|^2$$

follows from Parseval's relation. If the signal is real, then conjugate symmetry implies that $R_{xx}(e^{j\Omega}) = R_{xx}(e^{-j\Omega})$. Thus, one typically assumes $0 \le \Omega_1 \le \Omega_2 \le \pi$ and says the total energy in the positive frequency band $[\Omega_1, \Omega_2]$ is

$$E = E_{[-\Omega_2,-\Omega_1]} + E_{[\Omega_1,\Omega_2]} = 2E_{[\Omega_1,\Omega_2]}.$$

Thus, one must be careful to distinguish between these two conventions.

1.6 The Discrete Fourier Transform

The DTFT maps a discrete-time signal $x[n]$ to a continuous (but periodic) frequency domain $X(e^{j\Omega})$. The discrete Fourier transform (DFT) of length-N maps a discrete-time sequence $x[n]$ (supported on $0 \le n \le N-1$) to a discrete set of frequencies $X[k]$ (supported on $0 \le k \le N-1$) using the rule

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-2\pi jkn/N}.$$

It is easy to verify that the inverse DFT is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{2\pi jkn/N}.$$

This DFT is closely related to computing values of the DTFT on a regularly spaced grid. For example, let $\Omega_k = \frac{2\pi k}{N}$ and observe that

$$X[k] = X(e^{2\pi jk/N}) = X(e^{j\Omega_k}).$$

If the DT signal $x[n]$ is periodic, then this transform arises naturally because the DTFT consists of Dirac delta functions supported on a discrete set of harmonic frequencies $\Omega_k = \frac{2\pi k}{N}$ for $k = 0, 1, \ldots, N-1$. For real signals, conjugate symmetry (i.e., $X[k] = X^*[N-k]$) implies that the signal information is contained completely in the first $\lceil (N+1)/2 \rceil$ values of $X[k]$.

The output $X[k_0]$ is called the DFT for "frequency bin k_0" because the DFT can be seen to transform N time-domain samples into N frequency-domain samples. Moreover, this transform is unitary (except for an overall scale factor) and preserves the energy in the sense that

$$\begin{aligned}
E_x &= \sum_{n=0}^{N-1} |x[n]|^2 \\
&= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{i=0}^{N-1} X[i] e^{2\pi jin/N} \right)^* \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{2\pi jkn/N} \right) \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} X^*[i] X[k] \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi j(k-i)n/N} \\
&= \frac{1}{N} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} X^*[i] X[k] \delta[k-i] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2 .
\end{aligned}$$

Thus, the DFT decomposes the signal energy E_x into N frequency bins that expose the signal's energy distribution over frequency.

For a short duration signal $x[n]$ (supported on $0 \leq n \leq M-1$), one can compute the DFT on a denser set of frequencies by zero padding the signal up to length $N > M$. In practice, this computation can be done quickly by zero padding the sequence up to a power of 2 length $N = 2^m$ and applying the fast Fourier transform (FFT).

One can also use the DFT to numerically approximate the energy (or power) spectral density of a DT signal $x[n]$. Using a length-N DFT, the idea is to approximate the integrals in the previous section. For the energy spectral density, one gets

$$E_{[\Omega_1,\Omega_2]} \approx \frac{1}{N} \sum_{k=\lfloor \Omega_1 N/(2\pi) \rfloor}^{\lfloor \Omega_2 N/(2\pi) \rfloor} |X[k]|^2$$

and an additional scale factor of $1/N$ is used for the power spectral density.

2 FIR Filter Design via the Window Method

As we saw in the last section, the impulse response of an ideal DT low-pass filter can be computed in closed form. This filter cannot be implemented, however, because its impulse response extends to $n = -\infty$. To fix this, one can truncate impulse response to get

$$h_M[n] = \begin{cases} \frac{1}{\pi n}\sin(\Omega_c n) & \text{if } |n| \leq M \\ 0 & \text{if } |n| > M. \end{cases}$$

