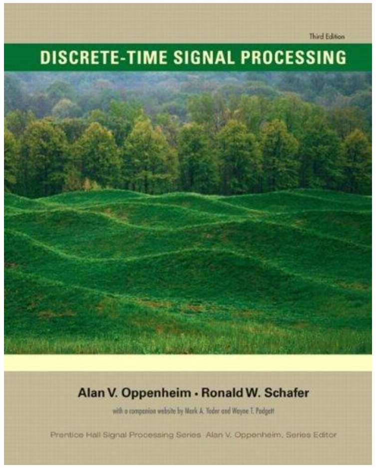


Lecture 09:

Applications of DFT

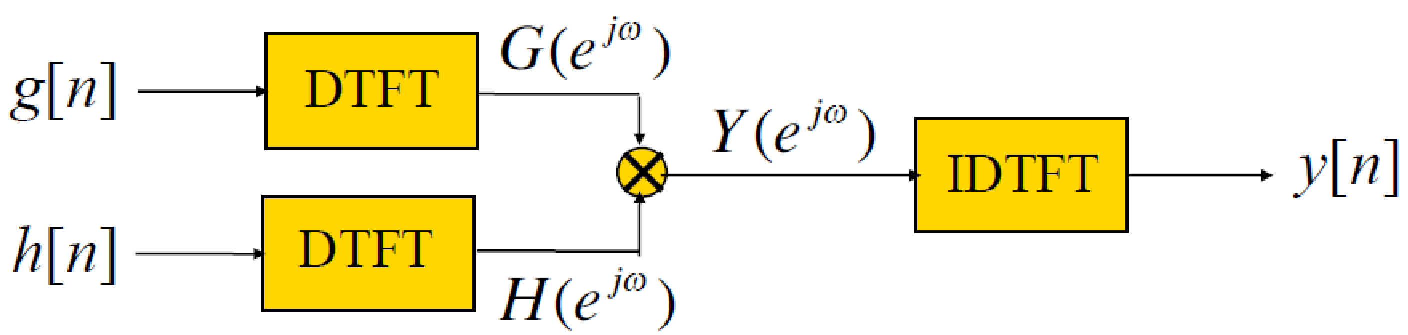
Outline

- Signal Processing
- Spectral Estimation
- Short-Time Fourier Transform

1. Convolution with DTFT

- Since $g[n] \circledast h[n] \quad \leftrightarrow \quad G(e^{j\omega})H(e^{j\omega})$

  we can calculate a convolution by:
  - finding DTFTs of $g, h \rightarrow G, H$
  - multiply them: $G \cdot H$
  - IDTFT of product is result, $g[n] \circledast h[n]$



DFT properties summary

- Circular convolution

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N] \quad \leftrightarrow \quad G[k]H[k]$$

- Modulation

$$g[n]\cdot h[n] \quad \leftrightarrow \quad \tfrac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$$

- Duality

$$G[n] \quad \leftrightarrow \quad N\cdot g[\langle -k\rangle_N]$$

- Parseval

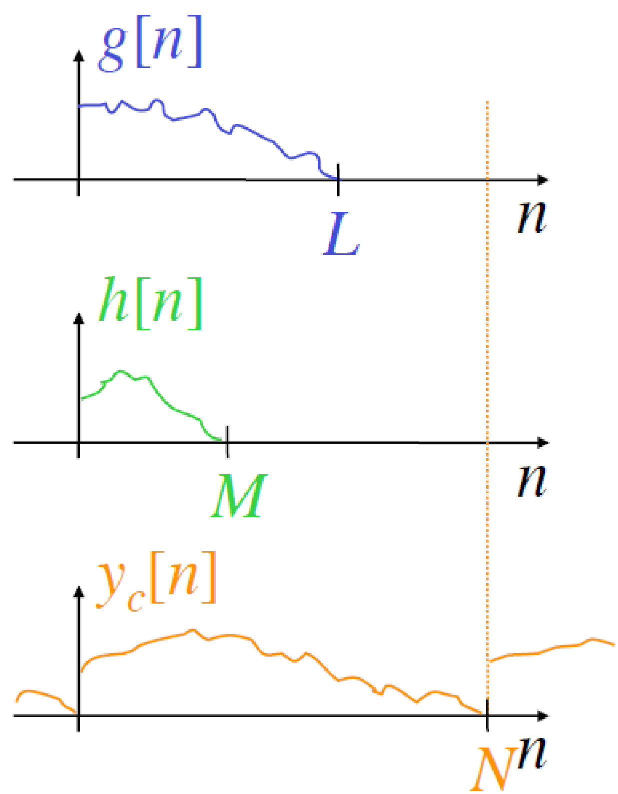
$$\sum_{n=0}^{N-1}\left|x[n]\right|^2 = \tfrac{1}{N}\sum_{k=0}^{N-1}\left|X[k]\right|^2$$

Linear convolution w/ the DFT

- DFT → fast circular convolution
- .. but we need linear convolution
- Circular conv. is time-aliased linear conv.; can aliasing be avoided?
- e.g. convolving L-pt $g[n]$ with M-pt $h[n]$: $y[n] = g[n] \circledast h[n]$ has $L+M-1$ nonzero pts
- Set DFT size $N \geq L+M-1$ → **no aliasing**

Linear convolution w/ the DFT

- Procedure ($N = L + M - 1$):
  - pad L-pt $g[n]$ with (at least) M-1 zeros
    $\rightarrow$ N-pt DFT $G[k]$, $k = 0..N$-1
  - pad M-pt $h[n]$ with (at least) L-1 zeros
    $\rightarrow$ N-pt DFT $H[k]$, $k = 0..N$-1
  - $Y[k] = G[k] \cdot H[k]$, $k = 0..N$-1
  - $\mathrm{IDFT}\{Y[k]\} = \sum_{r=-\infty}^{\infty} y_L[n + rN] = y_L[n]$ $(0 \le n < N)$



Overlap-Add convolution

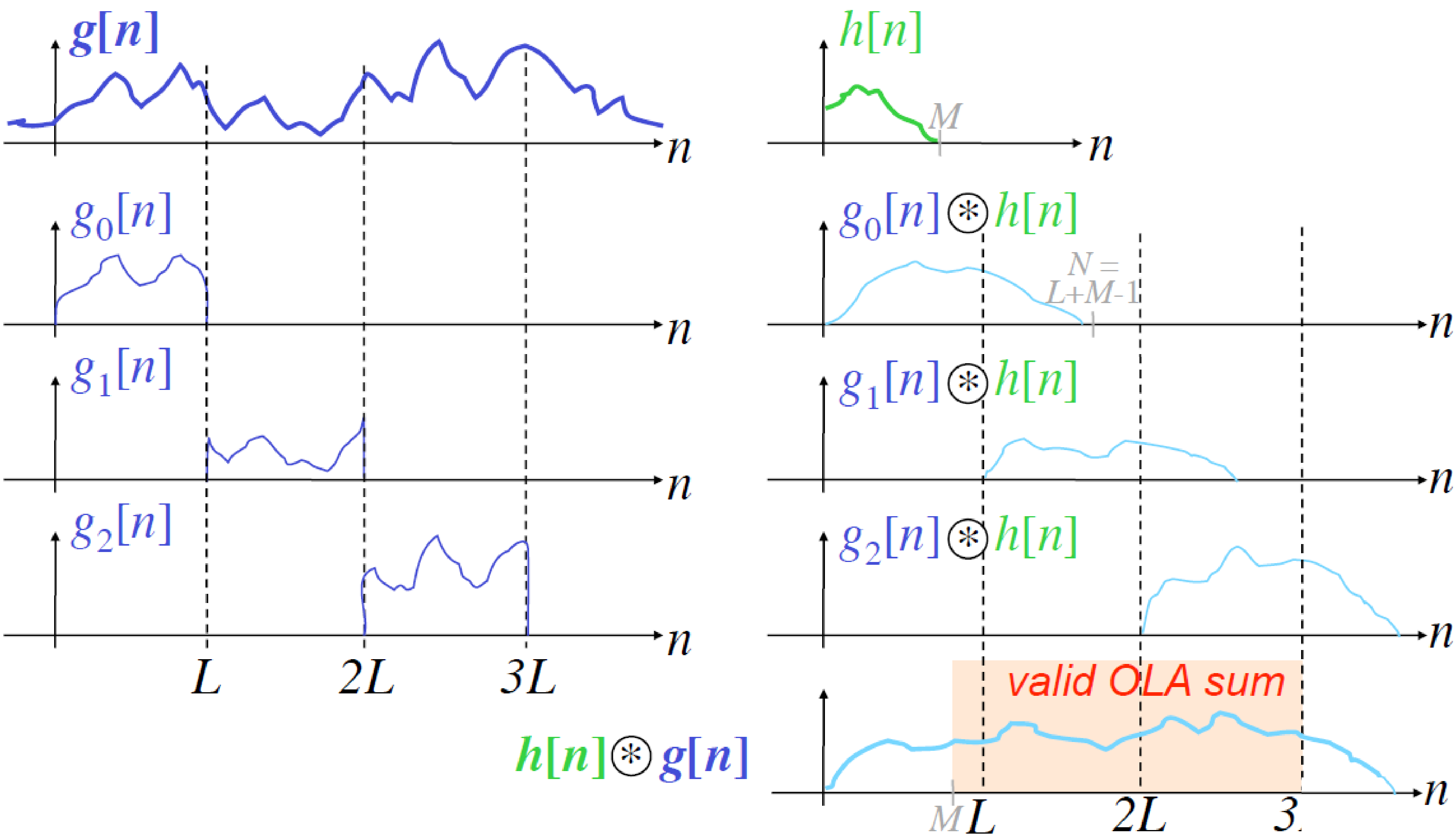
- Very long $g[n]$ → break up into segments, convolve piecewise, overlap → bound size of DFT, processing delay
- Make $g_i[n] = \begin{cases} g[n] & i \cdot N \leq n < (i+1) \cdot N \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \qquad g[n] = \textstyle\sum_i g_i[n]$$

