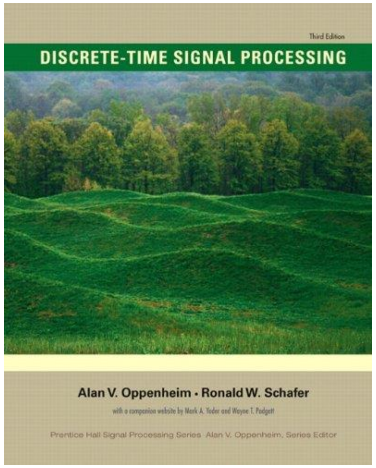


Lecture 08:

Discrete Fourier Transform

Outline

- Discrete Fourier Transform
- Convolution with the DFT
- Short-Time Fourier Transform

Fourier Transforms

	Time	*Frequency*
Fourier Series (FS)	Continuous periodic $\tilde{x}(t)$	Discrete infinite c_k
Fourier Transform (FT)	Continuous infinite $x(t)$	Continuous infinite $X(\Omega)$
Discrete-Time FT (DTFT)	Discrete infinite $x[n]$	Continuous periodic $X(e^{j\omega})$
Discrete FT (DFT)	Discrete finite/pdc $\tilde{x}[n]$	Discrete finite/pdc $X[k]$

Discrete FT (DFT)

Discrete FT (DFT)	Discrete finite/pdc $x[n]$	Discrete finite/pdc $X[k]$

- A *finite* or *periodic* sequence has only N unique values, $x[n]$ for $0 \leq n < N$
- Spectrum is completely defined by N distinct frequency samples
- Divide $0..2\pi$ into N equal steps,

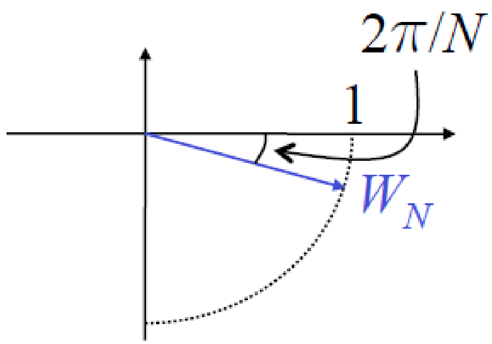
$$\{\omega_k\} = \frac{2\pi k}{N}$$

DFT and IDFT

- Uniform sampling of DTFT spectrum:

$$X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

- **DFT:** $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$



where $W_N = e^{-j\frac{2\pi}{N}}$ i.e. -1/Nth of a revolution

IDFT

- Inverse DFT: **IDFT** $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-nk}$
- Check:

$$x[n] = \frac{1}{N}\sum_k \left(\sum_l x[l] W_N^{kl}\right) W_N^{-nk}$$

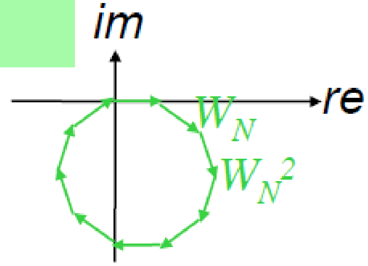
$$= \frac{1}{N}\sum_{l=0}^{N-1} x[l] \sum_{k=0}^{N-1} W_N^{k(l-n)}$$

$$= x[n] \checkmark$$

$0 \le n < N$

Sum of complete set of rotated vectors

$= 0$ if $l \neq n$; $= N$ if $l = n$



or finite geometric series

$= (1-W_N^{lN})/(1-W_N^{l})$

DFT examples

- *Finite* impulse $x[n] = \begin{cases} 1 & n = 0 \\ 0 & n = 1 \ldots N-1 \end{cases}$

$$\Rightarrow X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = W_N^0 = 1 \quad \forall k$$

- Periodic sinusoid:

$$x[n] = cos\left(\frac{2\pi r n}{N}\right) \quad (r \in \mathbb{Z}) \quad = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

$$\Rightarrow X[k] \quad = \tfrac{1}{2} \textstyle\sum_{n=0}^{N-1} (W_N^{-rn} + W_N^{rn}) W_N^{kn}$$

$(0 \le k < N)$

$$= \begin{cases} \frac{N}{2} & k = r, k = N - r \\ 0 & \text{otherwise} \end{cases}$$

DFT: Matrix form

- $X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{kn}$ as a matrix multiply:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

- i.e. $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

Matrix IDFT

- If $\mathbf{X} = \mathbf{D}_N \cdot \mathbf{x}$

  then $\mathbf{x} = \mathbf{D}_N^{-1} \cdot \mathbf{X}$
- i.e. inverse DFT is also just a matrix,

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$={}^1/_N D_N^{\,*}$

DFT and MATLAB

- MATLAB is concerned with *sequences* not continuous functions like $X(e^{j\omega})$
- Instead, we use the DFT to sample $X(e^{j\omega})$ on an (arbitrarily-fine) grid:
  - `X = freqz(x,1,w);` samples the DTFT of sequence `x` at angular frequencies in `w`
  - `X = fft(x);` calculates the N-point DFT of an N-point sequence `x`

