Chapter 3

System Properties and Solution techniques

In this chapter we discuss system properties and solution techniques for discrete-time systems. (For the continuous case, see Rowell & Wormley [4], chapter 8).

3.1 Singularity input functions

In Rowell & Wormley [4], chapter 8, some singularity input functions for continuous-time systems were defined. In this section we redefine some of the functions for discrete-time:

The Unit Step Function: The discrete-time unit step function $u_s(k)$, $k \in \mathbb{Z}$ is defined as:

$$u_s(k) = \begin{cases} 0 & \text{for } \; k < 0 \\ 1 & \text{for } \; k \geq 0 \end{cases} \tag{3.1}$$

The Unit Impulse Function: The discrete-time unit Impulse function $\delta(k)$, $k \in \mathbb{Z}$ is defined as:

$$\delta(k) = \begin{cases} 1 & \text{for } \; k = 0 \\ 0 & \text{for } \; k \neq 0 \end{cases} \tag{3.2}$$

Note that for discrete-time the definition of a function $\delta_T(k)$ is not appropriate. The discrete-time impulse function (or discrete-time Dirac delta function) has the property that

$$\sum_{m=-\infty}^{\infty} \delta(m) = 1$$

The unit ramp function: The discrete-time unit ramp function $u_r(k)$, $k \in \mathbb{Z}$ is defined to be a linearly increasing function of time with a unity increment:

$$u_r(k) = \begin{cases} 0 & \text{for } k \leq 0 \\ k & \text{for } k > 0 \end{cases} \tag{3.3}$$

Note that

$$\Delta\{u_r(k+1)\} \;=\; u_r(k+1) - u_r(k) = u_s(k) \tag{3.4}$$

and

$$\Delta\{u_s(k)\} \;=\; u_s(k) - u_s(k-1) = \delta(k) \tag{3.5}$$

and also in the reverse direction:

$$\Delta^{-1}\{\delta(k)\} \;=\; \sum_{m=0}^{k} \delta(k) = u_s(k) \tag{3.6}$$

and

$$\Delta^{-1}\{u_s(k)\} \;=\; \sum_{m=0}^{k} u_s(k) = u_r(k+1) \tag{3.7}$$

3.2 Classical solution of linear difference equations

In this section we briefly review the classical method for solving a linear nth-order ordinary difference equation with constant coefficients, given by

$$\begin{aligned} & y(k+n) + a_{n-1}\, y(k+n-1) + \ldots + a_0\, y(k) \\ = \;& b_m\, u(k+m) + b_{m-1}\, u(k+m-1) + \ldots + b_0\, u(k) \end{aligned} \tag{3.8}$$

where in general $m \leq n$. We define the forcing function

$$f(k) = b_m\, u(k+m) + b_{m-1}\, u(k+m-1) + \ldots + b_0\, u(k) \tag{3.9}$$

which is known, because $u(k)$ is known for all $k \in \mathbb{Z}$. Eq. (3.8) now becomes

$$y(k+n) + a_{n-1}\, y(k+n-1) + \ldots + a_0\, y(k) = f(k) \tag{3.10}$$

The task is to find a unique function $y(k)$ for $k \geq k_0$ that satisfies the difference equation (3.8) given the forcing function $f(k)$ and a set of initial conditions $y(k_0), y(k_0{+}1), \ldots, y(k_0{+}n-1)$.
The general solution to Eq. (3.8) may be derived as the sum of two solution components

$$y(k) = y_h(k) + y_p(k) \tag{3.11}$$

where y_h is the solution of the homogeneous equation (so for $f(k) = 0$) and $y_p(k)$ is a particular solution that satisfies (3.10) for the specific $f(k)$ (but arbitrary initial conditions).

3.2.1 Homogeneous solution of the difference equation

The homogeneous difference equation ($a_n \neq 0$) is given by

$$a_n\, y(k+n) + a_{n-1}\, y(k+n-1) + \ldots + a_0\, y(k) = 0 \tag{3.12}$$

The standard method of solving difference equations assumes there exists a solution of the form $y(k) = C\,\lambda^k$, where λ and C are both constants. Substitution in the homogeneous equation gives:

$$C\Big(a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0\Big)\lambda^k = 0 \tag{3.13}$$

For any nontrivial solution, C is nonzero and λ^k is never zero, we require that

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 = 0 \tag{3.14}$$

For an nth-order system with $a_n \neq 0$, there are n (complex) roots of the characteristic polynomial and n possible solution terms $C_i\,\lambda_i^k$ $(i = 1, \ldots, n)$, each with its associated constant. The homogeneous solution will be the sum of all such terms:

$$\begin{aligned} y_h(k) &= C_1\,\lambda_1^k + C_2\,\lambda_2^k + \ldots + C_n\,\lambda_n^k && (3.15)\\ &= \sum_{i=1}^{n} C_i\,\lambda_i^k && (3.16)\end{aligned}$$

