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BITS Pilani presentation

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Linearly Independent (LI) and Linearly dependent (LD) set

Let V be a vector space. *Then*

v1, v2, ... , vk € V are **LI** if

c1 v1 + c2 v2 + ... + ck vk = 0

=> ci = 0 for all i

Otherwise, vectors are Linearly Dependent.

Linearly dependent (LD) set

The vectors $v1, v2, \ldots, vk$ in a vector space **V** are **LD** if

there exist scalars c_i, $i = 1,2, \ldots, k$ **Not all zero at a time** such that

$$c1\,v1 + c2\,v2 + \ldots + ck\,vk = 0$$

Remark:

(a) A non-zero vector is always LI for if $u \neq 0_V$, then $\alpha u = 0_V \Rightarrow \alpha = 0$.

(b) Zero vector is always LD, for $1.0_V = 0_V$, and hence any set containing the zero vector is always *LD.*

(c) By convention, we take the empty set ϕ as LI.

Example 1

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ *and* $e_3 = (0,0,1)$. Then the set

$S = \{e_1, e_2, e_3\}$ is LI in R^3 over $\Re$ as

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0 = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$$

$$\Rightarrow \alpha_1 = 0 = \alpha_2 = \alpha_3 .$$

Example 2

- $v1 = (1, 2)$, $v2 = (5, 10)$ in R2 are not LI, i.e. **Linearly Dependent**.

Example 3

Is $S = \{ p1(x), p2(x), p3(x) \}$ LI? where

$p1(x) = x2 + x + 2$

$p2(x) = 2x2 + x$

$p3(x) = 3x2 + x + 2$

Soln:

Let a p1(x) +b p2(x) +c p3(x) =0

i.e. a(x2 + x + 2)+b(2x2 + x)+c(3x2 + x + 2)=0

Which gives the system of linear equations in *a, b & c:*

a+2b+3c=0

a+b+c=0

2a+2c=0

This implies a=0=b=c and hence S is LI

Example 4

Check whether the given set

$S=\{1-x,\ x-x^2, 1-x^2\}$ is LD or LI in P_2 over $\Re$.

Solution: Let a,b,c be scalars such that

$$a(1-x)+b(x-x^2)+c(1-x^2)=0 \text{, a zero polynomial.}$$

$$\Rightarrow (a+c)+(b-a)x-(b+c)x^2=0.$$

$$\Rightarrow a+c=0,\ b-a=0,\ b+c=0.$$

On solving, we get

$$b = a,$$

$$c = -a,$$

and a is arbitrary real number.

Thus, the system of equations (2.1) has infinitely many solution and therefore a nontrivial solution which implies that the set S is LD.

Basis and dimension

Let V be a vector space.

Let S = {v1, v2, ... ,vn} be a subset of V

Then S is Basis of V if

1. S is LI and

2. [S] = V.

Dimension

- **Definition.** The **dimension** of a vector space is the number of elements in a basis for V . If the number of elements in a basis for V is finite, the space is said to be **finite dimensional**.
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Examples

Ex 1: S = {(1, 0), (0, 1)} is a basis of R2

dim R2 = 2

Ex 2: S = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is a basis of R3

dim R3 = 3

Example 3

$e_1 = (1,0,0,\ldots,0),\ e_2 = (0,1,0,\ldots,0),\ \ldots,\ e_n = (0,0,\ldots\ldots 0,1)$ are

LI in $\Re^n$ over $\Re$. Let

$x = (x_1, x_2, \ldots, x_n) \in \Re^n$ be any element.

Then $x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$

$= (x_1, x_2, \ldots, x_n).$

This shows that the set $S = \{e_1, e_2, \ldots, e_n\}$ spans $\Re^n$

Hence S is a basis for $\Re^n$

$\dim \Re^n = \text{n}$

- Ex4. The set S = {1, x } is a basis for P1 .
- dim P1 = 2.
- Ex5. The set S = {1, x, x2 } is a basis for P2 .
- dim P2 = 3.
- Ex6. The set S = {1, x, x2, ….,xn } is a basis for Pn .
- dim Pn = n+1.
- Note: If V = { 0 }, Then dim V=0.

Useful results

1. If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.

2. Let V be a finite-dimensional vector space and dimV =n. Then any subset of V which contains more than n vectors is LD.

3. In an *n*-dimensional vector space V , any set of n linearly independent vectors is a basis for V.

4. Let W_1 and W_2 are finite-dimensional subspaces of a vector space V. Then $W_1 + W_2$ is finite-dimensional and $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Example

Let S = {$x2+x$, $x-1$, $x+1$}

Show that S is a basis of P2

Soln:

Let **α, β & γ be scalers such that**

$\alpha(x2+x)+\beta(x - 1)+\gamma(x + 1) =0$.

This implies $\alpha x2 + (\alpha+ \beta+ \gamma)x+ (- \beta +\gamma)=0$.

- Which gives

$\alpha=0$

$\alpha+\beta+\gamma=0$

$-\beta+\gamma=0.$

Thus, $\alpha=0=\beta=\gamma$.

Conclusions: (i) S is LI

(ii) dim P2 = 3 = no. of elements in S

Example

Suppose that we want to check whether the set

$$S = \{(1,2,1,),(-1,1,0),(5,-1,2)\}$$

is a basis for R^3 over $\Re$.

Step I: Check if S is LI

For this, we let $\alpha_1, \alpha_2, \alpha_3$ be scalars (reals) such that

$$\alpha_1(1,2,1) + \alpha_2(-1,1,0) + \alpha_3(5.-1,0) = (0,0,0).$$

$$\Rightarrow \alpha_1 - \alpha_2 + 5\alpha_3 = 0,$$

$$2\alpha_1 + \alpha_2 - \alpha_3 = 0,$$

$$\alpha_1 + 2\alpha_3 = 0.$$

This shows that $\alpha_1 = 0 = \alpha_2 = \alpha_3$.

Hence the given set S is LI.

- Step II: since dim R3 =3 = no. of elements in S
- This implies that S spans R3
- Conclusion: S is a basis for R3 .

Example

Let $S = \{(x_1, x_2, x_3) \in V_3 : x_1 + x_2 + x_3 = 0\}$ be a subspace of R^3 over $\Re$. Determine a basis for S and hence find *dimS*.

Solution: We have

$$
\begin{aligned}
S &= \{(x_1, x_2, x_3) \in R^3 : x_1 = -x_2 - x_3\} \\
&= \{(-x_2 - x_3, x_2, x_3) : x_2, x_3 \in \Re\} \\
&= \{(-x_2, x_2, 0) + (-x_3, 0, x_3) : x_2, x_3 \in \Re\} \\
&= \{x_2(-1,1,0) + x_3(-1,0,1) : x_2, x_3 \in \Re\}
\end{aligned}
$$

$$\Rightarrow S = [\{(-1,1,0), (-1,0,1)\}].$$

Clearly the set $B = \{(-1,1,0), (-1,0,1)\}$ spans S and it is easy to check that B is *LI*. Hence B is a basis for S and $dim\ S = 2$.