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BITS Pilani presentation

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Useful results

1. If V is a finite-dimensional vector space, then any two bases of V have the same number of elements.

2. Let V be a finite-dimensional vector space

   and dimV = n. Then

   any subset of V which contains more than n

   vectors is LD.

3. In an *n*-dimensional vector space V , any set of n linearly independent vectors is a basis for V.

4. Let W_1 and W_2 are finite-dimensional subspaces of a vector space V. Then $W_1 + W_2$ is finite-dimensional and $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

Example

Let S = {$x2+x$, $x-1$, $x+1$}

Show that S is a basis of P2

Soln:

Let α, **β & γ be scalers such that**

$\alpha(x2+x)+\beta(x - 1)+\gamma(x + 1) =0$.

This implies $\alpha x2 + (\alpha+ \beta+ \gamma)x+ (- \beta +\gamma)=0$.

- Which gives

  $\alpha=0$

  $\alpha+\beta+\gamma=0$

  $-\beta+\gamma=0.$

Thus, $\alpha=0=\beta=\gamma$.

Conclusions: (i) S is LI

(ii) dim P2 = 3 = no. of elements in S

Example

Suppose that we want to check whether the set

$$S = \{(1,2,1,),(-1,1,0),(5,-1,2)\}$$

is a basis for R^3 over $\Re$.

Step I: Check if S is LI

For this, we let $\alpha_1, \alpha_2, \alpha_3$ be scalars (reals) such that

$$\alpha_1(1,2,1) + \alpha_2(-1,1,0) + \alpha_3(5.-1,0) = (0,0,0).$$

$$\begin{aligned} \Rightarrow \alpha_1 - \alpha_2 + 5\alpha_3 &= 0, \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= 0, \\ \alpha_1 + 2\alpha_3 \qquad &= 0. \end{aligned}$$

This shows that $\alpha_1 = 0 = \alpha_2 = \alpha_3$.

Hence the given set S is LI.

- Step II: since dim R3 =3 = no. of elements in S
- This implies that S spans R3

Example

Let $S=\{(x_1,x_2,x_3)\in V_3 : x_1+x_2+x_3=0\}$ be a subspace of R^3 over $\Re$. Determine a basis for S and hence find *dimS*.

Solution: We have

$$\begin{aligned} S &= \{(x_1,x_2,x_3)\in R^3 : x_1=-x_2-x_3\} \\ &= \{(-x_2-x_3,x_2,x_3): x_2,x_3\in\Re\} \\ &= \{(-x_2,x_2,0)+(-x_3,0,x_3): x_2,x_3\in\Re\} \\ &= \{x_2(-1,1,0)+x_3(-1,0,1): x_2,x_3\in\Re\} \end{aligned}$$

$$\Rightarrow S=[\{(-1,1,0),(-1,0,1)\}].$$

Clearly the set $B=\{(-1,1,0),(-1,0,1)\}$ spans S and it is easy to check that B is *LI*. Hence B is a basis for S and *dim* $S = 2$.

Linear Transformations (L.T.)

- **Definition :**
  Let U and V be real vector spaces.
- **A map *T: U* $\rightarrow$ *V* is called a linear map, or Linear transformation iff**

(i) $T(u+v)=T(u)+T(v)$ for all u,v in U and

(ii) $T(\alpha u)=\alpha T(u)$ for all u in U and for all real numvers α .

Example 1

Let U be any real vector space. Then the identify transformation $T = I$ from U into U defined by $I(u) = u$ for all u in U is a linear transformation, because

$$I(u+v) = u + v = I(u) + I(v) \text{ for all } u, v \text{ in } U \text{ and}$$

$$I(\alpha u) = \alpha u = \alpha I(u) \text{ for all } u \text{ in } U \text{ and } \alpha \text{ in } \mathbb{R}.$$

Example 2

- **Reflection about x-axis $T(x,y)= (x,-y)$**
- **is a L.T.**

Theorem

- Let T: $U \rightarrow V$ be a linear map, then

i. $T(0U)=0V$

ii. $T(-u) = -T(u)$ *for all u in U*

- Hints:

- $T(ku)=k\ T(u)$ for any scalar k, since T is L.T

- Take $k=0$ *and* $k=-1$.

Range and Null Spaces

Definition. Let U and V be vector spaces over the field F and
let $T : U \to V$ be a linear transformation. Then

$$R(T) = \{v \in V : T(u) = v \text{ for some } u \in U\}$$

is called range space of T, and

$$N(T) = \{u \in U : T(u) = 0_V\}$$

is called null space of T.

