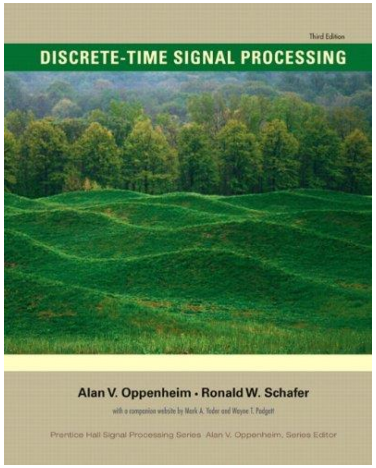


Lecture 13:

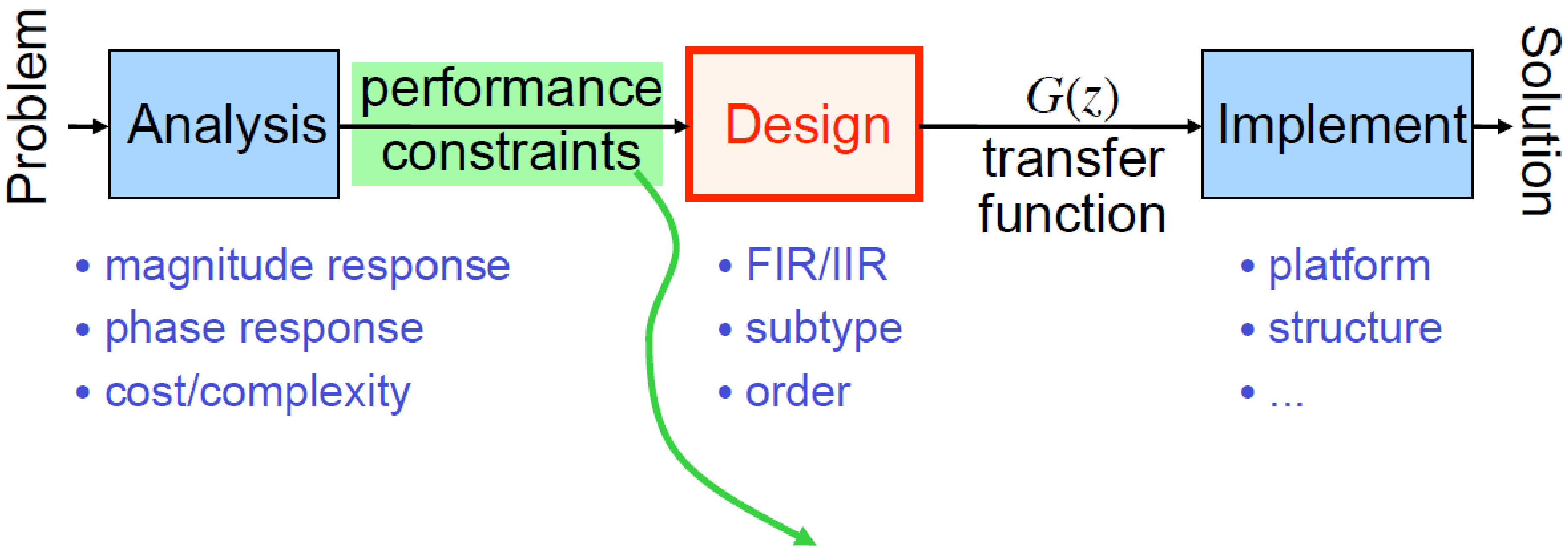
Filter Design – IIR

Outlines

- Filter Design Specifications
- Analog Filter Design
- Digital Filters from Analog Prototypes

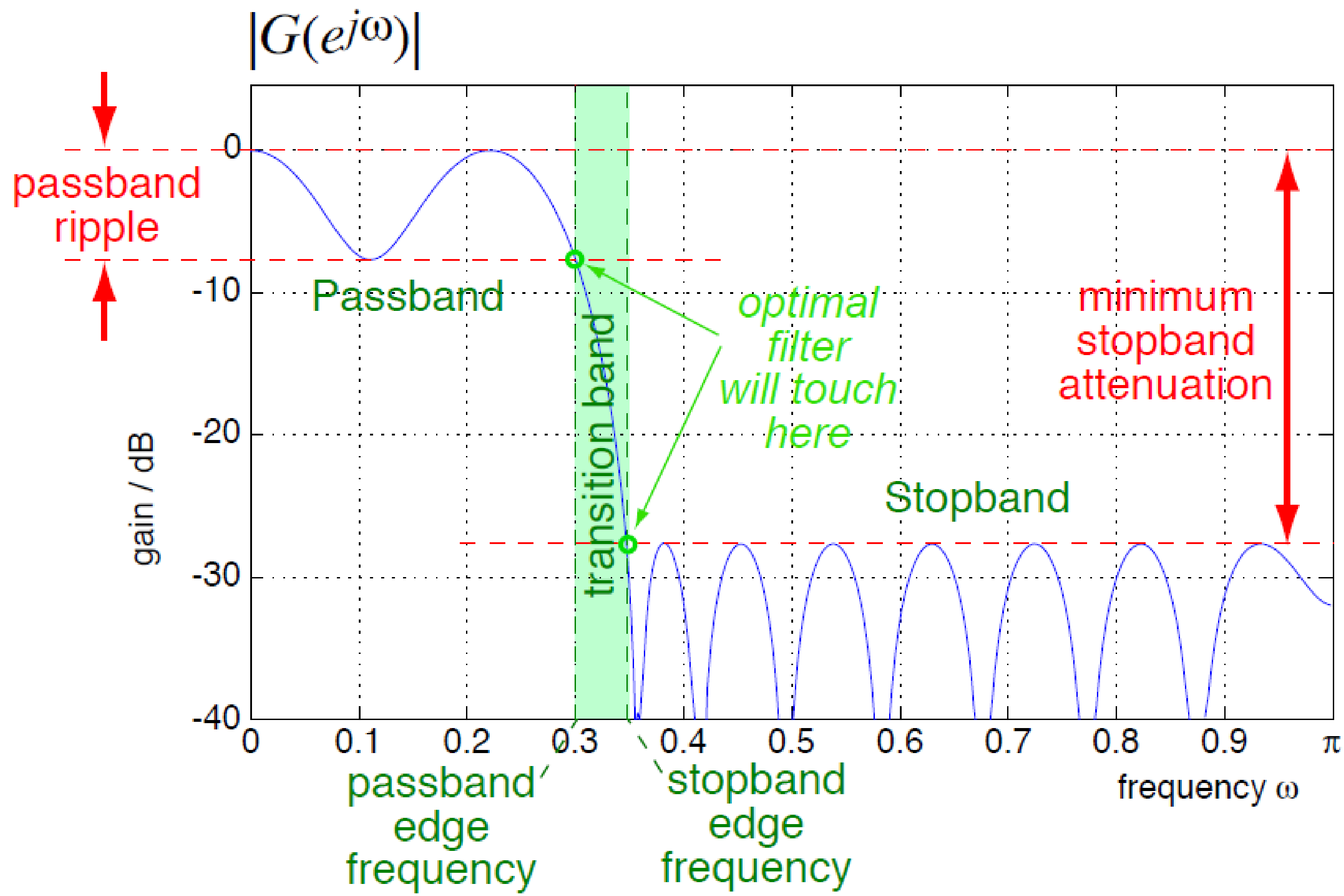
1. Filter Design Specifications

- The filter design process:



