Understanding Steering Vectors for Uniform Linear Arrays (ULA)

Beginner-Friendly Notes

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1 Motivation and Big Picture

Imagine you have a row of identical antennas placed along a straight line. You send the *same* signal from all of them, but you delay some antennas slightly. By doing this carefully, you can make the signal add up strongly in one direction and cancel out (or weaken) in others. This is the heart of *beamforming*.

If that feels abstract, you can picture a group of people clapping. If everyone claps at exactly the same time, the sound is loud in every direction. If instead each person waits a tiny bit longer than the previous one, you can make the sound travel as a *moving wave* across the group. An antenna array works in a very similar way, just with radio waves instead of sound.

The **steering vector** is a compact mathematical way to describe how much phase (and sometimes amplitude) each antenna element should use in order to "look" or "point" in a particular direction.

These notes focus on:

- What a Uniform Linear Array (ULA) is.
- How to derive the steering vector step by step.
- How the steering vector depends on direction, wavelength, and spacing.
- Several worked examples.
- Common pitfalls and intuitive explanations.

Tip

Think of the steering vector as the *"direction fingerprint"* of a plane wave: each entry is the complex phase seen at one antenna. Once you know this fingerprint, you can design beamformers, DoA estimators, and interference cancellers by simple linear algebra.

2 Uniform Linear Array (ULA) Basics

2.1 Geometry of the Array

Consider an array of M antenna elements placed in a straight line. We assume:

- The elements are equally spaced by distance d.
- The array lies along the x-axis.
- Element 0 is at the origin, element 1 at d, element 2 at $2d$, $\ldots$, element m at position md.

We describe a signal arriving from some angle θ. Throughout these notes:

- θ is measured from the array broadside (perpendicular to the array), unless otherwise stated.
- Broadside means $\theta = 0°$.
- Endfire means $\theta = 90°$ (along the array).

2.2 Wavelength and Wavenumber

Let the carrier frequency be f and the wave propagate at speed c (for radio waves in free space, $c \approx 3 \times 10^8$ m/s). Then the wavelength is

$$\lambda = \frac{c}{f},$$

and the wavenumber is

$$k = \frac{2\pi}{\lambda}.$$

The wavenumber k tells us how quickly the phase changes with distance. Over one full wavelength λ, the phase advances by 2π radians.

In more everyday language:

- λ answers "how long is one cycle in space?"
- k answers "how many radians of phase do we accumulate per meter?"

You do not need to memorize every detail — just remember that k is the bridge that converts distance into phase.

3 Propagation Delay Across the Array

To build intuition, imagine a plane wave coming towards the array from direction θ (broadside at $\theta = 0°$).

3.1 Extra Distance to Each Element

Because the wavefront is planar and the array is linear, different elements will "see" the wave at slightly different times. The element at the origin receives the wave first. Element m at position md experiences an *extra path length* compared to the reference element.

If the array lies on the x-axis and we define θ as the angle from broadside, the projection of the inter-element spacing d on the direction of arrival is $d \sin\theta$. Thus, the m-th element has an extra path length

$$\Delta r_m = md \sin\theta.$$

3.2 Phase Difference Between Elements

An extra path length corresponds to an extra phase shift. The phase shift for a path difference of Δr is

$$\Delta\phi = k\,\Delta r = \frac{2\pi}{\lambda}\,\Delta r.$$

Therefore, the m-th element sees an additional phase of

$$\phi_m = k\,\Delta r_m = \frac{2\pi}{\lambda}\,md \sin\theta.$$

If the signal at the first element (element 0) is represented as $s(t)$ or, in complex baseband form, as $e^{j\omega t}$, then the signal at element m is

$$s_m(t) \propto e^{j(\omega t - \phi_m)} = e^{j\omega t}\, e^{-j\frac{2\pi}{\lambda} md \sin\theta}.$$

