ICE503/IMPTE502/IICD524 DSP – Midterm

1. (10%) To determine the impulse response of an unknown causal, linear time-invariant (LTI) system, Kai applies the following input $x[n]$ to the system:

$$x[n] = 0, \text{ if } n < 0; \ x[n] = 1, \text{ if } n \geq 0.$$

The corresponding output $y[n]$ is given as: $y[n] = 0$ for $n < 0$; $y[n] = 8, 12, 14, 15, 15.5$ for $n = 0, 1, 2, 3, 4$, respectively; $y[n] = 15.75$ for $n \geq 5$.

   (a) (5%) Find the impulse response of this system.

   (b) (5%) Given any input $x[n]$, determine the input-output relationship for this system.

2. (35%) The block diagram of a causal LTI system is illustrated in Figure 1.

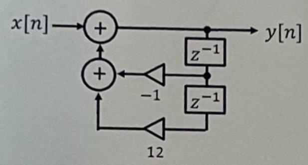


Figure 1: Block diagram of a causal LTI system.

   a) (5%) Derive the difference equation that characterizes the relationship between the input $x[n]$ and output $y[n]$.

   b ~~a)~~ (5%) Compute the Z-transform of the system, $H(z)$, and specify the Region of Convergence (ROC) for $H(z)$.

   c ~~b)~~ (5%) Sketch the poles and zeros of the system in the complex plane.

   d c) (5%) Determine whether the system is stable. Justify your answer.

   e ~~d)~~ (5%) Find the impulse response $h[n]$ of the system.

   f ~~e)~~ (10%) Given the system input $x[n] = \delta[n]$ and the initial conditions $y[-1] = -1$ and $y[-2] = -0.5$, compute the system's response $y[n]$ for $n \geq 0$.

3. (25%) Figure 2(a) shows the overall system for filtering a continuous-time signal using a discrete-time filter. The frequency responses of the reconstruction filter $H_r(j\Omega)$ and the discrete-time filter $H(e^{j\omega})$ are shown in Figure 2(b).

   ~~a)~~ (20%) For $X_c(j\Omega)$ as shown in Figure 2(d) and $1/T_1 = 10\text{kHz}$, sketch $X_p(j\Omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\Omega)$, and $Y_r(j\Omega)$.

   b) (5%) For a certain range of values of T_1, the overall system, with input $x_c(t)$ and output $y_r(t)$, is equivalent to a continuous-time lowpass filter with frequency response $H_{\text{eff}}(j\Omega)$ sketched in Figure 2(e).

   Determine the range of values of T_1 for which the information presented in (a) is true when $X_c(j\Omega)$ is bandlimited to $|\Omega| \leq 2\pi \times 5 \times 10^3$ as shown in Figure 2(d).

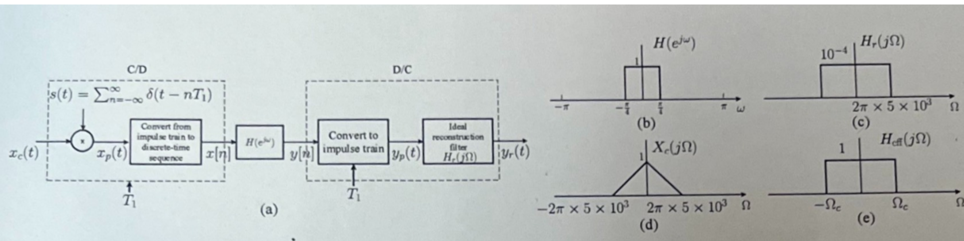


Figure 2: (a) A system. (b), (c), (d), (e) Frequency responses.

4. (15%) Figure 3(a) below illustrates two systems, System A and System B, each consisting of a compressor and an expander.