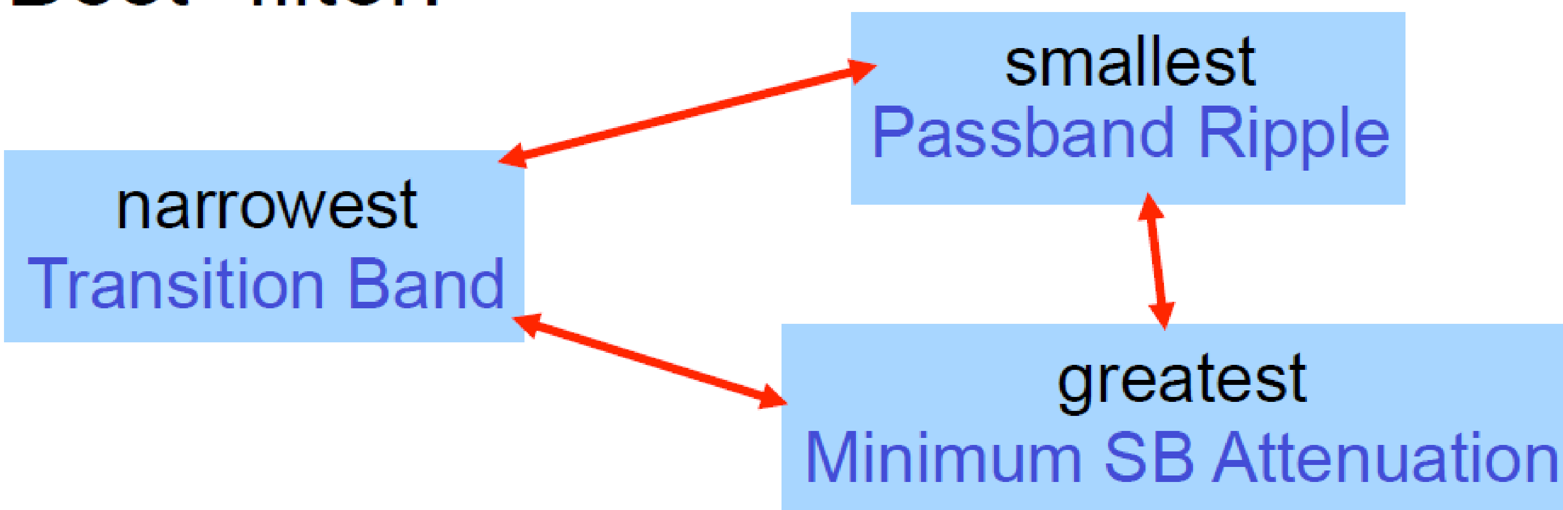
Performance Constraints

- .. in terms of magnitude response:



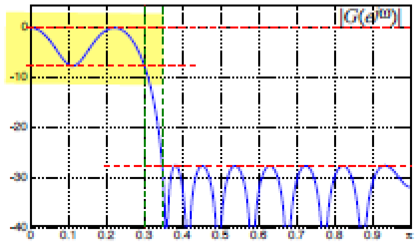
Performance Constraints

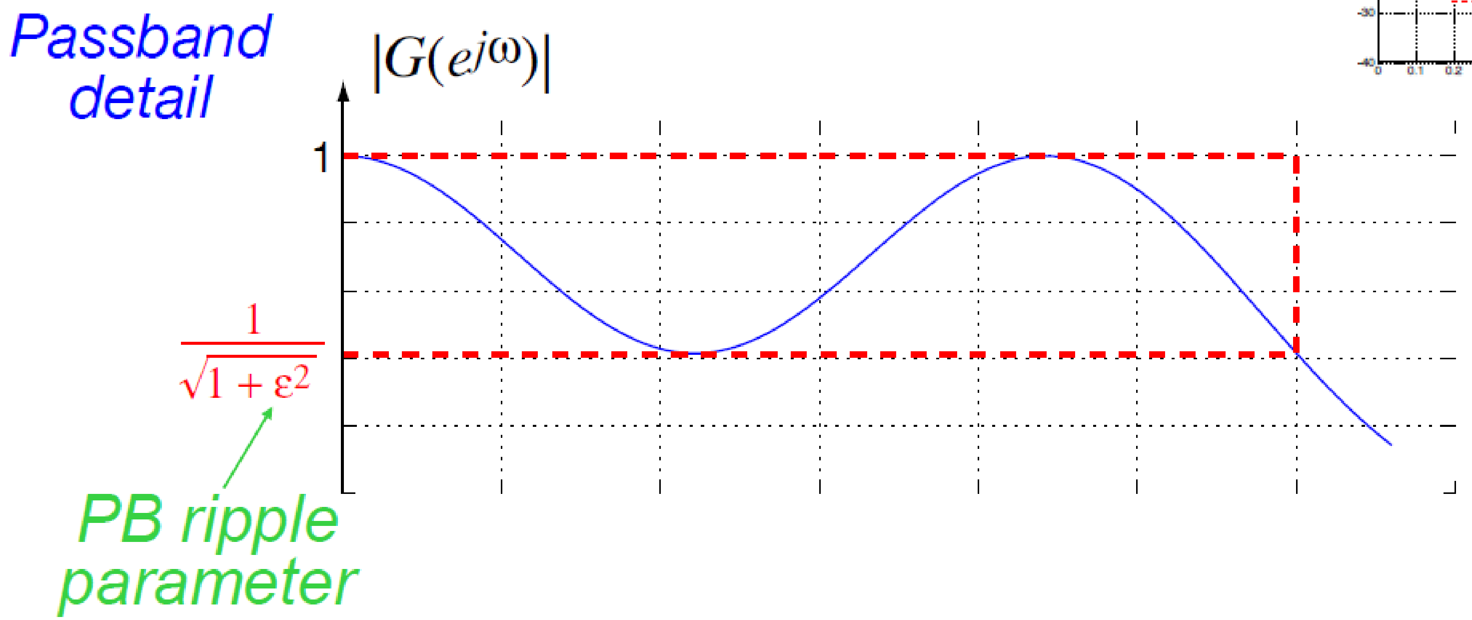
- "Best" filter:



  - improving one usually worsens others
- But: increasing filter order (i.e. cost) can improve all three measures

