Note on Feynman integral

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The problem The integral we would like to evaluate is

$$
\begin{aligned}
K^{\pm} &= \int \frac{d^d l_1}{\pi^{d/2}} \int \frac{d^d l_2}{\pi^{d/2}} \frac{1}{P_1 \cdots P_5} \\
&= \int \frac{d^d l_1}{\pi^{d/2}} \int \frac{d^d l_2}{\pi^{d/2}} \frac{1}{(\boldsymbol{l}_1 \cdot \boldsymbol{u} - i\delta_1)(\pm \boldsymbol{l}_2 \cdot \boldsymbol{u} - i\delta_2)((\boldsymbol{l}_1 + \boldsymbol{l}_2 - \boldsymbol{q})^2 - i\delta_3)((\boldsymbol{l}_1 - \boldsymbol{q})^2 - i\delta_4)((\boldsymbol{l}_2 - \boldsymbol{q})^2 - i\delta_5)} \\
&= \int \frac{d^d l_1}{\pi^{d/2}} \int \frac{d^d l_2}{\pi^{d/2}} \int_0^\infty dx_1 \int_0^\infty dx_2 \int_0^\infty dx_3 \int_0^\infty dx_4 \int_0^\infty dx_5 \ \delta\left(1 - \sum_{i=1}^5 x_i\right) \frac{4!}{\left[\sum_{i=1}^5 x_i P_i\right]^5}, \qquad (0.1)
\end{aligned}
$$

with $d = 3 - 2\epsilon$ and δ_i positive infinitesimal. The momenta are Euclidean. The external momenta satisfy $\boldsymbol{u}^2 = \boldsymbol{q}^2 = 1$ and $\boldsymbol{u} \cdot \boldsymbol{q} = 0$. It is known that $K^{\pm}$ are different by a factor of 2

$$
K^+ = -\frac{4\pi}{3} \frac{\Gamma_{1+2\epsilon} \Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}} = -\frac{4\pi}{\epsilon^2} + \frac{8\pi\gamma_E}{\epsilon} + \left(\frac{2\pi^3}{3} - 8\pi\gamma_E^2\right) + \mathcal{O}(\epsilon) \simeq -\frac{12.6}{\epsilon^2} + \frac{14.5}{\epsilon} + 12.3 + \mathcal{O}(\epsilon), \qquad (0.2)
$$

$$
K^- = -\frac{2\pi}{3} \frac{\Gamma_{1+2\epsilon} \Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}} = -\frac{2\pi}{\epsilon^2} + \frac{4\pi\gamma_E}{\epsilon} + \left(\frac{\pi^3}{3} - 4\pi\gamma_E^2\right) + \mathcal{O}(\epsilon) \simeq -\frac{6.28}{\epsilon^2} + \frac{7.25}{\epsilon} + 6.15 + \mathcal{O}(\epsilon). \qquad (0.3)
$$

Here $\Gamma_a \equiv \Gamma(a)$. Pysecdec does not seem to reproduce this factor, but instead returns a wrong value for K^+. Note that the $\pm$ sign cannot be factored out from Eq. (0.1) because of the $-i\delta$ inside the propagators.

Calculation For each propagator we introduce the Feynman parameters $x_1, \cdots, x_5$. We rewrite the denominator as

$$
\sum_{i=1}^5 x_i P_i = \sum_{l,m=1}^2 \boldsymbol{l}_l M_{lm} \boldsymbol{l}_m + 2 \sum_{l=1}^2 \boldsymbol{l}_l \boldsymbol{Q}_l + J - i \sum_{i=1}^5 x_i \delta_i, \qquad (0.4)
$$

with

$$
M = \begin{pmatrix} x_3 + x_4 & x_3 \\ x_3 & x_3 + x_5 \end{pmatrix}, \qquad \boldsymbol{Q} = \begin{pmatrix} x_1 \boldsymbol{u}/2 - (x_3 + x_4)\boldsymbol{q} \\ \pm x_2 \boldsymbol{u}/2 - (x_3 + x_5)\boldsymbol{q} \end{pmatrix}, \qquad J = x_3 + x_4 + x_5. \qquad (0.5)
$$