DFT and DTFT

DTFT $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

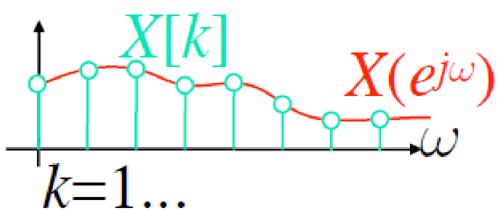
- *continuous freq* ω
- *infinite* $x[n]$, $-\infty < n < \infty$

DFT $$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

- *discrete freq* $k = N\omega/2\pi$
- *finite* $x[n]$, $0 \leq n < N$

- DFT 'samples' DTFT at discrete freqs:

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}}$$



DTFT from DFT

- N-point DFT completely specifies the continuous DTFT of the finite sequence

$$X(e^{j\omega}) = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}\right)e^{-j\omega n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{-j\left(\omega - \frac{2\pi k}{N}\right)n}$$

"periodic sinc"

$$\Delta\omega_k = \omega - \frac{2\pi k}{N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \cdot \frac{\sin N\frac{\Delta\omega_k}{2}}{\sin\frac{\Delta\omega_k}{2}} \cdot e^{-j\frac{(N-1)}{2}\cdot\Delta\omega_k}$$

interpolation

Periodic sinc

$$\sum_{n=0}^{N-1} e^{-j\Delta\omega_k n} = \frac{1-e^{-jN\Delta\omega_k}}{1-e^{-j\Delta\omega_k}}$$

factor out half the angle

$$= \frac{e^{-jN\Delta\omega_k/2}}{e^{-j\Delta\omega_k/2}} \cdot \frac{e^{jN\Delta\omega_k/2} - e^{-jN\Delta\omega_k/2}}{e^{j\Delta\omega_k/2} - e^{-j\Delta\omega_k/2}}$$

$$= \underbrace{e^{-j\frac{(N-1)}{2}\cdot\Delta\omega_k}}_{\text{pure phase}} \underbrace{\frac{\sin N\frac{\Delta\omega_k}{2}}{\sin\frac{\Delta\omega_k}{2}}}_{\text{pure real}}$$

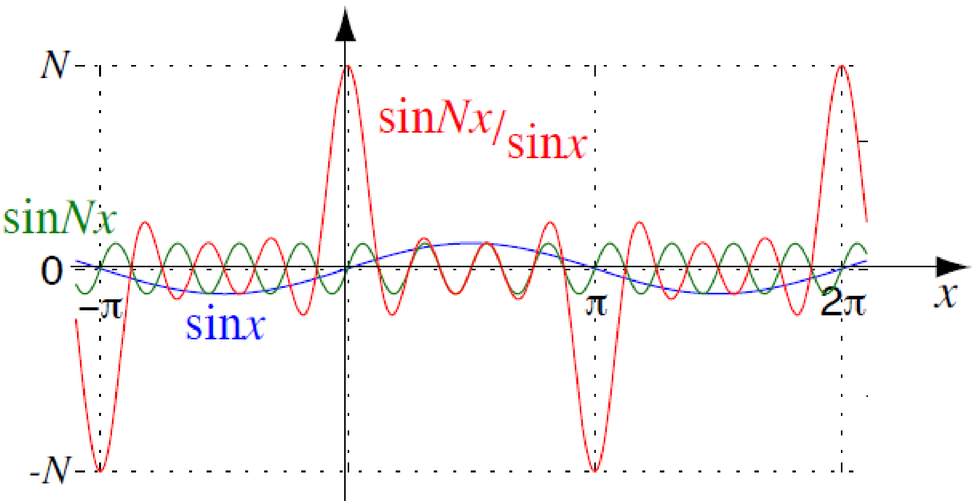
- $= N$ when $\Delta\omega_k = 0$; $= (\text{-})N$ when $\Delta\omega_k/2 = \pi$

  $= 0$ when $\Delta\omega_k/2 = r\cdot\pi/N,\ r = \pm 1, \pm 2, \ldots$

  other values in-between...