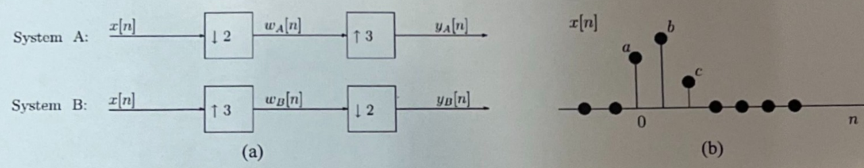


Figure 3: (a) System A and System B. (b) Sequence $x[n]$.

(a) (5%) For the sequence $x[n]$ shown in Figure 3(b), sketch the output sequences $y_A[n]$ and $y_B[n]$, assuming that $x[n] = 0$ outside the interval shown.

(b) (5%) Let $X(e^{j\omega})$ denote the Fourier transform of an arbitrary sequence $x[n]$. Derive and express $Y_B(e^{j\omega})$ in terms of $X(e^{j\omega})$. Your answer should be presented as an equation, not as a graph.

(c) (5%) For any arbitrary $x[n]$, will $y_A[n] = y_B[n]$? If your answer is **yes**, provide an algebraic justification. If your answer is **no**, clearly explain or provide a counterexample.

5. (20%) Consider a finite length sequence

$$x[n] = \begin{cases} 1, & n = 0, 1, \ldots, N/2 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

a) (10%) Compute the DTFT and N-point DFT of $x[n]$.

b) (10%) Sketch the magnitude of the DTFT and N-point DFT of $x[n]$ when $N = 8$.

6. (10%) Write a MATLAB function to reconstruct the continuous signal $x_a(t)$ from discrete-time signals $x[n]$. The format of the function should be function `[x_t] = D2A(x,T_s)`.

```
% Reconstruct the continuous signal from discrete-time signals
% [x_a] = D2A(x,T_s,t)
%
% x_a(t) = continuous signal at time t
% x[n] = discrete-time signals in vector
% T_s = sampling period
% t = time instance
```

1.

(a) $x[n] = \begin{cases} 0 & , n<0 \\ 1 & , n\geq 0 \end{cases}$

$y[n] = \sum_{k=0}^{n} h[k]$

$y[0]=8$, $y[1]=12$, $y[2]=14$, $y[3]=15$, $y[4]=15.5$, $y[n]=15.75$ $(n\geq 5)$

$\Rightarrow h[n] = y[n] - y[n-1]$

$h[0] = y[0] = 8$

$h[1] = y[1] - y[0] = 12-8 = 4$

$h[2] = y[2] - y[1] = 14-12 = 2$

$h[3] = y[3] - y[2] = 15-14 = 1$

$h[4] = y[4] - y[3] = 15.5-15 = 0.5$

$h[5] = y[5] - y[4] = 15.75-15.5 = 0.25$

when $n\geq 6$, $h[n] = y[n]-y[n-1] = 0$

$$\Rightarrow h[n] = \begin{cases} 8 & , n=0 \\ 4 & , n=1 \\ 2 & , n=2 \\ 1 & , n=3 \\ 0.5 & , n=4 \\ 0.25 & , n=5 \\ 0 & , n\geq 6 \end{cases} = 8\delta[n]+4\delta[n-1]+2\delta[n-2]+\delta[n-3]+0.5\delta[n-4]+0.25\delta[n-5]$$ #

(b)

$y[n] = x[n] * h[n] = \sum_{k=0}^{n} x[k]h[n-k]$

$y[0] = x[0]h[0] = 8x[0]$

$y[1] = x[0]h[1] + x[1]h[0] = 4x[0]+8x[1]$

$y[2] = 2x[0]+4x[1]+8x[2]$

$\vdots$

$y[5] = 0.25x[0]+0.5x[1]+x[2]+2x[3]+4x[4]+8x[5]$

$$\Rightarrow \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \\ 2 & 4 & 8 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 \\ 0.5 & 1 & 2 & 4 & 8 & 0 \\ 0.25 & 0.5 & 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \end{bmatrix}$$ #

2.