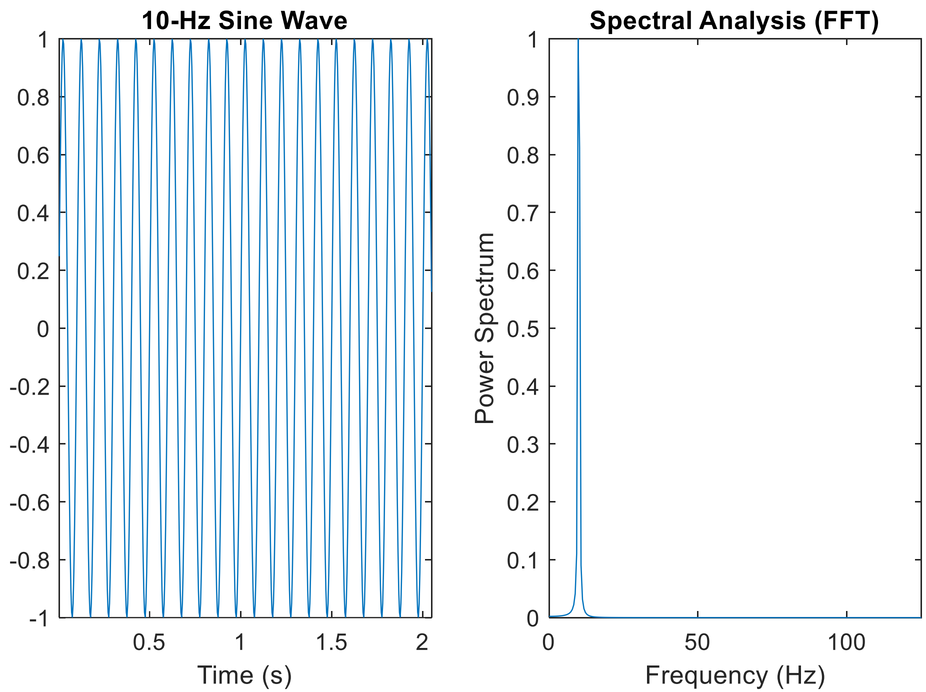
$$\Rightarrow \quad h[n] \circledast g[n] = \textstyle\sum_i h[n] \circledast g_i[n]$$

- Called Overlap-Add (**OLA**) convolution

Overlap-Add convolution



2. Spectral Estimation




```
%% initialize parameters
samplerate=250; % in Hz
N=512; % data length

sinefreq=10; % in Hz

%% generate a sine signals
t=[1:N]/samplerate;
sig=sin(2*pi*sinefreq*t);

figure,
subplot(1,2,1),plot(t,sig),xlim([t(1) t(end)])
title([num2str(sinefreq) '-Hz Sine Wave'])
xlabel('Time (s)')
```

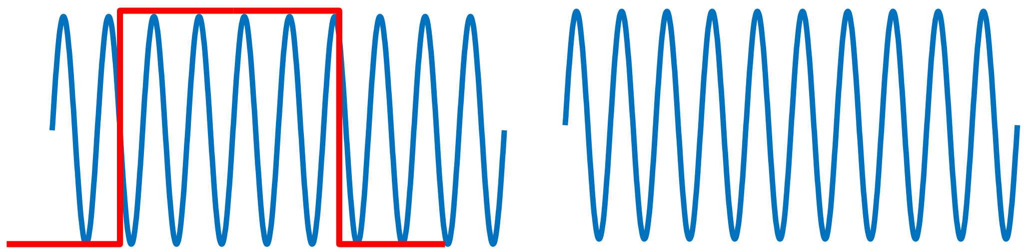
```
%% Spectral analysis (FFT)
nfft = 2^nextpow2(N); % Next power of 2 from length of y
sig_freq=fft(sig,nfft);
PS=abs(sig_freq).^2;
PS=PS/max(PS);  % normalize PS to its maximum
faxis=samplerate/2*linspace(0,1,nfft/2+1);

subplot(1,2,2),plot(faxis,PS(1:nfft/2+1))
xlim([faxis(1) faxis(end)])
title('Spectral Analysis (FFT)')
xlabel('Frequency (Hz)')
ylabel('Power Spectrum')
```

2. Spectral Estimation

- **What if the signal is not time-limited? We can think of limiting the sum to *N* points as a truncation of the signal:**

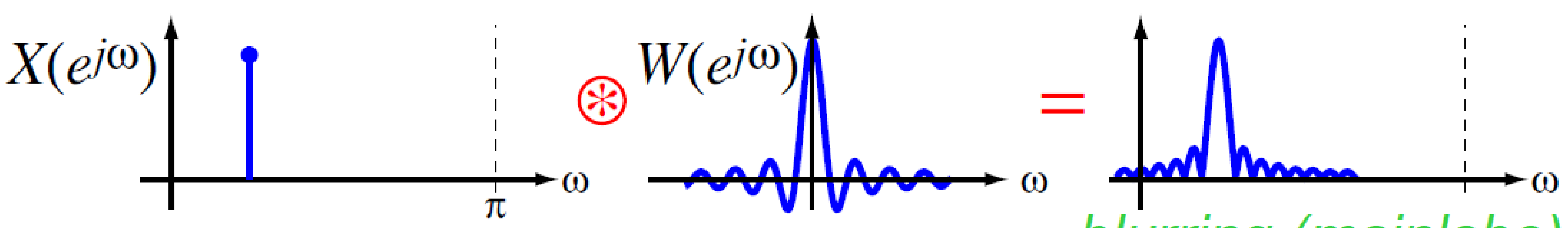
$$x_w[n] = w[n]x[n]$$

$$w[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots, N \\ 0, & otherwise \end{cases}$$



- **What are the implications of this in the frequency domain? (Hint: convolution)**

- e.g. if $x[n]$ is a pure sinusoid,



blurring (mainlobe)
+ ghosting (sidelobes)

