### Linear Algebra

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### **Linear Equations**

$$\mathbf{T} = \frac{\mathbf{T}}{\mathbf{T}} ???$$

Casell: 2 = 0 = 2 infinite number of solutions

Linear homogeneous equation:

$$\mathbf{m} = 0$$

## **System of Linear Equations**

A linear equation in the variables  $x_1, x_2, ..., x_n$  is an equation that can be written in the form  $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ ,

where  $a_1, a_2, ..., a_n$  and b are in F ( $\Re$  or C), usually known in advance.

## Examples

The equations

$$2x_1 - x_2 = 5$$
 and  $5x_1 + 2x_3 = 0$ 

are both linear.

The equations

$$x_1 - x_2 = x_1 x_2$$
 and  $\sqrt{x_1} + 2x_2 = 1$ 

are not linear because of the presence of the term

 $x_1 x_2$  in the first equation and  $\sqrt{x_1}$  in the second equation.

System of linear equations: collection of more than one linear equations involving the same variables  $x_1, x_2, ..., x_n$  (say).

• A general system of m linear equations, in n unknowns, is written as

 $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$ .

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix

$$B = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{vmatrix} = (A,b)$$

is called **augmented matrix** of the system (1).

The matrix notation of the linear system of equations given in equation (1) is written as

$$Ax = b. (2)$$

If b = 0, then equation (2) is called homogeneous otherwise inhomogeneous.

#### A system of linear equations has

- either no solution or
- exactly one solution or
- infinitely many solutions.

#### **Examples:**

(a) 
$$x + y = 1$$
,  
 $x + y = 2$ .

(b) 
$$x + y = 1$$
,  
 $x - y = 0$ .

(c) 
$$x + y = 1$$
,  
 $2x + 2y = 2$ .

## Consistency and inconsistency

A system of linear equations is said to be **consistent** if it has a solution (one solution or infinitely many solutions), a system is **inconsistent** if it has no solution.

**Remark:** A linear homogeneous system of equations Ax = 0

is always consistent as it has always a trivial solution  $0 = (0,0,....0)^T$ .

# How to solve a linear system of equations?

- We know that the solution of the system of linear equation does not change if we
- Interchange any two equations
- Multiply any equation by a non-zero scalar
- Replace a equation by the sum of itself and a scalar multiple of another equation.

## **System of Linear Equations**

Above three operations can be transformed for matrices and are called as elementary row operations on matrices

#### Elementary row operations on matrices

- 1. Multiplications of any row by a non-zero number
- 2. Interchanging any two rows.
- 3. Replacing a row by sum of itself and a scalar multiple of any other row.

Note: Above operations are similar to three operations done on system of equations

#### Some definitions

#### **Row-equivalent:**

Let A and B are  $m \times n$  matrices over the field F. Then B is called **row-equivalent** to A if B can be obtained from A by a finite sequence of elementary row operations.

We note that if a matrix B can be obtained from a matrix A, by the three elementary row operations defined above, then we can recover A from B by applying the inverse elementary row operations on B in the reverse order. Therefore, the two systems of linear equations  $Ax = b_1$  and  $Bx = b_2$  have the same solution set. Thus, we can state the following theorem.

 Theorem 1. If two system of equations are row-equivalent, then they have same set of solutions.

#### Leading entry of a row:

First non zero entry (from the left) of a non zero row is called **leading entry** of that row.

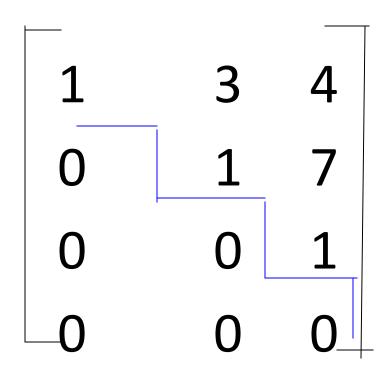
## Row Echelon Form (REF) of a matrix

- (i) All zero rows, if any, are at the bottom of matrix
- (ii) First nonzero entry (from the left) of a nonzero row is 1 (such a entry is called leading one of its row)
- (iii) If a column contains leading one of any row, then all other entries below the leading one in that column are zero
  - (iv)For each non-zero row, leading one comes to the right and below any leading one's in previous rows