To understand the effect of this truncation, we rewrite this as

$$h_M[n] = \frac{1}{\pi n}\sin(\Omega_c n) v_M[n],$$

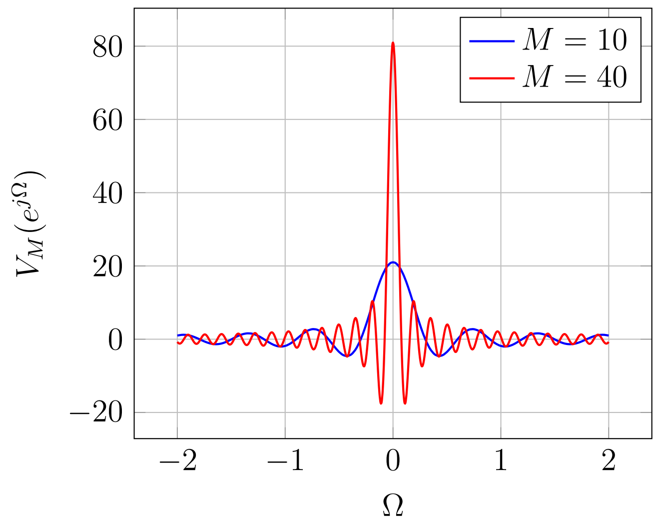
where $v_M[n] = \sum_{k=-M}^{M} \delta[n-k]$. From this, we observe that the frequency response is given by

$$H_M(e^{j\Omega}) = H(e^{j\Omega}) \circledast V_M(e^{j\Omega}).$$

From the DTFT examples, it is easy to see that

$$V_M(e^{j\Omega}) = \frac{\sin(\Omega(2M+1)/2)}{\sin(\Omega/2)}.$$

The portion of this function satisfying $|\Omega| \leq \Omega_0 = 2\pi/(2M+1)$ is called the main lobe of the window response. The section from $\Omega_0 \leq |\Omega| \leq 2\Omega_0$ is called the first sidelobe of the window response. This figure shows the response for two M values.



In the window method of filter design, the main lobe of the window smears the cutoff frequency into a transition band whose width approximately equals the width of the window's main lobe. Likewise, the height of the first side lobe typically limits the stop-band rejection. Choosing the window function to optimize this trade-off is known as the window method of filter design. If delay and complexity are not a problem, one can always choose a long enough window so that the error is quite small.

To achieve better stop-band rejection, one needs to choose a smoother window function. For example, the Hann window function

$$w[n] = \frac{1}{2}\left(1 + \cos\left(\frac{\pi n}{M}\right)\right)$$

(for $n = -M, -M+1, \ldots, M$) provides reasonable performance. For the Hann window, this filter design method is implemented by the MATLAB function `h=fir1(2*M,wc/pi,hann(2*M+1))`.

3 From Discrete Time to Continuous Time

The process of "filling in" the signal waveform between sample points is called interpolation. From a purely discrete-time perspective, the process of interpolation consists of designing a filter that results in a non-integer time delay.

Example 3. An *ideal delay* of n_0 samples maps $e^{j\Omega n}$ to $e^{j\Omega(n-n_0)}$. Thus, it can be represented by an LTI system whose DTFT frequency response is

$$H\left(e^{j(2\pi k+\Omega)}\right) = e^{-j\Omega n_0},$$

for $\Omega \in (-\pi, \pi]$ and integer k. From this, the inverse DTFT implies that

$$\begin{aligned} h[n] &= \frac{1}{2\pi} \int_{2\pi} H(e^{j\Omega}) e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j\Omega n_0} e^{j\Omega n} d\Omega \\ &= \frac{1}{2\pi j (n - n_0)} \left[e^{j\Omega (n-n_0)} \right]_{-\pi}^{\pi} \\ &= \frac{\sin(\pi(n - n_0))}{\pi(n - n_0)}. \end{aligned}$$

Since $y[n] = x[n] * h[n]$ implies that $y[0]$ is equal to the signal value at time $t = n_0$, the general interpolation formula is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(n - t))}{\pi(n - t)}.$$

In practice, one needs to window this filter to make its duration finite.

The same interpolation formula can also be derived as the result of filtering the CT signal $x(t) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - k)$ with the ideal CT low-pass filter

$$h(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega t} = \frac{\sin(\pi t)}{\pi t}.$$

4 Sample Rate Conversion

To achieve sample rate conversion by a rational factor $R = L/M$, the direct approach is to upsample by L, filter, and then downsample by M. The complexity of this approach can be quite high and there are a variety of ways to reduce the complexity.

