[redacted]

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1j}x_j + \cdots\cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2j}x_j + \cdots\cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots\cdots + a_{ij}x_j + \cdots\cdots + a_{in}x_n &= b_i \\
\vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mj}x_j + \cdots\cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

[redacted]

$$\begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & --- & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$

[redacted]

[redacted]

$$\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1j} & \cdots\cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2j} & \cdots\cdots & a_{2n} & b_2 \\ & \vdots & & \vdots & & \vdots & \\ a_{i1} & a_{i2} & \cdots\cdots & a_{ij} & \cdots\cdots & a_{in} & b_i \\ & \vdots & & \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mj} & \cdots\cdots & a_{mn} & b_m \end{bmatrix}$$

[redacted]

Definition: Number of non-zero rows in row echelon form (row reduced echelon form) of matrix is called row-rank of matrix .

[redacted]

Theorem: Let $Ax = b$ be a system.

Then

1. $R(A) \neq R(A:b)$

$\Longrightarrow$ **Solution doesn't exists**

i.e. INCONSISTENT (Example 3)

2. $R(A) = R(A:b)$ = No. of Unknowns

Implies

UNIQUE SOLUTION (Example1)

3. $R(A) = R(A:b) <$ No. of Unknowns

Implies

INFINITELY MANY SOLUTIONS (Example2)

Note : For consistency

$$R(A) = R(A:b)$$

Example 1: Consider the Linear Systems

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{aligned}
$$

Find the solution, if exists, reducing the augmented matrix to (i) row echelon form and (ii) row reduced echelon form

[redacted]

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\downarrow \qquad R_3 \to R_3 + 4R_1$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-3x_2 + 13x_3 &= -9
\end{aligned}
\quad \text{or} \quad
\left[\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
0 & -3 & 13 & -9
\end{array}\right]
$$

Let us take 2 as a pivot

$$\downarrow \qquad R_2 \to \frac{1}{2} R_2$$

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
-3x_2 + 13x_3 &= -9
\end{aligned}
\quad \text{or} \quad
\left[\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & -3 & 13 & -9
\end{array}\right]
$$

$$\downarrow \qquad R_3 \to R_3 + 3R_2$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(i) **This is the row echelon form.**

Is the system consistent?

The system is consistent since R(A) = R(A:b) =3= No. of Unknowns

[redacted]

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\left(x_1,\ x_2, x_3\right) = (29,\ 16, 3)$$

(ii) For row reduced echelon form

$R_2 \to R_2 + 4R_3$ and $R_1 \to R_1 + (-1)R_3$

$$
\begin{aligned}
x_1 - 2x_2 &= -3 \\
x_2 &= 16 \\
x_3 &= 3
\end{aligned}
\quad \text{or} \quad
\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$R_1 \to R_1 + 2R_2$

$$
\begin{aligned}
x_1 &= 29 \\
x_2 &= 16 \\
x_3 &= 3
\end{aligned}
\quad \text{or} \quad
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

(ii) This is the row reduced echelon form.

[redacted]

$$\left(x_1, x_2, x_3\right) = (29, 16, 3)$$

Example2

- Solve the system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] \downarrow R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \downarrow R_1 \rightarrow \frac{1}{2} R_1$$

$$\begin{bmatrix} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\downarrow \quad R_3 \rightarrow R_3 - 5R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{array}\right]$$

$$\downarrow \quad R_3 \to R_3 + \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array}\right]$$

[redacted]

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$x_2 - 4x_3 = 8$$

$$0 = 5/2$$

- [redacted]

(INCONSISTENT)

Example3

Solve the system of equations

$x + 2y - 3z = 6$

$2x - y + 4z = 2$

$4x + 3y - 2z = 14$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array}\right) \begin{array}{l} \\ R2=R2-2R1 \\ R3=R3-4R1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array}\right) \begin{array}{l} \\ R2=R2/-5 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & -5 & 10 & -10 \end{array}\right) \quad R3=R3+5R2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R1=R1-2R2$$