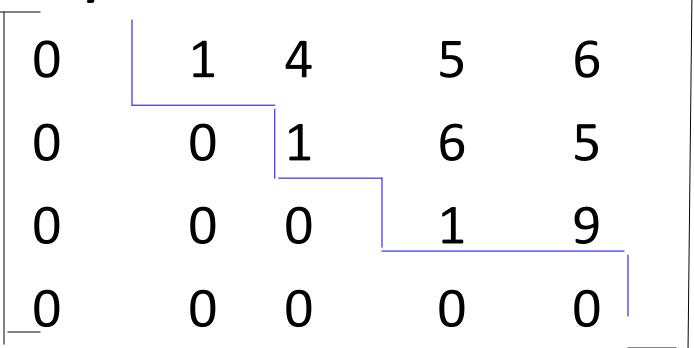
## Row- Reduced Echelon Form (RREF) of a matrix

- (i) All zero rows, if there are any, are at the bottom of matrix
- (ii) First nonzero entry (from the left) of a nonzero row is 1 (such a entry is called leading one of its row)
- (iii) If a column contains leading one of any row, then every other entries in that column is zero
  - (iv)For each non-zero row, leading one comes to the right and below any leading one's in previous rows

## Example-1: Echelon Matrix:



## Example-2: Echelon Matrix:



#### Ex: Row Reduced Echelon Matrix

0 1	3	0	0	4	0
0 0	0	1	0	3	0
0 0	0	0	1	2	0
0 0	0	0	0	0	1

#### Ex: Row Reduced Echelon Matrix

1. 
$$0_{m \times n} : m \times n$$
 zero matrix

2. 
$$I_n: n \times n$$
 identity matrix

3.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the starred entries (\*) may take any values including zero.

#### **Definitions:**

- **Pivot position:** is a position of a leading entry in an echelon form of the matrix.
- <u>Pivot:</u> a nonzero number that either is used in a pivot position to create 0's OR is changed into a leading 1, which in turn is used to create 0's.
- Pivot row/column: a row/column that contains a pivot position.

## PROBLEM: Find Row Reduced Echelon form of a Matrix

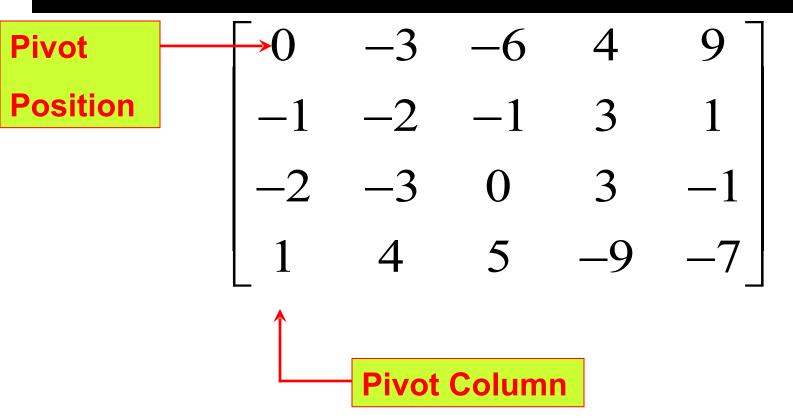
$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

• Step 1: Selecting Pivot column:
Begin with the leftmost nonzero column.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$
Pivot Column

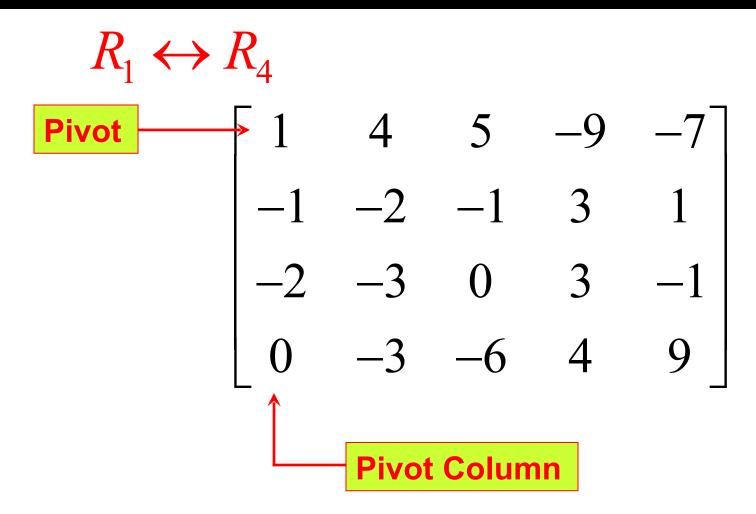
#### Step 2: Selecting Pivot Position:

Select a **nonzero entry** in the pivot column as a pivot. If necessary interchange rows to move this entry into the pivot position.