Example 4 (Upsampling by L)**.** Consider the signal

$$y[n] = \begin{cases} x[n/L] & \text{if } L \text{ divides } n \\ 0 & \text{otherwise,} \end{cases}$$

where $y[n]$ is formed by inserting $L - 1$ zeros between each sample of $x[n]$. The DTFT of $y[n]$ is given by

$$\begin{aligned} Y(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} y[n] e^{-j\Omega n} \\ &= \sum_{n=-\infty}^{\infty} y[Ln] e^{-j\Omega (Ln)} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-Lj\Omega n} \\ &= X(e^{jL\Omega}). \end{aligned}$$

Since $X(e^{j\Omega})$ is periodic with period 2π, the DTFT of $Y(e^{j\Omega})$ is periodic with period $2\pi/L$ and it contains L aliased copies (or images) of $X(e^{j\Omega})$. Passing $y[n]$ through an ideal low-pass filter with cutoff frequency $\Omega_c = \pi/L$ leaves only a single low-pass version and removes the extra "images". The result is a signal with the same frequency content as $x[n]$ but sampled L times faster.

Example 5 (Downsampling by M)**.** Consider the signal

$$y[n] = x[nM]$$

where $y[n]$ is formed by taking only every M-th sample. In the frequency domain, this operation is best understood as the combination of two steps. First, we let $v[n] = \sum_{k=-\infty}^{\infty} \delta[n-kM]$ and define $z[n] = x[n]v[n]$. Then, we let $y[n] = z[nM]$. Since the DTFT table shows that

$$V(e^{j\Omega}) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{M}\right),$$

the effect of the first step in the frequency domain is given by

$$Z(e^{j\Omega}) = X(e^{j\Omega}) \circledast V(e^{j\Omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j(\Omega - 2\pi k/M)}\right).$$

From this, we see that the DTFT $Z(e^{j\Omega})$ is periodic with period $2\pi/M$. Working backwards in the previous example, one can see that effect of the second step is given by

$$Z(e^{j\Omega}) = Y(e^{jM\Omega}).$$

Thus, we find that

$$Y(e^{j\Omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\Omega/M - 2\pi k/M}\right).$$

Finally, if $X(e^{j\Omega}) = 0$ for $\frac{2\pi}{M} \leq |\Omega| \leq \pi$, then $Y(e^{j\Omega}) = \frac{1}{M} X(e^{j\Omega/M})$.

5 The Z-Transform

5.1 Introduction

The Z-transform is a generalization of the discrete-time Fourier transform (DTFT) that can be applied to a larger class of signals. It is also the discrete-time analog of the Laplace transform. For a DT signal $x[n]$, the Z-transform $X(z) = \mathcal{Z}\{x[n]\}$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.$$

The region of convergence (ROC) for a signal $x[n]$ is the set of (complex) z-values for which the sum is absolutely convergent

$$\mathcal{R}\left\{x[n]\right\} = \left\{ z \in \mathbb{C} \;\middle|\; \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty \right\}.$$

The relationship between $x[n]$ and $X(z)$ is denoted by $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$. Ignoring the ROC, the Z-transform can be seen as a generalization of the DTFT because choosing $z = e^{jw}$ recovers the definition of the DTFT.

Let $x[n]$ be a *finite-duration* signal where $x[n] = 0$ for $n < n_1$ and $n > n_2$. Then,

$$X(z) = \sum_{n=n_1}^{n_2} x[n]z^{-n}$$

and $z^{-n_1}X(z^{-1})$ is a polynomial. Consider the case of $x[n] = \{1, 3, \underline{2}, 4, 7\}$, where the underline denotes the location associated with time $n = 0$. In this case, we have

$$X(z) = 1z^2 + 3z + 2 + 4z^{-1} + 7z^{-2}$$

and the ROC is $0 < |z| < \infty$. From this, we see that the Z-transform simply uses the power of the indeterminate, z, to keep track of the time index associated with a signal value. This representation is particularly useful for convolution because

$$\begin{aligned}
\mathcal{Z}\left\{h[n] * x[n]\right\} &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]x[n-k] \right) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{-k} \sum_{n=-\infty}^{\infty} x[n-k]z^{-n+k} \\
&= H(z)X(z).
\end{aligned}$$