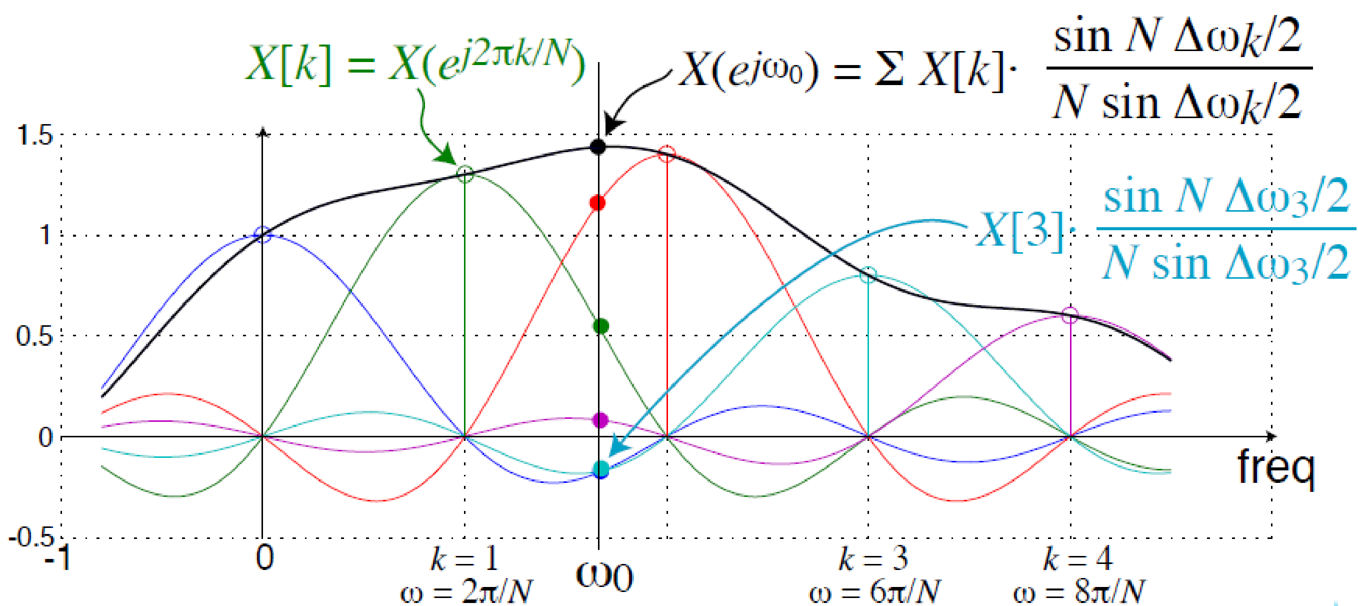
Periodic sinc

$$\frac{\sin Nx}{\sin x}$$



DFT→ DTFT
= interpolation
by periodic
sinc

$X[k] \rightarrow X(e^{j\omega})$



DFT from overlength DTFT

- If $x[n]$ has more than N points, can still form $X[k] = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}$
- IDFT of $X[k]$ will give N point $\tilde{x}[n]$
- How does $\tilde{x}[n]$ relate to $x[n]$?

DFT from overlength DTFT

$$x[n] \xrightarrow{\textit{DTFT}} X(e^{j\omega}) \xrightarrow{\textit{sample}} X[k] \xrightarrow{\textit{IDFT}} \tilde{x}[n]$$

$-A \leq n < B$ $\qquad\qquad$ $0 \leq n < N$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} \right) W_N^{-nk}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left(\boxed{\frac{1}{N} \sum_{k=0}^{N-1} W_N^{k(\ell-n)}} \right)$$

$=1$ for $n\text{-}l = rN$, $r \in \mathrm{I}$
$= 0$ otherwise

$$\Rightarrow \tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

$0 \leq n < N$

all values shifted by exact multiples of N pts to lie in $0 \leq n < N$

DFT from DTFT example

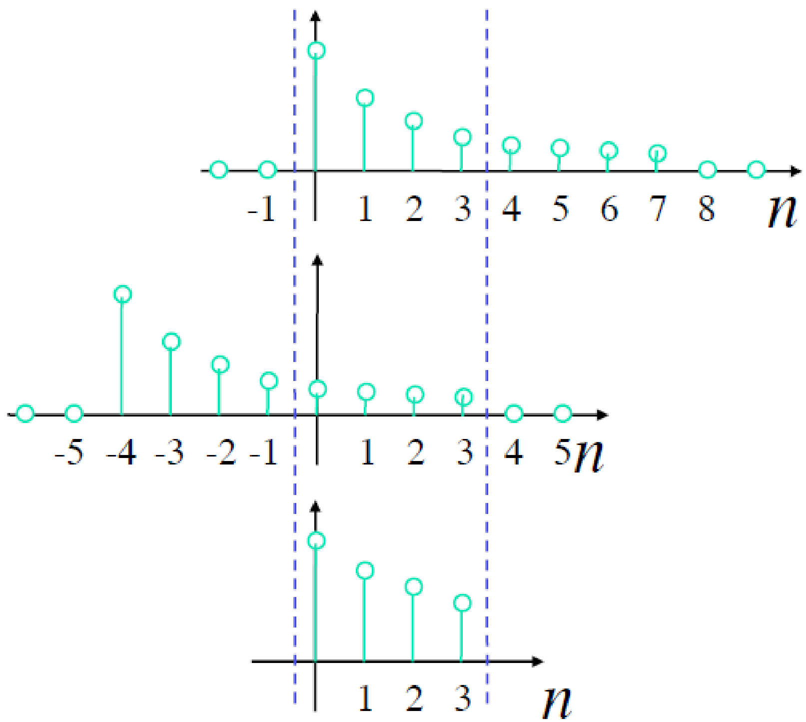
- If $x[n] = \{ 8, 5, 4, 3, 2, 2, 1, 1\}$ (8 point)
- We form $X[k]$ for $k = 0, 1, 2, 3$
  by sampling $X(e^{j\omega})$ at $\omega = 0, \pi/2, \pi, 3\pi/2$
- IDFT of $X[k]$ gives 4 pt $\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$
- Overlap only for $r = -1$: $(N = 4)$