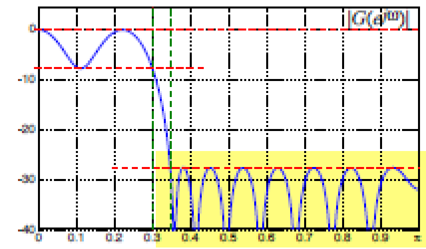
Passband Ripple

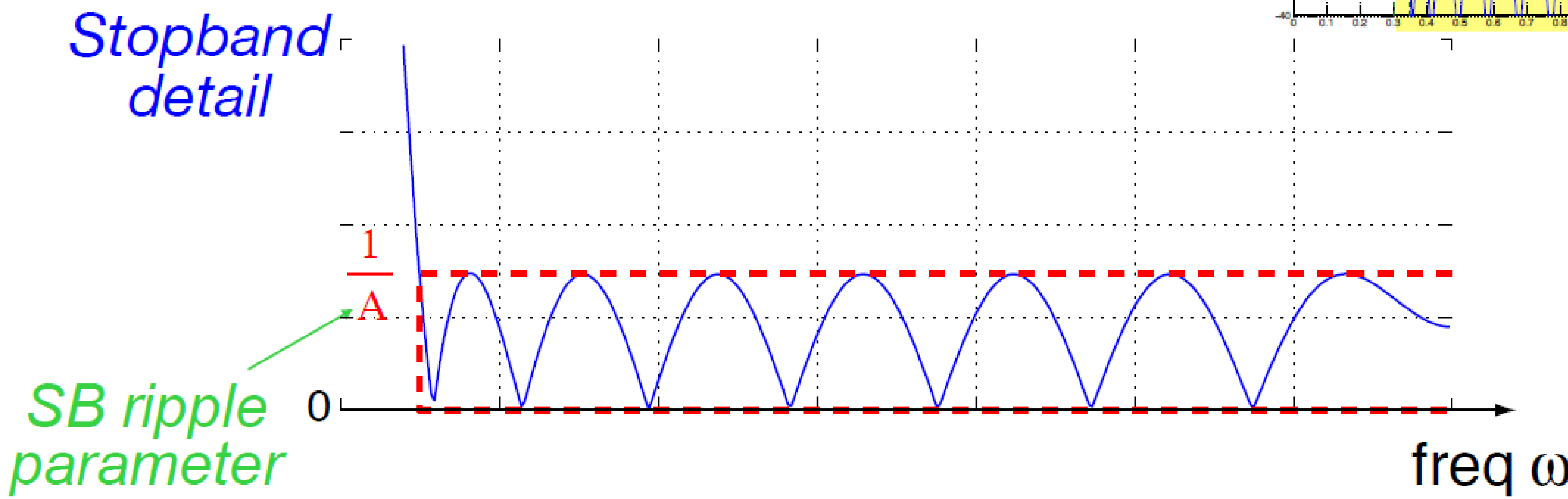




- Assume peak passband gain = 1
  then *minimum* passband gain $= \dfrac{1}{\sqrt{1+\varepsilon^2}}$
- Or, ripple $\alpha_{\max} = 20\log_{10}\sqrt{1+\varepsilon^2}$ dB

Stopband Ripple





- Peak passband gain is $A\times$ larger than peak stopband gain
- Hence, minimum stopband attenuation

$$\alpha_s = -20\log_{10}\frac{1}{A} = 20\log_{10} A \quad \text{dB}$$

Filter Type Choice : FIR vs. IIR

	FIR	IIR
	▪ No feedback (just zeros) ▪ Always stable ▪ Can be linear phase	▪ Feedback (poles & zeros) ▪ May be unstable ▪ Difficult to control phase
BUT	▪ High order (20-2000) ▪ Unrelated to continuous-time filtering	▪ Typ. < 1/10th order of FIR (4-20) ▪ Derive from *analog prototype*

FIR vs. IIR

- If you care about computational cost
  → use low-complexity IIR

  (computation no object → linear phase FIR)

- If you care about phase response
  → use linear-phase FIR

  (phase unimportant → go with simple IIR)

IIR Filter Design

- IIR filters are directly related to analog filters (continuous time)
  - via a mapping of $H(s)$ (CT) to $H(z)$ (DT) that preserves many properties
- Analog filter design is sophisticated
  - signal processing research since 1940s

→ Design IIR filters via *analog prototype*

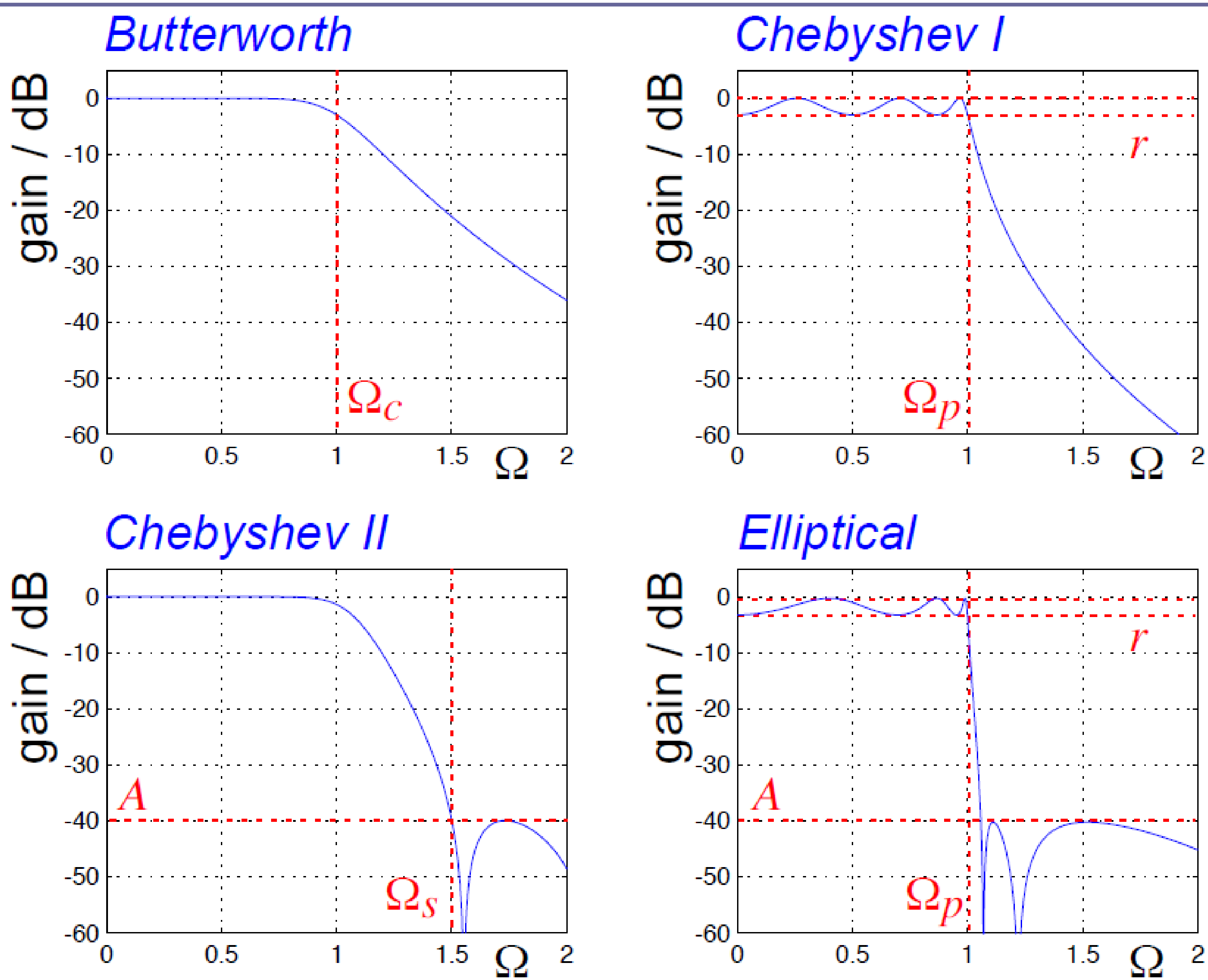
  - need to learn some CT filter design

2. Analog Filter Design

- Decades of analysis of transistor-based filters – sophisticated, well understood
- Basic choices:
  - ripples vs. flatness in stop and/or passband
  - more ripples → narrower transition band