The Symanzik polynomials become

$$
\mathcal{U} = \det M = x_3 x_4 + x_4 x_5 + x_5 x_3, \qquad (0.6)
$$

$$
\mathcal{F} = \det M \cdot (J - Q^T M^{-1} Q) = x_3 x_4 x_5 - \frac{1}{4}(x_1 \ \pm x_2) \begin{pmatrix} x_3 + x_5 & -x_3 \\ -x_3 & x_3 + x_4 \end{pmatrix} \begin{pmatrix} x_1 \\ \pm x_2 \end{pmatrix}. \qquad (0.7)
$$

We can complete the square as

$$
\sum_{i=1}^5 x_i P_i = \mathcal{U}^{1/2} \left[\sum_{l,m=1}^2 \tilde{\boldsymbol{l}}_l \tilde{M}_{lm} \tilde{\boldsymbol{l}}_m + \frac{\mathcal{F} - i\mathcal{U} \sum_{i=1}^5 x_i \delta_i}{\mathcal{U}^{3/2}} \right] = \mathcal{U}^{1/2} \left[\sum_{l,m=1}^2 \tilde{\boldsymbol{l}}_l \tilde{M}_{lm} \tilde{\boldsymbol{l}}_m + \frac{\mathcal{F} - i\delta}{\mathcal{U}^{3/2}} \right]. \qquad (0.8)
$$

with $\tilde{\boldsymbol{l}} \equiv \boldsymbol{l} + M^{-1}\boldsymbol{Q}$ being the shifted momentum and $\tilde{M} \equiv M/\mathcal{U}^{1/2}$ satisfying $\det \tilde{M} = 1$. Since $\mathcal{U}(= x_3 x_4 + x_4 x_5 + x_5 x_3)$ and x_i are positive, we rewrote the small imaginary part as $-i\delta$. The momentum integrations can be performed to give

$$
K^{\pm} = \Gamma_{2+2\epsilon} \int_0^\infty dx_1 \cdots dx_5 \ \delta\left(1 - (x_3 + x_4 + x_5)\right) \frac{\mathcal{U}^{1/2+3\epsilon}}{(\mathcal{F} - i\delta)^{2+2\epsilon}}. \qquad (0.9)
$$

Here we used the Cheng-Wu theorem so that x_1 and x_2 do not appear inside the δ function. In order to discuss K^+ and K^- in parallel, we define $y_2 \equiv -x_2$ for K^- with the integration range $(-\infty, 0]$. Note that this is just for convenience, in order to calculate $K^\pm$ in parallel analytically. Also note that the small imaginary part remains unaffected because each term in $\delta = \mathcal{U}\sum_{i=1}^{5} x_i\delta_i = \mathcal{U}[x_1\delta_1 + (-y_2)\delta_2 + x_3\delta_3 + x_4\delta_4 + x_5\delta_5]$ is still positive. Naming y_2 back to x_2, we can write the original integral as

$$K^\pm = \Gamma_{2+2\epsilon}\int_0^\infty dx_1 \left[\int_0^\infty dx_2 \quad (\text{for } K^+) \text{ or } \int_{-\infty}^0 dx_2 \quad (\text{for } K^-)\right]$$
$$\times \int_0^\infty dx_3 dx_4 dx_5\ \delta\left(1-(x_3+x_4+x_5)\right) \frac{(x_3x_4+x_4x_5+x_5x_3)^{1/2+3\epsilon}}{\left[x_3x_4x_5 - \frac{1}{4}\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}x_3+x_5 & -x_3\\ -x_3 & x_3+x_4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - i\delta\right]^{2+2\epsilon}}. \tag{0.10}$$

If we just would like to calculate the *sum* of K^+ and K^-, it is actually easy. In fact, we can replace the integration range with $(1/2)\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2$

$$K^+ + K^- = \Gamma_{2+2\epsilon} \times \frac{1}{2}\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \times (\text{second line of Eq. (0.10)}). \tag{0.11}$$