Example 6. Let $h[n] = \{1, \underline{0}, -1\}$and observe that $H(z) = z - z^{-1}$. Thus,

$$\begin{aligned}
\mathcal{Z}\left\{h[n] * x[n]\right\} &= H(z)X(z). \\
&= \left(z - z^{-1}\right)\left(1z^2 + 3z + 2 + 4z^{-1} + 7z^{-2}\right) \\
&= \left(z^3 + 3z^2 + 2z + 4 + 7z^{-1}\right) - \left(z + 3 + 2z^{-1} + 4z^{-2} + 7z^{-3}\right) \\
&= z^3 + 3z^2 + z + 1 + 5z^{-1} - 4z^{-2} - 7z^{-3}.
\end{aligned}$$

Thus, it is easy to see that $h[n] * x[n] = \{1, 3, 1, \underline{1}, 5, -4, -7\}$.

5.2 The Region of Convergence

To understand the importance of the ROC, we consider two simple examples.

Example 7. Consider the signal $x[n] = a^n u[n]$ whose Z-transform is given by

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= \frac{1}{1 - az^{-1}}, \text{ if } |z| > |a|. \end{aligned}$$

The last step follows from the geometric sum formula and the ROC is $|z| > |a|$.

Example 8. Consider the signal $x[n] = -a^n u[-n-1]$ whose Z-transform is given by

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -a^{-1} z \sum_{n=0}^{\infty} a^{-n} z^n \\ &= -\frac{a^{-1} z}{1 - a^{-1} z}, \text{ if } |z| < |a| \\ &= \frac{1}{1 - az^{-1}}, \text{ if } |z| < |a|. \end{aligned}$$

Although the signals in these two examples are quite different, the only difference between their Z-transforms is the ROC. Thus, if one ignores the ROC, then there are multiple signals with the same Z-transform.

Example 9. Consider the signal $x[n] = a^n u[n] + b^n u[-n-1]$. By linearity, we can compute the Z-transform using the two previous examples and we get

$$X(z) = \frac{1}{1 - az^{-1}} - \frac{1}{1 - bz^{-1}}, \text{ if } |a| < |z| < |b|.$$

In this case, the ROC is empty if $|b| \leq |a|$ and is a donut shaped region otherwise.

Definition 10. A signal $x[n]$ is *right-sided* if there is an n_0 such that $x[n] = 0$ for all $n < n_0$. For this n_0, it follows that the shifted signal $y[n] = x[n + n_0]$ is causal. Similarly, a signal is *left-sided* if there is an n_0 such that $x[n] = 0$ for all $n > n_0$. Any signal that is neither right-sided nor left-sided is called *two-sided*.

The ROC of a right-sided sequence is of the form $|z| > r$ for some $r \in [0, \infty]$. Similarly, the ROC of a left-sided sequence is of the form $|z| < r$ for some $r \in [0, \infty]$. In general, the ROC of a two-sided sequence is of the form $r_1 < |z| < r_2$ for some $r_1, r_2 \in [0, \infty]$. This matches exactly what we saw in the previous examples.

5.3 Block Diagrams

5.4 Properties of the Z-Transform

The following properties of the Z-transform are analogous to similar properties of the DTFT:

1) Linearity: $ax[n] + by[n] \xleftrightarrow{\mathcal{Z}} aX(z) + bY(z)$, ROC given by intersection

2) Time shift: $x[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}X(z)$, ROC unchanged except possibly $\{0, \infty\}$

3) Frequency shift: $a^n x[n] \xleftrightarrow{\mathcal{Z}} X(a^{-1}z)$, new ROC is $|a|r_1 < |z| < |a|r_2$

4) Conjugation: $x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*)$, ROC unchanged

5) Time flip: $x[-n] \xleftrightarrow{\mathcal{Z}} X(z^{-1})$, new ROC $1/r_2 < |z| < 1/r_1$

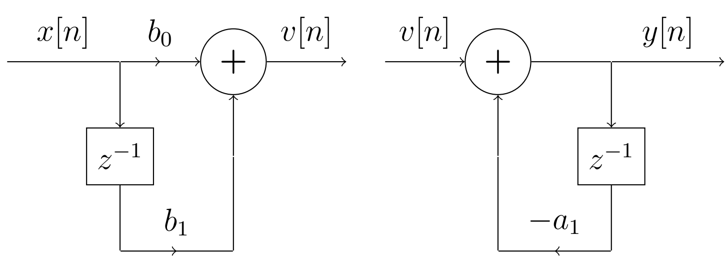
6) Convolution: $x[n] * y[n] \xleftrightarrow{\mathcal{Z}} X(z)Y(z)$, ROC includes intersection

7) Differentiation: $n\,x[n] \xleftrightarrow{\mathcal{Z}} -z\frac{\mathrm{d}}{\mathrm{d}z}X(z)$, ROC unchanged

In general, the inverse of the Z-transform can be computed using a contour integral. In practice, one typically uses partial fraction expansion to obtain a sum of simple terms and then table lookup is used to invert each term.