Family	*PassBand*	*StopBand*
Butterworth	flat	flat
Chebyshev I	ripples	flat
Chebyshev II	flat	ripples
Elliptical	ripples	ripples

Analog Filter Types Summary



$N = 6$

$r = 3$ dB

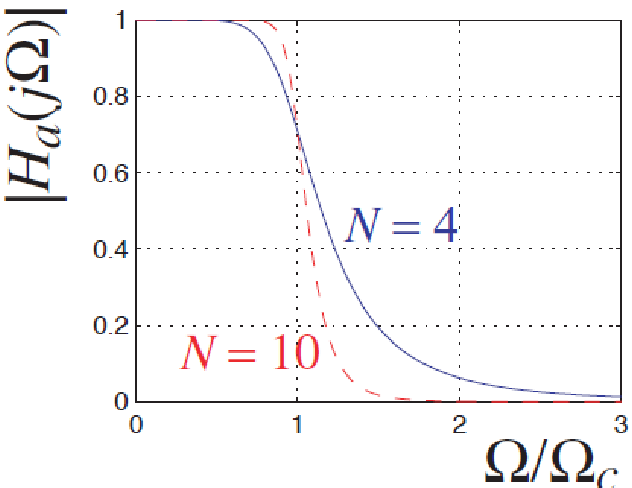
$A = 40$ dB

Butterworth Filters

Maximally flat in pass and stop bands

- Magnitude response (LP): $$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$ filter order N
  - $\Omega \ll \Omega_c$, $|H_a(j\Omega)|^2 \to 1$
  - $\Omega = \Omega_c$, $|H_a(j\Omega)|^2 = {}^1/_2$

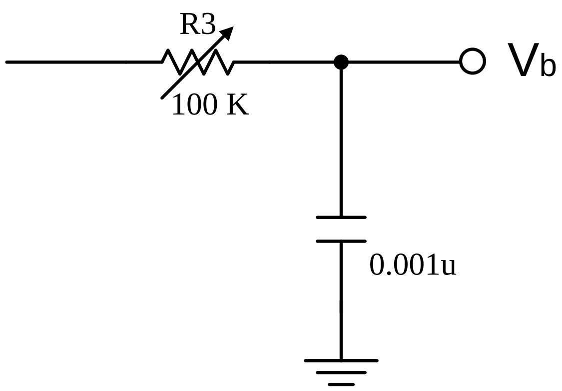
3dB point



analog fre

Butterworth Filters

Example: first-order



$$H(f) = \frac{V_b(f)}{V_a(f)} = \frac{1}{1 + j\frac{f}{f_0}}$$

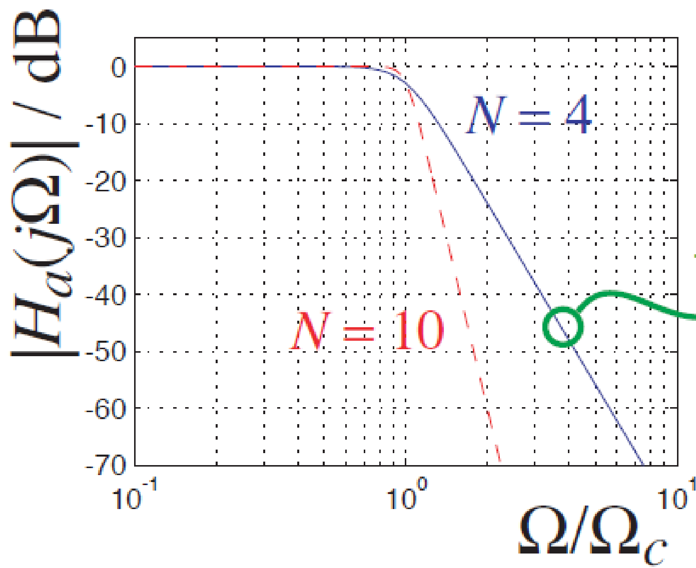
3dB cut-off frequency: $f_0 = \dfrac{1}{2\pi RC}$

Butterworth Filters

6N dB/oct rolloff

- $\Omega \gg \Omega_c, \quad |H_a(j\Omega)|^2 \rightarrow (\Omega_c/\Omega)^{2N}$

Log-log magnitude response



slope → $-6N$ dB/oct

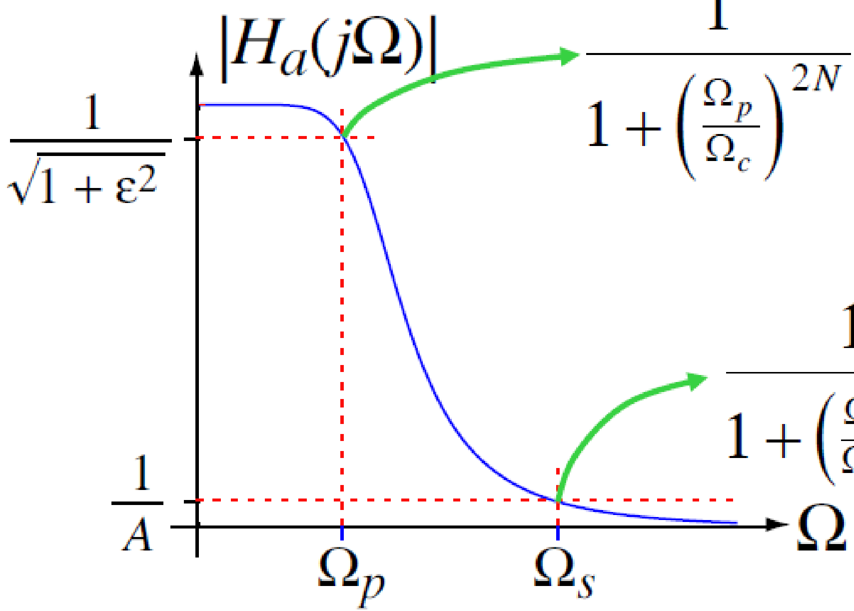
analog freq

- *flat* → $\dfrac{d^n}{d\Omega^n}\left|H_a(j\Omega)\right|^2 = 0$

@ $\Omega = 0$ for $n = 1 \,..\, 2N\text{-}1$

Butterworth Filters

- How to meet design specifications?



$$\frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \geq \frac{1}{1+\varepsilon^2}$$

$$\frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \leq \frac{1}{A^2}$$

Design Equation

$$N \geq \frac{1}{2}\frac{\log_{10}\left(\frac{A^2-1}{\varepsilon^2}\right)}{\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