The logic for this replacement is as follows. Starting from Eq. (0.10), the integration range for $K^+ + K^-$ is $x_1 \in [0,\infty)$ and $x_2 \in (-\infty,\infty)$. Since the denominator is unchanged (including the $-i\delta$ term) under $(x_1,x_2) \leftrightarrow (-x_1,-x_2)$, we can extend the integration range to $x_1 \in (-\infty,\infty)$ and $x_2 \in (-\infty,\infty)$ with a factor of $1/2$. Then x_1 and x_2 can be integrated out, leaving a factor coming essentially from the eigenvalues of the 2×2 matrix in the denominator of Eq. (0.10)

$$\begin{aligned}
&K^+ + K^- \\
&= \frac{\Gamma_{2+2\epsilon}}{2}\int_0^\infty dx_3 dx_4 dx_5\ \delta\left(1-(x_3+x_4+x_5)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{1/2+3\epsilon}}{(x_3x_4x_5)^{2+2\epsilon}} \\
&\qquad \times \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \frac{1}{\left[1 - \frac{1}{4}\frac{1}{x_3x_4x_5}\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}x_3+x_5 & -x_3\\ -x_3 & x_3+x_4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - i\delta\right]^{2+2\epsilon}} \\
&= \frac{\Gamma_{2+2\epsilon}}{2}\int_0^\infty dx_3 dx_4 dx_5\ \delta\left(1-(x_3+x_4+x_5)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{1/2+3\epsilon}}{(x_3x_4x_5)^{2+2\epsilon}} \times \frac{-4\pi}{1+2\epsilon}\frac{x_3x_4x_5}{\sqrt{x_3x_4+x_4x_5+x_5x_3}} \\
&= -2\pi\frac{\Gamma_{1+2\epsilon}\Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}}.
\end{aligned} \tag{0.12}$$

Here $\frac{-4\pi}{1+2\epsilon}\frac{x_3x_4x_5}{\sqrt{x_3x_4+x_4x_5+x_5x_3}}$ is the factor coming from x_1 and x_2 integration. Note that the minus sign comes from the $-i\delta$ prescription: $\int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2\ \frac{1}{[1-(z_1^2+z_2^2)-i\delta]^{2+2\epsilon}} = -\frac{\pi}{1+2\epsilon}$ for $2+2\epsilon > 1$. Now, the simplest way to see that K^+ and K^- do not share the same value is to observe that there is no symmetry between the two. To see this more explicitly, consider the eigenvectors and the eigenvalues of the matrix in the denominator of (0.10)

$$\begin{pmatrix}x_3+x_5 & -x_3\\ -x_3 & x_3+x_4\end{pmatrix}. \tag{0.13}$$

For positive x_3, x_4, and x_5, this matrix has a smaller eigenvalue in the direction of the first (or third) quadrant, and it has a larger eigenvalue in the direction of the second (or fourth) quadrant (see Fig. 1). Thus K^+ and K^- are not equivalent. A more detailed calculation gives $(2/3) : (1/3)$ splitting between the two (see Appendix A).

Numerical results with pysecdec We try 16 choices for each of $K^\pm$, from which we summarize only relevant ones in Table 1. The details are summarized in Appendix B. The Symanzik variables are indeed the same as Eqs. (0.6) and (0.7). Since the default prescription of pysecdec is $+i\delta$, we invert the signs of the propagators to accommodate $-i\delta$. Writing explicitly, the propagators for each prescription are

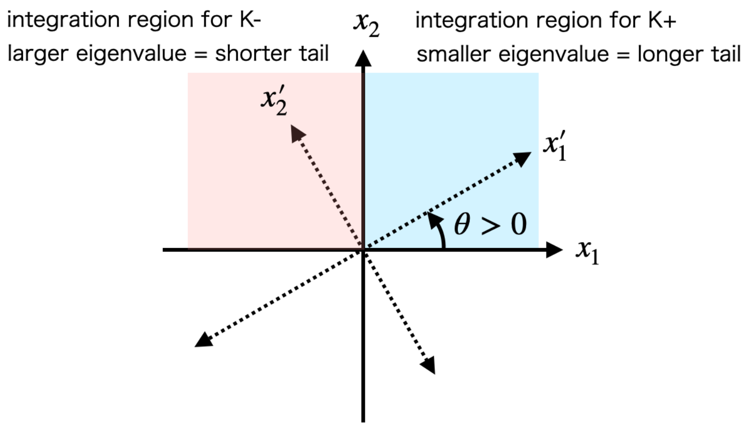

Figure 1: Illustration for how the difference between $K^{\pm}$ arises.

