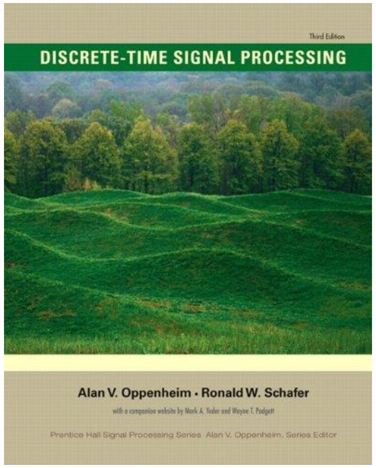


Lecture 02:

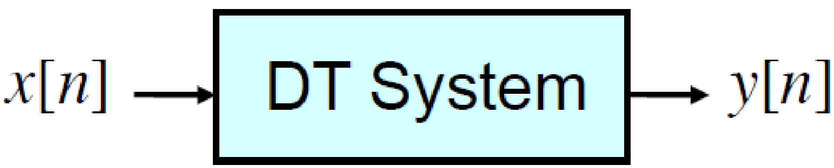
Discrete Time Systems (Part I)

Outlines

- ✓ 1. Discrete-time systems
- ✓ 2. Convolution
- 3. Linear Constant-Coefficient Difference Equations (LCCDEs)
- 4. Correlation

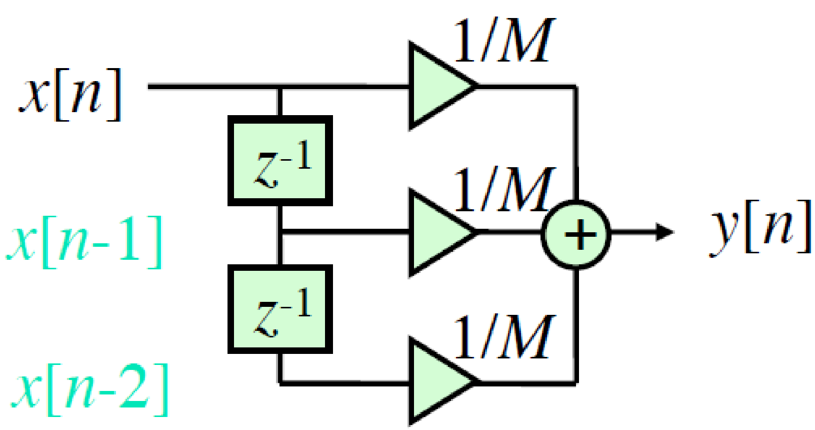
1. Discrete-time systems

- A **system** converts input to output:



$$\{y[n]\} = f(\{x[n]\})\ \forall n$$

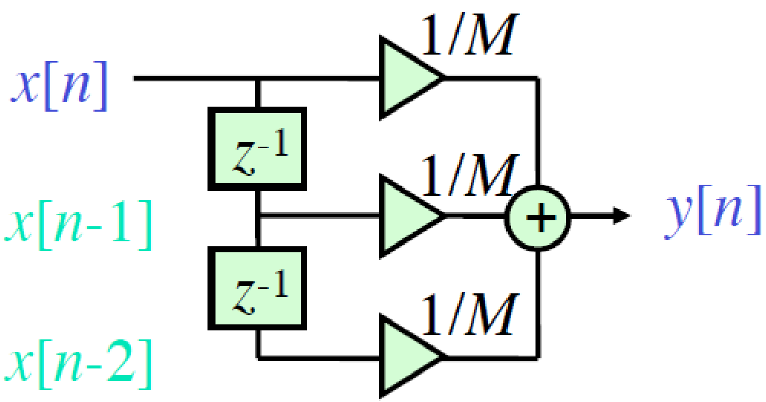
- E.g. Moving Average (MA):