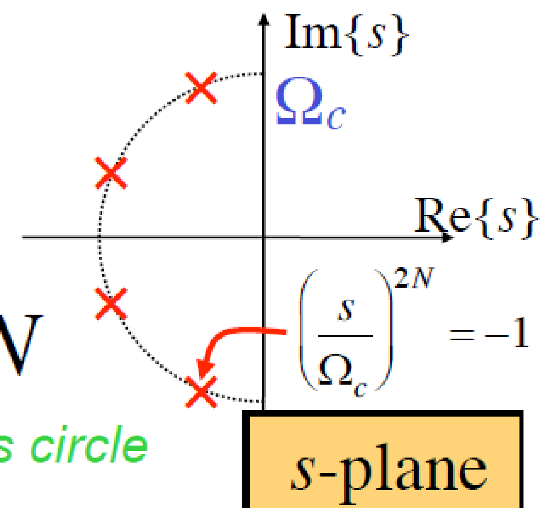
- $k_1 = \frac{\varepsilon}{\sqrt{A^2-1}}$ ="discrimination", $\ll 1$
- $k = \frac{\Omega_p}{\Omega_s}$ ="selectivity", < 1

Butterworth Filters

- $|H_a(j\Omega)|^2 = \dfrac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}}$ ... but what is $H_a(s)$?
- Traditionally, look it up in a table
  - calculate N → normalized filter with $\Omega_c = 1$
  - scale all coefficients for desired Ω_c
- In fact, $H_a(s) = \dfrac{1}{\prod_i (s - p_i)}$

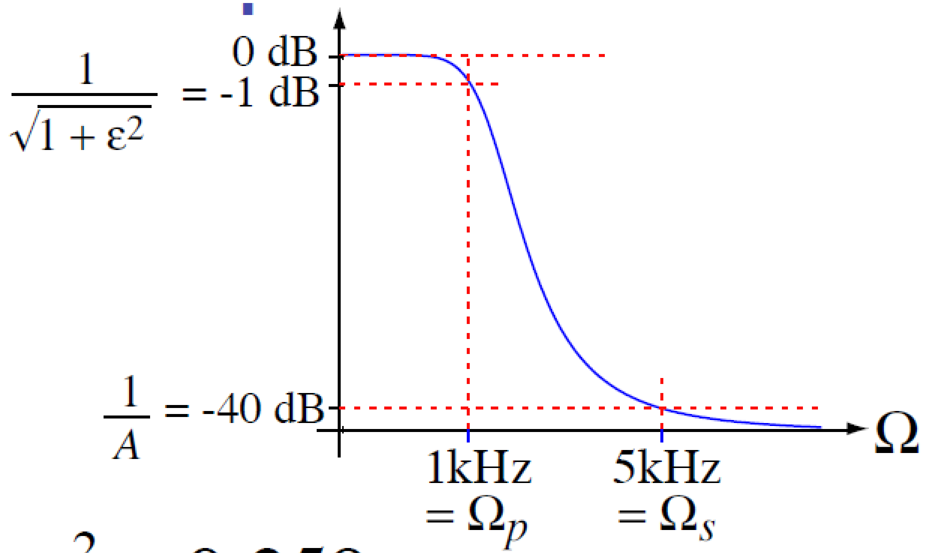
  where $p_i = \Omega_c e^{j\pi\frac{N+2i-1}{2N}} \quad i = 1..N$

  odd-indexed uniform divisions of Ω_c-radius circle



Butterworth Example

Design a Butterworth filter with 1 dB cutoff at 1kHz and a minimum attenuation of 40 dB at 5 kHz



$$-1\text{dB} = 20\log_{10}\frac{1}{\sqrt{1+\varepsilon^2}} \Rightarrow \varepsilon^2 = 0.259$$

$$-40\text{dB} = 20\log_{10}\tfrac{1}{A} \Rightarrow A = 100$$

$$\Omega_s \big/ \Omega_p = 5$$

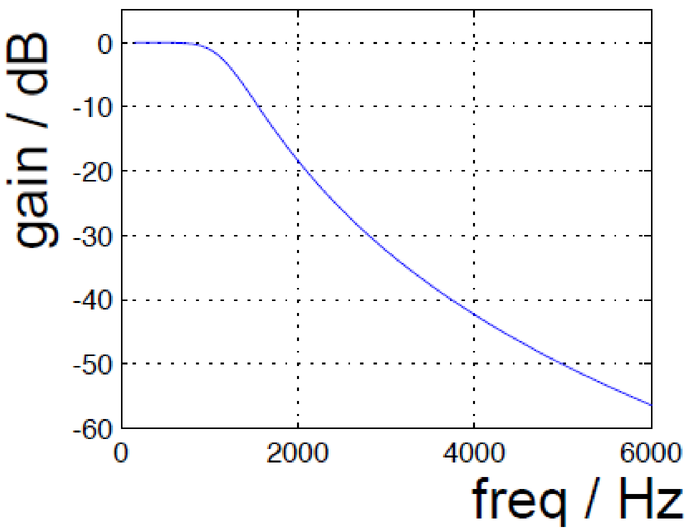
$$N \geq \frac{1}{2}\frac{\log_{10}\frac{9999}{0.259}}{\log_{10} 5}$$

$$\Rightarrow \boxed{N = 4} \geq 3.28$$

Butterworth Example

- Order $N = 4$ will satisfy constraints; What are Ω_c and filter coefficients?
  - from a table, $\Omega_{-1dB} = 0.845$ when $\Omega_c = 1$ $\Rightarrow \Omega_c = 1000/0.845 = 1.184$ kHz
  - from a table, get normalized coefficients for $N = 4$, scale by $1184 \cdot 2\pi$
- Or, use Matlab:

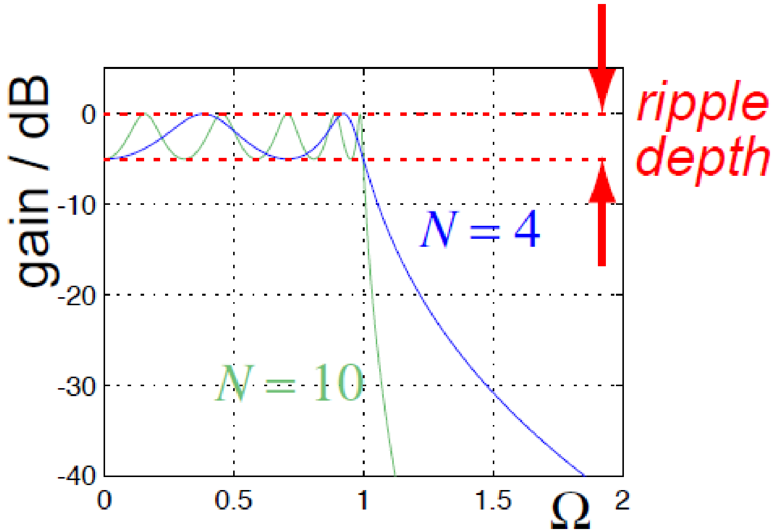
```
[B,A] =
butter(N,Wc,'s');
```



Chebyshev I Filter

- Equiripple in passband (flat in stopband)
  → minimize maximum error

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}$$



Chebyshev polynomial of order N →

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega) & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega) & |\Omega| > 1 \end{cases}$$