- K^+ $(+i\delta)$: `propagators = ['u*l1', 'u*l2', '(l1+l2-q)**2', '(l1-q)**2', '(l2-q)**2']`
- K^+ $(-i\delta)$: `propagators = ['-u*l1', '-u*l2', '-(l1+l2-q)**2', '-(l1-q)**2', '-(l2-q)**2']`
- K^- $(+i\delta)$: `propagators = ['u*l1', '-u*l2', '(l1+l2-q)**2', '(l1-q)**2', '(l2-q)**2']`
- K^- $(-i\delta)$: `propagators = ['-u*l1', 'u*l2', '-(l1+l2-q)**2', '-(l1-q)**2', '-(l2-q)**2']`

The numerical results are also shown in Table 1. For K^- they coincide with the analytic calculation (up to unimportant overall signs), while for K^+ they do not. Note that the 2:1 ratio between $K^{\pm}$ is already violated at the leading order ϵ^{-2}. As we discuss in Appendix B, pysecdec reproduces this 2:1 ratio for $d = 4 - 2\epsilon$. We also find that other seemingly incorrect replacement rules also give the correct value for K^-, though the signs in the Symanzik variables are different: see (3) K^- and (6) K^- in Table 5.

	dim.	$\pm i\delta$	rpl. rules	U, F from pysecdec
(1) K^+	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(8) K^+	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(1) K^-	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(8) K^-	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$

	ϵ^{-2}	ϵ^{-1}	ϵ^0
(1) K^+	6.29	$-97.7 + 8.39i$	$-7010 - 1330i$
(8) K^+	6.29	$-99.3 + 6.52i$	$-9470 + 1150i$
(1) K^-	6.29	-7.16	-5.51 (± 1.93)
(8) K^-	6.29	-7.18	-4.92 (± 2.02)

Table 1: Setup of pysecdec (top) and numerical results (bottom).

A Splitting between $K^\pm$

We derive the (2/3) : (1/3) splitting. Let x'_1 and x'_2 be the directions that diagonalize the matrix in the denominator

$$\frac{1}{4}\frac{1}{x_3x_4x_5}\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}x_3+x_5 & -x_3\\ -x_3 & x_3+x_4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}x'_1 & x'_2\end{pmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}x'_1\\ x'_2\end{pmatrix}, \tag{A.14}$$

with a rotation

$$\begin{pmatrix}x'_1\\ x'_2\end{pmatrix} = \begin{pmatrix}c_\theta & s_\theta\\ -s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}. \tag{A.15}$$

We take $-\pi/4 < \theta < \pi/4$ so that x'_1 and x'_2 are mostly x_1 and x_2, respectively. We get

$$\sin 2\theta = \frac{1}{2}\frac{1}{\lambda_2-\lambda_1}\frac{x_3}{x_3x_4x_5}, \qquad \cos 2\theta = \frac{1}{4}\frac{1}{\lambda_2-\lambda_1}\frac{x_4-x_5}{x_3x_4x_5}, \qquad \tan 2\theta = \frac{2x_3}{x_4-x_5}. \tag{A.16}$$

The eigenvalues satisfy

$$\lambda_1+\lambda_2 = \frac{1}{4}\frac{(x_3+x_5)+(x_3+x_4)}{x_3x_4x_5}, \qquad \lambda_1\lambda_2 = \frac{1}{16}\frac{x_3x_4+x_4x_5+x_5x_3}{(x_3x_4x_5)^2}. \tag{A.17}$$