$$\Rightarrow \tilde{x}[n] = \begin{Bmatrix} 8 & 5 & 4 & 3 \\ + & + & + & + \\ 2 & 2 & 2 & 1 \end{Bmatrix} = \{10 \quad 7 \quad 5 \quad 4\}$$

DFT from DTFT example

- $x[n]$
- $x[n+N]$

  $(r = -1)$
- $\tilde{x}[n]$



- $\tilde{x}[n]$ is the time aliased or 'folded down' version of $x[n]$.

Properties: Circular time shift

- DFT properties mirror DTFT, with twists:
- Time shift must stay within N-pt 'window'

$$g[\langle n - n_0 \rangle_N] \quad \leftrightarrow \quad W_N^{kn_0} G[k]$$

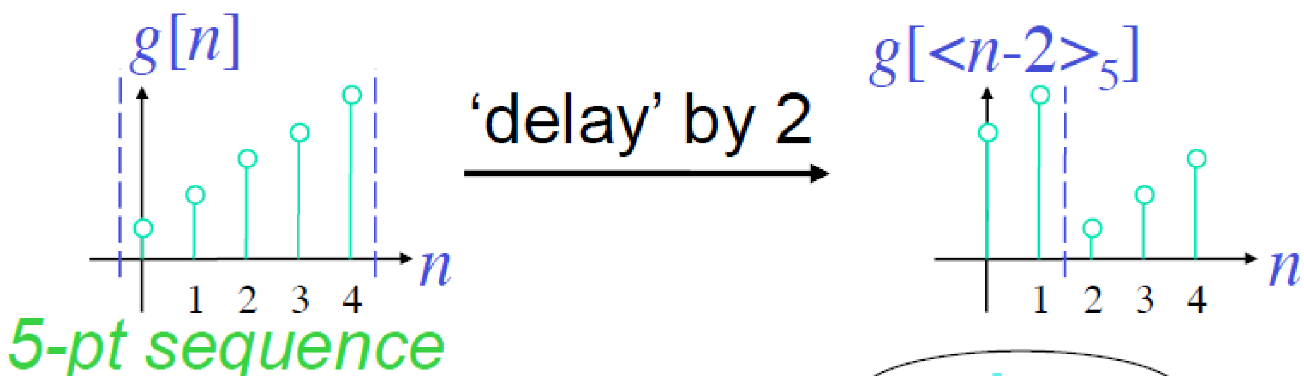
- Modulo-N indexing keeps index between 0 and N-1:

$$g[\langle n - n_0 \rangle_N] = \begin{cases} g[n - n_0] & n \geq n_0 \\ g[N + n - n_0] & n < n_0 \end{cases}$$

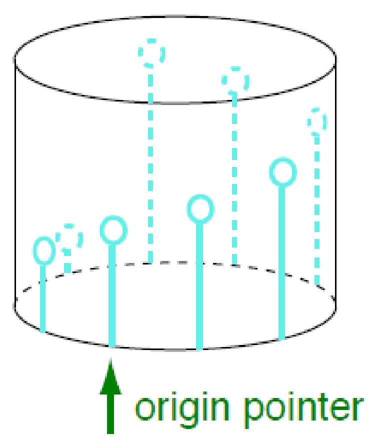
$0 \leq n_0 < N$

Circular time shift

- Points shifted out to the right don't disappear – they come in from the left



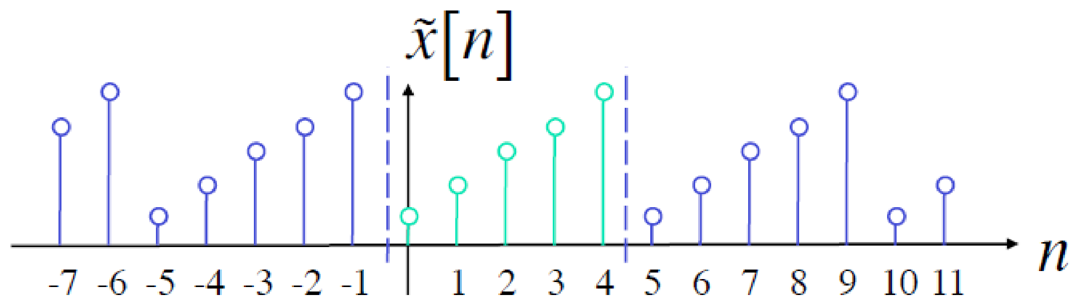
- Like a 'barrel shifter':