Chebyshev I Filter

- Design procedure:
  - desired passband ripple → ε
  - min. stopband atten., Ω_p, Ω_s → N :

$$\frac{1}{A^2}=\frac{1}{1+\varepsilon^2 T_N^2(\frac{\Omega_s}{\Omega_p})}=\frac{1}{1+\varepsilon^2\left[\cosh\left(N\cosh^{-1}\frac{\Omega_s}{\Omega_p}\right)\right]^2}$$

$$\Rightarrow N \geq \frac{\cosh^{-1}\left(\frac{\sqrt{A^2-1}}{\varepsilon}\right)}{\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$1/k_1$, *discrimination*

$1/k$, *selectivity*

$\cosh^{-1}$ *grows slower than* $\log 10$

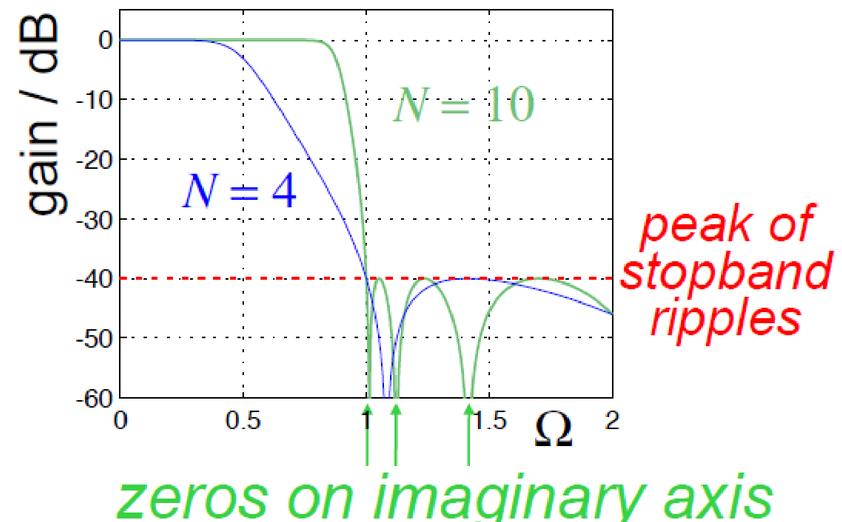
Chebyshev II Filter

- Flat in passband, equiripple in stopband

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\dfrac{T_N(\frac{\Omega_s}{\Omega_p})}{T_N(\frac{\Omega_s}{\Omega})}\right)^2}$$

constant

$\sim 1/T_N(1/\Omega)$



- Filter has poles and zeros (some)
- Complicated pole/zero pattern

Elliptical (Cauer) Filters

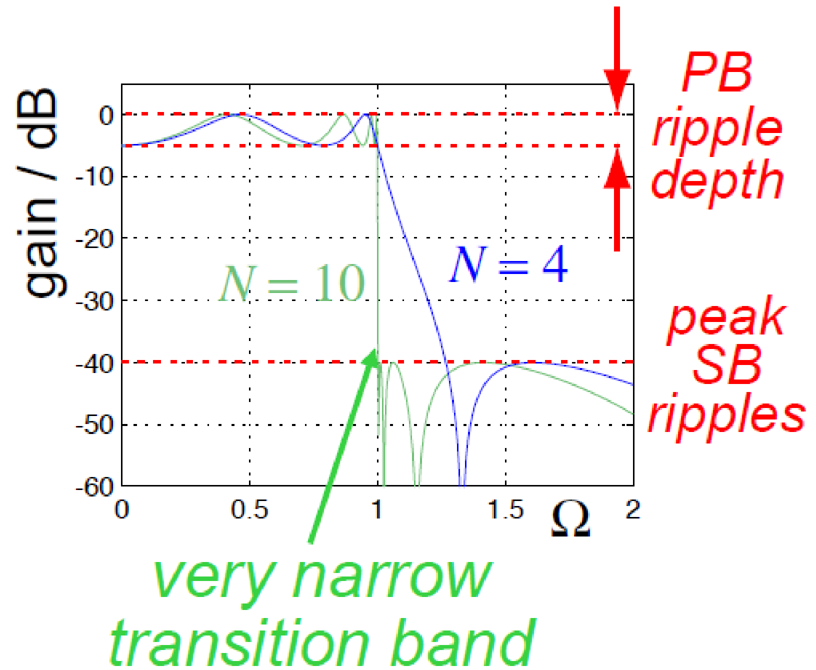
- Ripples in both passband and stopband

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\frac{\Omega}{\Omega_p})}$$

function; satisfies

$R_N(\Omega^{-1}) = R_N(\Omega)^{-1}$

zeros for $\Omega<1$ $\rightarrow$ *poles for* $\Omega>1$



- Complicated; not even closed form for N

Analog Filter Transformations

- All filters types shown as lowpass; other types (highpass, bandpass..) derived via transformations
- i.e. $\hat{s} = F^{-1}(s)$

Desired alternate response; still a ***rational polynomial***

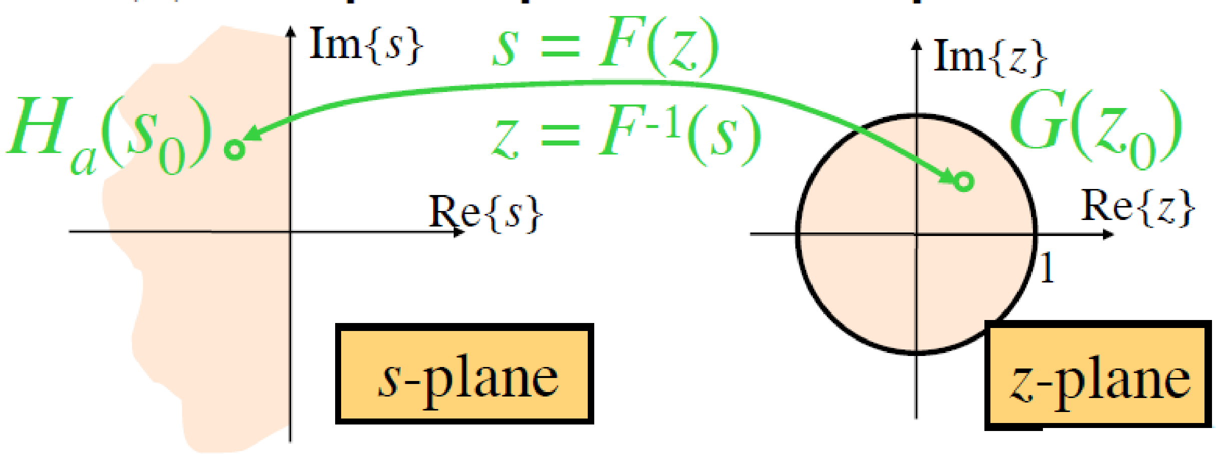
lowpass prototype $H_{LP}(s) \quad \rightarrow \quad H_D(\hat{s})$

- General mapping of s-plane *BUT* keep LHHP & $j\Omega \rightarrow j\hat{\Omega}$; poles OK, frequency response 'shuffled'