Impact on Truncation

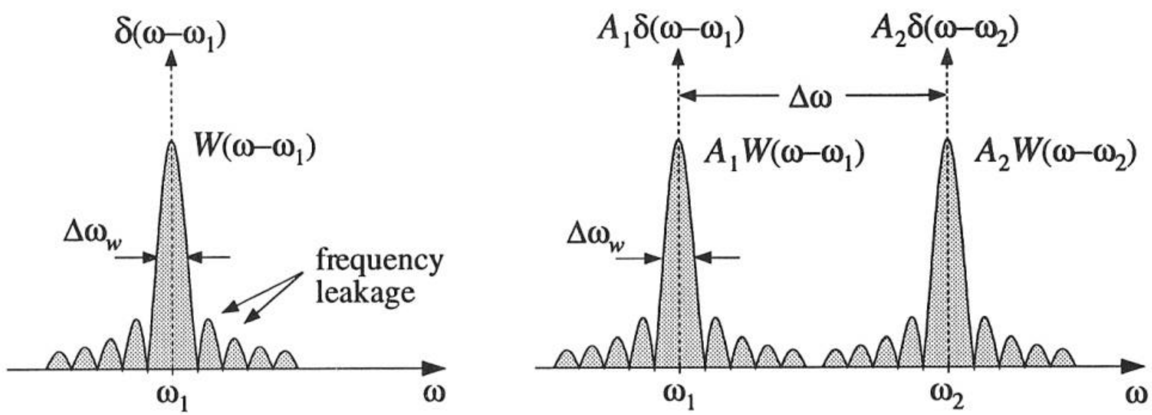


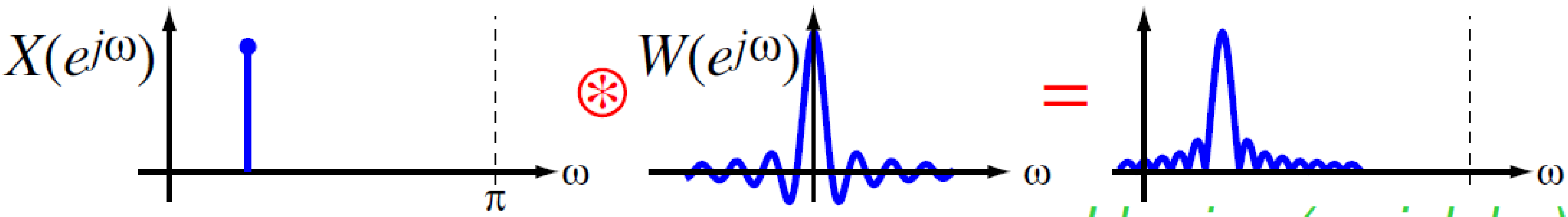
Fig. 9.1.3 Spectra of windowed single and double sinusoids.

- The spectrum of a windowed sinewave is the convolution of two impulse functions with the frequency response of the window.

- For two closely spaced sine waves, there is “leakage” between each sine wave’s spectrum.
- The impact of this leakage can be mitigated by using a window function with a narrower main lobe.

FT Window Shape

- e.g. if $x[n]$ is a pure sinusoid,

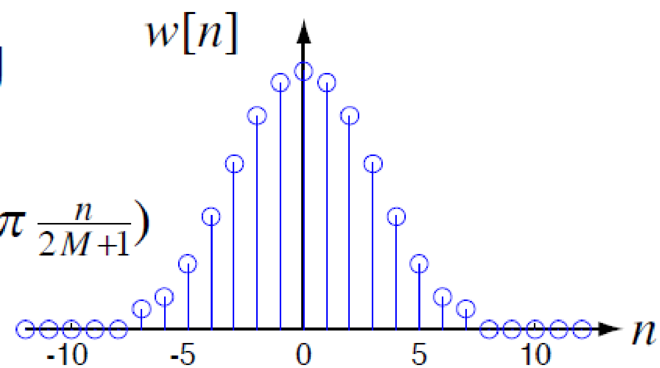


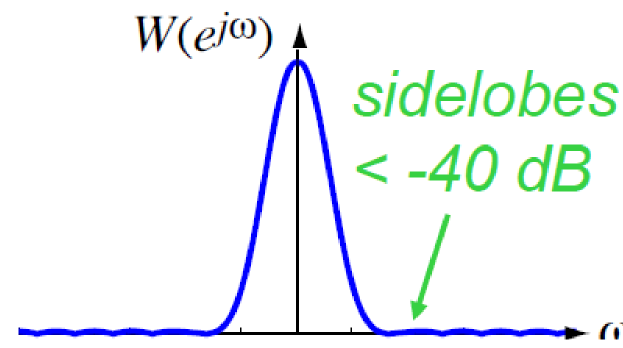
blurring (mainlobe)
+ ghosting (sidelobes)

- Hence, use tapered window for $w[n]$

e.g. Hamming

$$w[n] = 0.54 + 0.46\cos(2\pi \tfrac{n}{2M+1})$$





sidelobes
< -40 dB

Popular Windows

- **Popular Windows:**
  - **Rectangular:**

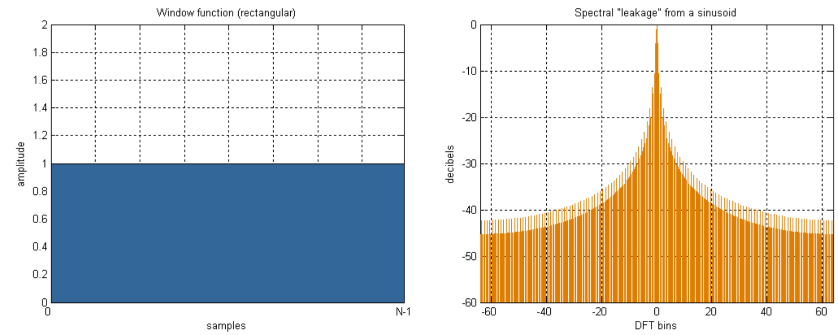
$$w[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots, N \\ 0, & otherwise \end{cases}$$

  - **Generalized Hanning:**

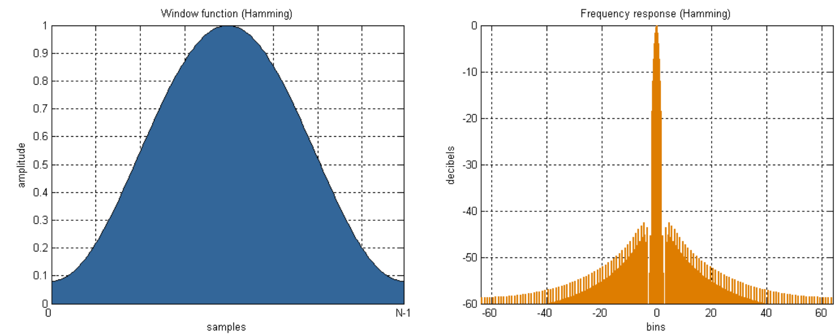
$$w[n] = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right)$$

  - **Triangular:**

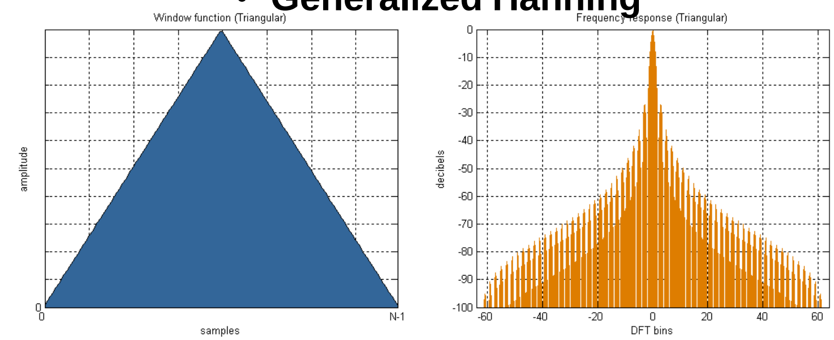
$$w[n] = \frac{2}{N}\left(\frac{N}{2} - \left|n - \frac{N-1}{2}\right|\right)$$