- **Is the system consistent?**

Here $x + z = 2$

$$y - 2z = 2$$

Or Let z=r (any real number)

$$x = 2 - r$$

$$y = 2 + 2r$$

Conclusion: System has Infinitely many solutions

Let $z = 1$ Then $x = 1, \ y = 4$

$z = 2$ Then $x = 0, \ y = 6$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vdots \qquad \vdots \qquad \vdots$$

Conclusion: System has Infinitely many solutions

HOMOGENEOUS SYSTEM

$A X = 0$

$X = 0$ is always a solution

called TRIVIAL SOLUTION

Homogeneous system is always consistent.

Results:

1. $R(A) =$ No. of Unknowns

   Then Unique Solution as $X = 0$.

2. $R(A) <$ No. of Unknowns

Exa :Find all the solutions to the given system

$$x + 2y + 3z = 0$$

$$x + y + z = 0$$

$$5x + 7y + 9z = 0$$

The coefficient matrix is $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 5 & 7 & 9 \end{pmatrix}$

Using R2=R2-R1 & R3=R3-5R1

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{pmatrix}$$

The RRE is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R(A) = 2 <$ No. of Unknowns $= 3$

Hence Infinitely many solutions.

Equivalent system is

$x - z = 0$ & $y + 2z = 0$

Let $z = r$ (any real number)

Then the solution is:

$x = r$, $y = -2r$ & $z = r$

Ex ample: Solve

$$
\begin{aligned}
x + y + z &= 0\\
x + z &= 0\\
2x + y - 2z &= 0\\
x + 5y + 5z &= 0
\end{aligned}
$$

Then

$$
\left[\begin{array}{ccc|c}
1 & 1 & 1 & \\
1 & 0 & 1 & \\
2 & 1 & -2 & \\
1 & 5 & 5 &
\end{array}\right]
$$

R2 = R2- R1

R3 = R 3- 2R1

R4 = R4 - R1

1	1	1	
0	-1	0	
0	-1	-4	R3 = R 3- R2
0	4	4	R4 = R4 + 4R2

1	1	1	
0	1	0	
0	0	-4	

$$\begin{array}{ccc|} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline \end{array}$$

Now $R(A) = 3$ = No. of Unknowns

Hence $x = 0$, $y = 0$, $z = 0$,

is trivial solution as Unique Solution.

The Inverse of a Matrix

- **An n x n matrix *A* is NON-SINGULAR / INVERTIBLE if there exists an n x n matrix B such that**

$$A\,B = B\,A = I$$

B is called inverse of A.

If there exists no such matrix B, Then A is SINGULAR.

Ex:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$$

since A B = B A = I

i.e. B is inverse of A

Hence A is NON-SINGULAR

Th: Inverse of a matrix is unique, if it exists.

Properties of the Inverse

Let A and B be nonsingular matrices

Then 1. $(A-1)-1 = A$

2. $(AB)-1 = B-1A-1$

3. $(AT)-1 = (A-1)T$

Th: Let A1, A2, ... , Am are n x n nonsingular matrices. Then

$(A1A2 \ldots Am)-1 = Am-1 \ldots A1-1$

Some Remarks:

(i) Row reduced echelon form of a non-singular matrix is an Identity Matrix

(ii) If at least one row of row reduced echelon form of a matrix is zero, then inverse of that matrix does not exist

Algorithm for finding A-1

Form the matrix [A | In]

- Apply the elementary row operations to the [A | In]
- Reduce A to row reduced echelon: [In | A-1]

Exa: Find inverse of A, if exists

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Same Operations to this matrix

Let us write

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ R2 - R1 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \quad R3=R3-R2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right) \quad R3= -1.R3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

R1-R3 & R2-2R3

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

R1-R2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

Hence

$$A\text{-}1 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$