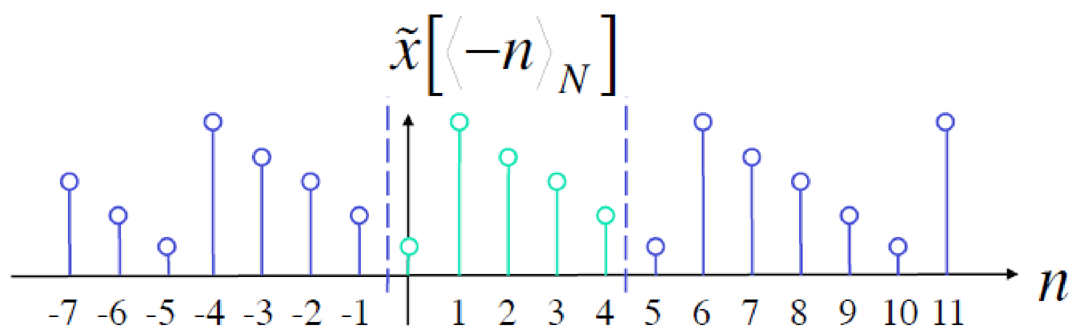
Circular time reversal

- Time reversal is tricky in ‘modulo-N’ indexing - **not** reversing the sequence:

5-pt sequence made periodic



Time-reversed periodic sequence



- Zero point stays fixed; remainder flips

Duality

- DFT and IDFT are very similar
  - both map an N-pt vector to an N-pt vector
- Duality:

$$\text{if} \quad g[n] \leftrightarrow G[k]$$

$$\text{then} \quad G[n] \leftrightarrow N \cdot g[\langle -k \rangle_N]$$

Circular time reversal

- i.e. if you treat DFT sequence as a time sequence, result is almost symmetric

2. Convolution with the DFT

- IDTFT of product of DTFTs of two N-pt sequences is their $2N$-1 pt convolution
- IDFT of the product of two N-pt DFTs can only give N points!
- Equivalent of $2N$-1 pt result time aliased:
  - i.e. $y_c[n] = \sum_{r=-\infty}^{\infty} y_l[n+rN]$ $(0 \leq n < N)$
  - must be, because $G[k]H[k]$ are exact samples of $G(e^{j\omega})H(e^{j\omega})$
- This is known as circular convolution

Circular convolution

- Can also do entire convolution with modulo-N indexing
- Hence, **Circular Convolution**:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N] \quad \leftrightarrow \quad G[k]H[k]$$

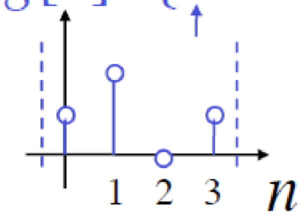
- Written as $g[n] \, Ⓝ \, h[n]$

Circular convolution example

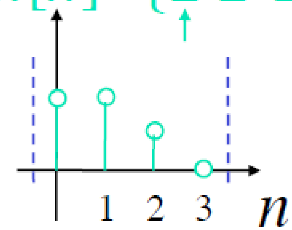
- 4 pt sequences:

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$$

$g[n]=\{1\ 2\ 0\ 1\}$



$h[n]=\{2\ 2\ 1\ 0\}$

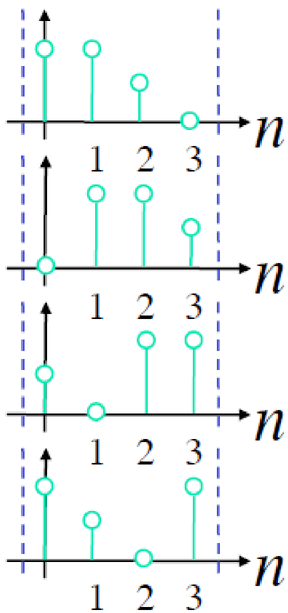


$h[\langle n-0\rangle_4] \cdot 1$

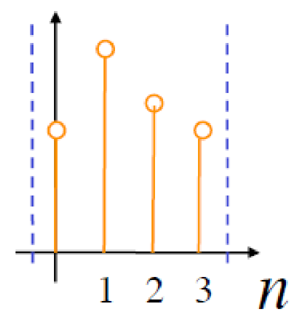
$h[\langle n-1\rangle_4] \cdot 2$

$h[\langle n-2\rangle_4] \cdot 0$

$h[\langle n-3\rangle_4] \cdot 1$



$g[n] \, ④ \, h[n]=\{4\ 7\ 5\ 4\}$



check: $g[n] \circledast h[n]$

$=\{2\ 6\ 5\ 4\ 2\ 1\ 0\}$

DFT properties summary

- Circular convolution

$$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N] \quad \leftrightarrow \quad G[k]H[k]$$