- **Rectangular**



- **Generalized Hanning**



- **Triangular**

Spectral Estimation with Different Windows

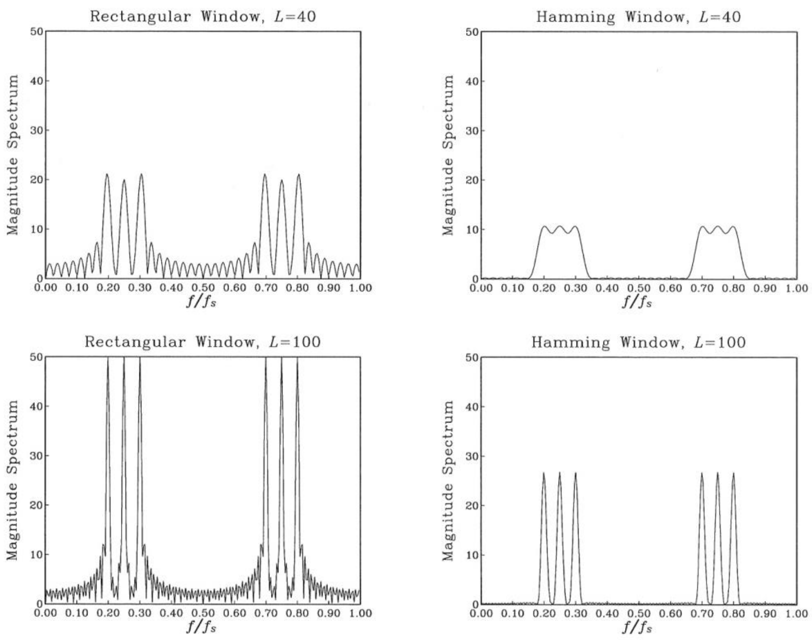


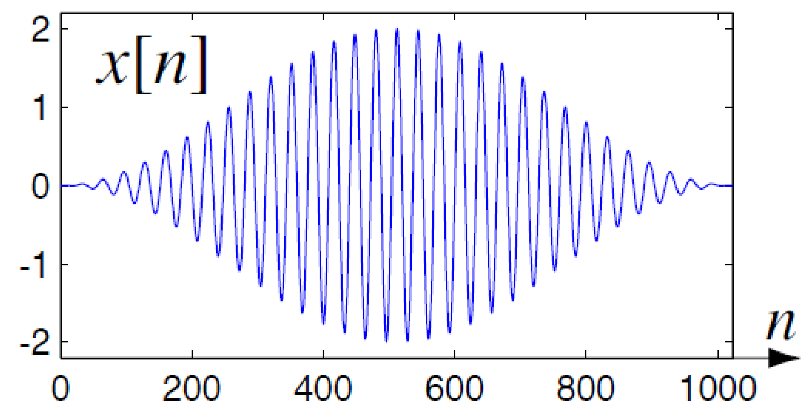
Fig. 9.1.9 Rectangular and Hamming spectra for $L = 40$ and 100.

- Consider the spectrum of three sine waves computed using a rectangular and a Hamming window.
- We see that for the same number of points, the spectrum produced by the Hamming window separates the sinewaves.
- What is the computational cost?

3. Short-Time Fourier Transform (STFT)

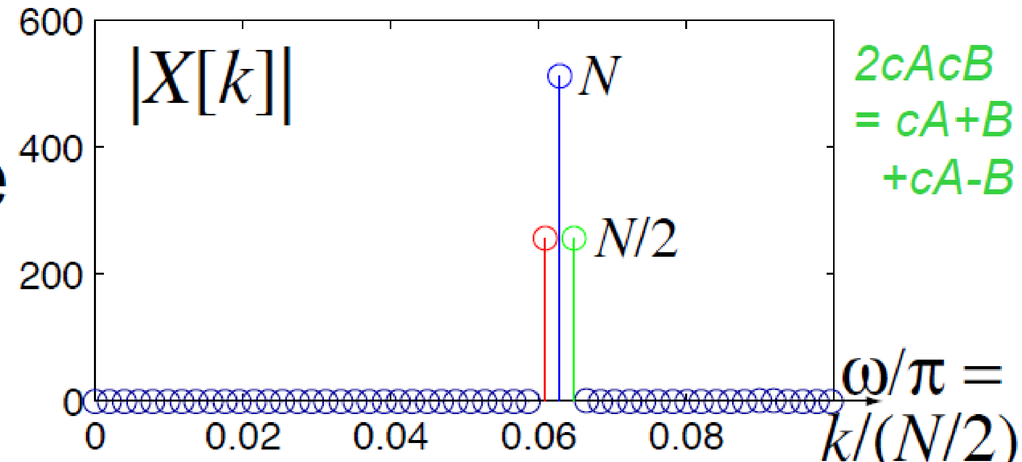
- Fourier Transform (e.g. DTFT) gives spectrum of an entire sequence:
- How to see a time-varying spectrum?
- e.g. slow AM of a sinusoid carrier:

$$x[n] = \left(1 - \cos\frac{2\pi n}{N}\right)\cos\omega_0 n$$



Fourier Transform of AM Sine

- Spectrum of whole sequence indicates modulation indirectly...
- ... as cancellation between closely-tuned sines

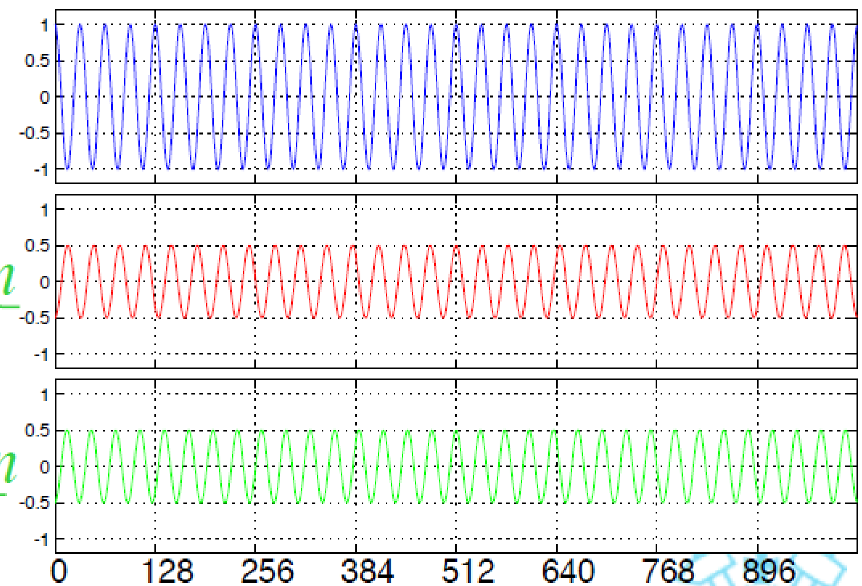


2cAcB
= cA+B
+cA-B

$N \sin 2\pi \frac{kn}{N}$

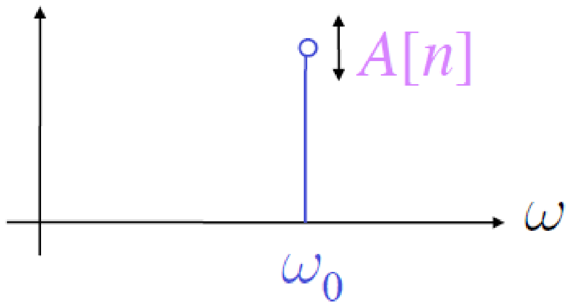
$-\frac{N}{2} \sin 2\pi \frac{(k-1)n}{N}$

$-\frac{N}{2} \sin 2\pi \frac{(k+1)n}{N}$



Fourier Transform of AM Sine

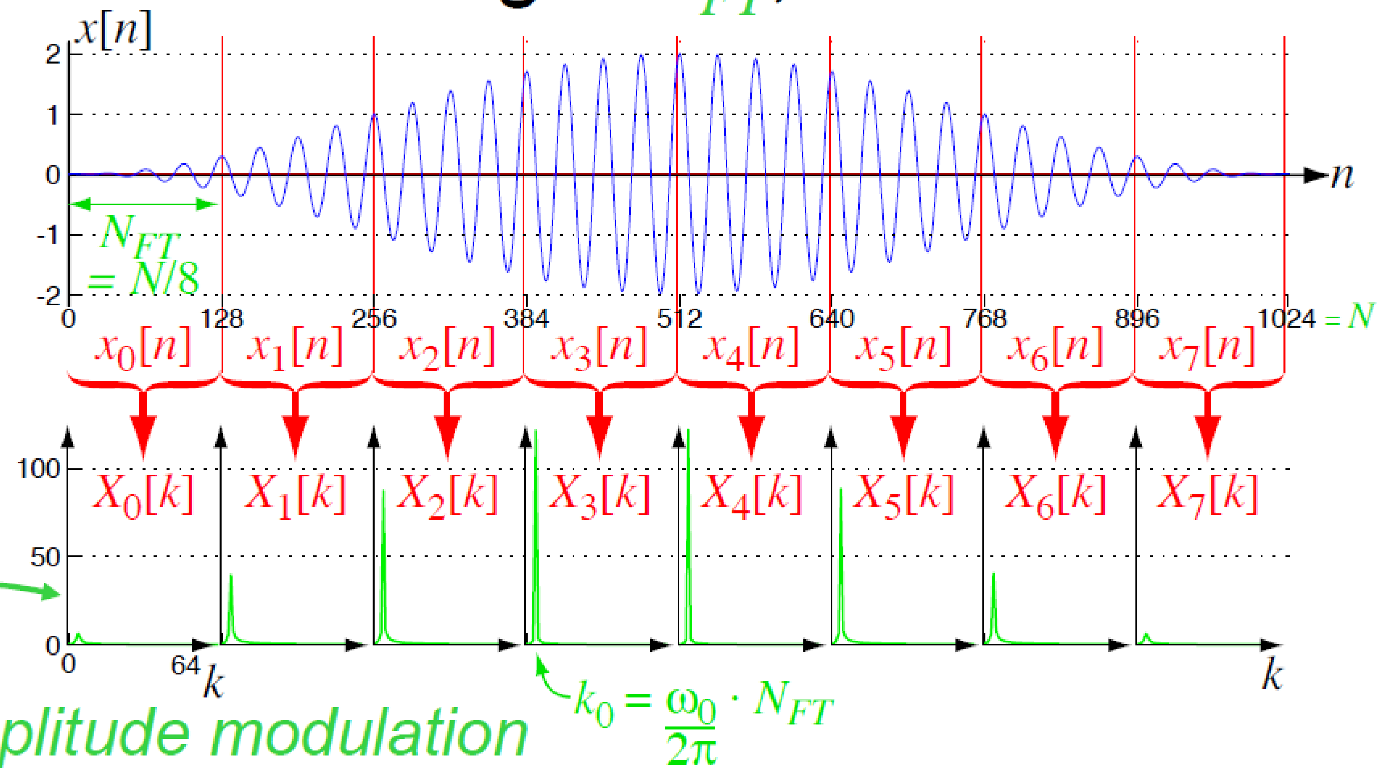
- Sometimes we'd rather separate modulation and carrier:
  $x[n] = A[n]\cos\omega_0 n$