We often ignore the common time factor $e^{j\omega t}$ and focus on the spatial phase term:

$$e^{-j\frac{2\pi}{\lambda} md \sin\theta}.$$

4 Definition of the Steering Vector

4.1 Receive Steering Vector

For an M-element ULA, the **receive steering vector** for a signal coming from angle θ is usually defined as

$$\boldsymbol{a}(\theta) = \begin{bmatrix} 1 \\ e^{-j\frac{2\pi}{\lambda}d\sin\theta} \\ e^{-j\frac{2\pi}{\lambda}2d\sin\theta} \\ \vdots \\ e^{-j\frac{2\pi}{\lambda}(M-1)d\sin\theta} \end{bmatrix}.$$

This is a column vector of length M. Each entry corresponds to one array element. The first element is taken as the reference (phase 0).

4.2 Transmit Steering Vector

For transmit beamforming, the steering vector looks similar, but the sign convention can change depending on how you define the Fourier transform and propagation direction. One common convention is to use

$$\boldsymbol{w}(\theta) = \begin{bmatrix} 1 \\ e^{+j\frac{2\pi}{\lambda}d\sin\theta} \\ e^{+j\frac{2\pi}{\lambda}2d\sin\theta} \\ \vdots \\ e^{+j\frac{2\pi}{\lambda}(M-1)d\sin\theta} \end{bmatrix}$$

for transmit weights that *steer* a beam towards θ.

The important idea is: **relative phases between elements control direction**. The exact sign (plus or minus) depends on your chosen convention, but the structure and dependence on d, λ, and θ are the same.

4.3 Compact Expression

Sometimes we write the steering vector more compactly using an index m:

$$[\boldsymbol{a}(\theta)]_m = e^{-j\frac{2\pi}{\lambda}md\sin\theta}, \quad m = 0, 1, \ldots, M-1.$$

5 Special Case: Half-Wavelength Spacing

In practice, a very common choice is

$$d = \frac{\lambda}{2}.$$

This spacing avoids spatial aliasing (grating lobes) for angles within the visible region.

Important

Choosing $d = \lambda/2$ is the *safe default* for ULAs: it keeps the main lobe unique over the full visible range and avoids strong unwanted beams (grating lobes) that can confuse detection or DoA estimation.

You can think of this as a "don't put antennas too far apart" rule, very similar to the Nyquist sampling rule in time. If elements are spaced more than half a wavelength, the array starts to “reuse” the same pattern in other directions, which shows up as extra beams that you usually do not want.

Substituting $d = \lambda/2$ into the steering vector gives

$$[\boldsymbol{a}(\theta)]_m = e^{-j\pi m \sin\theta}, \quad m = 0, 1, \ldots, M-1.$$

This is a very convenient form:

- The phase shift between neighboring elements is $-\pi \sin\theta$.
- The total phase shift from element 0 to element $M-1$ is $-\pi(M-1)\sin\theta$.

6 Worked Numerical Examples

In all examples below, we take $d = \lambda/2$ for simplicity. We focus on the receive steering vector $\boldsymbol{a}(\theta)$.

6.1 Example 1: 4-Element ULA, Broadside

Let $M = 4$ and $\theta = 0^\circ$ (broadside). Then $\sin\theta = \sin 0^\circ = 0$ and

$$[\boldsymbol{a}(0^\circ)]_m = e^{-j\pi m \cdot 0} = e^0 = 1 \quad \text{for all } m.$$

So the steering vector is simply

$$\boldsymbol{a}(0^\circ) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Interpretation: at broadside, all elements experience the incoming wave with the same phase. The array response is maximized when we add all elements in phase.

6.2 Example 2: 4-Element ULA, $\theta = 30^\circ$

Again, $M = 4$ and $d = \lambda/2$. Now $\theta = 30^\circ$, so $\sin\theta = \sin 30^\circ = 0.5$.