Taking K^+ as an example, the integral in question can be written as

$$\begin{aligned}
&\int_0^\infty dx_1 \int_0^\infty dx_2 \frac{1}{\left[1-\frac{1}{4}\frac{1}{x_3x_4x_5}\begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}x_3+x_5 & -x_3\\ -x_3 & x_3+x_4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} - i\delta\right]^{2+2\epsilon}}\\
&= \int dx'_1 \int dx'_2 \frac{1}{[1-\lambda_1 x_1'^2 - \lambda_2 x_2'^2 - i\delta]^{2+2\epsilon}} = \frac{1}{\sqrt{\lambda_1\lambda_2}}\int dx''_1 \int dx''_2 \frac{1}{[1-(x_1''^2+x_2''^2)-i\delta]^{2+2\epsilon}}.
\end{aligned} \tag{A.18}$$

Here, the integration range for x'_1, x'_2 is the light-blue region in Fig. 2, and we defined $x''_1 \equiv \sqrt{\lambda_1}x'_1$ and $x''_2 \equiv \sqrt{\lambda_2}x'_2$ in the last equality. The point is that the integration range for x''_1 and x''_2 is not square any more, as we see below. Since x_1 and x_2 have the same integration range $[0,\infty)$ for K^+, we can take $x_4 > x_5$ without losing generality, meaning $\lambda_2 > \lambda_1$ (see Eq. (A.14), and note that x'_1 and x'_2 are mostly x_1 and x_2, respectively). Let us define the angle $\alpha^\pm$ as in Fig. 2. These angles are no more $\pi/2$ but are calculated to be

$$\alpha^+ = \pi - \arctan\frac{\sqrt{x_3x_4+x_4x_5+x_5x_3}}{x_3}, \qquad \alpha^- = \arctan\frac{\sqrt{x_3x_4+x_4x_5+x_5x_3}}{x_3}. \tag{A.19}$$

These angles are the same for $x_4 < x_5$ as well. Therefore, $K^\pm$ is modified from the naive (1/2) : (1/2) splitting as

$$K^\pm = -\pi\Gamma_{1+2\epsilon}\int_0^\infty dx_3dx_4dx_5\ \delta\left(1-(x_3+x_4+x_5)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}} \times \frac{2}{\pi}\alpha^\pm. \tag{A.20}$$

The integration involving arctan is calculable, referring to Appendix A of Ref. [1]. The essence is the property of arctan for $x_3x_4+x_4x_5+x_5x_3 = 1$:

$$\begin{aligned}
&\arctan x_3 + \arctan x_4 + \arctan x_5\\
&= \mathrm{Im}\left[\ln(1+ix_3)+\ln(1+ix_4)+\ln(1+ix_5)\right]\\
&= \mathrm{Im}\left[\ln(1+i(x_3+x_4+x_5)-(x_3x_4+x_4x_5+x_5x_3)-ix_3x_4x_5\right]\\
&= \mathrm{Im}\left[\ln(i(x_3+x_4+x_5-x_3x_4x_5))\right]\\
&= \frac{\pi}{2}.
\end{aligned} \tag{A.21}$$

This and the Cheng-Wu theorem give

$$\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3+x_4+x_5)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\ \arctan\left(\frac{\sqrt{x_3x_4+x_4x_5+x_5x_3}}{x_3}\right)$$
$$=2\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3x_4+x_4x_5+x_5x_3)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\ \arctan\left(\frac{\sqrt{x_3x_4+x_4x_5+x_5x_3}}{x_3}\right)$$
$$=2\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3x_4+x_4x_5+x_5x_3)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\ \arctan\left(\frac{1}{x_3}\right)$$
$$=2\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3x_4+x_4x_5+x_5x_3)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\left[\frac{\pi}{2}-\arctan x_3\right]$$
$$=2\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3x_4+x_4x_5+x_5x_3)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\left[\frac{\pi}{2}-\frac{1}{3}(\arctan x_3+\arctan x_4+\arctan x_5)\right]$$
$$=2\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3x_4+x_4x_5+x_5x_3)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}\left[\frac{\pi}{2}-\frac{1}{3}\cdot\frac{\pi}{2}\right]$$
$$=\frac{\pi}{3}\int_0^\infty dx_{3\ldots5}\ \delta\left(1-(x_3+x_4+x_5)\right)\ \frac{(x_3x_4+x_4x_5+x_5x_3)^{3\epsilon}}{(x_3x_4x_5)^{1+2\epsilon}}$$
$$=\frac{\pi}{3}\frac{\Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}}. \tag{A.22}$$