3. Analog Protos → IIR Filters

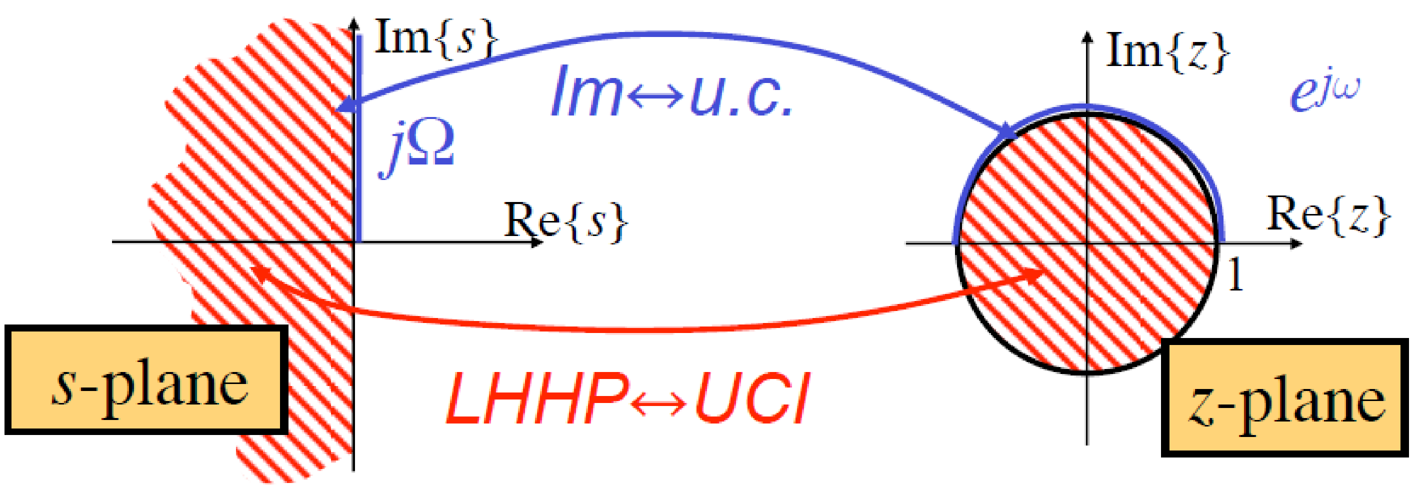
- Can we map high-performance CT filters to DT domain?
- Approach: **transformation** $H_a(s) \rightarrow G(z)$
  i.e. $G(z) = H_a(s)\big|_{s=F(z)}$
  where $s = F(z)$ maps s-plane ↔ z-plane:

Every value of $G(z)$ is a value of $H_a(s)$ somewhere on the s-plane & vice-versa



CT to DT Transformation

- Desired properties for $s = F(z)$:
  - s-plane $j\Omega$ axis $\leftrightarrow$ z-plane unit circle
    $\rightarrow$ preserves frequency response values
  - s-plane LHHP $\leftrightarrow$ z-plane unit circle interior
    $\rightarrow$ preserves stability of poles



Bilinear Transformation

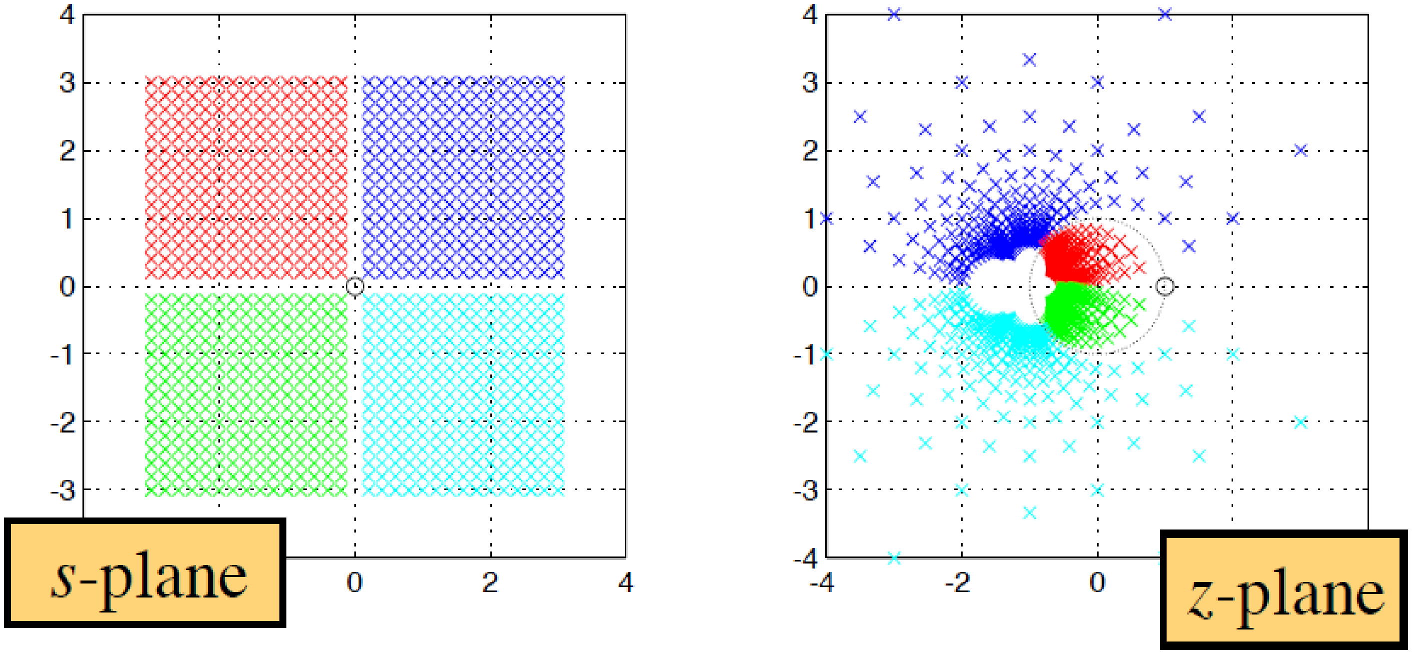
- Solution: $$s = \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{z-1}{z+1}$$ **Bilinear Transform**
- Hence inverse: $z = \dfrac{1+s}{1-s}$ *unique, 1:1 mapping*
- Freq. axis? $s = j\Omega \quad \rightarrow \quad z = \dfrac{1+j\Omega}{1-j\Omega}$ $|z| = 1$ *i.e. on unit circle*
- Poles? $s = \sigma + j\Omega \quad \rightarrow \quad z = \dfrac{(1+\sigma)+j\Omega}{(1-\sigma)-j\Omega}$

$$\Rightarrow |z|^2 = \frac{1+2\sigma+\sigma^2+\Omega^2}{1-2\sigma+\sigma^2+\Omega^2}$$

$\sigma < 0$
$\leftrightarrow |z| < 1$

Bilinear Trnsformation

- How can entire half-plane fit inside u.c.?