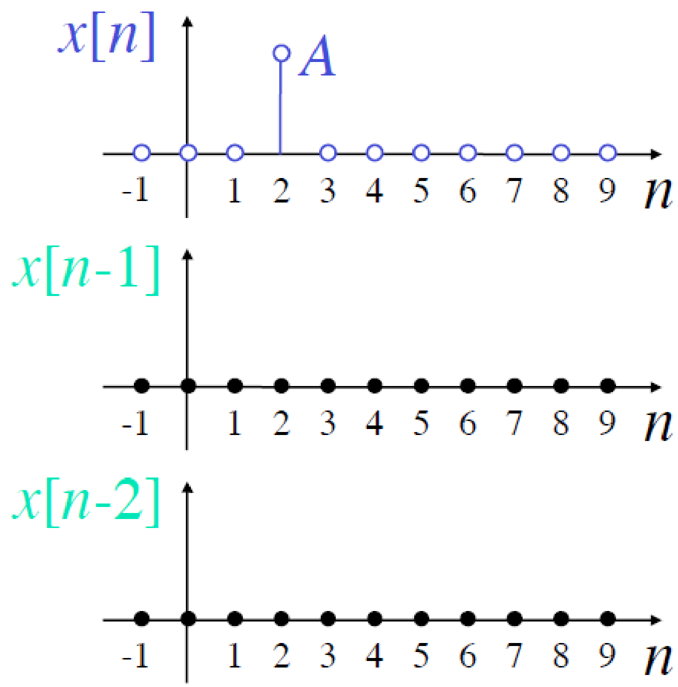


$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

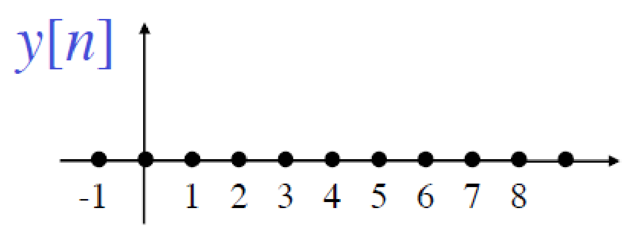
$$(M = 3)$$

Moving Average (MA)





$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$



MA Smoother

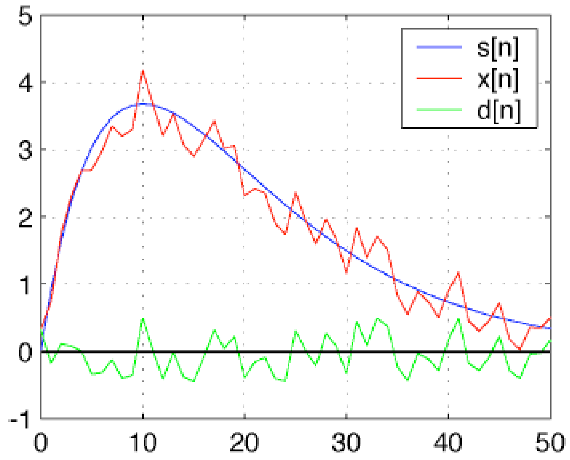
- MA smoothes out rapid variations (e.g. "12 month moving average")
- e.g.

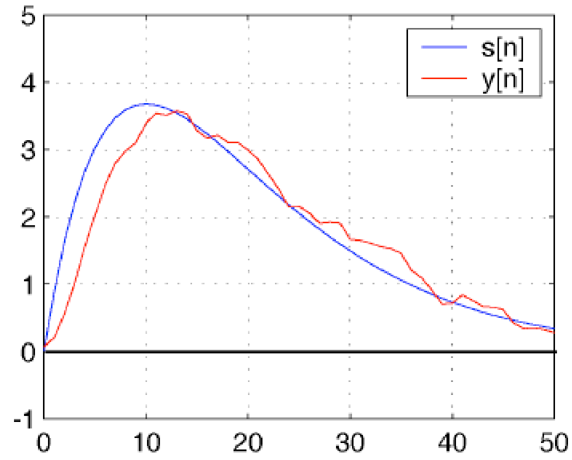
signal noise

$$x[n] = s[n] + d[n]$$

$$y[n] = \frac{1}{5}\sum_{k=0}^{4} x[n-k]$$

5-pt moving average





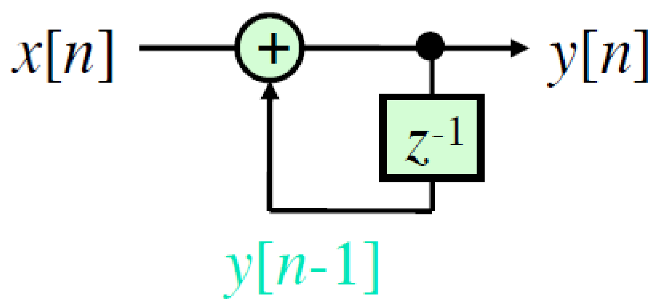
Accumulator

- Output accumulates all past inputs:

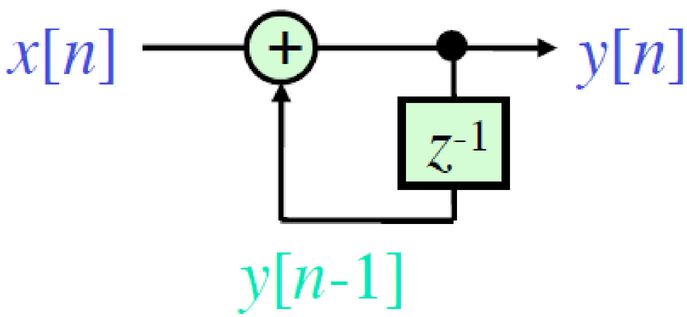
$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

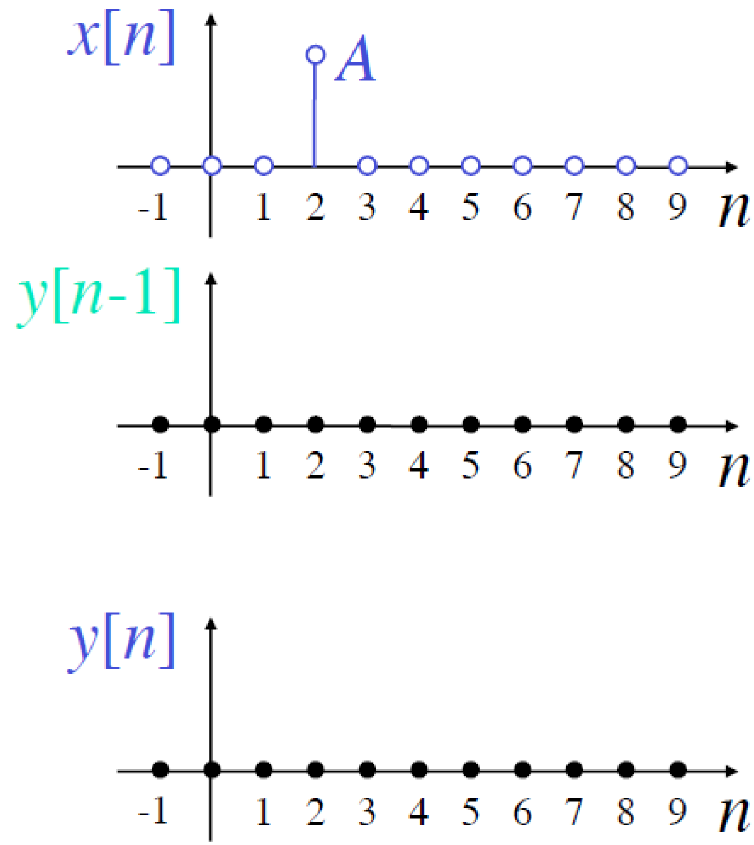
$$= \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n]$$

$$= y[n-1] + x[n]$$



Accumulator





Classes of DT systems

- **Linear** systems obey superposition:

$$x[n] \longrightarrow \boxed{\text{DT system}} \longrightarrow y[n]$$

  - if input $x_1[n] \rightarrow$ output $y_1[n]$, $x_2 \rightarrow y_2$ ...
  - given a linear combination

    of inputs: $x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$
  - then output $y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$

    for *all* α, β, x_1, x_2

    i.e. same linear combination of outputs

Linearity: Example 1

- Accumulator: $$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

$$x[n] = \alpha \cdot x_1[n] + \beta \cdot x_2[n]$$

$$\rightarrow \quad y[n] = \sum_{\ell=-\infty}^{n} (\alpha x_1[\ell] + \beta x_2[\ell])$$

$$= \sum (\alpha x_1[\ell]) + \sum (\beta x_2[\ell])$$

$$= \alpha \sum x_1[\ell] + \beta \sum x_2[\ell]$$

$$= \alpha \cdot y_1[n] + \beta \cdot y_2[n]$$

✓ Linear

Linearity Example 2:

- "Teager Energy operator":

$$y[n] = x^2[n] - x[n-1] \cdot x[n+1]$$

$$x[n] = \alpha \cdot x_1[n] + \beta \cdot x_2[n]$$

$$\rightarrow y[n] = (\alpha x_1[n] + \beta x_2[n])^2$$
$$-(\alpha x_1[n-1] + \beta x_2[n-1])$$
$$\cdot(\alpha x_1[n+1] + \beta x_2[n+1])$$
$$\neq \alpha \cdot y_1[n] + \beta \cdot y_2[n]$$

✗ **Nonlinear**

Linearity Example 3:

- 'Offset' accumulator: $y[n] = C + \sum_{\ell=-\infty}^{n} x[\ell]$

$$\Rightarrow y_1[n] = C + \sum_{\ell=-\infty}^{n} x_1[\ell]$$

but $y[n] = C + \sum_{\ell=-\infty}^{n} (\alpha x_1[\ell] + \beta x_2[\ell])$

$$\neq \alpha y_1[n] + \beta y_2[n]$$ ✗ ***Nonlinear***

.. unless $C = 0$

Property: Shift (time) invariance

- **Time-shift** of input causes same shift in output
- i.e. if $x_1[n] \rightarrow y_1[n]$
  then $x[n] = x_1[n - n_0]$
  $$\Rightarrow y[n] = y_1[n - n_0]$$
- i.e. process doesn't depend on absolute value of n

Shift-invariance counterexample

- Upsampler: $x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y[n]$

$$y[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$y_1[n] = x_1[n/L] \quad (n = r \cdot L)$$

$$x[n] = x_1[n - n_0]$$

$$\Rightarrow y[n] = x[n/L] = x_1[n/L - n_0]$$

Not shift invariant

$$= x_1\left[\frac{n - L \cdot n_0}{L}\right] = y_1[n - L \cdot n_0] \neq y_1[n - n_0]$$

Another counterexample

$$y[n] = n \cdot x[n]$$ *scaling by time index*

- Hence $y_1[n-n_0] = (n-n_0) \cdot x_1[n-n_0]$
- If $x[n] = x_1[n-n_0]$
  then $y[n] = n \cdot x_1[n-n_0]$

$\neq$

- Not shift invariant
  - parameters depend on n

Linear Shift Invariant (LSI)

- Systems which are both linear and shift invariant are easily manipulated mathematically
- This is still a wide and useful class of systems
- If discrete index corresponds to time, called Linear Time Invariant (LTI)

Causality

- If output depends only on past and current inputs (not future), system is called **causal**
- Formally, if $x_1[n] \rightarrow y_1[n]$ & $x_2[n] \rightarrow y_2[n]$

Causal $\rightarrow$ $x_1[n] = x_2[n] \quad \forall n < N$

$$\Leftrightarrow y_1[n] = y_2[n] \quad \forall n < N$$

Causality example

- Moving average: $y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$

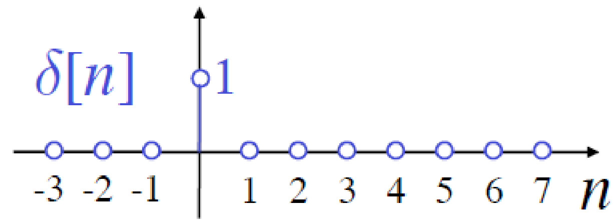
  $y[n]$ depends on $x[n\text{-}k]$, $k \geq 0$ → causal
- 'Centered' moving average

  $$y_c[n] = y[n + (M-1)/2]$$

  $$= \frac{1}{M}\left(x[n] + \sum_{k=1}^{(M-1)/2} x[n-k] + x[n+k]\right)$$
- .. looks forward in time → noncausal
- .. Can make causal by delaying

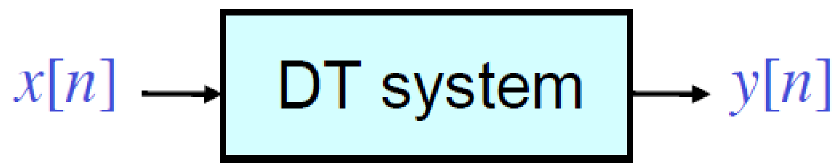
Impulse response (IR)

- Impulse $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



  (unit sample sequence)

- Given a system:



