

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

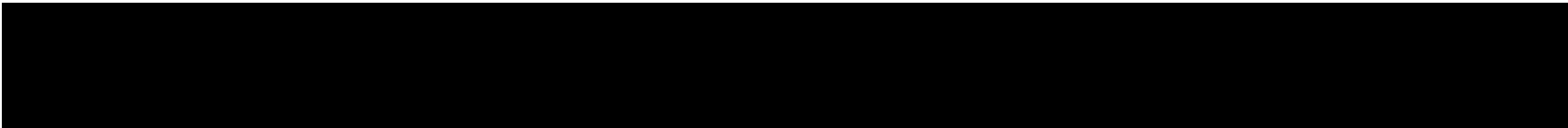
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i$$

$$\vdots$$
$$\vdots$$
$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n = b_m$$

| | | | | | | | |
|----------|----------|----------|----------|--|----------|-----|----------|
| a_{11} | a_{12} | $---$ | a_{1n} | | x_1 | | b_1 |
| a_{21} | a_{22} | $---$ | a_{2n} | | x_2 | | b_2 |
| \vdots | \vdots | \vdots | \vdots | | | | \vdots |
| a_{i1} | a_{i2} | $---$ | a_{in} | | \vdots | $=$ | b_i |
| \vdots | \vdots | \vdots | \vdots | | | | \vdots |
| a_{m1} | a_{m2} | $---$ | a_{mn} | | x_n | | b_m |





$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{bmatrix}$$

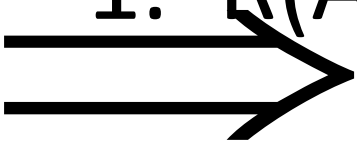


Definition: Number of non-zero rows in row echelon form (row reduced echelon form) of matrix is called row-rank of matrix .



Theorem: Let $Ax = b$ be a system.

Then

1. $R(A) \neq R(A:b)$
 **Solution doesn't exist**
i.e. INCONSISTENT (Example 3)

2. $R(A) = R(A:b) = \text{No. of Unknowns}$

Implies

UNIQUE SOLUTION (Example1)

3. $R(A) = R(A:b) < \text{No. of Unknowns}$

Implies

**INFINITELY MANY SOLUTIONS
(Example2)**

Note : For consistency

$$R(A) = R(A:b)$$

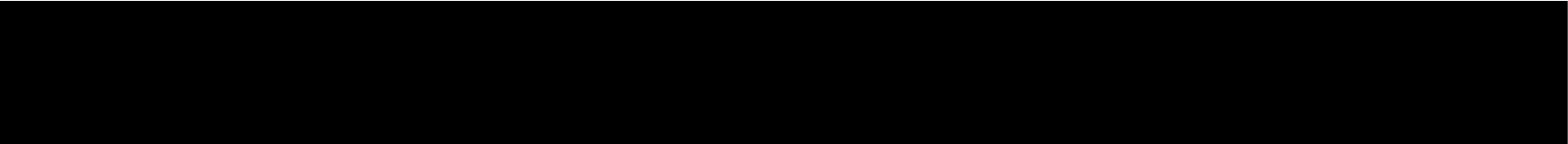
Example 1: Consider the Linear Systems

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

Find the solution, if exists, reducing the augmented matrix to (i) row echelon form and (ii) row reduced echelon form


$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$$\begin{array}{rcl}
 x_1 - 2x_2 + x_3 & = & 0 \\
 2x_2 - 8x_3 & = & 8 \\
 -4x_1 + 5x_2 + 9x_3 & = & -9
 \end{array}
 \quad \text{or} \quad
 \left[\begin{array}{ccc|c}
 1 & -2 & 1 & 0 \\
 0 & 2 & -8 & 8 \\
 -4 & 5 & 9 & -9
 \end{array} \right]$$



$$R_3 \rightarrow R_3 + 4R_1$$

$$\begin{array}{rcl}
 x_1 - 2x_2 + x_3 & = & 0 \\
 2x_2 - 8x_3 & = & 8 \\
 -3x_2 + 13x_3 & = & -9
 \end{array}
 \quad \text{or} \quad
 \left[\begin{array}{ccc|c}
 1 & -2 & 1 & 0 \\
 0 & 2 & -8 & 8 \\
 0 & -3 & 13 & -9
 \end{array} \right]$$

$$\begin{array}{rcl}
 x_1 - 2x_2 + x_3 & = & 0 \\
 2x_2 - 8x_3 & = & 8 \\
 -3x_2 + 13x_3 & = & -9
 \end{array}
 \quad \text{or} \quad
 \left[\begin{array}{ccc|c}
 1 & -2 & 1 & 0 \\
 0 & 2 & -8 & 8 \\
 0 & -3 & 13 & -9
 \end{array} \right]$$