Finally we get

$$K^+=-\frac{4\pi}{3}\frac{\Gamma_{1+2\epsilon}\Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}},\qquad K^-=-\frac{2\pi}{3}\frac{\Gamma_{1+2\epsilon}\Gamma_{-\epsilon}^3}{\Gamma_{-3\epsilon}}. \tag{A.23}$$

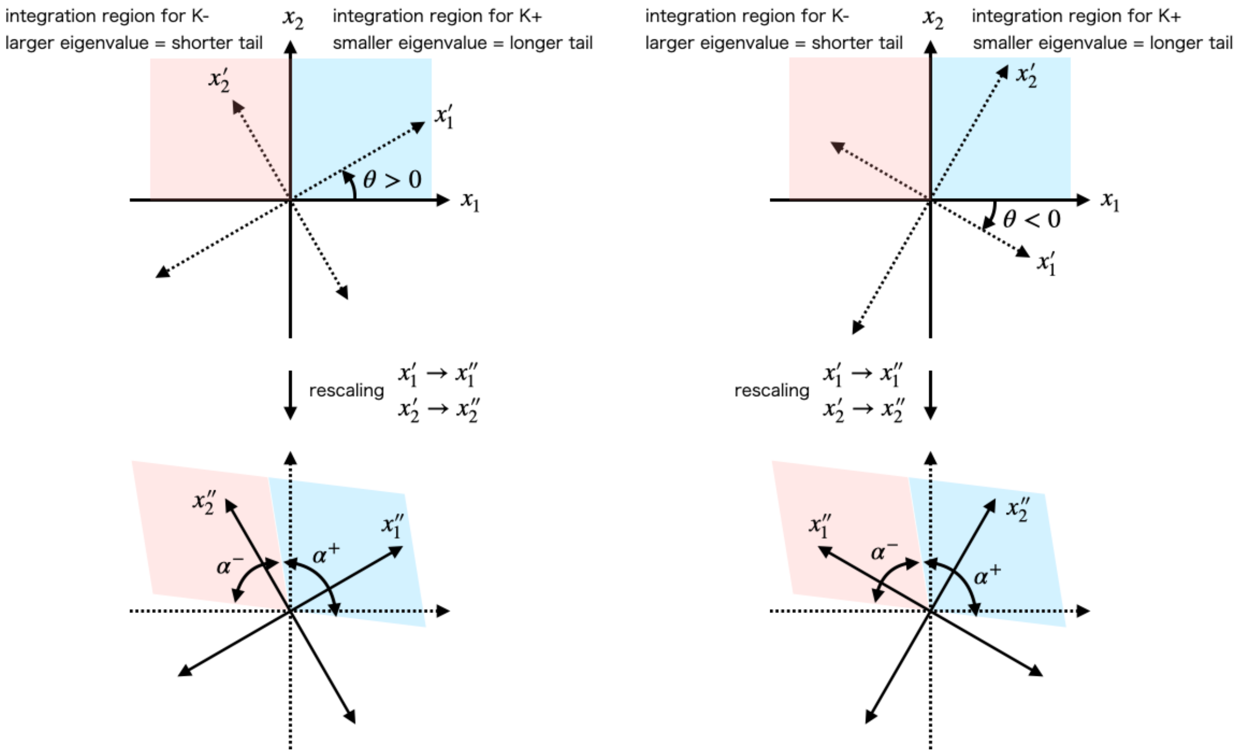

Figure 2: Rotation from (x_1, x_2) to (x_1', x_2') and rescaling to (x_1'', x_2'').

