ICE503 DSP-Homework#9

1. Given a sequence $x[n] = \cos\left(\frac{2\pi n}{N}\right)$, where N is an even integer, calculate the discrete Fourier transform (DFT) of this sequence.

$$x(n) = \cos\left(\frac{2\pi n}{N}\right) = \frac{1}{2}\left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right)$$

DFT is defined as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}\left(e^{j\frac{2\pi n}{N}} + e^{-j\frac{2\pi n}{N}}\right) e^{-j\frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}\left(e^{j\frac{2\pi n - 2\pi k n}{N}} + e^{-j\frac{2\pi n + 2\pi k n}{N}}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{N-1}\left(e^{-j\frac{2\pi(k-1)n}{N}} + e^{-j\frac{2\pi(k+1)n}{N}}\right)$$

- We will use the identity,

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi k n}{N}} = \begin{cases} N & k = 0, \pm N, \pm 2N, \dots \\ 0 & \text{otherwise} \end{cases}$$

- Hence, $$\sum_{n=0}^{N-1} e^{-j\frac{2\pi(k-1)n}{N}} = \begin{cases} N & k-1 = 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} N & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

- $$\sum_{n=0}^{N-1} e^{-j\frac{2\pi(k+1)n}{N}} = \begin{cases} N & k+1 = 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} N & k = -1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} N & k = N-1 \\ 0 & \text{otherwise} \end{cases}$$

- $$X(k) = \begin{cases} \frac{N}{2} & k = 1, N-1 \\ 0 & \text{otherwise} \end{cases}$$

2. The two 8-point sequence $x_1[n]$ and $x_2[n]$ shown in Figure 1. have DFTs $X_1[k]$ and $X_2[k]$, respectively.

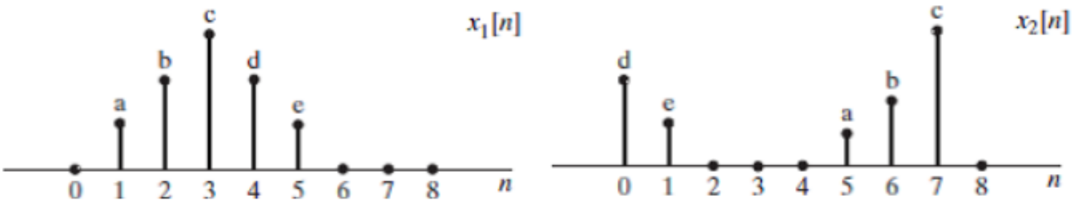

Figure 1. $x_1[n]$ and $x_2[n]$

(a) Determine the relationship between $X_1[k]$ and $X_2[k]$.

(b) Plot the sequence $x_3[n]$ whose DFT is $X_3[k] = W_8^{-5k} X_1[k]$.

(a) From Fig-1

$x_1(n) = \{0, a, b, c, d, e, 0, 0\}$

$x_2(n) = \{d, e, 0, 0, 0, a, b, c\}$

Hence, $x_2(0) = x_1(4) = x_1((4-4) \bmod 8)$

$x_2(1) = x_1(5) = x_1((5-4) \bmod 8)$

$\therefore x_2(n) = x_1((n-4) \bmod 8)$

$\therefore X_2(k) = \sum_{n=0}^{8-1} x_2(n) W_8^{kn}$ where $W_N^{kn} = e^{-j2\pi \frac{kn}{N}}$

$= \sum_{n=0}^{8-1} x_1((n-4) \bmod 8) W_8^{kn}$ $\Rightarrow W_8^{kn} = e^{-j2\pi \frac{kn}{8}}$