The n coefficients C_i are arbitrary constants and must be found from the n initial conditions.
If the characteristic polynomial has repeated roots, that is $\lambda_i = \lambda_j$ for $i \neq j$, there are not n linearly independent terms in the general homogeneous response (3.16). In general if a root λ of multiplicity m occurs, the m components in the general solution are

$$C_1\,\lambda^k \;\;,\;\; C_2\,k\,\lambda^k \;\;, \ldots ,\;\; C_m\,k^{m-1}\,\lambda^m$$

If one or more roots are equal to zero ($\lambda_i = 0$), the solution $C\,\lambda^k = 0$ has no meaning. In that case we add a term $C\,\delta(k-k_0)$ to the general solution. If the root $\lambda = 0$ has multiplicity m, the m components in the general solution are

$$C_1\,\delta(k-k_0) \;\;,\;\; C_2\,\delta(k-k_0-1) \;\;, \ldots ,\;\; C_m\,\delta(k-k_0-m+1)$$

3.2.2 Particular solution of the difference equation

Also for discrete-time systems, the method of undetermined coefficients can be used (see [4], page 255-257), where we use table 3.1 to find the particular solutions for some specific forcing functions.

Terms in $u(k)$	Assumed form for $y_p(k)$	Test value
α	β_1	0
$\alpha\, k^n$, $(n = 1, 2, 3, \ldots)$	$\beta_n\, k^n + \beta_{n-1}\, k^{n-1} + \ldots + \beta_1\, k + \beta_0$	0
$\alpha\, \lambda^k$	$\beta\, \lambda^k$	λ
$\alpha\, \cos(\omega k)$	$\beta_1\, \cos(\omega k) + \beta_2\, \sin(\omega k)$	$j\omega$
$\alpha\, \sin(\omega k)$	$\beta_1\, \cos(\omega k) + \beta_2\, \sin(\omega k)$	$j\omega$

Figure 3.1: Definition of Particular $y_p(k)$ using the method of undetermined coefficients

Example 9 (Solution of a difference equation)
Consider the system described by the difference equation

$$2\, y(k+3) + y(k+2) = 7\, u(k+1) - u(k)$$

Find the system response to a ramp input $u(k) = k$ *and initial conditions given by* $y(0) = 2$, $y(1) = -1$ *and* $y(2) = 2$.

Solution:
homogeneous solution: The characteristic equation is:

$$2\, \lambda^3 + \lambda^2 = 0$$

which has a root $\lambda_1 = -0.5$ *and a double root* $\lambda_{2,3} = 0$. *The general solution of the homogeneous equation is therefore*

$$y_h(k) = C_1\, (-0.5)^k + C_2\, \delta(k) + C_3\, \delta(k-1)$$

particular solution: From table 3.1 we find that for $u(k) = k$, *the particular solution is selected:*

$$y_p = \beta_1\, k + \beta_0$$

Testing gives:

$$2\, (\beta_1\, (k+3) + \beta_0) + (\beta_1\, (k+2) + \beta_0) = 7\, (k+1) - k = 6\, k + 7$$

We find $\beta_1 = 2$ *and* $\beta_0 = -3$ *and so the particular solution is*

$$y_p(k) = 2\, k - 3$$

complete solution: The complete solution will have the form:

$$y(k) = y_h(k) + y_p(k) = C_1\, (-0.5)^k + C_2\, \delta(k) + C_3\, \delta(k-1) + 2\, k - 3$$

Now the initial conditions are evaluated in $k = 0$, $k = 1$ and $k = 2$.

$$
\begin{aligned}
y(0) &= C_1\,(-0.5)^0 + C_2\,\delta(0) + C_3\,\delta(0-1) + 2\,0 - 3 \\
&= C_1 + C_2 - 3 = 2 \\
y(1) &= C_1\,(-0.5)^1 + C_2\,\delta(1) + C_3\,\delta(1-1) + 2\,1 - 3 \\
&= -C_1\,0.5 + C_3 - 1 = -1 \\
y(2) &= C_1\,(-0.5)^2 + C_2\,\delta(2) + C_3\,\delta(2-1) + 2\,2 - 3 \\
&= C_1\,0.25 + 1 = 2
\end{aligned}
$$

We find a solution for $C_1 = 4$, $C_2 = 1$, $C_3 = 2$ and so the final solution becomes:

$$y(k) = 4\,(-0.5)^k + \delta(k) + 2\,\delta(k-1) + 2\,k - 3$$

3.3 Discrete-time Convolution

In this section we derive the computational form of the discrete-time system $H\{u(k)\}$, defined in [4], chapter 7, and section 2.1 of these lecture notes, that is based on a system's response to an impulse input. We assume that the system is initially at rest, that is, all initial conditions are zero at time $t = 0$, and examine the discrete-time domain forced response $y(k)$, $k \in \mathbb{Z}$ to a discrete-time waveform $u(k)$.
To start with, we assume that the system response to $\delta(k)$ is a known function and is designated $h(k)$ as shown in figure 3.2.