  - $A[n]$ varies on a different (slower) timescale
- One approach:
  - chop $x[n]$ into short sub-sequences ..
  - .. where slow modulator is ~ constant
  - DFT spectrum of pieces → show variation

FT of Short Segments

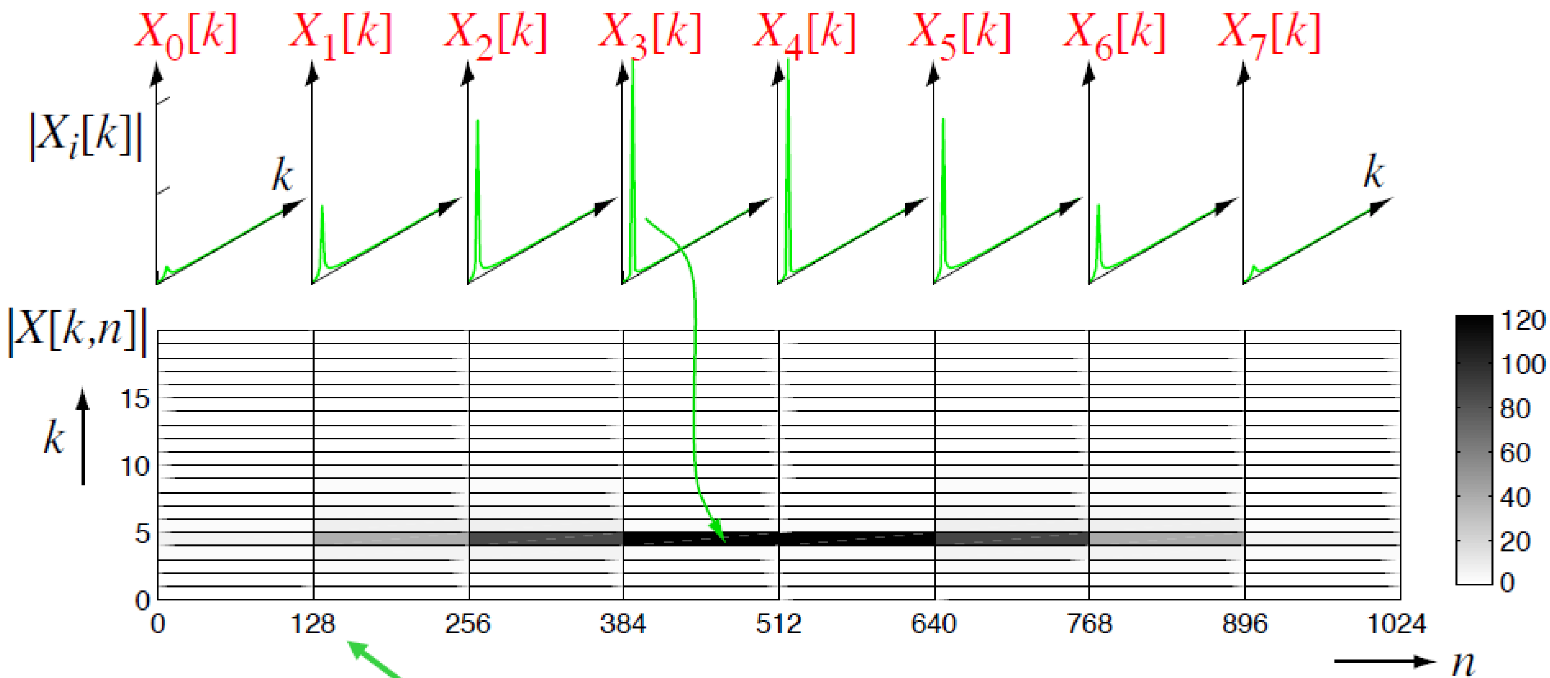
- Break up x[n] into successive, shorter chunks of length N_{FT}, then DFT each:



Shows amplitude modulation of ω_0 energy

The Spectrogram

- Plot successive DFTs in time-frequency:



*time **hopsize** (between successive frames) = 128 points*

- This image is called the Spectrogram

Short-Time Fourier Transform

- Spectrogram = STFT magnitude plotted on time-frequency plane
- STFT is (DFT form):

$$X[k, n_0] = \sum_{n=0}^{N_{FT}-1} x[n_0 + n] \cdot \quad \cdot e^{-j\frac{2\pi kn}{N_{FT}}}$$

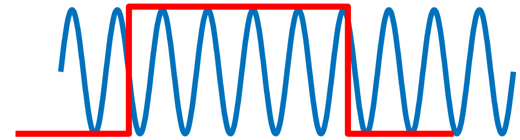
frequency index

time index

N_{FT} points of x starting at n_0

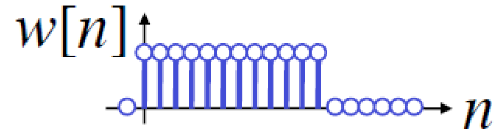
DFT kernel

- intensity as a function of time & frequency



STFT Window Shape

- $w[n]$ provides 'time localization' of STFT
  - e.g. rectangular



  selects $x[n]$, $n_0 \le n < n_0 + N_W$

- <u>But:</u> resulting spectrum has same problems as windowing for FIR design:

DTFT form of STFT

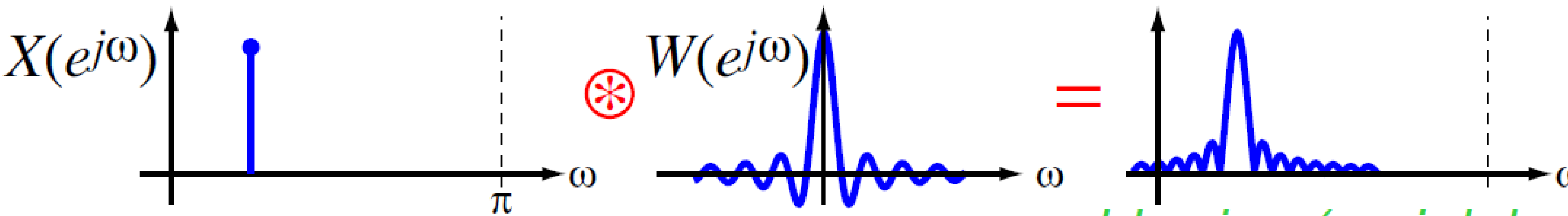
$$X\left(e^{j\omega}, n_0\right) = \mathrm{DTFT}\{x[n_0 + n] \cdot w[n]\}$$

$$= \int_{-\pi}^{\pi} e^{j\theta n_0} X\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta$$