- Modulation

$$g[n]\cdot h[n] \quad \leftrightarrow \quad \tfrac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$$

- Duality

$$G[n] \quad \leftrightarrow \quad N\cdot g[\langle -k\rangle_N]$$

- Parseval

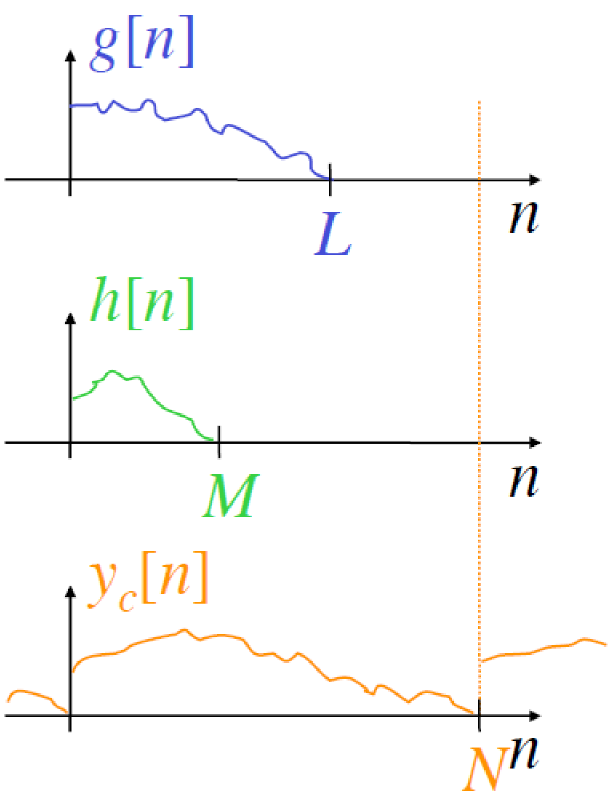
$$\sum_{n=0}^{N-1} \left|x[n]\right|^2 = \tfrac{1}{N}\sum_{k=0}^{N-1} \left|X[k]\right|^2$$

Linear convolution w/ the DFT

- DFT → fast circular convolution
- .. but we need linear convolution
- Circular conv. is time-aliased linear conv.; can aliasing be avoided?
- e.g. convolving L-pt $g[n]$ with M-pt $h[n]$: $y[n] = g[n] \circledast h[n]$ has $L+M$-1 nonzero pts
- Set DFT size $N \geq L+M$-1 → **no aliasing**

Linear convolution w/ the DFT

- Procedure ($N = L + M - 1$):
  - pad L-pt $g[n]$ with (at least) M-1 zeros $\rightarrow$ N-pt DFT $G[k]$, $k = 0..N$-1
  - pad M-pt $h[n]$ with (at least) L-1 zeros $\rightarrow$ N-pt DFT $H[k]$, $k = 0..N$-1
  - $Y[k] = G[k] \cdot H[k]$, $k = 0..N$-1
  - IDFT{$Y[k]$} $= \sum_{r=-\infty}^{\infty} y_L[n + rN] = y_L[n]$ $\quad (0 \le n < N)$