$= \sum_{m=0}^{8-1} x_1(m \bmod 8) W_8^{k(m+4)}$ Put $m = n-4$

$= \sum_{m=0}^{8-1} x_1(m) W_8^{km} \times W_8^{k4}$

$= W_8^{4k} \sum_{m=0}^{8-1} x_1(m) W_8^{km}$

$= W_8^{4k} X_1(k)$

(b) $X_3(k) = W_8^{-5k} X_1(k)$

$= W_8^{k(-5)} X_1(k)$

$\therefore x_3(k) = x_1((n-(-5)) \bmod 8)$

$= x_1((n+5) \bmod 8)$

But given

$x_1(n) = \{0, a, b, c, d, e, 0, 0\}$

$\therefore x_3(k) = x_1(\{5, 6, 7, 8, 9, 10, 11, 12\} \bmod 8)$

$= x_1(\{5, 6, 7, 0, 1, 2, 3, 4\})$

$= \{e, 0, 0, 0, a, b, c, d\}$

3. MATLAB simulation:

   Generate a cosine wave for 1 second

$$x(t) = \cos(2\pi 5t).$$

   Then, sample the cosine wave $x(t)$ with 100Hz to obtain $x[n]$.

   (a) Compute the DFT of $x[n]$ with DFT matrix to obtain $X[k]$.

   (b) Compute the IDFT of $X[k]$ with DFT matrix to obtain $x[n]$.

   (c) Compute the DFT of $x[n]$ with fft function to obtain $X[k]$.

   (d) Compute the IDFT of $X[k]$ with ifft function to obtain $x[n]$.