Let us take 2 as a pivot

$$\begin{array}{rcl}
 x_1 - 2x_2 + x_3 & = & 0 \\
 x_2 - 4x_3 & = & 4 \\
 -3x_2 + 13x_3 & = & -9
 \end{array}
 \quad \text{or} \quad
 \left[\begin{array}{ccc|c}
 1 & -2 & 1 & 0 \\
 0 & 1 & -4 & 4 \\
 0 & -3 & 13 & -9
 \end{array} \right]$$

$\downarrow \quad R_2 \rightarrow \frac{1}{2}R_2$



$$R_3 \rightarrow R_3 + 3R_2$$

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 0 \\ x_2 - 4x_3 & = & 4 \\ x_3 & = & 3 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(i) This is the row echelon form.
Is the system consistent?

The system is consistent since $R(A) = R(A:b) = 3 =$


No. of Unknowns


$$\begin{aligned}
 x_1 - 2x_2 + x_3 &= 0 \\
 x_2 - 4x_3 &= 4 \\
 x_3 &= 3
 \end{aligned}
 \quad \text{or} \quad
 \left[\begin{array}{ccc|c}
 1 & -2 & 1 & 0 \\
 0 & 1 & -4 & 4 \\
 0 & 0 & 1 & 3
 \end{array} \right]$$

$$(x_1, x_2, x_3) = (29, 16, 3)$$

(ii) For row reduced echelon form

$$R_2 \rightarrow R_2 + 4R_3 \quad \text{and} \quad R_1 \rightarrow R_1 + (-1)R_3$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$


$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$


$R_1 \rightarrow R_1 + 2R_2$

(ii) This is the row reduced echelon form.

echelon form and then find the solution

$$(x_1, x_2, x_3) = (29, 16, 3)$$

Example2

- Solve the system

$$x_2 - 4x_3 = 8$$

$$2x_1 - 3x_2 + 2x_3 = 1$$

$$5x_1 - 8x_2 + 7x_3 = 1$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array} \right] \quad \downarrow R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array} \right] \quad \downarrow R_1 \rightarrow \frac{1}{2}R_1$$

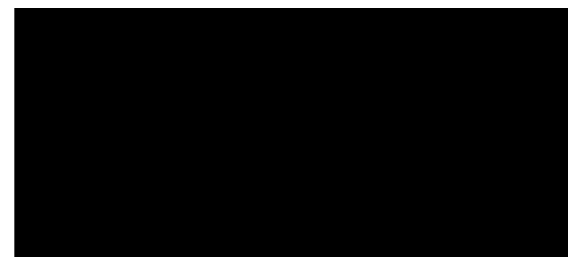
$$\begin{bmatrix} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

$$\downarrow \quad R_3 \rightarrow R_3 - 5R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{array} \right]$$

$$\downarrow \quad R_3 \rightarrow R_3 + \frac{1}{2}R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array} \right]$$



$$2x_1 - 3x_2 + 2x_3 = 1$$

$$x_2 - 4x_3 = 8$$

$$0 = 5/2$$



(INCONSISTENT)

Example3

Solve the system of equations

$$x + 2y - 3z = 6$$

$$2x - y + 4z = 2$$

$$4x + 3y - 2z = 14$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 2 & -1 & 4 & 2 \\ 4 & 3 & -2 & 14 \end{array} \right) \begin{array}{l} \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & -5 & 10 & -10 \\ 0 & -5 & 10 & -10 \end{array} \right) \begin{array}{l} \\ R_2 = R_2 / -5 \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & -5 & 10 & -10 \end{array} \right] R_3 = R_3 + 5R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad R1=R1-2R2$$

- **Is the system consistent?**

Here $x + z = 2$

$$y - 2z = 2$$

Or Let $z=r$ (any real number)

$$x = 2 - r$$

$$y = 2 + 2r$$

Conclusion: System has Infinitely many solutions

Let $z = 1$ Then $x = 1, y = 4$

$z = 2$ Then $x = 0, y = 6$

:

:

:

:

:

:

:

:

:

Conclusion: System has Infinitely many solutions

HOMOGENEOUS SYSTEM

$$A X = 0$$

$X = 0$ is always a solution

called TRIVIAL SOLUTION

Homogeneous system is always consistent.