spectrum of short-time window is convolved with (twisted) parent spectrum

STFT Window Shape

- e.g. if $x[n]$ is a pure sinusoid,

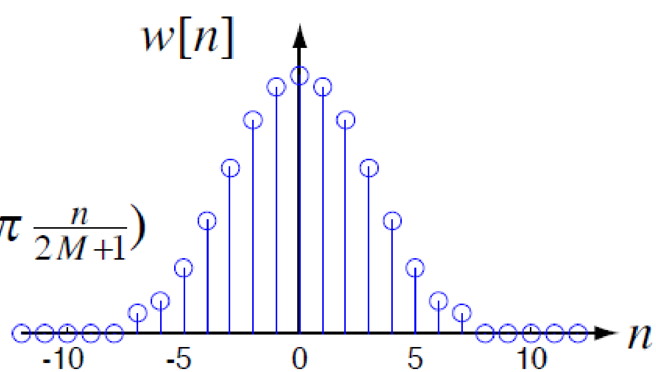


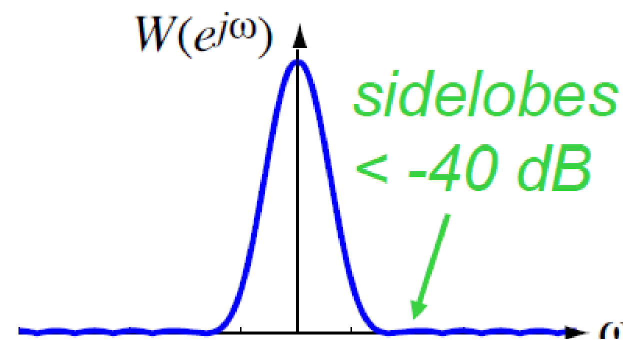
blurring (mainlobe)
+ ghosting (sidelobes)

- Hence, use tapered window for $w[n]$

e.g. Hamming

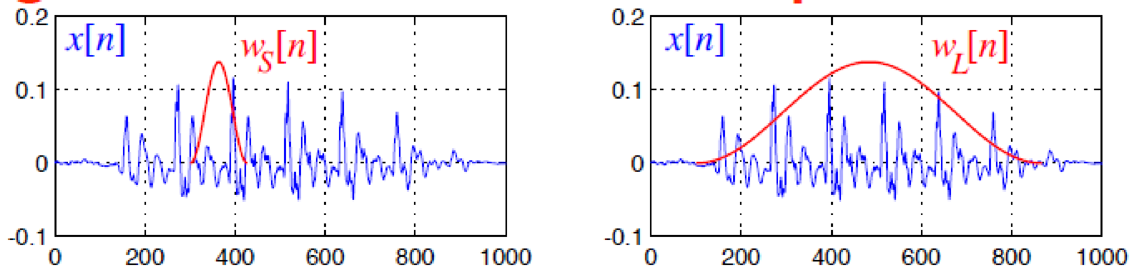
$$w[n] = 0.54 + 0.46\cos(2\pi \tfrac{n}{2M+1})$$





STFT Window Length

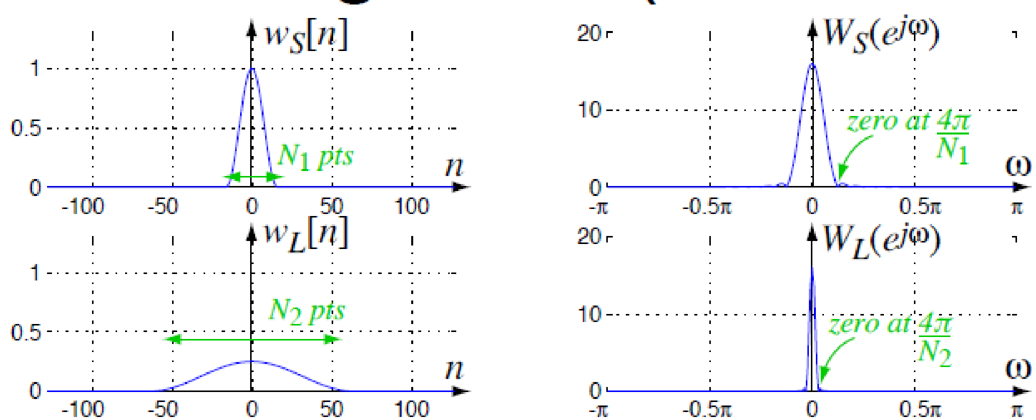
- Length of $w[n]$ sets temporal resolution



short window measures only local properties

longer window averages spectral character

- Window length $\propto$ 1/(Mainlobe width)

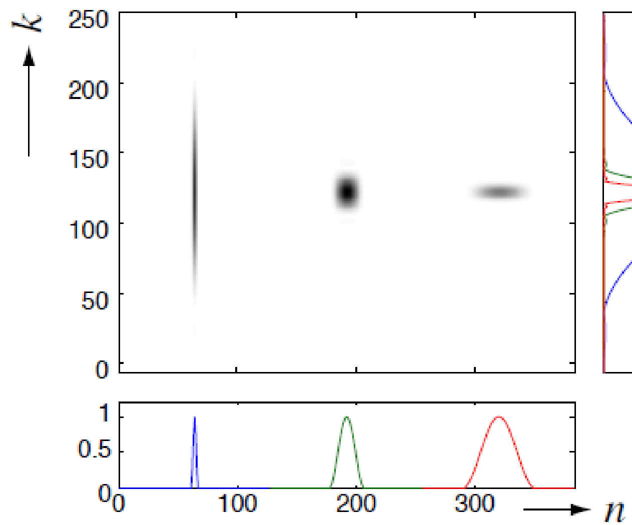


shorter window → more blurred spectrum

- *more* time detail ↔ *less* frequency **detail**

STFT Window Length

- Can illustrate time-frequency tradeoff on the time-frequency plane:

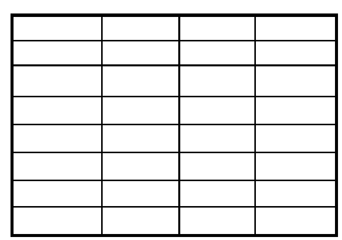


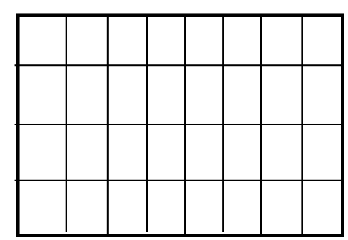
disks show 'blurring' due to window length; ***area*** *of disk is constant*