6 Filter Implementation

6.1 Block Diagrams



Consider the system

$$y[n] = b_0x[n] + b_1x[n-1] - a_1y[n-1]$$

and notice that it can be implemented as the cascade of the block diagrams shown above:

$$\begin{aligned} v[n] &= b_0x[n] + b_1x[n-1] \\ y[n] &= -a_1y[n-1] + v[n]. \end{aligned}$$

Taking the Z-transform of these systems, one gets

$$\begin{aligned} V(z) &= b_0 X(z) + b_1 X(z) z^{-1} \\ Y(z) &= -a_1 Y(z) z^{-1} V(z). \end{aligned}$$

These can be rewritten as

$$\begin{aligned} V(z) &= (b_0 + b_1 z^{-1}) X(z) \\ V(z) &= (a_0 + a_1 z^{-1}) Y(z), \end{aligned}$$

with $a_0 = 1$. Thus, the Z-transform of the impulse response is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{a_0 + a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

and it is a rational function of z^{-1}.

Currently, this system requires two delays. Can it be implemented with only one delay?

6.2 Rational Z-Transforms

The set of signals with rational Z-transforms,

$$H(z) = \frac{B(z)}{A(z)} = \frac{b[0] + b[1]z^{-1} + \cdots + b[M]z^{-M}}{a[0] + a[1]z^{-1} + \cdots + a[N]z^{-N}},$$

define a very important family of signals. If we factor the numerator and denominator, then we can rewrite this as

$$H(z) = \frac{b[0]}{a[0]} z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)},$$

where $z_1, \ldots, z_M$ are the zeros of $B(z)$ and $p_1, \ldots, p_N$ are the zeros of $A(z)$ (i.e., the poles of $1/A(z)$).

The pole-zero representation of a Z-transform, $H(z)$, is particularly useful when the signal $h[n]$ is the impulse response of a filter. This is because the zeros identify the signals (e.g., of the form $x[n] = z_k^n$) that are mapped to the zero signal and the poles identify the signals (e.g., of the form $x[n] = p_k^n$) that experience infinite gain.

Example 11. Consider $x[n] = a^n u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{1}{1-az^{-1}}$. The pole-zero decomposition shows that

$$X(z) = \frac{(z-0)}{(z-a)}$$

has zero at $z_1 = 0$ and a pole at $p_1 = a$.

Example 12. Consider a *finite impulse response* (FIR) filter defined by

$$y[n] = \sum_{k=0}^{M} b[k]x[n-k].$$

Thus, the output satisfies

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{M} b[k]x[n-k]z^{-n} \\ &= \sum_{k=0}^{M} b[k]z^{-k} \sum_{n=-\infty}^{\infty} x[n-k]z^{-n+k} \\ &= B(z)X(z), \end{aligned}$$

where $B(z) = z^{-M} \prod_{k=1}^{M}(z - z_k)$ is a filter with M zeros and M trivial poles at $z = 0$.

Example 13. Consider an *all-pole* filter defined by

$$y[n] = x[n] - \sum_{k=1}^{N} a[k]y[n-k].$$

Similarly, the output satisfies $Y(z) = X(z)/A(z)$ where $1/A(z) = 1/\left(z^{-N} \prod_{k=1}^{N}(z - p_k)\right)$ is a filter with N poles and N trivial zeros at $z = 0$.

Example 14. Consider the *linear constant-coefficient difference equation* (LCCDE) defined by

$$\sum_{k=0}^{N} a[k]y[n-k] = \sum_{k=0}^{M} b[k]x[n-k].$$

In this case, the Z-transform $A(z)Y(z) = B(z)X(z)$ implies that $H(z) = Y(z)/X(z) = B(z)/A(z)$ has a rational Z-transform. We note that solving an LCCDE using the Z-transform implicitly assumes that the system was initially at rest.