Overlap-Add convolution

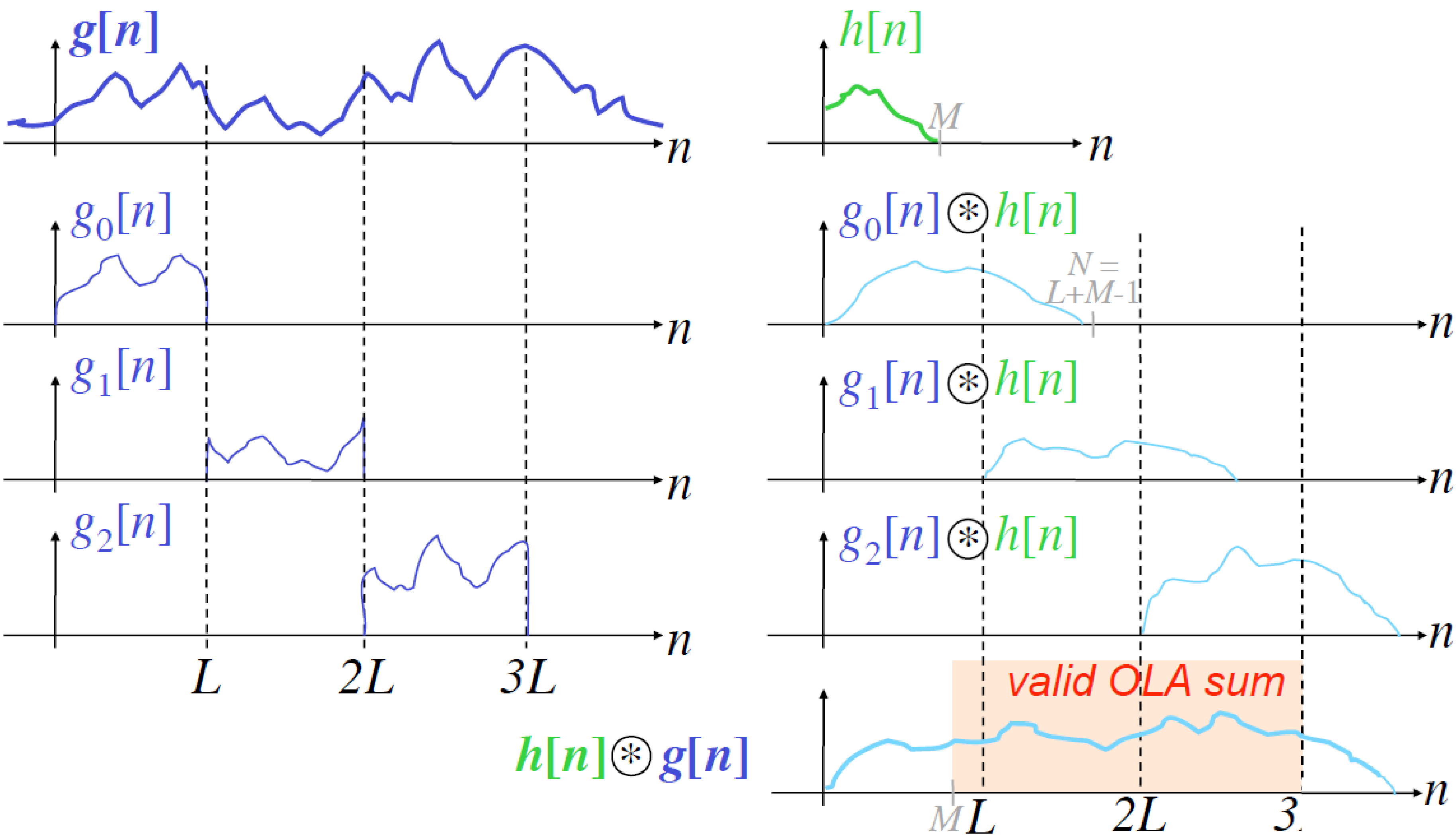
- Very long $g[n]$ → break up into segments, convolve piecewise, overlap
  → bound size of DFT, processing delay
- Make $g_i[n] = \begin{cases} g[n] & i \cdot N \leq n < (i+1) \cdot N \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow \qquad g[n] = \textstyle\sum_i g_i[n]$$

$$\Rightarrow \quad h[n] \circledast g[n] = \textstyle\sum_i h[n] \circledast g_i[n]$$

- Called Overlap-Add (**OLA**) convolution

Overlap-Add convolution



DFT of Truncated Signals

- **What if the signal is not time-limited? We can think of limiting the sum to *N* points as a truncation of the signal:**

$$x_w[n] = w[n]x[n]$$

$$w[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots, N \\ 0, & otherwise \end{cases}$$

- **What are the implications of this in the frequency domain? (Hint: convolution)**

- **Popular Windows:**
  - **Rectangular:**

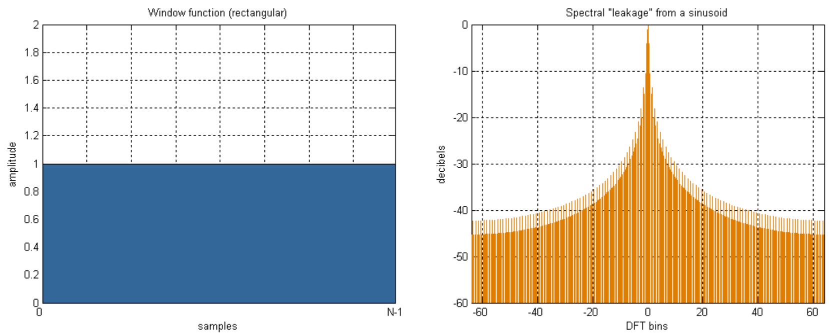
$$w[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots, N \\ 0, & otherwise \end{cases}$$

  - **Generalized Hanning:**

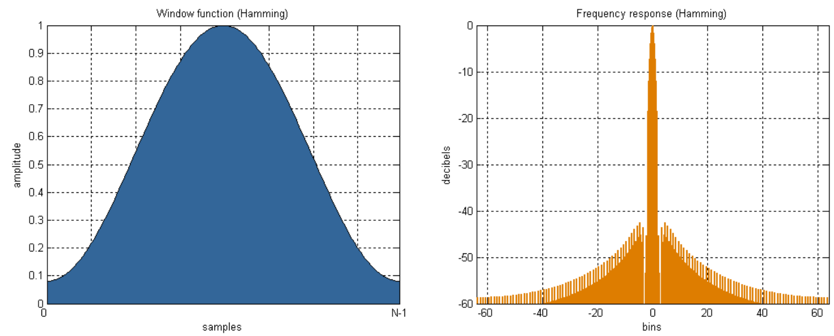
$$w[n] = \alpha + (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right)$$