Remark: Null space of T is also called kernel of T and is denoted by ker (T).

Theorem. Let U and V be vector space over the field $\mathbb{R}$,

and $T: U \rightarrow V$ is a linear transformation. Then

(a) $R(T)$ is a subspace of V, and

(b) $N(T)$ is a subspace of U.

(c) T is one-one iff $N(T) = \{0_U\}$.

Example

Let $T : R^2 \to R^3$ be a linear transformation defined by

$$T(x_1, x_2) = (x_1, x_1 - x_2, x_2).$$

Suppose that we wish to know whether T is one – one or not.

We have

$$\begin{aligned} N(T) &= \{x = (x_1, x_2) \in R^2 : T(x) = T(x_1, x_2) = 0_{V_3}\} \\ &= \{(x_1, x_2) \in R^2 : (x_1, x_1 - x_2, x_2) = (0,0,0)\} \\ &= \{(x_1, x_2) \in R^2 : x_1 = 0, x_1 - x_2 = 0, x_2 = 0\} \\ &= \{(0,0)\} \\ &= \{0_{R^2}\}. \end{aligned}$$

Hence T is one – one.

Definition. Let U and V be vector spaces over the field $\mathbb{R}$ and $T : U \to V$ be a linear transformation. If U is finite dimensional, then

$$\text{rank of } T = \dim R(T), \text{ and}$$
$$\text{nullity of } T = \dim N(T).$$

Rank-nullity Theorem

Let U and V be vector spaces over the field R and let T be a linear transformation from U into V. Suppose that U is finite dimensional. Then

$$\dim R(T) + \dim N(T) = \dim U.$$

Remark 1. Suppose $T : U \to V$ is a linear transformation and $\dim U < \infty$. Then $\dim R(T) \le \dim U$, because $\dim R(T) + \dim N(T) = \dim U < \infty$.

Remark 2. If $T : U \to V$ is a linear transformation and $\dim U < \infty$. Then $\dim R(T) \le \min\{\dim U,\ \dim V\}$.

Proof : Since $R(T)$ is a subspace of V

$$\Rightarrow \dim R(T) \le \dim V.$$

From Remark 1, $\dim R(T) \le \dim U$

Hence $\dim R(T) \le \min\{\dim U,\ \dim V\}$.

Remark 3. If $T : U \to V$ is a linear transformation and $\dim U < \infty$. Then $\dim R(T) = \dim U \Leftrightarrow T$ is one-one.

Proof: $\Theta \dim R(T) + \dim N(T) = \dim U$

$$
\begin{aligned}
\therefore \dim R(T) = \dim U &\Leftrightarrow \dim N(T) = 0 \\
&\Leftrightarrow N(T) = \{0_V\} \\
&\Leftrightarrow T \text{ is one} - \text{one.}
\end{aligned}
$$

Remark 4. If $\dim R(T) < \dim U$, then T is not one-one.

It follows from Remark 3.

Remark 5. T is onto $\Leftrightarrow R(T) = V$

$\Leftrightarrow \dim R(T) = \dim V$

Remark 6. $\dim R(T) = \dim U = \dim V \Leftrightarrow T$ is one – one and onto.

Example

Let T: $R^4 \rightarrow P_3$ be a linear map defined as

$$T(x_1, x_2, x_3, x_4) = x_1 + (x_2 - x_3)t + (x_1 - x_3)t^3$$

(i) Find ker(T) or N(T) and a basis of N(T)

(ii) Find range (T) and a basis of range (T)

(iii) **Verify Rank-Nullity Theorem**

Soln:

$N(T) = \{x = (x_1, x_2, x_3, x_4) \in R^4 : T(x) = x_1 + (x_2 - x_3)x + (x_1 - x_3)x^2 = 0\}$

$$= \{(x_1, x_2, x_3, x_4) \mid x_1 = 0,\ x_2 - x_3 = 0 \,\&\, x_1 - x_3 = 0\}$$

$$= \{(0,0,0,x4) \mid x4 \quad \text{is any real number}\}$$

$$= [\{(0,0,0,1)\}]$$

basis of N(T) is $\{(0,0,0,1)\}$

dim(N(T))=1

Range space of T:

R(T) = $\{p \in P3 \mid p=T(x), \text{ for } x \in R4\}$

$= \{(x1+(x2-x3)t+(x1-x3)t3 : (x1,x2,x3,x4) \in R4\}$

$= [\{1, t, t3\}]$

Since $\{1, t, t3\}$ *is LI*

basis for R(T) is $\{1, t, t3\}$

$dim(R(T))=3$