Example 15. Determine the impulse response of

$$y[n] = \frac{1}{2}y[n-1] + 2x[n].$$

The Z-transform implies that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2}z^{-1}}.$$

Based on our previous calculations, this implies that

$$h[n] = 2\left(\frac{1}{2}\right)^{n} u[n].$$

6.3 Understanding Filters via Pole-Zero Plots

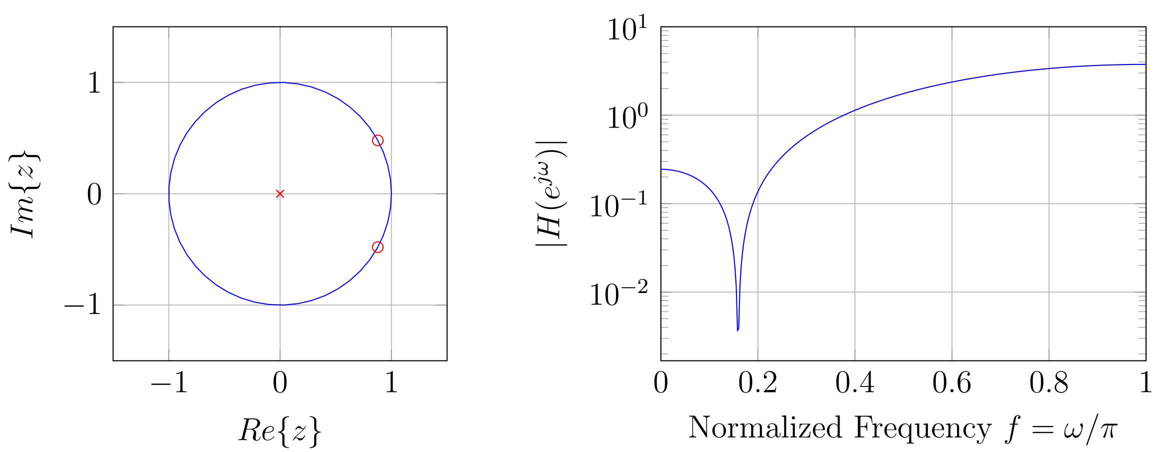
A filter, $h[n]$, with a rational Z-transform, $H(z)$, is completely defined by the locations of its poles and zeros in the complex plane. A pole-zero plot shows these locations in the complex plane with zeros represented by circles and poles are represented by x's. If there is a zero at some location z, then the filter maps the input signal, $x[n] = z^n$, mapped to the zero signal. If there is a pole at some point z, then the filter maps the input signal, $x[n] = z^n$, to an unbounded output and the ROC does not include z.

Since $y[n] = H(z)x[n]$ for $x[n] = z^n$, we see that the unit circle corresponds to $|z| = 1$ and $x[n] = e^{j\omega n}$ for some $\omega \in [-\pi, \pi]$. For z inside the unit circle (i.e., $|z| < 1$), the signal $x[n]$ is decaying exponentially in n while, z outside the unit circle (i.e., $|z| > 1$), the signal $x[n]$ is growing exponentially in n.

Recall that a filter, $h[n]$, is bounded-input bounded-output (BIBO) stable iff $\sum_{n=-\infty}^{\infty} |h[n]|$. From the definition of the ROC, it follows that $h[n]$ is BIBO stable iff only the ROC of its Z-transform, $H(z)$, includes $z = 1$. Thus, for a right-sided BIBO-stable filter, this implies the ROC must include $|z| \geq 1$ and the Z-transform must have all its poles inside the unit circle.

6.4 Notch Filters

Notch filters are band-stop filters with a very narrow stop band. For example, analog notch filters are commonly used to remove the 60 Hz hum from audio amplifiers that can leak into audio systems from the power supply. In terms of the pole-zero diagram, one can design a digital FIR notch filter that maps $x[n] = \cos(\omega_0 n + \phi)$ to zero by placing filter zeros at $z = e^{\pm j\omega_0}$.