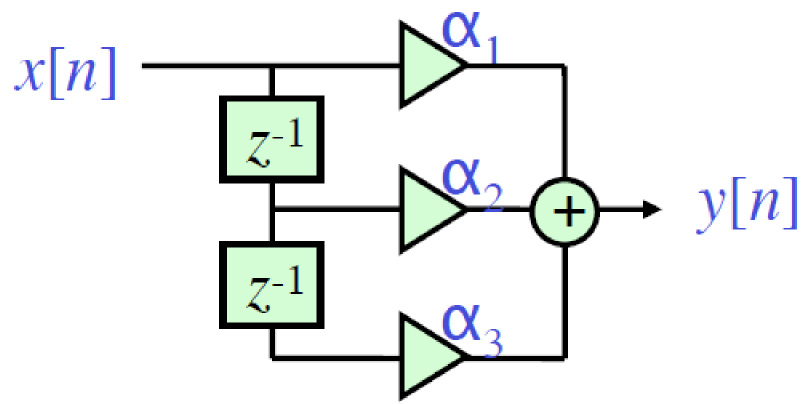
  if $x[n] = \delta[n]$ then $y[n] \triangleq h[n]$

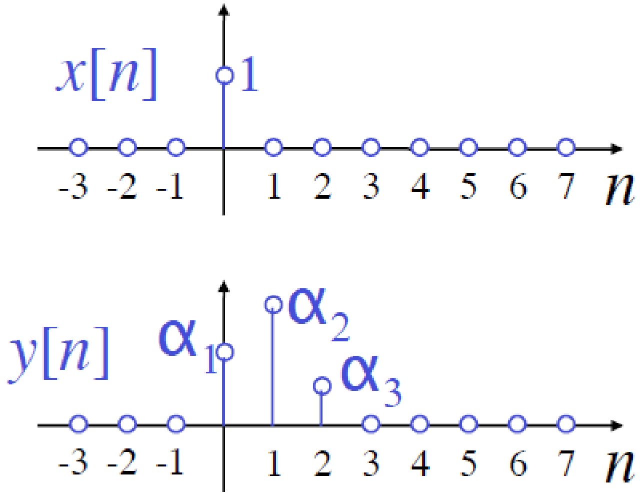
  "impulse response"

- LSI system completely specified by $h[n]$

Impulse response example

- Simple system:





$x[n] = \delta[n]$ impulse

↓

$y[n] = h[n]$ impulse response

2. Convolution

- Impulse response: $\delta[n] \rightarrow \boxed{\text{LSI}} \rightarrow h[n]$
- Shift invariance: $\delta[n-n_0] \rightarrow \boxed{\text{LSI}} \rightarrow h[n-n_0]$
- + Linearity: $\alpha \cdot \delta[n-k] + \beta \cdot \delta[n-l] \rightarrow \boxed{\text{LSI}} \rightarrow \alpha \cdot h[n-k] + \beta \cdot h[n-l]$
- Can express any sequence with δs:

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2]..$$

Convolution sum

- Hence, since $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- For LSI, $\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]}$ **Convolution sum**

  written as $y[n] = x[n] \circledast h[n]$
- Summation is symmetric in x and h

  i.e. $l = n - k \rightarrow$

  $$x[n] \circledast h[n] = \sum_{l=-\infty}^{\infty} x[n-l]h[l] = h[n] \circledast x[n]$$

Convolution properties

- **LSI System output $y[n]$ = input $x[n]$ convolved with impulse response $h[n]$ → $h[n]$ completely describes system**
- Commutative: $x[n] \circledast h[n] = h[n] \circledast x[n]$
- Associative:

$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

- Distributive:

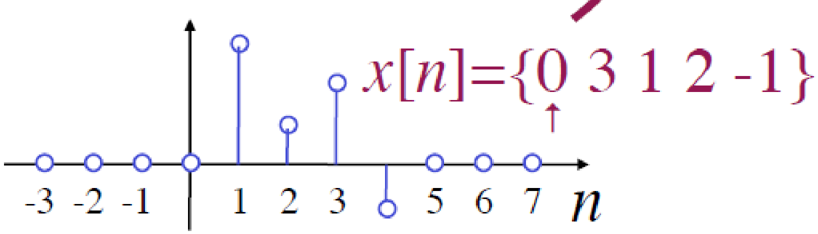
$$h[n] \circledast (x[n] + y[n]) = h[n] \circledast x[n] + h[n] \circledast y[n]$$