- Highly nonuniform warping!
  - "Moebius Transformations Revealed" http://www.youtube.com/watch?v=G87ehdmHeac

Bilinear Transformation

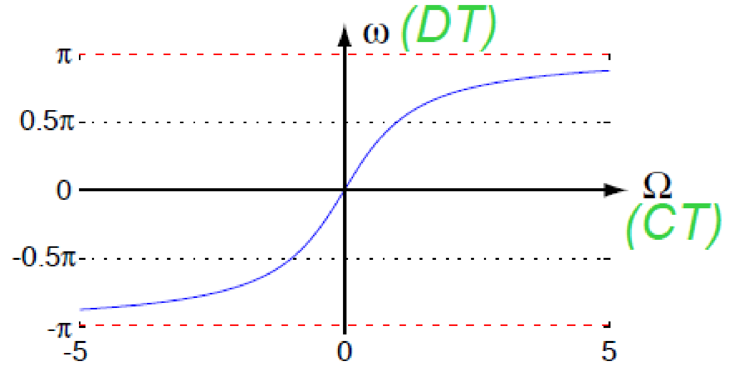
- What is CT↔DT freq. relation $\Omega \leftrightarrow \omega$?

$$z = e^{j\omega} \Rightarrow s = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2j\sin\omega/2}{2\cos\omega/2} = j\tan\frac{\omega}{2}$$

u.circle im.axis

- i.e.

$$\boxed{\begin{aligned}\Omega &= \tan\left(\tfrac{\omega}{2}\right)\\ \omega &= 2\tan^{-1}\Omega\end{aligned}}$$



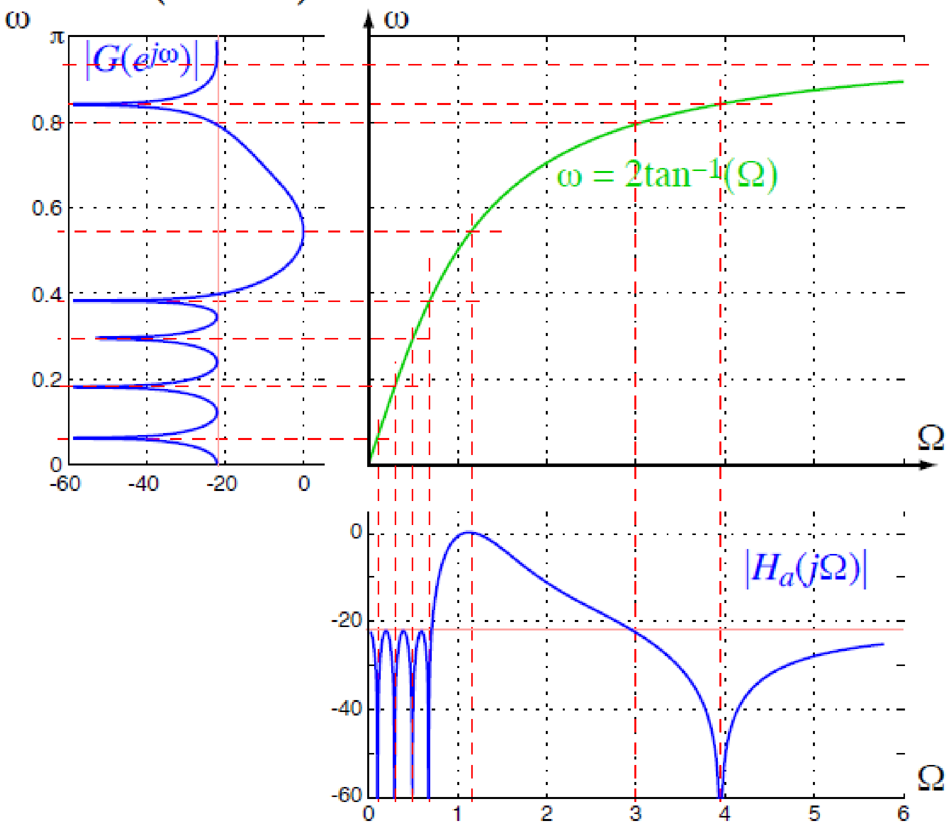
- *infinite* range of CT frequency $-\infty < \Omega < \infty$ maps to *finite* DT freq. range $-\pi < \omega < \pi$
- nonlinear; $\frac{d}{d\omega}\Omega \to \infty$ as $\omega \to \pi$

pack it all in!

Frequency Warping

- Bilinear transform makes

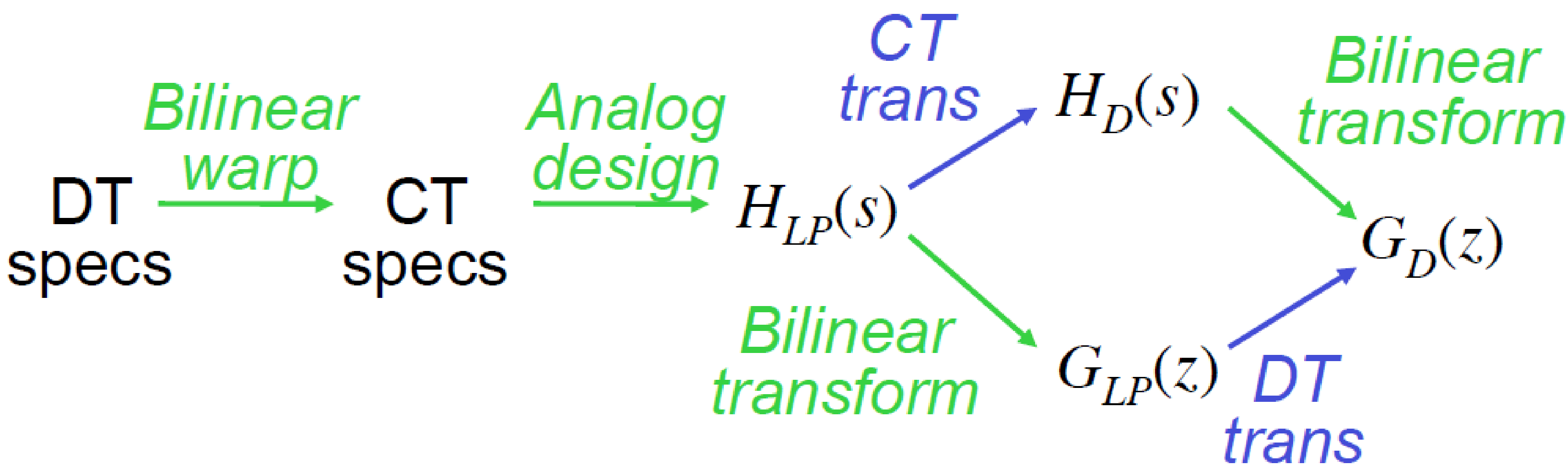
$$G\left(e^{j\omega}\right) = H_a(j\Omega)\Big|_{\omega = 2\tan^{-1}\Omega} \quad \text{for all } \omega, \Omega$$



- Same gain & phase (ε, A...), in same 'order', but with *warped* frequency axis

Other Filter Shapes

- Example was IIR LPF from LP prototype
- For other shapes (HPF, bandpass,...):



- Transform LP→X in CT *or* DT domain...