For example, the first plot in the above figure shows the pole-zero diagram of the filter

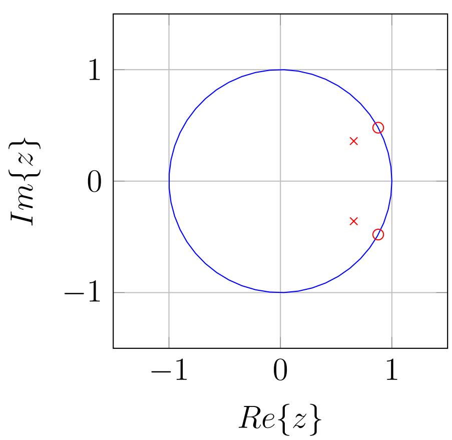
$$H_1(z) = z^{-2}(z - e^{j/2})(z - e^{-j/2}) = 1 - 2\cos(\tfrac{1}{2})z^{-1} + z^{-2},$$

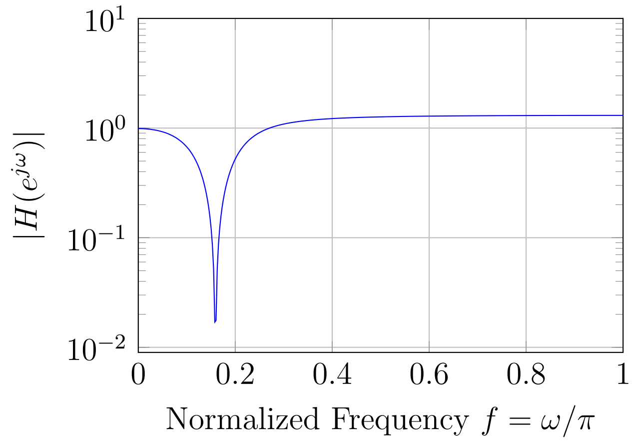
with zeros at $e^{\pm j/2}$ and two poles at 0. The frequency response of this filter is given by

$$\begin{aligned} H_1(e^{j\omega}) &= e^{j\omega}(e^{-j\omega} - 2\cos(\tfrac{1}{2}) + e^{j\omega}) \\ &= 2(\cos(\omega) - \cos(\tfrac{1}{2}))e^{j\omega}. \end{aligned}$$

The second plot shows the magnitude of this response. Notice that the response at $\omega = \frac{1}{2}$ is zero.

The main problem with FIR notch filters is that the response is very flat in the pass band. This problem can be rectified by designing an IIR notch filter that includes poles that are close to the zeros (e.g., at $z = re^{\pm j\omega_0}$ for some $r < 1$). For the input signal $x[n] = \cos(\omega n + \phi)$ where ω is not too close to ω_0, these poles cancel the effect of the zeros.





The first plot in the above figure show the pole-zero plot of the filter

$$H_2(z) = \frac{z^{-2}(z - e^{j/2})(z - e^{-j/2})}{z^{-2}(z - \frac{3}{4}e^{j/2})(z - \frac{3}{4}e^{-j/2})} = \frac{1 - 2\cos(\frac{1}{2})z^{-1} + z^{-2}}{1 - \frac{3}{2}\cos(\frac{1}{2})z^{-1} + \frac{9}{16}z^{-2}},$$

with zeros at $e^{\pm j/2}$ and poles at $\frac{3}{4}e^{\pm j/2}$. The magnitude-squared frequency response of this filter is given by

$$\begin{aligned} \left|H_2(e^{j\omega})\right|^2 &= \frac{\left|2\left(\cos(\omega) - \cos(\frac{1}{2})\right) e^{j\omega}\right|^2}{\left|(1 - \frac{3}{2}\cos(\frac{1}{2})e^{-j\omega} + \frac{9}{16}e^{-2j\omega})(1 - \frac{3}{2}\cos(\frac{1}{2})e^{j\omega} + \frac{9}{16}e^{2j\omega})\right|^2} \\ &= \frac{4\left(\cos(\omega) - \cos(\frac{1}{2})\right)^2}{1 + \frac{9}{4}\cos^2(\frac{1}{2}) + \frac{81}{256} - \frac{75}{32}\cos(\frac{1}{2})\cos(\omega) + \frac{81}{256}\cos(2\omega)}. \end{aligned}$$

The second plot in the above figure shows the magnitude of the frequency response. Notice that the response at $\omega = \frac{1}{2}$ is still zero but matching the zeros with poles makes the overall response much flatter.

One can also design digital IIR notch filters by first designing an optimal analog notch filter and then transforming it into the digital domain via the bilinear transform. In this class, we will not cover this method. But, that is the approach taken by the MATLAB function `iirnotch`.