Results:

1. $R(A) = \text{No. of Unknowns}$

Then Unique Solution as $X = 0$.

2. $R(A) < \text{No. of Unknowns}$

Exa :Find all the solutions to the given system

$$x + 2y + 3z = 0$$

$$x + y + z = 0$$

$$5x + 7y + 9z = 0$$

The coefficient matrix is

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 5 & 7 & 9 \end{pmatrix}$$

Using $R_2 = R_2 - R_1$ & $R_3 = R_3 - 5R_1$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -3 & -6 \end{pmatrix}$$

The RRE is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(A) = 2 < \text{No. of Unknowns} = 3$$

Hence Infinitely many solutions.

Equivalent system is

$$x - z = 0 \text{ \& \; } y + 2z = 0$$

Let $z = r$ (any real number)

Then the solution is:

$$x = r, y = -2r \text{ \& \; } z = r$$

Example: Solve

$$x + y + z = 0$$

$$x + z = 0$$

$$2x + y - 2z = 0$$

$$x + 5y + 5z = 0$$

Then

| | | |
|---|---|---|
| 1 | 1 | 1 |
|---|---|---|

| | | |
|---|---|---|
| 1 | 0 | 1 |
|---|---|---|

 $R_2 = R_2 - R_1$

| | | |
|---|---|----|
| 2 | 1 | -2 |
|---|---|----|

 $R_3 = R_3 - 2R_1$

| | | |
|---|---|---|
| 1 | 5 | 5 |
|---|---|---|

 $R_4 = R_4 - R_1$

| | | |
|---|---|---|
| 1 | 1 | 1 |
|---|---|---|

| | | |
|---|----|---|
| 0 | -1 | 0 |
|---|----|---|

| | | | |
|---|----|----|-------------------|
| 0 | -1 | -4 | $R_3 = R_3 - R_2$ |
|---|----|----|-------------------|

| | | | |
|---|---|---|--------------------|
| 0 | 4 | 4 | $R_4 = R_4 + 4R_2$ |
|---|---|---|--------------------|

| | | |
|---|---|---|
| 1 | 1 | 1 |
|---|---|---|

| | | |
|---|---|---|
| 0 | 1 | 0 |
|---|---|---|

| | | |
|---|---|----|
| 0 | 0 | -4 |
|---|---|----|

| | | |
|---|---|---|
| 0 | 0 | 4 |
|---|---|---|

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |
| 0 | 0 | 0 |

Now $R(A) = 3 = \text{No. of Unknowns}$

Hence $x = 0, y = 0, z = 0,$

is trivial solution as Unique Solution.

The Inverse of a Matrix

- An $n \times n$ matrix A is
NON-SINGULAR / INVERTIBLE
if there exists an $n \times n$ matrix B such
that

$$A B = B A = I$$

B is called inverse of A .

If there exists no such matrix B , Then A
is **SINGULAR**.

Ex:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$$

since $AB = BA = I$

i.e. B is inverse of A

Hence A is **NON-SINGULAR**

Th: Inverse of a matrix is unique, if it exists.

Properties of the Inverse

Let A and B be nonsingular matrices

Then 1. $(A^{-1})^{-1} = A$

2. $(AB)^{-1} = B^{-1}A^{-1}$

3. $(A^T)^{-1} = (A^{-1})^T$

Th: Let A_1, A_2, \dots, A_m are $n \times n$ nonsingular matrices. Then

$$(A_1 A_2 \dots A_m)^{-1} = A_m^{-1} \dots A_1^{-1}$$

Some Remarks:

(i) Row reduced echelon form of a non-singular matrix is an Identity Matrix

(ii) If at least one row of row reduced echelon form of a matrix is zero, then inverse of that matrix does not exist

Algorithm for finding A^{-1}

Form the matrix $[A \mid I_n]$

- Apply the elementary row operations to the $[A \mid I_n]$
- Reduce A to row reduced echelon: $[I_n \mid A^{-1}]$

Exa: Find inverse of A, if exists

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Same Operations
to this matrix

Let us write

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad \begin{array}{l} \\ R2 - R1 \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \quad \left. \vphantom{\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}} \right\} R3=R3-R2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array} \right) \quad \left. \vphantom{\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}} \right\} R3= -1.R3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array} \right)$$

R1-R3 & R2-2R3

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array} \right)$$

R1 R2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array} \right)$$

Hence

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$$