(a) $y[n] = x[n] - y[n-1] + 12y[n-2]$

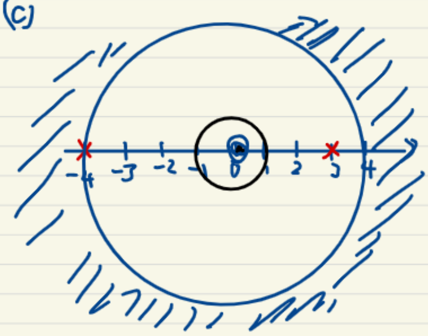
$\Rightarrow y[n] + y[n-1] - 12y[n-2] = x[n]$

(b) $\xrightarrow{Z.T.} Y(z)(1 + z^{-1} - 12z^{-2}) = X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+z^{-1}-12z^{-2}} = \frac{1}{(1+4z^{-1})(1-3z^{-1})}$

$= \frac{4}{7}\frac{1}{1+4z^{-1}} + \frac{3}{7}\frac{1}{1-3z^{-1}}$, ROC: $|z|>4$ (causal)

(c)



pole: $-4, 3$

zero: $0, 0$

(d) The system is not stable, because the ROC didn't cover unit circle.

(e) $H(z) \overset{Z.T.}{\longleftrightarrow} h[n] = \frac{4}{7}(-4)^n u[n] + \frac{3}{7}(3)^n u[n]$

(f) $y[n] = y_p[n] + y_c[n]$

for complementary solution, $y[n] = \lambda^n$, $x[n] = 0$

$\lambda^n + \lambda^{n-1} - 12\lambda^{n-2} = 0$, $\lambda = -4, 3 \Rightarrow y_c[n] = \alpha_1(-4)^n + \alpha_2(3)^n$

for particular solution $y[n] = \beta$, $x[n] = 0$

$\beta + \beta - 12\beta = 0$, $\beta = 0$

for $n=0$, $y[0] + y[-1] - 12y[-2] = x[0]$

$\alpha_1 + \alpha_2 - 1 - 12(-0.5) = 1$

$\alpha_1 + \alpha_2 = -4$ — ①

for $n=1$, $y[1] + y[0] - 12y[-1] = x[1]$

$-3\alpha_1 + 4\alpha_2 = -12$ — ②

①② $\Rightarrow \alpha_2 = -\frac{24}{7}$, $\alpha_1 = -\frac{4}{7}$

$y[n] = -\frac{4}{7}(-4)^n + (-\frac{24}{7})(3)^n$, $n \geq 0$

3. (25%) Figure 2(a) shows the overall system for filtering a continuous-time signal using a discrete-time filter. The frequency responses of the reconstruction filter $H_r(j\Omega)$ and the discrete-time filter $H(e^{j\omega})$ are shown in Figure 2(b).

a) (20%) For $X_c(j\Omega)$ as shown in Figure 2(d) and $1/T_1 = 10\text{kHz}$, sketch $X_p(j\Omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\Omega)$, and $Y_r(j\Omega)$.

b) (5%) For a certain range of values of T_1, the overall system, with input $x_c(t)$ and output $y_r(t)$, is equivalent to a continuous-time lowpass filter with frequency response $H_{\text{eff}}(j\Omega)$ sketched in Figure 2(e).

Determine the range of values of T_1 for which the information presented in (a) is true when $X_c(j\Omega)$ is bandlimited to $|\Omega| \le 2\pi \times 5 \times 10^3$ as shown in Figure 2(d).