   (e) Use stem function to plot the amplitude of $X[k]$ and $x[n]$ for (a) ~ (d).

```
% Homework 9
% Q. 3

% ------------ clear all ----------------
close all;
clear all;
clc;


% ------------ generate x(t) ------------
samplerate = 100; % Hz
Tsample = 1/samplerate; % s.
Nsamples = samplerate * 1; % sec

t = 0: Tsample: (1-Tsample); % time sample
n_t = t/Tsample;
x = cos(2*pi*5*t); % samples

% ------------- twiddle matrix ----------
W = zeros(Nsamples, Nsamples);

for m = 1: Nsamples
    for n = 1: Nsamples
        W(m,n)= exp(-1j * 2*pi/Nsamples * (m-1)*(n-1) );
    end
end

% -------------- twiddle inverse --------
Winv = 1/Nsamples * conj(W);


% -------------- a ----------------------
X_DFT = W * x.';
X_DFT = X_DFT.';

% -------------- b ----------------------
x_IDFT= Winv * X_DFT.';
x_IDFT= x_IDFT.';

% -------------- c ----------------------
X_FFT = fft(x);

% -------------- d ----------------------
x_IFFT = ifft(X_FFT);

% -------------- e ----------------------
f = figure(1);
f.Position = [200,100, 1200, 800];

subplot(5,1,1);
stem(n_t, x, 'linewidth', 2);
grid on
```

```
xlim([0, max(n_t)])
ylim([-1.2, 1.2])
xlabel('n')
ylabel('x(n)')
title("(a)", 'Units', 'normalized', 'Position', [0.5, -0.5, 0])


subplot(5,1,2);
stem(n_t, abs(X_DFT), 'linewidth', 2);
grid on
xlim([0, max(n_t)])
ylim([min(abs(X_DFT)), max(abs(X_DFT))+10])
xlabel('k')
ylabel('$X_{DFT}$(k)', 'interpreter', 'latex')
title("(b)", 'Units', 'normalized', 'Position', [0.5, -0.5, 0])

subplot(5,1,3);
stem(n_t, real(x_IDFT), 'linewidth', 2);
grid on
xlim([0, max(n_t)])
ylim([-1.2, 1.2])
xlabel('n')
ylabel('$X_{IDFT}$(n)', 'interpreter', 'latex')
title("(c)", 'Units', 'normalized', 'Position', [0.5, -0.5, 0])


subplot(5,1,4);
stem(n_t, abs(X_FFT), 'linewidth', 2);
grid on
xlim([0, max(n_t)])
ylim([min(abs(X_FFT)), max(abs(X_FFT))+10])
xlabel('k')
ylabel('$X_{FFT}$(k)', 'interpreter', 'latex')
title("(d)", 'Units', 'normalized', 'Position', [0.5, -0.5, 0])


subplot(5,1,5);
stem(n_t, real(x_IFFT), 'linewidth', 2);
grid on
xlim([0, max(n_t)])
ylim([-1.2, 1.2])
xlabel('n')
ylabel('$X_{IFFT}$(n)', 'interpreter', 'latex')
title("(e)", 'Units', 'normalized', 'Position', [0.5, -0.5, 0])


saveas(f, 'hw09_3.eps', 'epsc');
```

```
Couldn't create JOGL canvas--using painters
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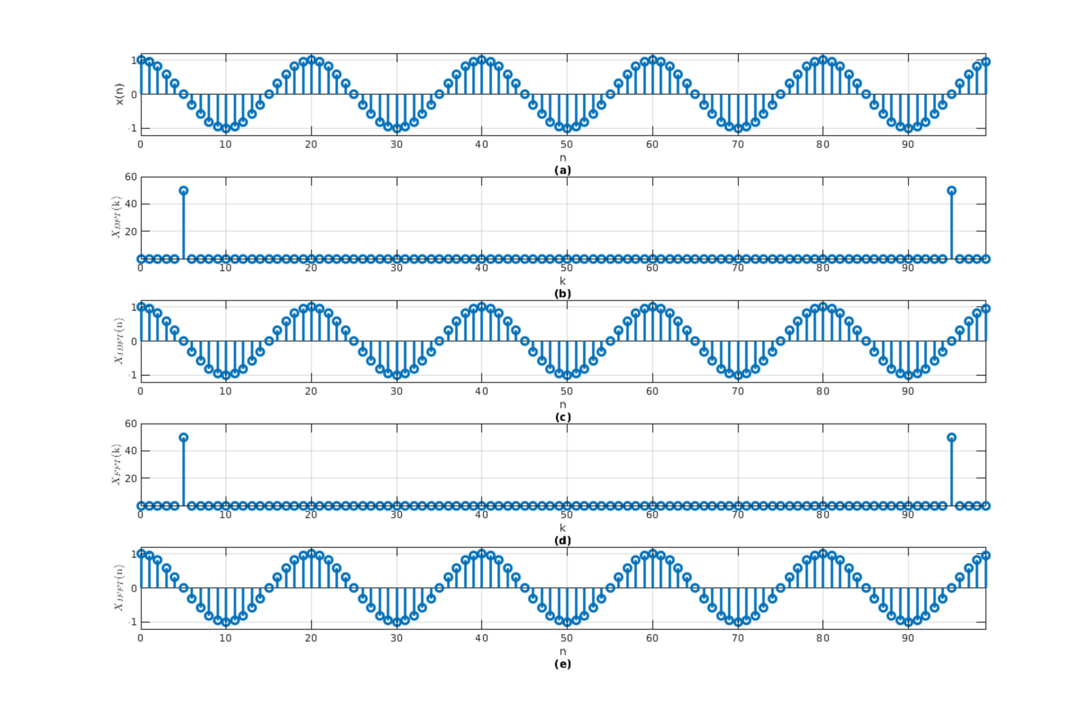

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ICE503 Homework-09

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Q. 3

(a) Given sequence is $x(t) = \cos(2\pi 5t)$. To find the DFT, the **twiddle matrix W** is calculated. The (m, n)-th cell of twiddle matrix $\mathbf{W}$ have the value $e^{-j\frac{2\pi}{N}mn}$, where $N = 100$ represents the number of samples of $x(n)$ sampled in 1 seconds. Thereafter, the DFT is computed by the formula $X(k) = W(k)x(n)$.

The sequence $x(n)$ is shown in Fig. (a) and the corresponding amplitude of the DFT $X(k)$ is shown in Fig. (b).

(b) Compute the inverse DFT matrix as $\hat{\mathbf{W}} = \frac{1}{N}\mathbf{W}^*$, where * represents the conjugate operation. Hence, the IDFT is written as $\hat{x}(n) = \hat{\mathbf{W}}X(k)$. The sequence $\hat{x}$ is shown in Fig. (c) as $x_{DFT}(n)$.

(c) With the $fft(x)$ function the FFT of $x(n)$ is computed with the Fast Fourier Transform (FFT) method. The corresponding amplitude is plotted in Fig. (d).

(d) With the $ifft(x)$ function the IFFT of $X(k)$ is computed with the Fast Fourier Transform (IFFT) method. The corresponding amplitude is plotted in Fig. (e).

(e)