Linear Algebra

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Linear Equations

[illegible] = [illegible], [illegible] & [illegible]

[illegible] = [illegible] / [illegible] ? ? ?

Case I: [illegible] ≠ 0: [illegible] *h* [illegible] = [illegible] / [illegible], unique solution

CaseII: [illegible] = 0 = [illegible] infinite number of solutions

CaseIII: [illegible] = 0, [illegible] ≠ 0: No solution

Linear homogeneous equation:

[illegible] = 0

System of Linear Equations

A **linear equation** in the variables $x_1, x_2, \ldots\ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are in $F\ (\Re \quad or \quad C)$, usually known in advance.

Examples

The equations

$$2x_1 - x_2 = 5 \quad and \quad 5x_1 + 2x_3 = 0$$

are both linear.

The equations

$$x_1 - x_2 = x_1 x_2 \quad and \quad \sqrt{x_1} + 2x_2 = 1$$

are not linear because of the presence of the term

$x_1 \, x_2$ in the first equation and $\sqrt{x_1}$ in the second equation.

System of linear equations: collection of more than one linear equations involving the same variables $x_1, x_2, \ldots\ldots, x_n$ (say).

- A general system of m linear equations, in n unknowns, is written as

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1, \\
&a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1) \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m.
\end{aligned}
$$

Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},\ x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ b=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix

$$B=\begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} & b_1 \\ a_{21} & a_{22} \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} \dots & a_{mn} & b_m \end{bmatrix}=(A,b)$$

is called **augmented matrix** of the system (1).

The matrix notation of the linear system of equations given in equation (1) is written as

$$Ax = b\,. \tag{2}$$

If $b = 0$, then equation (2) is called homogeneous otherwise inhomogeneous.

A system of linear equations has

- **either no solution or**
- **exactly one solution or**
- **infinitely many solutions.**

Examples:

(a) $\begin{aligned} x + y &= 1, \\ x + y &= 2. \end{aligned}$

(b) $\begin{aligned} x + y &= 1, \\ x - y &= 0. \end{aligned}$

(c) $\begin{aligned} x + y &= 1, \\ 2x + 2y &= 2. \end{aligned}$

Consistency and inconsistency

A system of linear equations is said to be **consistent** if it has a solution (one solution or infinitely many solutions), a system is **inconsistent** if it has no solution.

Remark: A linear homogeneous system of equations

$$Ax = 0$$

is always consistent as it has always a trivial solution $0 = (0,0,....0)^T$.

How to solve a linear system of equations ?

We know that the solution of the system of linear equation **does not change** if we

- Interchange any two equations
- Multiply any equation by a non-zero scalar
- Replace a equation by the sum of itself and a scalar multiple of another equation.

System of Linear Equations

Above three operations can be transformed for matrices and are called as elementary row operations on matrices

Elementary row operations on matrices

1. Multiplications of any row by a non-zero number
2. Interchanging any two rows.
3. Replacing a row by sum of itself and a scalar multiple of any other row.

Note: Above operations are similar to three operations done on system of equations

Some definitions

Row-equivalent :

Let A and B are $m \times n$ matrices over the field F. Then B is called **row-equivalent** to A if B can be obtained from A by a finite sequence of elementary row operations.

We note that if a matrix B can be obtained from a matrix A, by the three elementary row operations defined above, then we can recover A from B by applying the inverse elementary row operations on B in the reverse order. Therefore, the two systems of linear equations $Ax = b_1$ and $Bx = b_2$ have the same solution set. Thus, we can state the following theorem.

- **Theorem 1.** If two system of equations are row-equivalent, then they have same set of solutions.

Leading entry of a row:

First non zero entry (from the left) of a non zero row is called **leading entry of that row** .

Row Echelon Form (REF) of a matrix

(i) All zero rows, if any, are at the bottom of matrix

(ii) First nonzero entry (from the left) of a nonzero row is 1 (such a entry is called leading one of its row)

(iii) If a column contains leading one of any row, then all other entries below the leading one in that column are zero

(iv) For each non-zero row, leading one comes to the right and below any leading one's in previous rows

Row- Reduced Echelon Form (RREF) of a matrix

(i) All zero rows, if there are any, are at the bottom of matrix

(ii) First nonzero entry (from the left) of a nonzero row is 1 (such a entry is called leading one of its row)

(iii) If a column contains leading one of any row, then every other entries in that column is zero

(iv) For each non-zero row, leading one comes to the right and below any leading one's in previous rows

Example-1: Echelon Matrix :

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example-2: Echelon Matrix :

$$\begin{bmatrix} 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 6 & 5 \\ 0 & 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Ex: Row Reduced Echelon Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Ex: Row Reduced Echelon Matrix

1. $0_{m\times n}$: $m\times n$ zero matrix

2. $I_n : n\times n$ identity matrix

3.
$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the starred entries (*) may take any values including zero.

Definitions:

- ***Pivot position:*** is a position of a leading entry in an echelon form of the matrix.
- ***Pivot:*** a nonzero number that either is used in a pivot position to create 0's OR is changed into a leading 1, which in turn is used to create 0's.
- ***Pivot row/column:*** a row/column that contains a pivot position.

PROBLEM : Find Row Reduced Echelon form of a Matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

- **Step 1: Selecting Pivot column :**

Begin with the leftmost nonzero column.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot Column

Step 2: Selecting Pivot Position:

Select a **nonzero entry** in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position.

Pivot Position

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Pivot Column

- **Step 2: (Continued):**

Using the elementary row operation

$$R_1 \leftrightarrow R_4$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot Column

- **Step 3:** Use elementary row operations to create zeros in all positions below the pivot.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Apply Elementary row operation

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + 2R_1$

Step 3: (Continued):

After a few computations we get

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- **Step 4:** Ignore the row containing the pivot position and cover all rows, if any, above it.

Apply steps 1-3 to the remaining submatrix. Repeat the process until there are no more nonzero rows to modify.

Possible Pivots

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Let us take pivot as

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Make this pivot element to 1 and then apply

$R_3 \to R_3 - 5R_2 \qquad R_4 \to R_4 + 3R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Apply

$$R_3 \leftrightarrow R_4$$

And this pivot -5 to 1

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note: This is a row echelon form

- **Step 5:** Beginning with the **rightmost** pivot and working upward and to the left, create zeros above each pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \rightarrow R_1 + 9R_3$$

$$R_2 \rightarrow R_2 + 3R_3$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot as this is first non-zero element in 2nd row

$$R_1 \to R_1 - 4R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

[redacted]

Results :

1. Echelon form of a matrix may not be unique.
2. Row reduced echelon form of a matrix is unique
3. Every matrix is row equivalent to its echelon form.
4. Every matrix is row equivalent to its row reduced echelon form