→ ***Uncertainty principle:***

$$\delta f \cdot \delta t \geq k$$

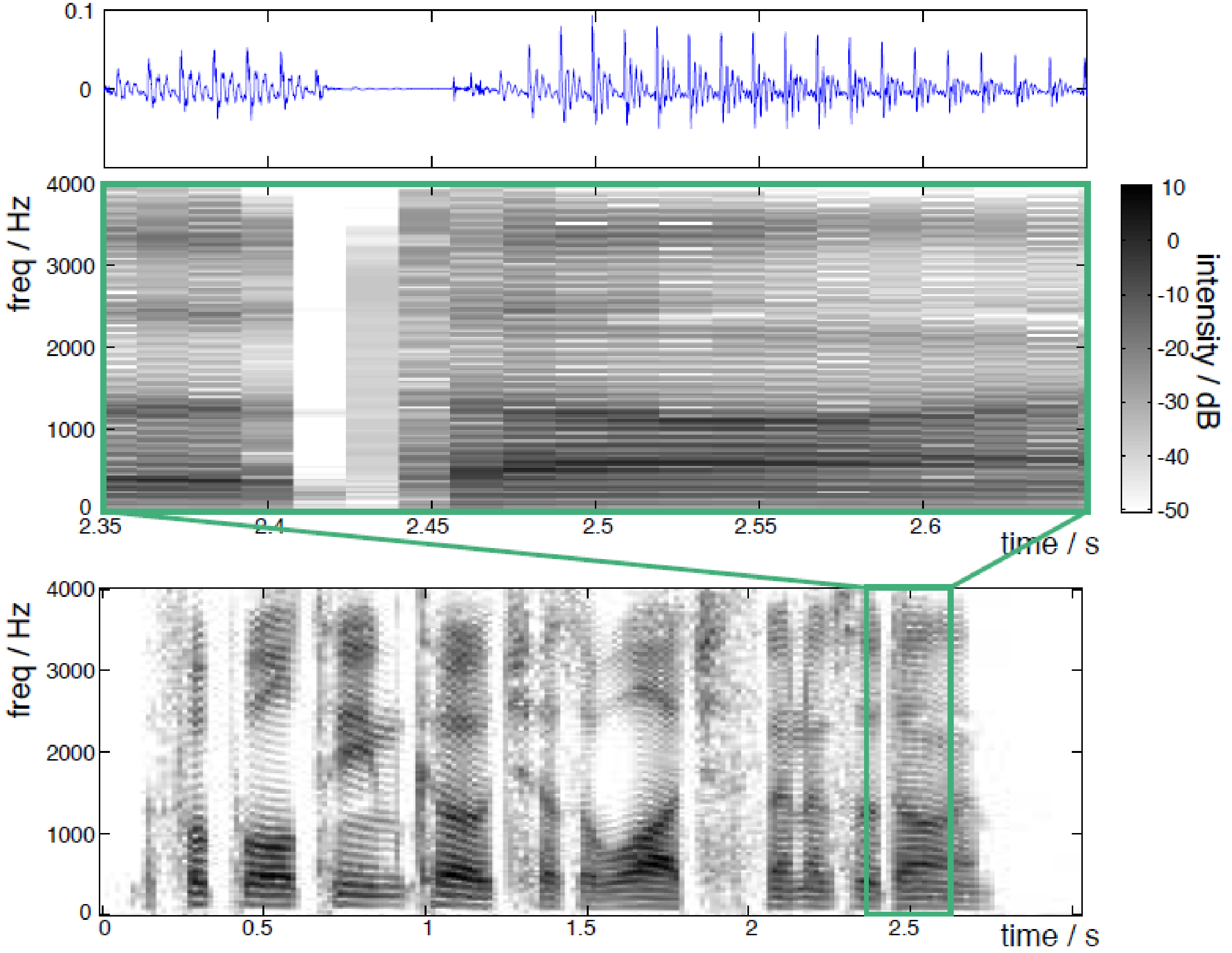
- Alternate tilings of time-freq:





half-length window → half as many DFT samples

Spectrograms of Real Sounds



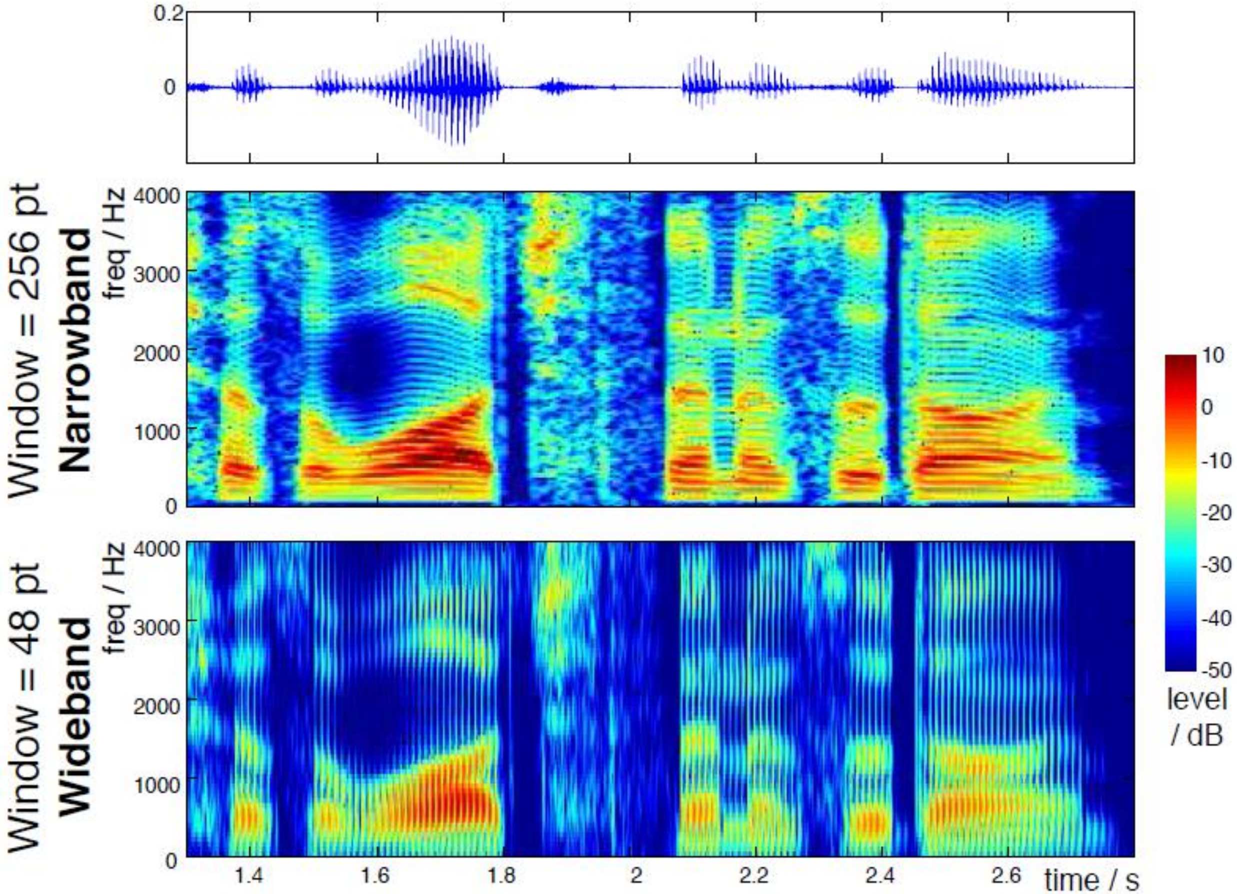
time-domain

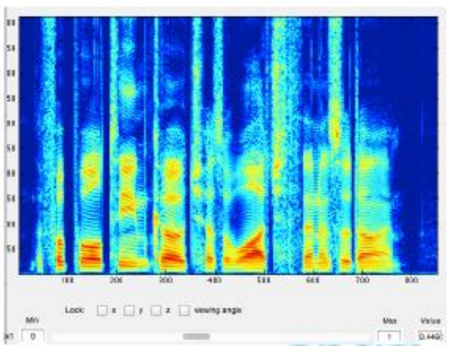
successive short DFTs

individual t-f cells merge into continuous image

Narrowband vs. Wideband

- **Effect of varying window length:**



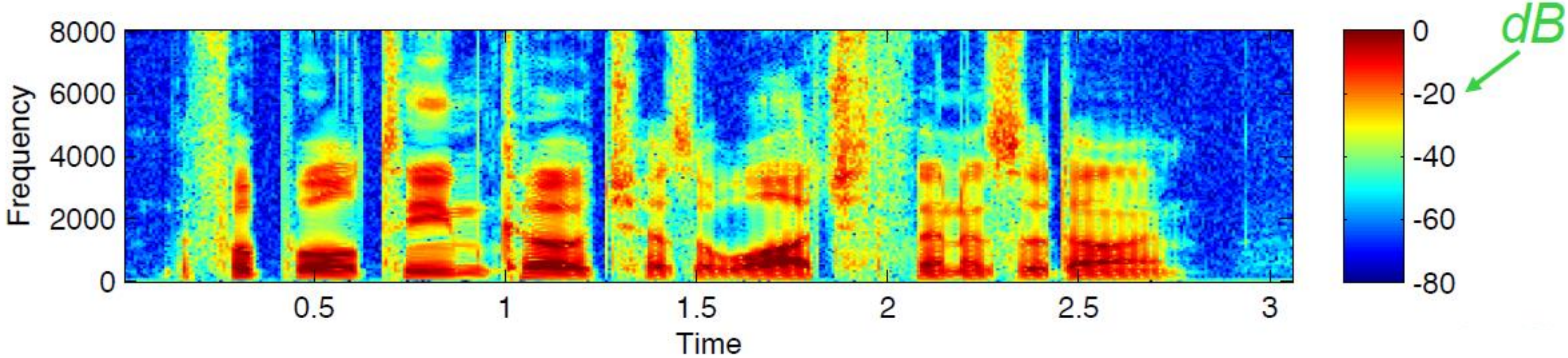


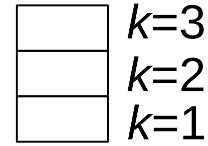
Spectrogram in MATLAB

```
>> [d,sr]=wavread('mpgr1_sx419.wav');
>> Nw=256;
>> specgram(d,Nw,sr)
>> caxis([-80 0])
>> colorbar
```