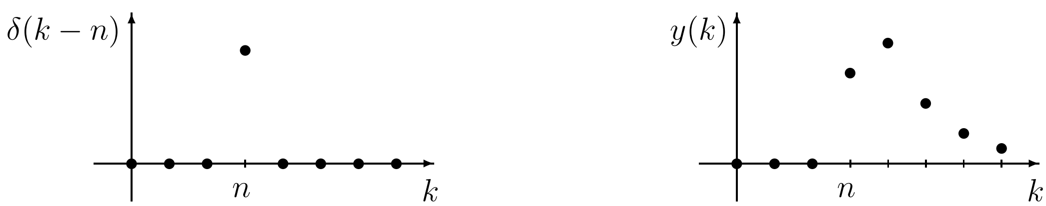

Figure 3.2: System response to a delayed unit pulse

Then if the system is linear and time-invariant, the response to a delayed unit pulse, occurring at time n is simply a delayed version of the pulse response:

$$u(k) = \delta(k-n) \quad \text{gives} \quad y(k) = h(k-n) \tag{3.17}$$

Multiplication with a constant α gives:

$$u(k) = \alpha\,\delta(k-n) \quad \text{gives} \quad y(k) = \alpha\,h(k-n) \tag{3.18}$$

The input signal $u(k)$ may be considered to be the sum of non-overlapping delayed pulses $p_n(k)$:

$$u(k) = \sum_{n=-\infty}^{\infty} p_n(k) \tag{3.19}$$

where

$$p_n(k) = \begin{cases} u(k) & \text{for} \quad k = n \\ 0 & \text{for} \quad k \neq n \end{cases} \tag{3.20}$$

Each component $p_n(k)$ may be written in terms of a delayed unit pulse $\delta(k)$, that is

$$p_n(k) = u(n)\delta(k-n)$$

From equation (3.18) it follows that for a delayed version of the pulse response, multiplied by a constant $u(n)$ we obtain the output $y(k) = u(n)\, h(k-n)$.
For an input

$$u(k) = \sum_{n=-\infty}^{\infty} p_n(k)$$

we can use the principle of superposition and the output can be written as a sum of all responses to $p_n(k)$, so:

$$y(k) = \sum_{n=-\infty}^{\infty} u(n)\, h(k-n) \tag{3.21}$$

This sum is denoted as the convolution sum for discrete-time systems.
For physical systems, the pulse response $h(k)$ is zero for time $k < 0$, and future components of the input do not contribute to the sum. So the convolution sum becomes:

$$y(k) = \sum_{n=-\infty}^{k} u(n)\, h(k-n) \tag{3.22}$$

The convolution operation is often denoted by the symbol $*$:

$$y(k) = u(k) * h(k) = \sum_{n=-\infty}^{k} u(n)\, h(k-n) \tag{3.23}$$

Example 10 (Discrete-time convolution)
The first-order exponential smoother of example 6 is described by the difference equation

$$y(k+1) = a\, y(k) + (1-a)\, u(k+1)$$

By repeated substitution it easily follows that if $y(k) = 0$ *for* $k < n_0$*, the output of the system is given by*

$$y(k) = (1-a) \sum_{n=n_0}^{k} a^{k-n}\, u(n) \qquad k \geq n_0$$

assuming that the input $u(k)$ *is such that the sum converges, and let* n_0 *approach to* $-\infty$*. Then the response of the system takes the form*

$$y(k) = (1-a) \sum_{n=-\infty}^{k} a^{k-n}\, u(n) \qquad k \in \mathbb{Z}$$

Define the function h *such that*

$$\begin{aligned} h(k) &= \begin{cases} 0 & \textit{for} \quad k < 0 \\ (1-a)a^k & \textit{for} \quad k \geq 0 \end{cases} \\ &= (1-a)a^k\, u_s(k) \end{aligned}$$

we see that on the infinite time axis the system may be represented as the convolution sum

$$y(k) = \sum_{n=-\infty}^{\infty} h(k-n)\, u(k)$$

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