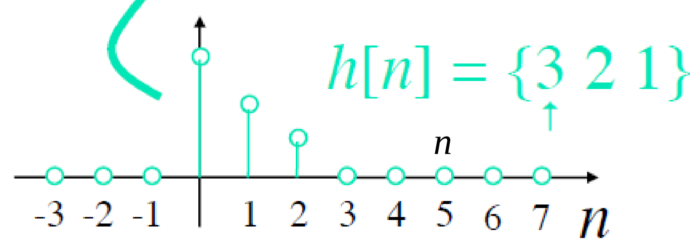
Interpreting convolution

- Passing a signal through a (LSI) system is equivalent to convolving it with the system's impulse response

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] \circledast h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$



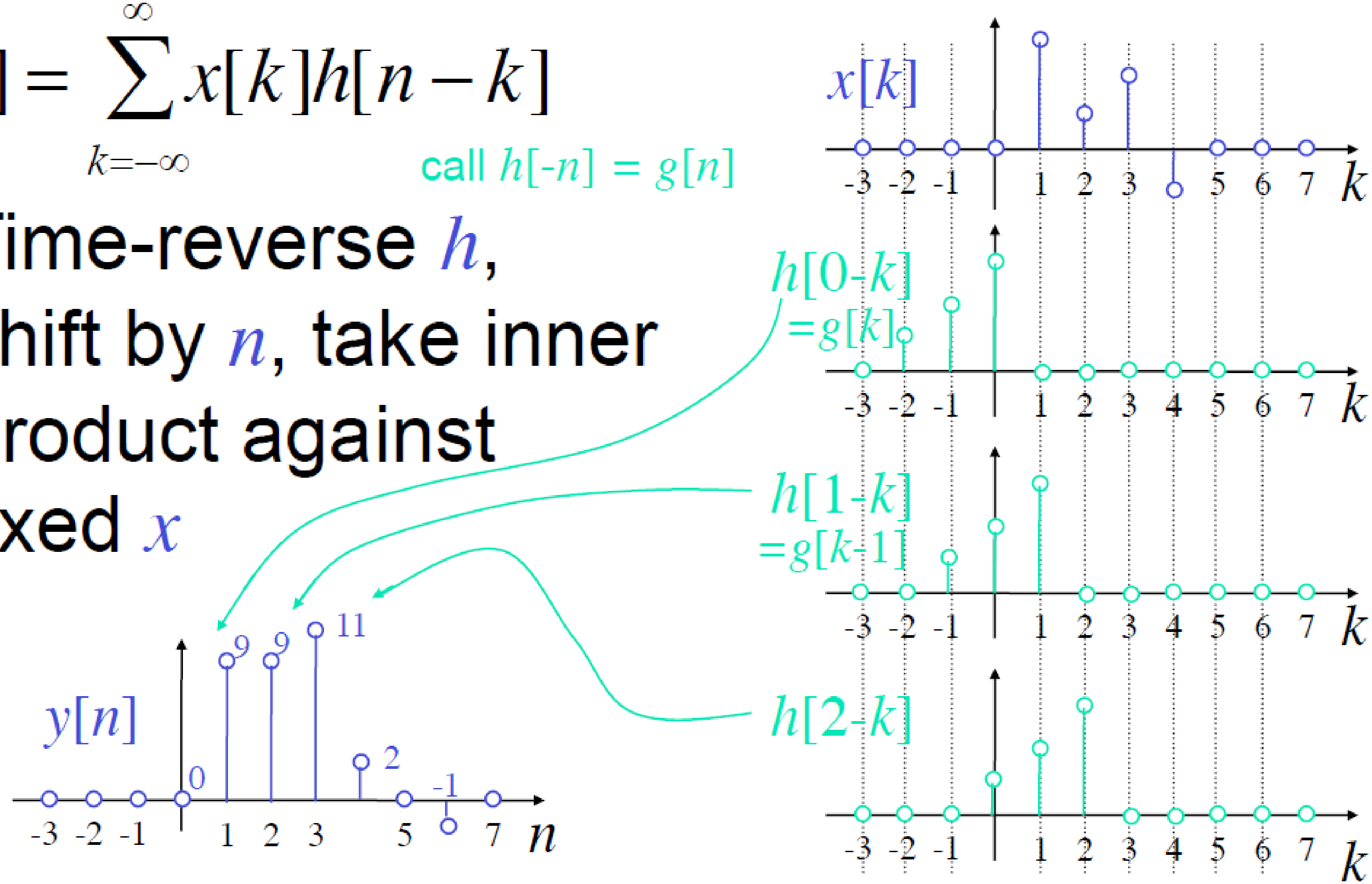


Convolution interpretation 1

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

call $h[-n] = g[n]$

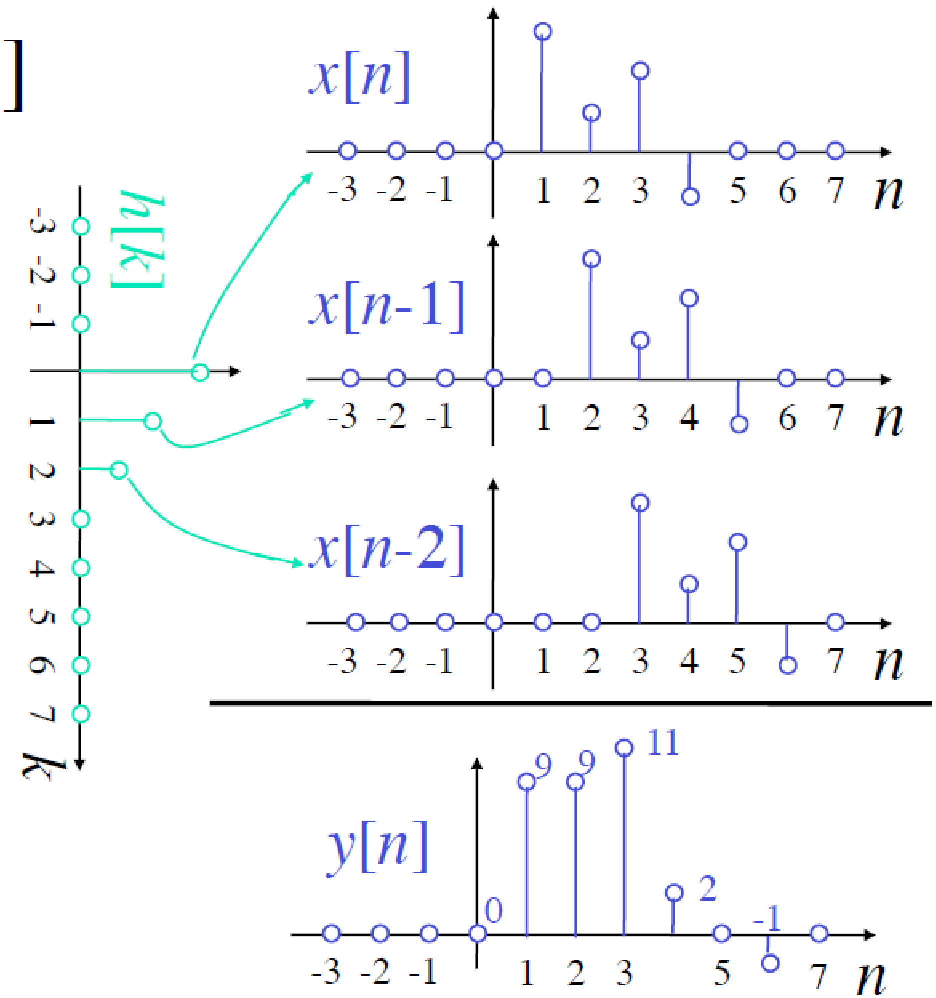
- Time-reverse h, shift by n, take inner product against fixed x



Convolution interpretation 2

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

- Shifted x's weighted by points in h
- Conversely, weighted, delayed versions of h ...



Matrix interpretation

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \dots \end{bmatrix} = \begin{bmatrix} x[0] & x[-1] & x[-2] \\ x[1] & x[0] & x[-1] \\ x[2] & x[1] & x[0] \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \\ h[2] \end{bmatrix}$$

- Diagonals in **X** matrix are equal

Convolution notes

- Total nonzero length of convolving N and M point sequences is $N+M-1$
- Adding the indices of the terms within the summation gives n :

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \qquad k + (n-k) = n$$

i.e. summation indices move in opposite senses

Convolution in MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences
- If $a = [0 \quad 3 \quad 1 \quad 2 \quad -1]$