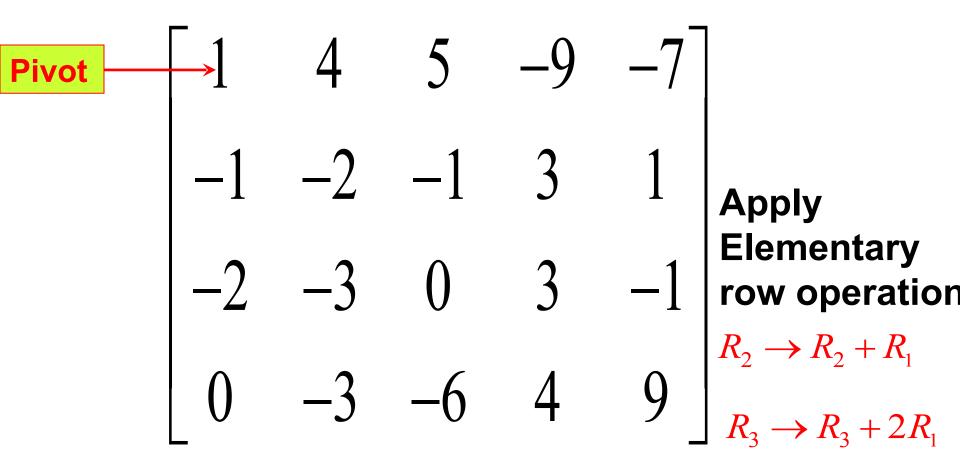


Step 2: (Continued):

#### Using the elementary row operation



 Step 3: Use elementary row operations to create zeros in all positions below the pivot.



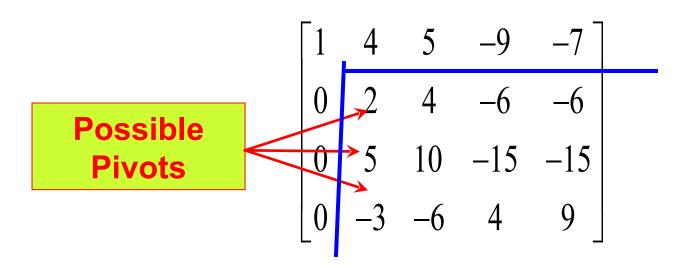
## Step 3: (Continued):

## After a few computations we get

	4	5	<b>-9</b>	-7
0	2	4	-6	-6 -15 9
0	5	10	-15	-15
0	-3	-6	4	9

• Step 4: Ignore the row containing the pivot position and cover all rows, if any, above it.

Apply steps 1-3 to the remaining submatrix. Repeat the process until there are no more nonzero rows to modify.



#### Let us take pivot as

#### Make this pivot element to 1 and then apply

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \end{bmatrix}$$

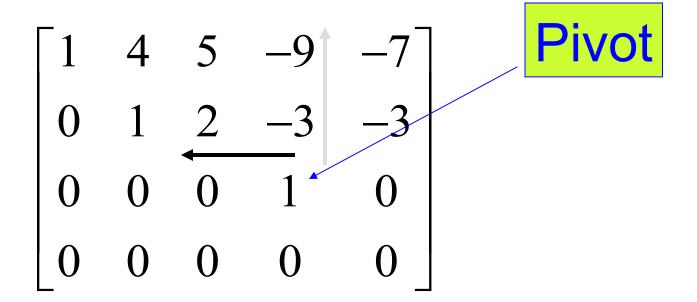
**Apply** 

 $R_3 \longleftrightarrow R_4$ 

And this pivot -5 to 1

Note: This is a row echelon form

 Step 5: Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.



$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 + 9R_3$$

$$\begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} Pivot$$

$$\begin{bmatrix}
1 & 4 & 5 & 0 & -7 \\
0 & 1 & 2 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
R_1 \rightarrow R_1 - 4R_2 \\
1 & 0 & -3 & 0 & 5 \\
0 & 1 & 2 & 0 & -3
\end{bmatrix}$$

Pivot as this is first non-zero element in 2<sup>nd</sup> row

 $\begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 

Reduced Row Echelon Form

# Results:

- 1. Echelon form of a matrix may not be unique.
- 2. Row reduced echelon form of a matrix is <u>unique</u>
- 3. Every matrix is row equivalent to its echelon form.
- 4. Every matrix is row equivalent to its row reduced echelon form