(hann) window length

actual sampling rate (to label time axis)





STFT as a Filterbank

- Consider one 'row' of STFT:

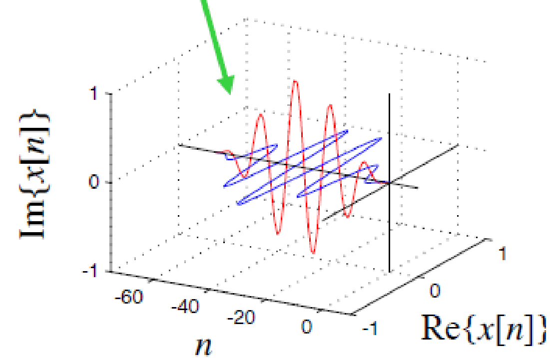
$$X_k[n_0] = \sum_{n=0}^{N-1} x[n_0+n] \cdot w[n] \cdot e^{-j\frac{2\pi kn}{N}}$$

just one freq.

$$= \sum_{m=0}^{-(N-1)} h_k[m]\, x[n_0-m]$$

convolution with complex IR

where $h_k[n] = w[-n] \cdot e^{j\frac{2\pi kn}{N}}$



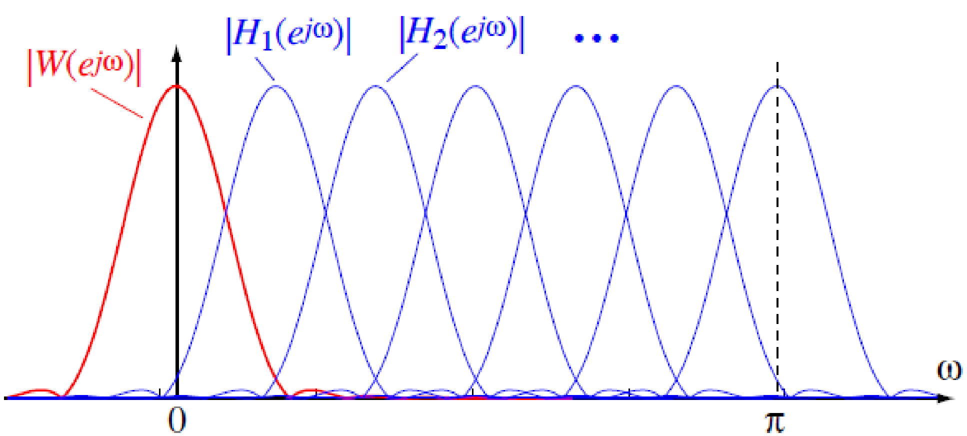
- Each STFT row is output of a filter (subsampled by the STFT hop size)

STFT as a Filterbank

- If $h_k[n] = w[(-)n] \cdot e^{j\frac{2\pi kn}{N}}$

  then $H_k(e^{j\omega}) = W\left(e^{(-)j\left(\omega - \frac{2\pi k}{N}\right)}\right)$ *shift-in-ω*

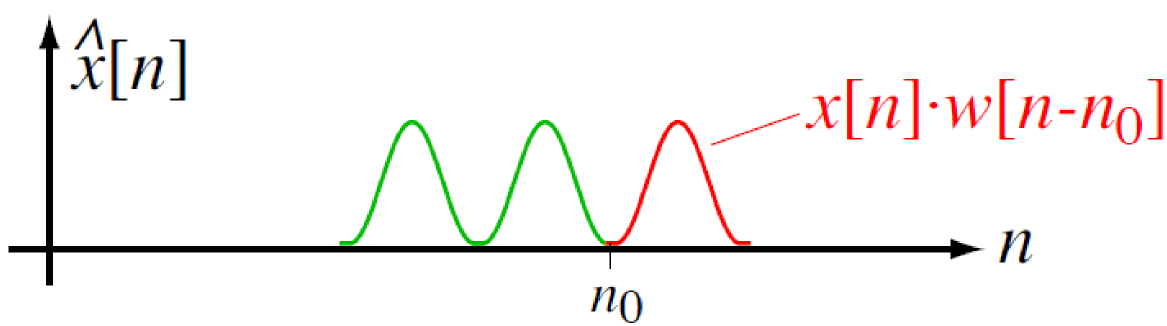
- Each STFT row is the same bandpass response defined by $W(e^{j\omega})$, frequency-shifted to a given DFT bin:



A bank of identical, frequency-shifted bandpass filters: ***"filterbank"***

STFT Analysis-Synthesis

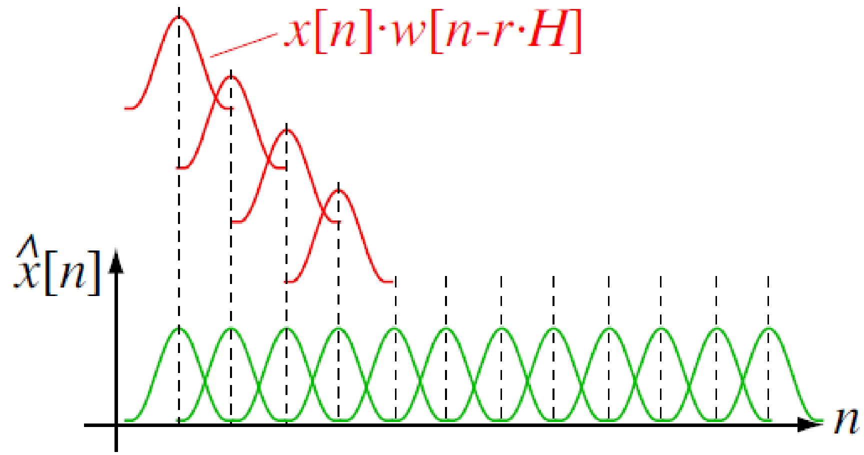
- IDFT of STFT frames can reconstruct (part of) original waveform
- e.g. if $X[k,n_0] = \mathrm{DFT}\{x[n_0+n]\cdot w[n]\}$
  then $\mathrm{IDFT}\{X[k,n_0]\} = x[n_0+n]\cdot w[n]$
- Can shift by n_0, combine, to get $\hat{x}[n]$:



- Could divide by $w[n\text{-}n_0]$ to recover $x[n]$...

STFT Analysis-Synthesis

- Dividing by small values of $w[n]$ is bad



- Prefer to overlap windows:

i.e. sample $X[k,n_0]$

at $n_0 = \mathrm{r}\cdot H$ where $H = N/2$ (for example)

hopsize window length

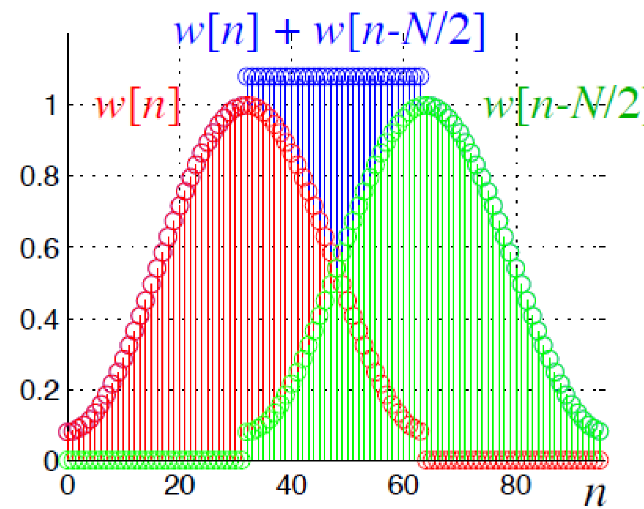
- Then $\hat{x}[n] = \sum_r x[n]w[n - rH]$

$$= x[n] \quad \text{if} \quad \sum_{\forall r} w[n - rH] = 1$$

STFT Analysis-Synthesis

- Hann or Hamming windows with 50% overlap sum to constant

$$\left(0.54+0.46\cos(2\pi\tfrac{n}{N})\right)$$
$$+\left(0.54+0.46\cos(2\pi\frac{n-\frac{N}{2}}{N})\right)=1.08$$



- Can modify individual frames of $X[k,n]$ and then reconstruct
  - complex, time-varying modifications
  - tapered overlap makes things OK

STFT Analysis-Synthesis

- e.g. Noise reduction:

STFT of original speech

Speech corrupted by white noise

Energy threshold mask