The phase increment between neighboring elements is

$$\Delta\phi = -\pi \sin\theta = -\pi \cdot 0.5 = -\frac{\pi}{2}.$$

Thus,

$$\boldsymbol{a}(30^\circ) = \begin{bmatrix} e^{-j0} \\ e^{-j\frac{\pi}{2}} \\ e^{-j\pi} \\ e^{-j\frac{3\pi}{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix}.$$

Interpretation: the phases rotate by -90° from element to element. If we want to *combine* the received signals as if they were all coming from 30°, we can compensate this phase progression using the conjugate weights.

6.3 Example 3: 8-Element ULA, $\theta = -20^\circ$

Now take $M = 8$, $d = \lambda/2$, and $\theta = -20^\circ$. Then $\sin\theta = \sin(-20^\circ) \approx -0.342$.

The phase increment is

$$\Delta\phi = -\pi \sin\theta \approx -\pi(-0.342) \approx +1.075 \text{ rad}.$$

So every step along the array adds about $+1.075$ radians of phase. The steering vector is

$$[\boldsymbol{a}(-20^\circ)]_m \approx e^{+j1.075m}, \quad m = 0, \ldots, 7.$$

In practice, we would compute these values numerically (e.g., in MATLAB or Python).

7 From Steering Vector to Array Factor

The **array factor** describes how the array responds (in gain or amplitude) to signals from different angles, given a set of weights $\boldsymbol{w}$.

7.1 Array Response

For a receive array with weight vector $\boldsymbol{w}$ (size $M \times 1$), the response to an incoming signal from angle θ is proportional to

$$H(\theta) = \boldsymbol{w}^{\mathrm{H}} \boldsymbol{a}(\theta),$$

where $(\cdot)^{\mathrm{H}}$ denotes the conjugate transpose.

If we choose the simple "steer-and-sum" weights

$$\boldsymbol{w} = \boldsymbol{a}(\theta_0),$$

then the response becomes

$$H(\theta) = \boldsymbol{a}(\theta_0)^{\mathrm{H}} \boldsymbol{a}(\theta).$$

This creates a main lobe around θ_0 and sidelobes at other angles. The precise shape of the beam depends on the number of elements M and the spacing d.

7.2 Interpretation

The steering vector essentially encodes the spatial "signature" of a plane wave from angle θ. By taking inner products with steering vectors, we can:

- Point the array in a given direction (beamforming).
- Estimate the direction of arrival (DoA) of signals.
- Form nulls in interfering directions.

8 Common Pitfalls and Clarifications

8.1 Angle Convention: Broadside vs. Array Axis

Different textbooks and papers sometimes measure θ differently. Two common conventions are:

1. θ measured from *broadside* (perpendicular to array).
2. θ measured from the *array axis*.

In these notes, we used broadside as $0°$. If a reference uses another convention, the formula for the steering vector may use $\cos\theta$ instead of $\sin\theta$.

Always check the geometry sketch or text: **from which direction is θ measured?**

Important

Before using any steering-vector formula from a paper or book, first check:

1. Is θ measured from the array *broadside* or the *array axis*?
2. Does the formula use $\sin\theta$ or $\cos\theta$?
3. Which element index is taken as the phase reference?

A quick sketch usually saves a lot of sign mistakes.

8.2 Sign Conventions

You might see steering vectors written as

$$e^{-jkmd\sin\theta} \quad \text{or} \quad e^{+jkmd\sin\theta}.$$

Both can be correct depending on:

- Whether you are modeling receive or transmit.
- The definition of the complex exponential for waves (e.g., $e^{j(\omega t-kx)}$ vs. $e^{j(\omega t+kx)}$).

The physical behavior is unchanged as long as you stay consistent within your system.

8.3 Element 0 at the Center vs. at One End

In some arrays, we place element 0 at the array center instead of at one end. Then element indices run from $-(M-1)/2$ to $(M-1)/2$ (for odd M).

In that case, the steering vector becomes

$$[\boldsymbol{a}(\theta)]_m = e^{-j\frac{2\pi}{\lambda}x_m\sin\theta},$$

where x_m is the actual position of element m. The structure is the same; only the reference point changes.