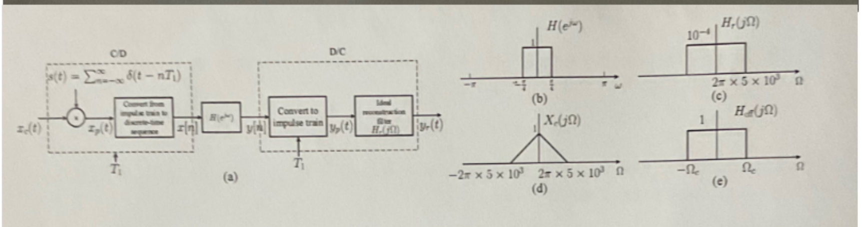
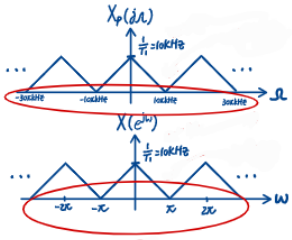
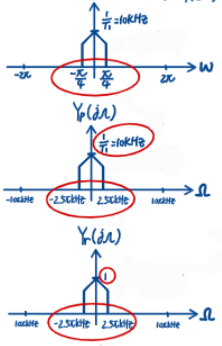


Figure 2: (a) A system. (b), (c), (d), (e) Frequency responses.

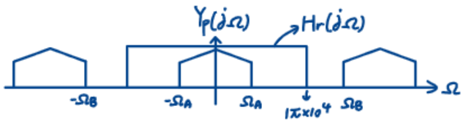
3. (a) $\frac{1}{T_1} = 10kHz$, $\Omega \cdot T = \omega = 2\pi \longrightarrow \frac{2\pi}{T_1} = 20\pi\, kHz$



$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$



(b) Find the range of T_1, $\Omega T_1 = \omega$



$$\boxed{\begin{cases} \Omega_A = \frac{\pi}{4} \cdot \frac{1}{T_r} \\ \Omega_B = \frac{2\pi}{T_r} - \Omega_A = \frac{7}{4}\pi \cdot \frac{1}{T_r} \end{cases}}$$, where $\Omega_A < 1\pi \times 10^4 < \Omega_B$

$$\longrightarrow \frac{1}{4 \times 10^4} < T_1 < \frac{7}{4 \times 10^4}$$

4. (15%) Figure 3(a) below illustrates two systems, System A and System B, each consisting of a compressor and an expander.

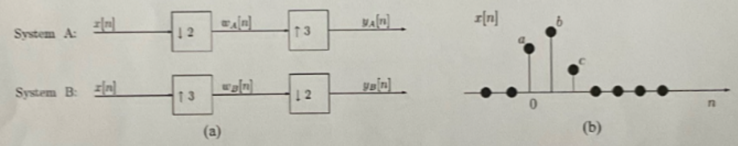


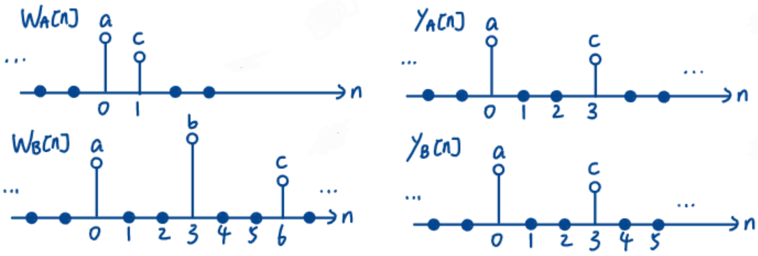
Figure 3: (a) System A and System B. (b) Sequence $x[n]$.

(a) (5%) For the sequence $x[n]$ shown in Figure 3(b), sketch the output sequences $y_A[n]$ and $y_B[n]$, assuming that $x[n] = 0$ outside the interval shown.

(b) (5%) Let $X(e^{j\omega})$ denote the Fourier transform of an arbitrary sequence $x[n]$. Derive and express $Y_B(e^{j\omega})$ in terms of $X(e^{j\omega})$. Your answer should be presented as an equation, not as a graph.

(c) (5%) For any arbitrary $x[n]$, will $y_A[n] = y_B[n]$? If your answer is **yes**, provide an algebraic justification. If your answer is **no**, clearly explain or provide a counterexample.