B Numerical check

We summarize our setups for $K^{\pm}$ (Table 2 and 3) and the results from these setups (Table 4 and 5). Small imaginary parts are neglected, and errors are put only when they are relevant. Note that the 2:1 ratio between $K^{\pm}$ is reproduced for $d = 4 - 2\epsilon$.

	dim.	$\pm i\delta$	rpl. rules	U, F from pysecdec
(1) K^+	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(2) K^+	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(3) K^+	$3-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(4) K^+	$3-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(5) K^+	$3-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(6) K^+	$3-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(7) K^+	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(8) K^+	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(9) K^+	$4-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(10) K^+	$4-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(11) K^+	$4-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(12) K^+	$4-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(13) K^+	$4-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(14) K^+	$4-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(15) K^+	$4-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(16) K^+	$4-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$

Table 2: Setup of pysecdec for K^+.

	dim.	$\pm i\delta$	rpl. rules	U, F from pysecdec
(1) K^-	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(2) K^-	$3-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(3) K^-	$3-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(4) K^-	$3-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(5) K^-	$3-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(6) K^-	$3-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(7) K^-	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(8) K^-	$3-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(9) K^-	$4-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(10) K^-	$4-2\epsilon$	$+$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(11) K^-	$4-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(12) K^-	$4-2\epsilon$	$+$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(13) K^-	$4-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(14) K^-	$4-2\epsilon$	$-$	$(u,u)=-1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 + \frac{1}{4}x_1^2x_3 + \frac{1}{4}x_1^2x_2 + \frac{1}{2}x_0x_1x_2 + \frac{1}{4}x_0^2x_4 + \frac{1}{4}x_0^2x_2$
(15) K^-	$4-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=-1$	$F = -x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$
(16) K^-	$4-2\epsilon$	$-$	$(u,u)=+1$	$U = x_3x_4 + x_2x_4 + x_2x_3$
			$(q,q)=+1$	$F = x_2x_3x_4 - \frac{1}{4}x_1^2x_3 - \frac{1}{4}x_1^2x_2 - \frac{1}{2}x_0x_1x_2 - \frac{1}{4}x_0^2x_4 - \frac{1}{4}x_0^2x_2$

Table 3: Setup of pysecdec for K^-.

	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
(1) K^+	6.29	$-97.7 + 8.39i$	$-7010 - 1330i$
(2) K^+	-6.25	$118 - 42.2i$	$-296 + 368i$
(3) K^+	-6.25	$123 + 2.53i$	$-398 + 56.5i$
(4) K^+	6.29	$-111 + 33.4i$	$-8300 + 1160i$
(5) K^+	6.29	$-111 + 33.7i$	$-8030 + 786i$
(6) K^+	-6.29	$122 + 2.54i$	$-421 + 65.6i$
(7) K^+	`nan`	`nan`	`nan`
(8) K^+	6.29	$-99.3 + 6.52i$	$-9470 + 1150i$
(9) K^+	0	13.1	63.4
(10) K^+	0	13.1	$63.7 + 82.2i$
(11) K^+	0	-13.1	-63.7
(12) K^+	0	-13.1	$-64.0 - 81.5i$
(13) K^+	0	-13.1	$-64.0 - 81.5i$
(14) K^+	0	-13.1	-63.7
(15) K^+	0	13.1	$63.7 + 82.2i$
(16) K^+	0	13.1	63.4

Table 4: Output of pysecdec for K^+.

	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
(1) K^-	6.29	-7.16	-5.51 (± 1.93)
(2) K^-	-6.25	$7.30 - 38.8i$	$130 + 46.2i$
(3) K^-	-6.25	7.32	6.29 (± 0.50)
(4) K^-	6.29	$-7.15 + 39.8i$	$-129 - 45.3i$
(5) K^-	6.29	$-7.15 + 39.8i$	$-129 - 45.3i$
(6) K^-	-6.25	7.32	6.29 (± 0.50)
(7) K^-	-6.25	$7.30 - 38.8i$	$130 + 46.2i$
(8) K^-	6.29	-7.18	-4.92 (± 2.02)
(9) K^-	0	6.58	31.9
(10) K^-	0	6.58	$31.9 + 41.4i$
(11) K^-	0	-6.58	-31.9
(12) K^-	0	-6.58	$-31.9 - 41.3i$
(13) K^-	0	-6.58	$-31.9 - 41.3i$
(14) K^-	0	-6.58	-31.9
(15) K^-	0	6.58	$31.9 + 41.4i$
(16) K^-	0	6.58	31.9

Table 5: Output of pysecdec for K^-.

References

[1] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "The Eikonal Approach to Gravitational Scattering and Radiation at $\mathcal{O}(G^3)$," arXiv:2104.03256 [hep-th].