9 Connecting to Linear Algebra

9.1 Vector View of the Array

We can think of the signals at each antenna as entries of a vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{M-1}(t) \end{bmatrix}.$$

For a single plane wave from angle θ with complex envelope $s(t)$, we can write

$$\boldsymbol{x}(t) = \boldsymbol{a}(\theta)\, s(t) + \boldsymbol{n}(t),$$

where $\boldsymbol{n}(t)$ is noise.

9.2 Beamforming as Inner Product

The beamformer output is

$$y(t) = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{x}(t).$$

If we choose $\boldsymbol{w}$ to match (or be closely related to) $\boldsymbol{a}(\theta_0)$, then the beamformer is effectively projecting the received vector onto the steering vector of the desired direction.

This shows how the steering vector plays a central role in array signal processing.

10 Questions and Answers

Q&A

Q: Why does the steering vector use $\sin\theta$?
A: In these notes, θ is measured from the *broadside* direction. The extra path between adjacent elements is the projection of the spacing d onto the wave direction, which gives $d\sin\theta$. If you measured θ from the array axis instead, the projection would involve $\cos\theta$ and the formulas would look slightly different.

Q&A

Q: What happens if I choose $d > \lambda/2$?
A: When the spacing exceeds half a wavelength, the array response starts to exhibit *grating lobes*: additional strong beams at other angles that are indistinguishable from the main lobe. This can cause the array to "see" a source in the wrong direction or respond strongly to interferers that should have been weak.

Q&A

Q: How many elements do I need for a narrow beam?
A: Roughly speaking, the more elements you have over a fixed aperture length, the narrower your main lobe becomes and the better your angular resolution. For a ULA with M elements and spacing d, the effective aperture is about $(M-1)d$, and the main-lobe width (in radians) scales approximately like $\lambda/[(M-1)d]$.

Q&A

Q: Is there an easy way to remember the steering-vector formula?
A: One mnemonic is:

$$[\boldsymbol{a}(\theta)]_m = e^{-jk\,x_m},$$

where x_m is the projection of the m-th element position onto the wave direction. For a ULA on the x-axis with spacing d and broadside angle convention, this reduces to $x_m = md\sin\theta$, which gives the familiar $\exp(-j2\pi md\sin\theta/\lambda)$.

Q&A

Q: All these constraints and conventions feel overwhelming. How should I think about them?
A: It helps to separate them into three simple checks:

1. *Geometry check:* draw the array and angle; make sure you know from where θ is measured.
2. *Spacing check:* keep $d \leq \lambda/2$ unless you have a very good reason not to.
3. *Sign check:* pick one convention for the complex exponential and stick to it consistently.

If you follow these three checks, most "mysterious" sign errors and aliasing problems simply disappear.

11 Summary

- A Uniform Linear Array (ULA) places M antennas along a line with spacing d.
- A plane wave from angle θ reaches each element with a different phase.
- The steering vector $\boldsymbol{a}(\theta)$ collects these phase shifts in a vector.
- For broadside ($\theta = 0°$), all elements are in phase.
- For other angles, phase progresses linearly along the array.
- Choosing appropriate weights based on steering vectors enables beamforming, direction-of-arrival estimation, and interference suppression.

Tip

When you feel lost, come back to three core ideas: *(i)* geometry gives you the path differences, *(ii)* path differences become phase differences via $k = 2\pi/\lambda$, and *(iii)* stacking these phases into a vector gives the steering vector that drives almost all array-processing algorithms.

These notes are only a starting point. Once you are comfortable with the basic steering vector, you can explore:

- Non-uniform arrays (e.g., sparse arrays).
- Planar and volumetric arrays.
- Adaptive beamforming algorithms (e.g., MVDR, LCMV).
- High-resolution DoA methods (e.g., MUSIC, ESPRIT).