4. (a)



(b)

$W_B(e^{jw}) = X_B(e^{j3w})$

$\rightarrow Y_B(e^{jw}) = \frac{1}{2} W_B(e^{j\frac{w}{2}} + e^{j(\frac{w}{2}-\pi)})$

$\Rightarrow Y_B(e^{jw}) = \frac{1}{2} X_B(e^{j\frac{3}{2}w} + e^{j(\frac{3w}{2}-3\pi)})$

(c)

$W_A(e^{jw}) = \frac{1}{2} X_A(e^{j\frac{w}{2}} + e^{j(\frac{w}{2}-\pi)})$

$\rightarrow Y_A(e^{jw}) = W_A(e^{j3w})$

$\Rightarrow Y_A(e^{jw}) = \frac{1}{2} X_A(e^{j\frac{3}{2}w} + e^{j(\frac{3w}{2}-3\pi)})$

Yes, 由上式 $Y_A(e^{jw}) = Y_B(e^{jw})$

$\therefore Y_A[n] = Y_B[n]$

5.

(a)

DTFT : $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\frac{N}{2}-1} e^{-j\omega n}$

$$= \frac{1-e^{-j\omega\left(\frac{N}{2}\right)}}{1-e^{-j\omega}}$$

N-point DFT : $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} W_N^{kn}$ $\quad \omega = \frac{2\pi k}{N}$

$$= \frac{1-e^{-j\frac{2\pi}{N}k\left(\frac{N}{2}\right)}}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= \frac{1-e^{-j\pi k}}{1-e^{-j\frac{2\pi}{N}k}}$$

$$= \begin{cases} \frac{N}{2} & , k=0 \\ \frac{2}{1-e^{-j\frac{2\pi}{N}k}} & , k=\text{odd} \\ 0 & , k=\text{even} \end{cases} \quad \times$$

(b) When $N=8$, $x[n]=[1,1,1,1,0,0,0,0]$

$X[k] = \sum_{n=0}^{3} W_8^{kn}$

$X[0] = \sum_{n=0}^{3} W_8^{0} = 1+1+1+1 = 4$

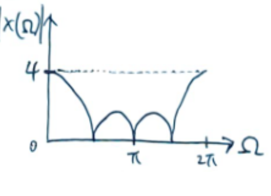
$X[1] = \sum_{n=0}^{3} W_8^{1} = 1+e^{-j\frac{\pi}{4}}+e^{-j\frac{\pi}{2}}+e^{-j\frac{3\pi}{4}} = 1+\left(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\right)+(-j)+\left(-\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\right) = 1-j(1+\sqrt{2})$

$X[2] = \sum_{n=0}^{3} W_8^{2} = 1+e^{-j\frac{\pi}{2}}+e^{-j\pi}+e^{-j\frac{3\pi}{2}} = 1-j-1+j = 0$

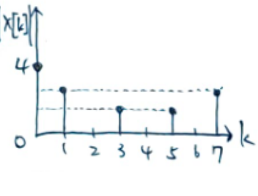
$X[3] = \sum_{n=0}^{3} W_8^{3} = 1+e^{-j\frac{3\pi}{4}}+e^{-j\frac{3\pi}{2}}+e^{-j\frac{9\pi}{4}} = 1+\left(-\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\right)+j+\left(\frac{\sqrt{2}}{2}-j\frac{\sqrt{2}}{2}\right) = 1+j(1-\sqrt{2})$

$X[4] = \sum_{n=0}^{3} W_8^{4} = 1+e^{-j\pi}+e^{-j2\pi}+e^{-j3\pi} = 1-1+1-1 = 0$

DTFT : $|X(\Omega)|$



8-point DFT : $|X[k]|$



6.

```
rate = T/t ;
m= 0: 1/t : 1/T ;

for ii = 1:length(X[n])
    y_c(ii,:) = X[n](ii)*sinc((-ii+1)*1/t+m);
end

x_a = sum(y_c);
```