  $b = [3 \quad 2 \quad 1]$

  then `conv(a,b)` yields

  $[0 \quad 9 \quad 9 \quad 11 \quad 2 \quad 0 \quad -1]$

Connected systems

- **Cascade** connection:

$$\longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow \quad \equiv \quad \longrightarrow \boxed{h_1[n] \circledast h_2[n]} \longrightarrow$$

Impulse response $h[n]$ of the cascade of two systems with impulse responses $h_1[n]$ and $h_2[n]$ is $h[n] = h_1[n] \circledast h_2[n]$

- By commutativity,

$$\longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow \quad \equiv \quad \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow$$

Inverse systems

- $\delta[n]$ is identity for convolution

  i.e. $x[n] \circledast \delta[n] = x[n]$

- Consider

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow y[n] \longrightarrow \boxed{h_2[n]} \longrightarrow z[n]$$

$$z[n] = h_2[n] \circledast y[n] = h_2[n] \circledast h_1[n] \circledast x[n]$$

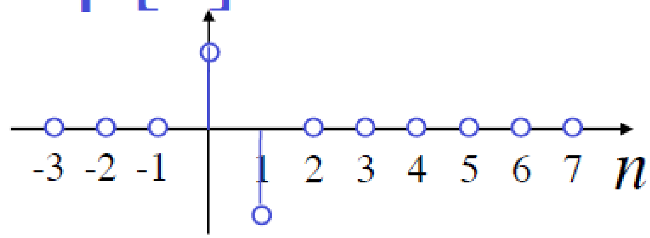
$$= x[n] \quad \textbf{if} \quad h_2[n] \circledast h_1[n] = \delta[n]$$

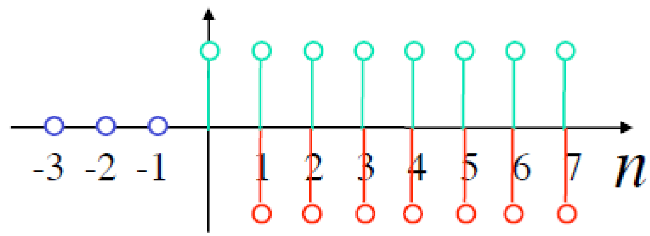
- $h_2[n]$ is the **inverse system** of $h_1[n]$

Inverse systems

- Use inverse system to recover input $x[n]$ from output $y[n]$ (e.g. to undo effects of transmission channel)
- Only sometimes possible - e.g. cannot 'invert' $h_1[n] = 0$
- In general, attempt to solve $h_2[n] \circledast h_1[n] = \delta[n]$

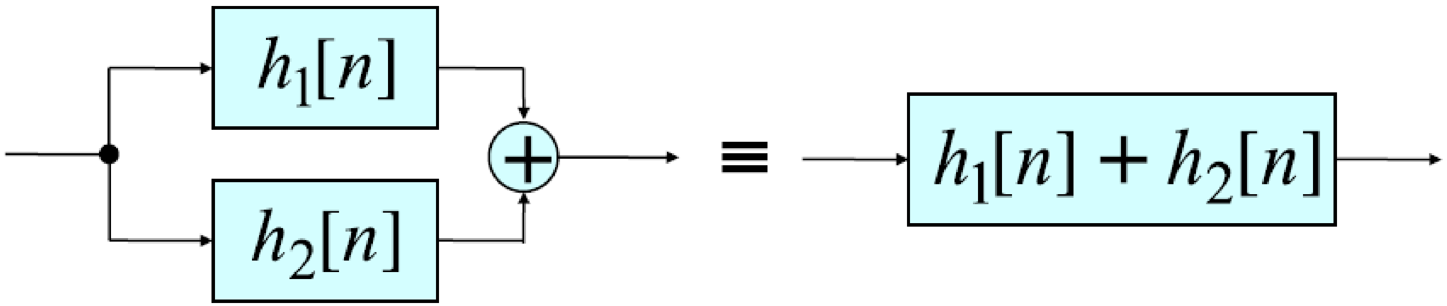
Inverse system example

- Accumulator:
  Impulse response $h_1[n] = \mu[n]$
- 'Backwards difference'
  $$h_2[n] = \delta[n] - \delta[n-1]$$
  

  .. has desired property:
  $$\mu[n] - \mu[n-1] = \delta[n]$$
  

- Thus, 'backwards difference' is inverse system of accumulator.

Parallel connection



- Impulse response of two parallel systems added together is:

$$h[n] = h_1[n] + h_2[n]$$