  - **Triangular:**

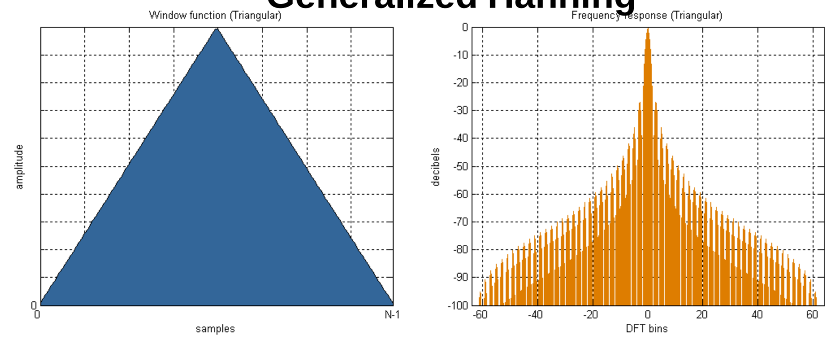
$$w[n] = \frac{2}{N}\left(\frac{N}{2} - \left|n - \frac{N-1}{2}\right|\right)$$



- **Rectangular**



- **Generalized Hanning**



- **Triangular**

Impact on Spectral Estimation

- The spectrum of a windowed sinewave is the convolution of two impulse functions with the frequency response of the window.
- For two closely spaced sinewaves, there is "leakage" between each sinewave's spectrum.

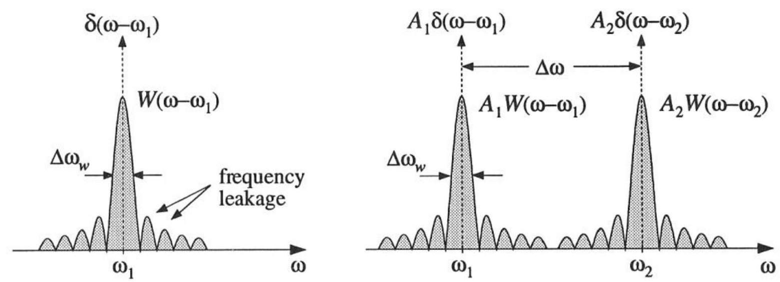


Fig. 9.1.3 Spectra of windowed single and double sinusoids.

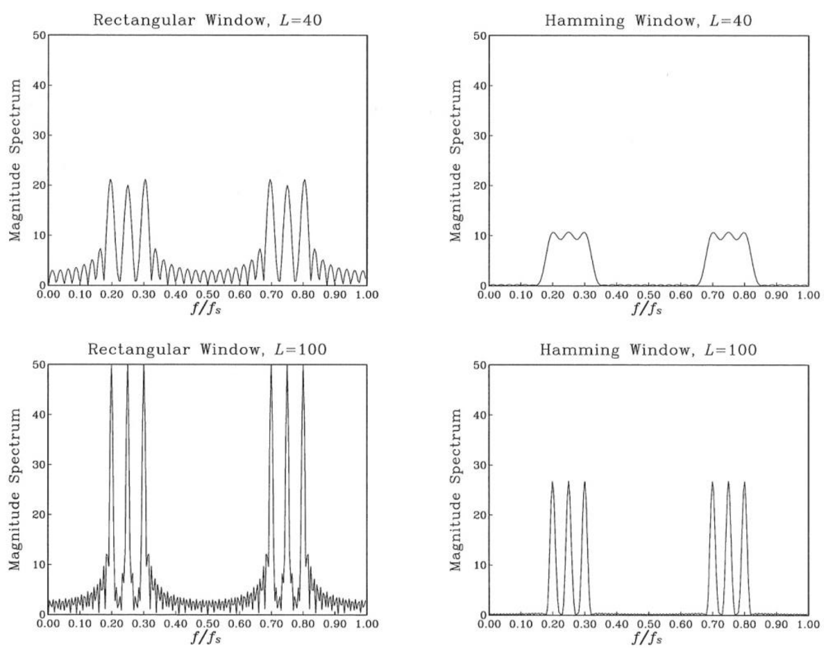


Fig. 9.1.9 Rectangular and Hamming spectra for $L = 40$ and 100.

- The impact of this leakage can be mitigated by using a window function with a narrower main lobe.
- For example, consider the spectrum of three sinewaves computed using a rectangular and a Hamming window.
- We see that for the same number of points, the spectrum produced by the Hamming window separates the sinewaves.
- What is the computational cost?